COLUMBIA
BUSINESS
SCHOOL

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B9801-33: Stochastic Processing Networks
409 Uris Hall

Homework #3 – Dynamic control of fluid models (Due 4/2/99)

1. *Optimality of $c\mu$ rule.* Consider a single server system with K classes of jobs with no feedback; that is, once a class k job completes service it exits the system ($P = 0$). Each class has its own exogeneous renewal arrival process with arrival rate α_k. Service times for class k jobs are i.i.d. according to some general distribution with mean $m_k = 1/\mu_k$. There is a linear holding cost c_k for class k jobs, and you may assume that system parameters are chosen and classes are ordered so that $c_1\mu_1 > c_2\mu_2 > \cdots > c_K\mu_K$.

   (a) What are the fluid model equations for the system?

   (b) Consider the performance criterion
   $$\int_0^T c' z(t) dt.$$
   Show that starting from any initial condition $z(0) = q$, the $c\mu$ rule that assigns priorities $1 > 2 > \cdots > K$ (i.e., according to $c_k\mu_k$) is optimal.

   (c) Show that the $c\mu$ rule is "greedy" in that $v(t)$ is chosen at any point in time in order to minimize $c'\dot{z}(t)$.

2. *Greed: some good news and some bad news.* Consider now a general multiclass network (as described in section 1.2 of Dai's notes) under the greedy control policy
   $$v(t) \in \text{argmin}\{c'\dot{z}(t) : v \in \mathcal{V}(z(t))\}.$$

   (a) Prove that this policy is stable.

   (b) Consider now a decentralized version of this policy where each station applies a variant of this greedy control law based only on queue lengths information of the classes served at that station. This is the $c\mu$ rule applied at each station. Construct a counterexample that illustrates that this policy can be unstable.

   Remark: This problem illustrates the difficulty in designing decentralized control policies; the most natural alternative –i.e., each station selects the optimal policy when considered in isolation– is unstable.

3. *Large deviations estimates.*

   (a) Let $N(t)$ be a Poisson arrival process with rate $\alpha > 1$. Show that

   $$P(N(t) \leq t) \leq e^{-th(\alpha)}.$$

   Find the function in the exponent $h(\alpha)$.

   (b) The sequence $v(i)$ denotes i.i.d. service time uniform random variables in $[.75, 1.25]$; in this system there is only one job class. Given a processing plan for p jobs over the current review period of length l, let T_{exe} denote the actual execution time for this plan; assume that it is executed from jobs present at the beginning of the review period. Provide a bound for the probability that $P(T_{exe} > (1+\epsilon)l)$.

   Hint. Use the value for p that yields the worse upper bound.

4. *Airline yield management.* An airline operates a network of flights as follows. It has a hub airport 0. The flights are $(i, 0)$ from city i to the hub and (i, j) via the hub, i.e., (i, j) represents two connecting flights $(i, 0)$ and $(0, j)$. Assume that there is a time horizon T. Customers that travel from i to j and pay full fare arrive at a rate $\lambda_{ij}^{f}(t)$. Economy fare customers arrive at a rate $\lambda_{ij}^{e}(t)$. The fares are r_{ij}^{f} and r_{ij}^{e} respectively. Assume that the capacity of flight $(i, 0)$ is C_{i0} and of flight $(0, j)$ is C_{0j}. Cancellation happen at a rate $\mu_{ij}^{f}(t)$ and $\mu_{ij}^{e}(t)$. The question is to decide how many economy customers will be accepted in order to maximize revenue within the time horizon $(0, T)$. Formulate the fluid control problem.

5. *Multiclass communication networks.* Consider a loss network that is represented by a complete directed graph $G = (V, A)$ with $N = |V|$ nodes and $|A| = N(N-1)$ ordered pairs of nodes (i, j). Calls of type (i, j) needs to be routed from node i to node j and carry a reward w_{ij}. Arriving calls of type (i, j) may be routed directly on link (i, j) or on a route $r \in R(i, j)$ (a path in G), where $R(i, j)$ is the set of alternative routes for calls of type (i, j). Note that if the network is not fully connected then the missing links have capacities $C_{ij} = 0$. Let $S(i, j) = \{(i, j)\} \cup R(i, j)$ be the set of routes for calls of type (i, j). When a call of type (i, j) arrives at node i, it can be routed through route r only if there is at least one free circuit on each link of the route. If it is accepted it generates a revenue w_{ij} and simultaneously it holds one circuit on each link on the route r for the holding period of the call. Incoming calls of type (i, j) arrive at the network according to a Poisson process λ_{ij}, while their holding period is asusmed exponentially distributed with rate μ_{ij} and independent of earlier arrivals and holding times. The problem is to find an admission control and dynamic routing policy to maximize the total expected reward in steady state.

   (a) Formulate the fluid optimal control problem.

   (b) How would you translate the admission control and routing decisions of the fluid control policy in the stochastic network?

Problems 4 and 5 are courtesy of Professor Bertsimas from a networks course he taught in the past.