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DERIVATIVES AND THE EFFICIENT ALLOCATION OF PRICE RISKS IN A GENERAL EQUILIBRIUM WORLD

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Derivatives and the Efficient Allocation of Price Risks in a General Equilibrium World

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Abstract

We establish an extension of the classical general equilibrium treatment of uncertainty to the phenomenon of price uncertainty. Traders do not know the prices at which trade will occur, but have expectations over possible prices. They trade derivatives, price-contingent securities, to insure against the risks arising from this uncertainty. We establish three results: one is a set of necessary and sufficient conditions for the existence of equilibrium, called an equilibrium with price insurance, in such a framework; another is the fact that equilibria with price insurance are Pareto efficient and agents insure themselves optimally against the price uncertainty represented by their price expectations; and finally we show that in this framework agents' price expectations matter, in the sense that they affect the equilibrium allocation of resources. Completeness of the securities markets requires an uncountable number of securities, one contingent on each possible goods price vector.

Key Words: derivatives, price uncertainty, endogenous uncertainty, general equilibrium, Hilbert space, price expectations.

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1 Introduction

Uncertainty about future prices has a real welfare cost, just as does uncertainty about exogenous states such as the weather: future prices may place a great value on our endowments or leave us impoverished. There is naturally a demand for insurance against this uncertainty. Many markets exist to meet this demand: options and futures and swap markets, indeed derivative markets in general, are used to insure against price uncertainty. In advanced economies uncertainty about prices is probably more extensive and more costly than uncertainty about states of nature. In this respect reality has been ahead of economic theory: until recently there have been few attempts to incorporate this type of uncertainty within the mainstream of resource allocation theory and in particular into the classical general equilibrium model, although in the finance literature there has been extensive discussion of the use of derivatives to hedge price risks. Recent work has suggested that in many respects uncertainty about prices and other endogenous variables is fundamentally di¤erent from uncertainty about exogenous states of nature.[1] We show here that there is in fact a very natural extension of the classical general equilibrium framework to price uncertainty, involving the use of derivative contracts conditioned on the price index. This extension provides a satisfactory account of the welfare role of derivative markets in hedging price uncertainty in a complete general equilibrium framework.

We consider an exchange economy augmented by the possibility of trading price contingent securities which pay a specified sum if and only if the equilibrium price vector assumes a specified value (called PCSs).[2] Prices play the role played by "states of nature" in the Savage framework, so the determination of the state is endogenous. Agents have expectations about possible prices, and trade price contingent securities to insure price risks. They use these securities to exchange income in price states in which they are well off for income in states in which they are badly off. Before an equilibrium price emerges agents trade these securities, shifting income across price states. This changes their endowments in the exchange economy, which now become price contingent.

[1] For more details, see Chichilnisky [7], Chichilnisky [6], Chichilnisky and Heal [12], Chichilnisky Dutta and Heal (CDH) [10], Hahn [19], Chichilnisky Heal Streufert and Swinkles [16] and Chichilnisky Heal and Tsomocos [15] and the references therein.

[2] Such securities were discussed in the context of a temporary equilibrium model by Svensson [28], and earlier in a growth model by Stigum [27]. More recently they have been discussed inWalrasian general equilibrium models by the authors cited above, and have been discussed as well as by Kurz [23], Kurz and Wu [24] and Henrotte [21]. None of these latter authors consider these instruments in a standard Walrasian model. Kurz and Wu, for example, use an overlapping generations model. An approach within the mainstream general equilibrium framework is in Detemple and Selden [17], which considers the interactions between markets for securities and for options on those securities.

We derive an indirect utility function relating agents' expected utilities from equilibrium consumption to their holdings of securities: agents select their portfolios of securities so as to maximize this function. Together with securities prices, this generates demands for and supplies of securities: the net endowment is zero. Market clearing prices in the securities markets equate supply and demand. Then, given portfolios of price contingent securities which determine endowments in each price state, agents trade goods and services. Here they are behaving as in a standard exchange economy except for the dependence of endowments on prices.[3]

An equilibrium of the entire system, called an equilibrium with price insurance, is an equilibrium in both the goods and the PCS markets. We establish the following results: (1) a set of conditions necessary and sufficient for the existence of such an equilibrium,[4] (2) that the equilibria are Pareto effocient and provide agents with insurance against the uncertainty represented by their price expectations and (3) that agents' price expectations influence the equilibrium allocation of resources. This is of course in the context of a complete set of markets, and so is different from the phenomena which arise in models with sunspots or temporary equilibria. In section 6 we give two examples which illustrate the role of price expectations in determining equilibrium prices.

The message from the earlier papers in this area is that using prices as states and trading contingent on such states leads to a more complex and richer set of possibilities than trading contingent on purely exogenous variables, the traditional "states of nature". One cannot trade in the same market both goods and securities contingent on the prices of those goods:[5] there has to be an arbitrage-free structure to trade.[6] If goods and contracts contingent on their prices are traded in the same markets, then arbitrage will prevent the securities markets from playing a positive role. Once the prices of goods are announced by an auctioneer, then it is immediately clear that only certain price-contingent securities can have non-zero prices. Chichilnisky [6] shows that this impossibility goes deep: she shows that asking whether a market can insure against the risks which it produces, is like asking Russel's question about whether the set of all sets is a member of itself: it leads to paradoxes which are resolved only by careful use of category theory. To avoid this, one needs to separate the markets and prevent arbitrage. In real markets, options and other PCSs are traded before goods are traded: this is an example of such an arbitrage-free structure.

[3] For an earlier discussion of equilibrium when endowments depend on prices, see Chichilnisky and Heal [13].

[4] The conditions we give for the existence of such an equilibrium are based on the necessary and su¢cient conditions for existence of a competitive equilibrium introduced by Chichilnisky, "limited arbitrage", [3] [5] [4] and their subsequent extension by Chichilnisky and Heal in [14] to infinite dimensional economies. The equilibria of this economy need not of course be unique, in contrast to the outcome in Chichilnisky Dutta and Heal [10].

[5] See Chichilnisky [6] and Chichilnisky Hahn and Heal [11].

[6] As in Arrow [1] or Radner [25].

2 The model

The economy $\mathcal{E}$ which we consider can be defined formally as follows. There are n markets for goods and services, and in addition a complete set of markets for price contingent securities, i.e., markets for securities which will pay a unit of account contingent on the realization of any goods price vector in the simplex $\Delta \subset \Re^n_+$. A portfolio of securities is an element of the Hilbert space L_2 of functions from the price simplex to the real line: it assigns to each price vector a value. Securities prices are elements of the dual to the Hilbert space L_2 of functions from the price simplex to the real line. As this space is self-dual, both portfolios and securities prices are in L_2.

There are I agents, $i \in I$. Agents are characterized by endowments - $_i \in \Re^n_+$, preferences $u_i : \Re^n_+ \to \Re$, price expectations $e_i : 2^{¢} \subset \Re^n_+ \to \Re$, and portfolios of PCSs. Here $2^{¢}$ is the σ-algebra of subsets of Δ and $e_i(p)$ is a probability measure representing agent i's expectations over possible prices in the simplex. $e_i(p)$ is measurable with respect to the σ-algebra. Consumption sets are the non-negative orthant $\Re^n_+$ for goods and L_2 for securities. So unbounded short sales are allowed for securities but not for commodities. This seems a natural combination, although all other possible combinations could equally well be covered.

Agents are unsure of the prices at which trade in goods in $\Re^n$ will occur. They face no other uncertainty. Prior to trading goods they trade price-contingent securities (PCSs) which allow them to shift income between alternative states, where states are defined as possible trading prices in the goods markets. Securities pay in units of numeraire, and the securities market is complete in that it is possible to trade contingent on any price vector in the simplex. The payoff from PCSs purchased by an agent i if trade in the goods market occurs at price vector p is $s_i(p)$ **Q** 0, where we use s_i to denote the holdings by agent i of price contingent securities. The securities traded have payoffs contingent on price vectors, and thus (by Lemma 1 section VII of [10]) can be interpreted as contingent on a price index, and indeed are a form of "exotic option" (Rubinstein [26]).

The utilities are assumed to be continuous, strictly concave and to satisfy the following condition of uniformly bounded rates of increase:

Assumption 1 Each trader i has a preference represented by a strictly concave continuously differentiable function $u_i : N(\Re^n_+) \to \Re$ (where $N(\Re^n_+)$ is a neighborhood of $\Re^n_+$) such that $u_i(0) = 0$, $\sup_{x \in \Re^n_+} u_i(x) = \infty$, and $\exists \epsilon, \delta > 0$ such that $\forall x \in N(\Re^n_+)$:

$$0 < \epsilon \leq \|Du_i(x)\| \leq \delta$$

where $Du_i(x)$ is the ...rst derivative of the function u_i at x.

Assumption 2 If an indifference surface through a positive consumption bundle c intersects a boundary ray r of $\Re^n_+$, then every indifference surface containing points preferred to c also intersects the ray r.

Assumption 2 is satisfied by many standard preferences on $\Re^n_+$, such as Cobb-Douglas, CES, linear preferences, most Leontief preferences, and smooth utilities defined on a neighborhood of the positive orthant which are transversal to its boundary.

3 Behavior

3.1 The goods market

Consider first the goods market with a known price vector $p \in \Delta$. Agent i solves the following problem

$$\max u_i(c_i) \text{ subject to } p.c_i \leq p.\text{-}_i + s_i(p) \tag{1}$$

Here with a given price vector utility is maximized subject to a budget constraint of the standard type augmented by the payo¤ from PCSs: these securities have been purchased to move income across price states. Denote the maximum of utility subject to this constraint by $w_i(p, s_i(p), \text{-}_i)$. As -_i will be considered fixed in the following, we suppress dependence on this and write

$$w_i(p, s_i(p)) = \max u_i(c_i) : p.c_i \leq p.\text{-}_i + s_i(p) \tag{2}$$

$w_i(p, s_i(p))$ denotes the maximum utility attainable if the goods market price is p, as a function of the payoff in this state from PCSs purchased by agent i. It is a "securities utility function" relating welfare to prices and holdings of PCSs. It can be written $w_i^p(s_i(p))$ to show that it can also be interpreted as a state-dependent utility function when the states are defined by price vectors. Clearly this is a variant on the standard concept of an indirect utility function, used to reflect the dependence of utilities on the portfolios of securities chosen as well as on market prices. The role of wealth in the standard construction is augmented by the payoffs from price-contingent securities.

3.2 The securities market

As the price p is perceived as a random variable, the agent seeks to pick securities so as to maximize the expected utility derived from trading goods, given the wealth resulting from securities holdings, i.e., to maximize with respect to securities portfolios the expectation of $w_i(p, s_i(p))$ over all possible prices p. The expectation is with respect to the probability distribution representing the agents' expectations of prices. Agents therefore maximize

$$W_i(s_i(p)) \equiv \int_{p \in ¢} w_i^p(s_i(p))\, de_i$$

This is the expected utility from consumption in the goods market, given the agent's price expectations and holdings of securities. This maximization is with respect to the function $s_i(p) : \Delta \to \Re$ defining the portfolio of price-contingent securities held by the agent as a result of transaction in PCSs. The constraint on the choice of price-contingent securities is a budget constraint. Let $\pi(p)$ be the price of a security which pays a unit of account if and only if the goods price vector is p: as agents' initial endowments of PCSs are zero, the budget constraint is now $\int_{¢} \pi(p) s_i(p) dp \equiv \langle \pi(p), s_i(p) \rangle = 0$ where $\langle , \rangle$ denotes the inner product of a portfolio vector with a price vector in the space of portfolios and securities prices, see below. In addition, an agent cannot sell more income in a state than he or she has in that state: $p.\text{-}_i + s_i(p) \geq 0$. This is a "no short sales" condition. Hence the agent's optimal choice of PCSs solves the problem

$$\left.\begin{array}{c} \max\limits_{s_i(p)} \int_{p \in ¢} w_i(p, s_i(p)) de_i \\ \text{subject to} \int_{¢} \pi(p) s_i(p) dp = 0 \text{ and } p.\text{-}_i + s_i(p) \geq 0. \end{array}\right\} \tag{3}$$

This is a generalization to a high dimensional domain of a classic problem in the calculus of variations, an isoperimetric problem.[7] A solution to such a problem is a set of holdings of PCSs, i.e. a portfolio of securities, which transfers income between price states so as to maximize the expected utility from consuming goods and services. Let $s_i^*(p)$ be the solution: then in the goods market the agent's optimal behavior solves

$$\max u_i(c_i) \text{ subject to } p.c_i \leq p.\text{-}_i + s_i^*(p) \tag{4}$$

4 The economy

Agents' portfolios of price contingent securities are functions $s_i(p)$ from the price simplex to the real line, indicating the amount of PCS purchased for each possible price vector in the simplex: $s_i(p) : \Delta \to \Re$. They are solutions to the variational problem (3). As such, they satisfy di¤erential equations, the Euler conditions [18], and so are C^1 functions from $\Delta \subset \Re_+^n$ to $\Re$. The space of portfolios of PCSs therefore has to contain C^1 functions from $\Delta \subset \Re_+^n$ to $\Re$. We assume this space to be L_2, the Hilbert space of measurable square integrable functions from $\Delta \in \Re_+^n$ to the real line. A Hilbert space has the norm

$$\|s\| = \left[\int_{¢} s(p)^2 dp\right]^{1/2}$$

and inner product $\langle x, y \rangle = \left[\int_{¢} x(p) y(p) dp\right]$. Prices of price contingent securities are then the dual to the portfolio space: if π is a set of PCS prices, then π is a continuous linear function on the space of PCSs, so that $\pi : L_2 \to \Re$ and π is in the dual of L_2,

[7] For a characterization of the solutions of such problems, see Gelfand and Fomin [18], chapter 7.

which is self dual, so that securities prices lie in the same space as securities. We can therefore use the inner product of a portfolio vector and a price vector to give the value of a portfolio.

The expected utility from securities, W_i, is assumed to satisfy conditions equivalent to Assumption 1.

Assumption 3 The indirect utility function $w_i^p(s_i(p))$ is normalized so that the expected utility $W_i(s_i(p))$ satis...es $W_i(0) = 0$ and $\sup_{s \in L_2} W_i = \infty$.[8]

Assumption 4 The expected indirect utility function W_i satisfies one of the following two conditions: either each indifference surface is bounded below or the set of supports to each indifference surface is a closed set.

Note the following:

(1) by standard arguments for each agent i, the real-valued function of a real variable $w_i^p(s_i(p))$ is concave in s_i.

(2) from Lemma 2 of Chichilnisky and Heal [14] (see also [9]) $W_i(s)$ has a uniformly bounded rate of increase, i.e., $\forall \epsilon > 0 \exists N, \delta \geq 0$ such that $\forall s \ \exists y(s) : W_i(s+y) \geq W_i(s) + \epsilon\delta$, $\|y(s)\| \leq \epsilon$, and $W_i(s+y) \geq W_i(s) + \epsilon\delta \Rightarrow \|y\| \geq \epsilon/N$. This condition of a uniformly bounded rate of increase is equivalent to that in Assumption 1, but applied to a function which is not necessarily differentiable.

We will use extensively the condition of limited arbitrage introduced in Chichilnisky [3] and developed most extensively in [5] and [14]. This condition will be applied in the goods markets, where it will be called limited goods arbitrage, and in the securities markets, where it will be called limited securities arbitrage. The definitions for the goods market follow (for a discussion see Chichilnisky [3] or [5]):

Definition 1 The goods global cone A_i of agent i consists of all directions in the space of goods along which utility increases without bound:

$$A_i(\text{-}_i) = \{z \epsilon \Re^n : \forall y \in \Re^n, \exists \lambda > 0 : u_i(\text{-}_i + \lambda z) > u_i(y)\}$$

Definition 2 The cone G_i is the set of directions in the space of goods along which utility never ceases to increase:

$$G_i(\text{-}_i) = \{z \in \Re^n : \forall \mu \exists \lambda : u_i(\text{-}_i + \lambda z) > u_i(\text{-}_i + \mu z) \text{ if } \lambda > \mu \geq 0\}$$

Under assumption 1 the cone $G_i(\text{-}_i)$ contains the cone $A_i(\text{-}_i)$ and in addition contains some of the boundary rays of $A_i(\text{-}_i)$.

[8] The exact value of the common supremum of all securities utility functions EW_i does not matter: it is taken to the infinity for definitiveness only. What matters is merely that there is such a common supremum.

Definition 3 The goods market cone ∂D_i is de…ned as follows. Let D_i be the dual to the cone G_i:

$$D_i = \left\{ p \in \Re^n_+ : \forall z \in G_i, p.z > 0 \right\}$$

Then

$$\partial D_i = D_i \cap S(A) \text{ if } S(A) \subset N \text{ and } \partial D_i = D_i \text{ otherwise}$$

where $S(A)$ is the set of prices which support efficient individually rational and affordable allocations and N the set of prices which assign some trader zero wealth.

Formally,

$$A = \left\{ \begin{array}{c} c_i \in \Re^n_+ : \sum_i c_i \leq \sum_i \text{-}_i, \forall i, u_i(c_i) \geq u_i(\text{-}_i), \nexists y_i \in \Re^n_+, \\ \sum y_i \leq \sum_i \text{-}_i \ \& \ \forall i, u_i(y_i) \geq u_i(c_i) \ \& \exists j : u_j(y_j) > u_j(c_j) \end{array} \right\}$$

$$S(A) = \{ p \in \Delta : p \text{ supports allocations in } A \text{ s.t. } p.c_i = p.\text{-}_i + s_i(p) \}$$

$$N = \left\{ v \in \Delta \subset \Re^n_+ : \exists j \ s.t. \ v.\text{-}_j + s_j(v) = 0 \right\}$$

In words, if at all supports to feasible efficient affordable individually rational allocations, some agent j has zero income (including income from securities transactions), then D_j consists of all those supporting prices at which only limited increases in utility can be afforded from initial endowments. Otherwise it is the dual of the set D_i of directions in which utility never ceases to increase.

Definition 4 The economy satisfies limited arbitrage in the goods market if there is a price vector at which no agent can always derive increased utility from arbitrarily large zero-cost trades in goods: formally, there exists a price $p \in \Re^n_+$ assigning strictly positive value to all vectors in G_i, for $i = 1, ..., I$, i.e.,

$$\bigcap_{1 \leq i \leq I} \partial D_i \neq \emptyset$$

The condition of limited arbitrage states, roughly, that if all supporting prices to efficient individually rational affordable allocations imply zero income for some trader, then there is one supporting price at which only limited increases in utility are a¤ordable from initial endowments. Otherwise, there exists a price at which only limited increases in utility are affordable.

The definitions of global cone, market cone and limited arbitrage given for the exchange economy have to be modified slightly to apply to the trading of securities in the economy with price insurance. The commodity spaces are different, in fact infinite dimensional, and the consumption sets are unbounded below. For the securities market these definitions are:

Definition 5 The securities global cone $\widetilde{A}_i$ of agent i consists of all directions in the space of securities along which expected utility W_i increases without bound:

$$\widetilde{A}_i(s_i) = \{ z \epsilon L_2 : \forall y \in L_2, \exists \lambda > 0 : W_i(s_i + \lambda z) > W_i(y) \}$$

Definition 6 The cone $\widetilde{G}_i$ is the set of directions in securities space along which expected utility W_i never ceases to increase:

$$\widetilde{G}_i(s_i) = \{z \in L_2 : \forall \mu \exists \lambda : W_i(s_i + \lambda z) > W_i(s_i + \mu z) \text{ if } \lambda > \mu \geq 0\}$$

Definition 7 The securities market cone $\widetilde{D}_i$ is the set of securities prices assigning positive value to all trades in directions along which expected utility W_i never ceases to increase:

$$\widetilde{D}_i = \{\pi \in L_2 : \forall z \in G_i, \langle \pi, z \rangle > 0\}$$

Definition 8 The economy satisfies limited arbitrage in securities markets if there is a securities price vector at which no agent can always derive increased expected utility from arbitrarily large zero-cost securities trades: formally, there exists a price $\pi \in L_2$ assigning strictly positive value to all vectors in $\widetilde{G}_i$, for $i = 1, ..., I$, i.e.,

$$\bigcap_{1 \leq i \leq I} \widetilde{D}_i \neq \emptyset$$

Note that while limited goods arbitrage places restrictions on differences between agents' preferences for goods, limited securities arbitrage places restrictions on differences between preferences for goods and on differences between price expectations. Even if limited goods arbitrage is satisfied, limited securities arbitrage may fail if agents' price expectations are very different. This is similar in concept to the condition of overlapping or similar expectations used as a sufficient condition for the existence of equilibrium in temporary equilibrium models and securities markets models (see Hart and Hammond). The examples at the end of this paper illustrate the role of preferences and expectations in determining demands for securities and the conditions for market equilibrium. A difference between our use of limited goods arbitrage and that in Chichilnisky [5][3] is in the set N, the set of prices at which some agent has zero income. Here this set is endogenous, in the sense that an agent's income at a given price vector depends not only on endowments but also on his or her position in the PCS market, and so on expectations and the agent's optimal income transfers across price states. To appreciate the difference, note that if agent i assigns probability zero to price vector p', then he will set $s_i(p') = -p'.\text{-}_i$ and sell all of his income in price state p'. If the agent were wrong in these expectations and the equilibrium price were p' then his income would be zero: this could prevent p' from being an equilibrium price, although Chichilnisky [3][5] shows that under certain conditions there may be a competitive equilibrium at which some agents have zero income.

It is proven in Chichilnisky [5] and Chichilnisky and Heal [14] that if the economy satisfies the condition (Assumption 1) of uniformly bounded preferences and also Assumption 4 and an equivalent assumption on the indifference surfaces of preferences over goods, then the cones $A_i, \widetilde{A}_i, D_i, \widetilde{D}_i, G_i$ and $\widetilde{G}_i$ are the same for any endowment vectors: they are translation invariant.

4.1 Equilibrium

An equilibrium with price insurance in the economy $\mathcal{E}$ is

(a) a set of PCS prices at which PCS markets clear, and such that the associated PCS holdings for each agent maximize the agent's expected utility according to her price expectations in goods markets, as indicated in (3), and

(b) a set of goods prices and goods consumptions such that goods markets clear and agents maximize utility as in (1). Formally,

Definition 9 An equilibrium with price insurance in the economy $\mathcal{E}$ is a set of PCS prices $\pi^* \epsilon L_2$, of goods prices $p^* \in \Delta$, of PCS holdings $s_i^*(p) \epsilon L_2, i = 1, ..., I$, and consumptions $c_i^* \in \Re_+^n, i = 1, .., I$ such that (1) c_i^* maximizes $u_i(c_i)$ subject to $p^* c_i^* \leq p^* \text{-}_i + s_i^*(p^*)$, (2) $\sum_i c_i^* = \sum \text{-}_i$, (3) $s_i^*(p)$ maximizes $\int_{¢} w_i(p, s_i^*(p))\, de_i(p)$ subject to $\int_{¢} s_i^*(p)\, \pi^*(p)\, dp = 0$ and $p.\text{-}_h + s_i^*(p) \geq 0$ and (4) $\sum_i s_i^*(p) = 0 \forall p \in \Delta$.

5 Existence and Efficiency of Equilibrium

5.1 Existence

The main theorem, theorem 1, establishes that limited arbitrage in goods and securities markets is necessary and su¢cient for the existence of an equilibrium as defined above for the economy $\mathcal{E}$. Limited arbitrage was shown in Chichilnisky [3] [5] [4] to be necessary and sufficient for the existence of a competitive equilibrium and for the non-emptiness of the core in an exchange economy, and to be necessary and sufficient for the existence of social choice rules of both the Arrowian and Chichilnisky types (see Chichilnisky [8]). More recently, Chichilnisky and Heal [14] have shown the same condition to be necessary and sufficient for the existence of equilibrium and the core in economies with infinitely many commodities.

Lemma 10 Let $s_i^*(p)$ be the solution to the problem (3)

$$\max_{s_i(p)} \int_{p \in ¢} w_i(p, s_i(p))\, de_i(p)$$

$$\text{where} \int_{p \in ¢} \pi(p)\, s_i(p) = 0 \text{ and } p.\text{-}_h + s_i(p) \geq 0$$

Then $s_i^*(p)$ is a continuous function of the goods price vector p.

Proof. This follows immediately from the fact that $s_i^*(p)$, being a solution to a variational problem, is the solution of a differential equation and so continuous. ■

Theorem 11 Under Assumptions 1, 2 and 3, the economy $\mathcal{E}$ has an equilibrium with price insurance if and only if it satisfies limited arbitrage in goods and securities markets.

Proof. There are two stages to the proof. First, we prove that for a given set of PCS holdings $s_i^*(p), i = 1, ..., I$ the goods economy has a competitive equilibrium if and only if it satisfies limited goods arbitrage. This is a relatively simple extension of the proof in Chichilnisky [3]: the economy satisfies limited arbitrage in goods, which is necessary and sufficient for the existence of an equilibrium in an exchange economy. The only difference from a standard competitive exchange economy is that agents have holdings of price-contingent securities which augment their endowments.

The second stage of the proof is to show that the entire economy has an equilibrium with price insurance if in addition the condition of limited securities arbitrage is satisfied.

Lemma 12 For a given set of PCS holdings, the economy has a competitive equilibrium if and only if it satisfies limited goods arbitrage.

Proof. We can use the proof in Chichilnisky [3] with one modification. On page 97, in the proof of Lemma 3, the correspondence φ from the simplex Δ^I in the space of utility levels to the set $T = \left\{ y \in \Re^I : \sum_{i=1}^{I} y_i = 0 \right\}$ must be modified so that for each $r \in \Delta^I$,

$$\varphi(r) = \{p.[\text{-}_1 + s_1(p) - x_1(r)], .., p.[\text{-}_I + s_I(p) - x_I(r)] : p \in P(r)\}$$

Given that $s_i(p)$ is continuous (Lemma 1 above), upper semi continuity of φ can now be established exactly as in Chichilnisky [3] page 97, and the proof of existence of an equilibrium in that paper can be used to establish the Lemma. ■

We can now complete the proof of the main result. This is a direct application of Theorem 5 page 12 of Chichilnisky and Heal [14]. Consider the exchange economy $\mathcal{H}$ with commodity space and consumption sets L_2, preferences $W_i(s_i(p)) : L_2 \to \Re$, and initial endowments $0 \in L_2$. This is an exchange economy which satisfies all the conditions of Theorem 5 of [14] and for which existence of a competitive equilibrium is therefore (by Theorem 5 of [14]) equivalent to limited arbitrage, which in the present context is precisely what has been called limited securities arbitrage. Hence under the conditions of the main theorem the economy $\mathcal{H}$ has a competitive equilibrium. This means that there exists a set of prices for price-contingent securities $\pi^*(p)$ and a set of securities holdings for each agent $s_i^*(p)$ such that markets for price-contingent securities clear, $\sum_i s_i^*(p) = 0 \forall p \in \Delta$, and all agents are optimizing subject to their budget constraints, i.e., for all i, $s_i^*(p)$ maximizes $\int_{¢} w_i^p(s_i^*(p)) e_i(p) dp$ subject to $\int_{¢} s_i^*(p) \pi^*(p) dp = 0$ and $p.\text{-}_h + s_i(p) \geq 0$.

Now, by Lemma 12, given any set of PCS holdings, and so in particular the equilibrium holdings $s_i^*(p)$ in the economy $\mathcal{H}$, the goods economy has a competitive equilibrium. Let the prices at this equilibrium be p^* and the consumptions $c_i^*, i = 1, ..., I$. So the securities holdings $s_i^*(p), i = 1, ...I$, securities prices $\pi^*(p)$, goods prices p^* and goods consumption vectors c_i^* satisfy all the conditions of the definition of an

equilibrium with price-contingent securities. This completes the proof of theorem 1. ■

Note the following important points:

Remark 1: The equilibrium with price insurance is in general not a Walrasian equilibrium of the exchange economy defined by preferences $u_i : \Re^n_+ \to \Re$, endowments - $_i \in \Re^n_+$, the commodity space $\Re^n$ and consumption sets $\Re^n_+$. The examples which follow below illustrate this.

Remark 2: The equilibrium with price insurance depends upon agents' price expectations $e_i(p)$, as these influence their positions in the PCS markets and thus their endowments in price-contingent states. So price expectations matter and affect the ultimate allocation of resources. This occurs in a context of complete markets and perfect competition in the usual Walrasian framework, as formalized by Arrow and Debreu. We have simply added the possibility of agents insuring against uncertainty about equilibrium prices, an uncertainty which in any realistic analysis of a competitive system they must face.

5.2 Efficiency

What are the welfare properties of an equilibrium? Firstly, note that it is Pareto efficient by the usual arguments:

Theorem 13 Any equilibrium with price insurance in the economy $\mathcal{E}$ is Pareto efficient.

Proof. This follows from the standard arguments. ■

In general, the equilibria with price insurance are neither unique not fully insured: there may be multiple equilibria, so that trading PCSs as described here does not remove price uncertainty and produce a unique outcome, and in this respect is weaker than the framework for trading such securities in Chichilnisky Dutta and Heal [10]. The key point is that the assumptions there are stronger: they involve (1) common price expectations, (2) price expectations restricted to the set of Walrasian equilibrium price, and (3) the trading of many levels of derivatives. Finally, note that whatever price occurs, agents have insured optimally against it according to their price expectations. However, if agents have common price expectations, i.e., if $e_i(p) = e_j(p) \forall i, j$, then all equilibria are fully insured:

Theorem 14 Equilibria with price insurance in the economy $\mathcal{E}$ are fully insured if agents have common price expectations, i.e., if $e_i(p) = e_j(p) \forall i, j$.

Proof. This follows immediately from Lemma 2 of Chichilnisky Dutta and Heal [10]. ■

6 Example

In this section we illustrate how the introduction of price contingent securities can a¤ect the equilibria of a Walrasian economy. The illustration uses a two-person two-good economy which can be understood via the geometry of an Edgeworth box. Note that this example does not satisfy the assumption that preferences are smooth, made above. This assumption was not central to the analysis and can be dropped at the cost of some technical complexity. Here the use of Leontief preferences facilitates the computation of solutions.

The economy we consider has two risk-averse agents. There are infinitely many equilibrium allocations when this economy is considered as an exchange economy: the introduction of PCSs, traded before goods prices are determined, leads to a unique equilibrium allocation in the trading of goods.

There are two agents with preferences $u_i = \log \min\{c_{1i}, c_{2i}\}, i = 1, 2$ where c_{ji} is the consumption of good j by agent i. The total endowment of each good is one unit, so that $c_{11} + c_{12} = 1$ and $c_{21} + c_{22} = 1$. Agents' endowments are - $_1 = (1, 0)$ and - $_2 = (0, 1)$, placing the initial endowments at the lower right corner of the Edgeworth box. At given prices agents' choice problems are

$$\max u_i = \log \min\{c_{1i}, c_{2i}\} \text{ s.t. } p_1 c_{1i} + p_2 c_{2i} = p_i + s_i(p_1, p_2)$$

where $s_i(p_1, p_2)$ is agent i's holding of price contingent securities. Demands are given by

$$c_{21} = c_{11} = \frac{p_1 + s_1}{p_1 + p_2}, c_{22} = c_{12} = \frac{p_2 + s_2}{p_1 + p_2}$$

Given that $p_1 + p_2 = 1$ and that $s_1(p_1, p_2) + s_2(p_1, p_2) = 0 \forall p_1, p_2$, it is clear that these demands clear markets at any prices. Consider first the case without PCSs so that $s_1 = s_2 = 0$. In this case, any point on the diagonal of the Edgeworth box is a competitive equilibrium (see figure 1).

In the case of markets with PCSs, the securities holdings are chosen to maximize expected utility given agents' price expectations. Assume these expectations to be uniform, so that the ratios $\frac{p_i}{p_1 + p_2}, i = 1, 2,$ are uniformly distributed between zero and one. Agent one now seeks to maximize

$$\int_0^1 \log(p_1 + s_1(p_1))\, dp_1 \text{ subject to } \int_0^1 s_1(p_1)\, \pi(p_1)\, dp_1 = 0$$

$$\text{and } p_1 + s_1 \geq 0$$

recalling that $p_1 + p_2 = 1$. Provided that the non-negativity constraint is never binding (which we assume from now on), the solution to this is

$$s_1(p_1) = K_1 - p_1 - (\lambda_1 \pi(p_1))^{-1}$$

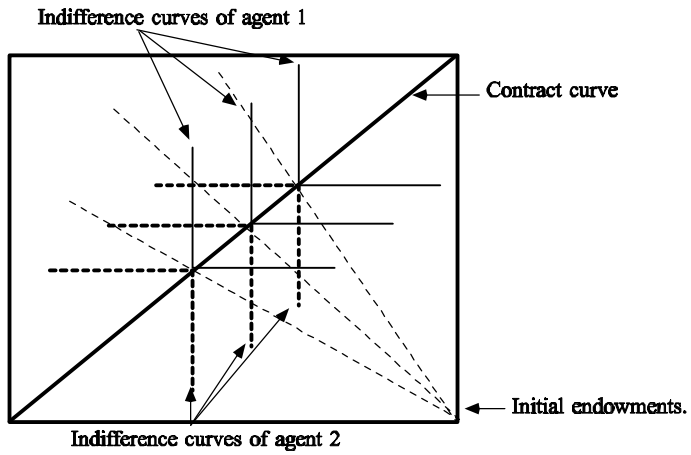

Figure 1: all points on the contract curve of the Edgeworth box are Walrasian equilibria of the exchange economy.

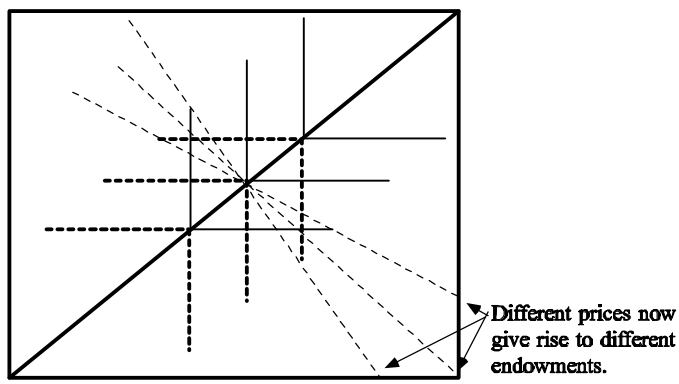

Figure 2: after trading PCSs, there is a unique consumption equilibrium at the mid point of the contract curve. Consumption is independent of prices, and payoffs from price-contingent securities vary so as the move the budget lines and maintain consumption constant.

where K_1 is a positive constant and λ_1 the shadow price on the integral constraint. This expresses agent 1's demand for PCSs as a function of their price π and the goods price p_1. By analogous arguments $s_2(p_1) = K_2 - 1 + p_1 - (\lambda_2 \pi(p_1))^{-1}$ and the condition that $s_1 + s_2 = 0$ implies that $\pi(p_1)$ is a constant. In this case holdings of PCSs are linear in the price p_1 for both agents. Clearly $s_1(p_1) + p_1$ is a constant, implying that consumption is independent of prices: Theorem 3 above applies and agents are fully insured at an equilibrium. There is now a unique equilibrium consumption allocation with price insurance, independent of goods prices, involving each agent consuming a half unit of each good. This unique consumption allocation is realized whatever the equilibrium prices, and endowments vary with prices, via the trading of PCSs, so as to make this possible, as illustrated in figure 2. So the introduction of PCSs has radically changed the set of equilibrium allocations in the goods markets, in this

particular case narrowing it from a continuum to a singleton.

7 Rational expectations?

So far, I have taken expectations to be exogenous, part of the data describing individuals. A natural progression is to endogenize agents' beliefs. That remains a task for further work: here I make some simple observations on this general matter.

It is natural to enquire about rational expectations in the present framework: what is a rational expectation here? One interpretation would be a set of point price expectations (probability measures putting all weight at one price vector in the simplex, Dirac measures) which are fulfilled at the equilibrium. Intuitively, we expect that a Walrasian equilibrium of the underlying exchange economy (the exchange economy defined by preferences $u_i : \Re^n_+ \rightarrow \Re$ and endowments - $_i \in \Re^n_+$) will be realized if expected with probability one by all agents, so that this equilibrium and the expectations giving probability one to it form a rational expectations equilibrium. This can be shown to be correct, although we should note that if all agents expect the same price vector p' to be realized with probability one, there will be no trade in the PCS market.

A more general and interesting case arises if there are several Walrasian equilibria of the underlying economy, and agents have expectations over these, and only over these: the set of Walrasian equilibria is the support of the distribution of price expectations, as in [10]. In this case, none of the Walrasian equilibria will be realized, even if all agents have identical expectations. This is proven in Chichilnisky Dutta and Heal [10]. It is easy to see why this is: agents will trade income between possible equilibrium prices, and in so doing will change their endowments and so the equilibria of the underlying economy. The key point is that in this case there will be activity in the PCS market, and this will change endowments. This suggests that with multiple Walrasian equilibria, there is no rational expectations equilibrium, not a new conclusion (see also Hahn [19]).

Finally let us ask what happens in the model of this paper if agents hold general non-point expectations, the operation of the economy is repeated many times, and agents revise their expectations between repetitions in the light of the equilibria which are realized. A general treatment must await another paper, but the following observation may be of interest. Take an extreme case of expectation revision in the light of outcomes, and assume that once an equilibrium price p_1 has been realized in the first operation of the economy, agents revise their expectations to give probability one to p_1. This is a case of myopic expectations. Will the expected price vector p_1 now be realized in further operations of the economy? As all agents hold the same point expectation, there will be no trade in the PCS market. Hence agents' endowments will be unaltered from the underlying exchange economy, and so the only possible competitive equilibria are those of that economy. Hence the expected p_1 is realized if and only if it is a Walrasian equilibrium of the underlying economy. Assume that

p_1 is not such a Walrasian equilibrium: then the outcome, call this p_2, must be a Walrasian equilibrium, and if all agents now once more form another myopic point expectation that p_2 will be the next equilibrium price, then this expectation will be fulfilled. So in the extreme case of myopic expectation formation after the first operation of the economy, it will converge rapidly to a Walrasian equilibrium of the underlying economy. It is of course an open question whether this will happen for more general expectation processes.

8 Conclusions

A relatively simple extension of the conventional general equilibrium framework allows it to incorporate uncertainty about the values of equilibrium prices, a form of uncertainty which is faced by most participants in modern economies yet which is not captured in the current formulations of general equilibrium models. The extension is achieved by the introduction of price-contingent securities, which are traded within a framework which prevents arbitrage between the market for these and the goods markets. Several extensions of this work seem natural. The present model is one of complete markets, in the sense that it is possible to trade securities contingent on any possible price vector. Incompleteness in this context will take the form of PCSs restricted to a subset of possible prices, or perhaps of PCSs whose payoffs have a different functional relationship with prices than those used here. The PCSs considered in this paper have payoffs represented by Dirac delta-functions: they pay only at one point in the price simplex. Conventional options, which are also PCSs, pay varying but non-zero amounts on a set of prices of positive measure. It would be interesting to know whether efficiency could be attained by the use of PCSs of this form. Another interesting question concerns the determinants of the prices of PCSs: how are the prices of these derivatives related to the prices of underlying goods and the expectations that agents hold over these?

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