Intermediate Financial Theory
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Solutions to Exercises

Chapter 1

1.1. U is a utility function, i.e., $U(x) > U(y) \Leftrightarrow x \succ y$
f(.) is an increasing monotone transformation, $f(a) > f(b) \Leftrightarrow a > b$;
then $f(U(x)) > f(U(y)) \Leftrightarrow U(x) > U(y) \Leftrightarrow x \succ y$

1.2. Utility function $U(c_1, c_2)$:
FOC: $U_1/U_2=p_1/p_2$
Let f=f(U(.)) be a monotone transformation.
Apply the chain rule for derivatives:
FOC: $f_1/f_2=f'U_1/f'U_2=p_1/p_2$ (prime denotes derivation).
Economic interpretation: f and U represent the same preferences, they must lead to the same choices.

1.3. When an agent has very little of one given good, he is willing to give up a big quantity of another good to obtain a bit more of the first.
MRS is constant when the utility function is linear additive (that is, the indifference curve is also linear):

$$U(c_1, c_2) = \alpha c_1 + \beta c_2$$
$$MRS = \frac{\alpha}{\beta}$$

Not very interesting; for example, the optimal choice over 2 goods for a consumer is always to consume one good only (if the slope of the budget line is different from the MRS) or an indefinite quantity of the 2 goods (if the slopes are equal).
Convex preferences can exhibit indifference curves with flat spots, strictly convex preferences cannot. The utility function is not strictly quasi-concave here.

Pareto set: 2 cases.

- Indifference curves for the agents have the same slope: Pareto set is the entire box;
- Indifference curves do not have the same slope: Pareto set is the lower side and the right side of the box, or the upper side and the left side, depending on which MRS is higher.

1.4. a. $U_1 = 6^{0.5}4^{0.5} = 4.90$
$U_2 = 14^{0.5}16^{0.5} = 14.97$

$$MRS_j = \frac{\partial U_j / \partial c_1^j}{\partial U_j / \partial c_2^j} = \frac{\alpha c_2^j}{(1-\alpha)c_1^j}$$

with $\alpha = 0.5$, $MRS_j = \frac{c_2^j}{c_1^j}$

$$MRS_1 = \frac{4}{6} = 0.67$$

$$MRS_2 = \frac{16}{14} = 1.14$$

$MRS_1 \neq MRS_2$, not Pareto Optimal; it is possible to reallocate the goods and make one agent (at least) better off without hurting the other.

b. PS = { $c_1^j = c_2^j$, $j = 1,2 : c_1^i + c_2^i = 20, i = 1,2$ }, the Pareto set is a straight line (diagonal from lower-left to upper-right corner).

c. The problem of the agents is

$\text{Max}U^j$ s.t. $p_1 e_1^j + e_2^j = p_1 c_1^j + c_2^j$.

The Lagrangian and the FOC's are given by

$$L^j = \left(c_1^j\right)^{1/2}\left(c_2^j\right)^{1/2} + y\left(p_1 e_1^j + e_2^j - p_1 c_1^j - c_2^j\right)$$

$$\frac{\partial L^j}{\partial c_1^j} = \frac{1}{2}\left(\frac{c_2^j}{c_1^j}\right)^{1/2} - yp_1 = 0$$

$$\frac{\partial L^j}{\partial c_2^j} = \frac{1}{2}\left(\frac{c_1^j}{c_2^j}\right)^{1/2} - y = 0$$

$$\frac{\partial L^j}{\partial y} = p_1 e_1^j + e_2^j - p_1 c_1^j - c_2^j = 0$$

Rearranging the FOC's leads to $p_1 = \frac{c_2^j}{c_1^j}$. Now we insert this ratio into the budget constraints of agent 1 $p_1 6 + 4 - 2p_1 c_1^1 = 0$ and after rearranging we get $c_1^1 = 3 + \frac{2}{p_1}$. This expression can be interpreted as a demand function. The remaining demand functions can be obtained using the same steps.

$$c_2^1 = 3p_1 + 2$$

$$c_1^2 = 7 + \frac{8}{p_1}$$

$$c_2^2 = 7p_1 + 8$$

To determine market equilibrium, we use the market clearing condition $c_1^1 + c_1^2 = 20, c_2^1 + c_2^2 = 20$.

Finally we find $p_1 = 1$ and $c_1^1 = c_2^1 = 5, c_1^2 = c_2^2 = 15$.

The after-trade MRS and utility levels are:

$U_1 = 5^{0.5}5^{0.5} = 5$

$U_2 = 15^{0.5}15^{0.5} = 15$

$$MRS_1 = \frac{5}{5} = 1$$

$$MRS_2 = \frac{15}{15} = 1$$

Both agents have increased their utility level and their after-trade MRS is equalized.

d. $U_j(c_1^j, c_2^j) = \ln\left(\left(c_1^j\right)^{\alpha} \cdot \left(c_2^j\right)^{1-\alpha}\right) = \alpha \ln c_1^j + (1-\alpha)\ln c_2^j$,

$$MRS_j = \frac{\partial U_j / \partial c_1^j}{\partial U_j / \partial c_2^j} = \frac{\alpha c_2^j}{(1-\alpha)c_1^j}$$

Same condition as that obtained in a). This is not a surprise since the new utility function is a monotone transformation (logarithm) of the utility function used originally.

$U_1 = \ln\left(6^{0.5}4^{0.5}\right) = 1.59$

$U_2 = \ln\left(14^{0.5}16^{0.5}\right) = 2.71$

MRS's are identical to those obtained in a), but utility levels are not. The agents will make the same maximizing choice with both utility functions, and the utility level has no real meaning, beyond the statement that for a given individual a higher utility level is better.

e. Since the maximizing conditions are the same as those obtained in a)-c) and the budget constraints are not altered, we know that the equilibrium allocations will be the same too (so is the price ratio). The after-trade MRS and utility levels are:

$U_1 = \ln\left(5^{0.5}5^{0.5}\right) = 1.61$

$U_2 = \ln\left(15^{0.5}15^{0.5}\right) = 2.71$

$$MRS_1 = \frac{5}{5} = 1$$

$$MRS_2 = \frac{15}{15} = 1$$

1.5. Recall that in equilibrium there should not be excess demand or excess supply for any good in the economy. If there is, then prices change accordingly to restore the equilibrium. The figure shows excess demand for good 2 and excess supply for good 1, a situation which requires p_2 to increase and p_1 to decrease to restore market clearing. This means that p_1/p_2 should decrease and the budget line should move counter-clockwise.

Chapter 3

3.1. Mathematical interpretation:

We can use Jensen's inequality, which states that if f(.) is concave, then

$$E(f(X)) \leq f(E(X))$$

Indeed, we have that

$$E(f(X)) = f(E(X)) \Leftrightarrow f'' = 0$$

As a result, when f(.) is not linear, the ranking of lotteries with the expected utility criterion might be altered.

Economic interpretation:

Under uncertainty, the important quantities are risk aversion coefficients, which depend on the first and second order derivatives. If we apply a non-linear transformation, these quantities are altered. Indeed, $R_A(f(U(.))) = R_A(U(.)) \Leftrightarrow$ f is linear.

a. L = (B, M, 0.50) = 0.50×U(B) + 0.50×U(M) = 55 > U(P) = 50. Lottery L is preferred to the "sure lottery" P.

b. f(U(X)) = a+b×U(X)

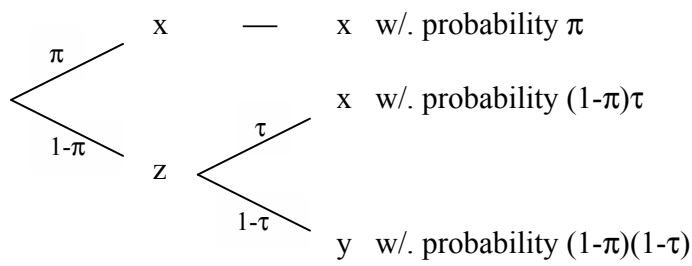
L_f = (B, M, 0.50)$_f$ = 0.50×(a+bU(B)) + 0.50×(a+bU(M)) = a + b55 > f(U(P)) = a+bU(P) = a + b50. Again, L is preferred to P under transformation f.

g(U(X)) = lnU(X)

L_g = (B, M, 0.50)$_g$ = 0.50×lnU(100) +0.50×lnU(10) = 3.46 < g(U(P)) = lnU(50) = 3.91. P is preferred to L under transformation g.

3.2. Lotteries:

We show that (x,z,π) = (x,y,π + (1-π)τ) if z = (x,y, τ).



The total probabilities of the possible states are

$$\pi(x) = \pi + (1-\pi)\tau$$

$$\pi(y) = (1-\pi)(1-\tau)$$

Of course, $\pi(x) + \pi(y) = \pi + (1-\pi)\tau + (1-\pi)(1-\tau) = 1$. Hence we obtain lottery (x,y,π + (1-π)τ).

Could the two lotteries (x,z,π) and $(x,y,\pi + (1-\pi)\tau)$ with $z = (x,y, \tau)$ be viewed as non-equivalent ? Yes, in a non-expected utility world where there is a preferences for gambling. Yes, also, in a world where non-rational agents might be confused by the different contexts in which they are requested to make choices. While the situation represented by the two lotteries is too simple to make this plausible here, the behavioral finance literature building on the work of Kahneman and Tversky (see references in the text) point out that in more realistic experimental situations similar 'confusions' are frequent.

3.3 U is concave. By definition, for a concave function f(.)

$$f(\lambda a + (1-\lambda)b) \geq \lambda f(a) + (1-\lambda)f(b), \lambda \in [0,1]$$

Use the definition with f = U, a = c_1, b = c_2, $\lambda = 1/2$

$$U\left(\frac{1}{2}c_1 + \frac{1}{2}c_2\right) \geq \frac{1}{2}U(c_1) + \frac{1}{2}U(c_2)$$

$$U(\bar{c}) \geq \frac{1}{2}U(c_1) + \frac{1}{2}U(c_2)$$

$$2U(\bar{c}) \geq U(c_1) + U(c_2)$$

$$V(\bar{c},\bar{c}) \geq V(c_1 + c_2)$$

Chapter 4

4.1 Risk Aversion: (Answers to a), b), c), and d) are given together here)

(1) $U(Y) = -\frac{1}{Y}$

$U'(Y) = \frac{1}{Y^2} > 0$

$U''(Y) = -\frac{2}{Y^3} < 0$

$R_A = \frac{2}{Y}$

$R_R = 2$

$\frac{\partial R_A}{\partial Y} = -\frac{2}{Y^2} < 0$

(2) $U(Y) = \ln Y$

$U'(Y) = \frac{1}{Y} > 0$

$U''(Y) = -\frac{1}{Y^2} < 0$

$R_A = \frac{1}{Y}$

$R_R = 1$

$\frac{\partial R_A}{\partial Y} = -\frac{1}{Y^2} < 0$

(3) $U(Y) = -Y^{-\gamma}$

$U'(Y) = \gamma Y^{-\gamma-1} > 0 \Leftrightarrow \gamma > 0$

$U''(Y) = -\gamma(\gamma+1)Y^{-\gamma-2} < 0$

$R_A = \frac{\gamma+1}{Y}$

$R_R = \gamma+1$

$\frac{\partial R_A}{\partial Y} = -\frac{\gamma+1}{Y^2} < 0$

(4) $U(Y) = -\exp(-\gamma Y)$

$U'(Y) = \gamma\exp(-\gamma Y) > 0 \Leftrightarrow \gamma > 0$

$U''(Y) = -\gamma^2 \exp(-\gamma Y) < 0$

$R_A = \gamma$

$R_R = \gamma Y$

$\frac{\partial R_R}{\partial Y} = \gamma > 0$

(5) $U(Y) = \frac{Y^\gamma}{\gamma}$

$U'(Y) = Y^{\gamma-1}$

$U''(Y) = (\gamma-1)Y^{\gamma-2} < 0 \Leftrightarrow \gamma < 1$

$R_A = \frac{1-\gamma}{Y}$

$R_R = 1-\gamma$

$\frac{\partial R_A}{\partial Y} = \frac{\gamma-1}{Y^2} < 0$

(6) $U(Y) = \alpha Y - \beta Y^2, \alpha > 0, \beta > 0$

$U'(Y) = \alpha - 2\beta Y > 0 \Leftrightarrow Y < \frac{\alpha}{2\beta}$

$U''(Y) = -2\beta < 0$

$R_A = \frac{2\beta}{\alpha - 2\beta Y} > 0$

$R_R = \frac{2\beta}{\alpha - 2\beta Y} Y > 0$

$\frac{\partial R_A}{\partial Y} = \frac{4\beta^2}{(\alpha - 2\beta Y)^2} > 0$

$\frac{\partial R_R}{\partial Y} = \frac{\partial}{\partial Y}(R_A Y) = \frac{\partial R_A}{\partial Y} Y + R_A > 0$

Note that γ controls for the degree of risk aversion. We check it with the derivative of R_A and R_R w.r.t. γ.

$$U(Y) = -Y^{-\gamma} \qquad \frac{\partial R_A}{\partial \gamma} = \frac{1}{Y} \qquad \frac{\partial R_R}{\partial \gamma} = 1$$

$$U(Y) = -\exp(-\gamma Y) \qquad \frac{\partial R_A}{\partial \gamma} = 1 \qquad \frac{\partial R_R}{\partial \gamma} = Y$$

$$U(Y) = \frac{Y^{\gamma}}{\gamma} \qquad \frac{\partial R_A}{\partial \gamma} = -\frac{1}{Y} \qquad \frac{\partial R_R}{\partial \gamma} = -1$$

In the last utility function above, we should better use $\gamma \equiv 1\text{-}\theta$, so that $U(Y) = \frac{Y^{\gamma}}{\gamma} \equiv \frac{Y^{1-\theta}}{1-\theta}$, and $\frac{\partial R_A}{\partial \theta} = \frac{1}{Y}, \frac{\partial R_R}{\partial \theta} = 1$ (look at $R_R = 1\text{-}\gamma = \theta$). After this change, every derivative w.r.t. θ is positive. If we increase θ, we increase the level of risk aversion (both absolute and relative).

4.2. Certainty equivalent .

The problem to be solved is: find x such that $\pi_1 U(Y_1^1) + \pi_2 U(Y_2^1) \equiv U(x)$

where Y_i^1 denotes outcome of lottery L_1 in state i and π_i denotes the probability of state i.

If U is bijective, it can be "inverted", so the solution is

$x = U^{-1}\{\pi_1 U(Y_1^1) + \pi_2 U(Y_2^1)\}$

where U^{-1} is the inverse function of U.

(1) $U(Y) = -\frac{1}{Y}, U^{-1}(Y) = -\frac{1}{Y}$

$$x = -\left(\frac{1}{2}\left(-\frac{1}{50000} - \frac{1}{10000}\right)\right)^{-1} = 16666.67$$

(2) $U(Y) = \ln Y, U^{-1}(Y) = \exp(Y)$

$$x = \exp\left(\frac{1}{2}\left(\ln\frac{1}{50000} + \ln\frac{1}{10000}\right)\right) = 22360.68$$

(3) $U(Y) = \frac{Y^{\gamma}}{\gamma}, U^{-1}(Y) = (\gamma Y)^{1/\gamma}$

$$x = \left(\gamma\left(\frac{1}{2}\left(\frac{Y_1^{\gamma}}{\gamma} + \frac{Y_2^{\gamma}}{\gamma}\right)\right)\right)^{1/\gamma}$$

with $\gamma = .25, x = 24232.88$

with $\gamma = .75, x = 28125.91$

Increasing γ leads to an increase in the value of x. This is because $1\text{-}\gamma$ (and not γ!) is the coefficient of relative risk aversion for this utility function. Therefore, increasing γ decreases the level of risk aversion, and the certainty equivalent is higher, i.e. the risk premium is lower.

4.3. Risk premium .

The problem to be solved (indifference between insurance and no insurance) is

$$EU(Y) = \sum_i \ln Y_i \pi(i) = \ln(100{,}000 - P)$$

where P is the insurance premium, Y_i is the worth in state i and $\pi(i)$ is the probability of state i. The solution to the problem is

$$P = 100{,}000 - \exp(EU(Y))$$

The solutions under the three scenarios are

$$\text{Scenario A}: \ P = 13312.04$$

$$\text{Scenario B}: \ P = 13910.83$$

$$\text{Scenario C}: \ P = 22739.27$$

Starting from scenario A, in scenario B and C we have transferred 1 percent from state 3 to another state (to state 2 in scenario B and to state 1 in scenario C). However, the outcome is very different: The premium is slightly bigger in scenario B, while it is a lot higher in C. This could have been expected because the logarithmic utility function is very curved at low values, and flattens out rapidly, i.e. ln 1 is very different from ln 100000, but ln 50000 is only slightly different from ln 100000. Also, logarithmic utility function is DARA.

4.4. Simply take the expected utility (initial wealth is normalized to 1)

$$E(U(1+\tilde{r}_A)) \equiv E(U(1+\tilde{r}_B+\tilde{\xi})) \geq E(U(1+\tilde{r}_B))$$

and apply Theorem 3.2. All individuals with increasing utility functions prefer A to B.

4.5 a. Let $\tilde{x}_A$ and $\tilde{x}_B$ be two probability distributions. The notion that $\tilde{x}_A$ FSD $\tilde{x}_B$ is the idea that $\tilde{x}_A$ assigns greater probability weight to higher outcome values; equivalently, it assigns lower outcome values a lower probability relative to $\tilde{x}_B$. Notice that there is no concern for "relative riskiness" in this comparison: the outcomes under $\tilde{x}_A$ could be made more 'spread out' in the region of higher values than $\tilde{x}_B$.

b. The notion that $\tilde{x}_A$ SSD $\tilde{x}_B$ is the sense that $\tilde{x}_B$ is related to $\tilde{x}_A$ via a "pure increase in risk". This is the sense of $\tilde{x}_B$ being defined from $\tilde{x}_A$ via a mean preserving spread. $\tilde{x}_B$ is just $\tilde{x}_A$ where the values have been spread out. Of course, any risk averse agent would prefer $\tilde{x}_A$.

c. Only two moments of a distribution are relevant for comparison: the mean and the variance. Agents like the former and dislike the latter. Thus, given two distributions with the same mean, the one with the higher variance is less desirable; similarly, given two distributions with the same variance, the one with the greater mean return is preferred.

d. (i) Compare first under mean variance criterion.

$$E\tilde{x}_A = \frac{1}{4}(2) + \frac{1}{2}(4) + \frac{1}{4}(9) = 4.75$$

$$E\tilde{x}_B = \frac{1}{3}(1+6+8) = 5$$

$$\sigma_A^2 = \frac{1}{4}(2-4.75)^2 + \frac{1}{2}(4-4.75)^2 + \frac{1}{4}(9-4.75)^2 = 6.6875$$

$$\sigma_B^2 = \frac{1}{3}(1-5)^2 + \frac{1}{3}(6-5)^2 + \frac{1}{3}(8-5)^2$$
$$= 26/3 = 8\frac{2}{3}.$$

So, $E\tilde{x}_A < E\tilde{x}_B$, $\sigma_A^2 < \sigma_B^2$

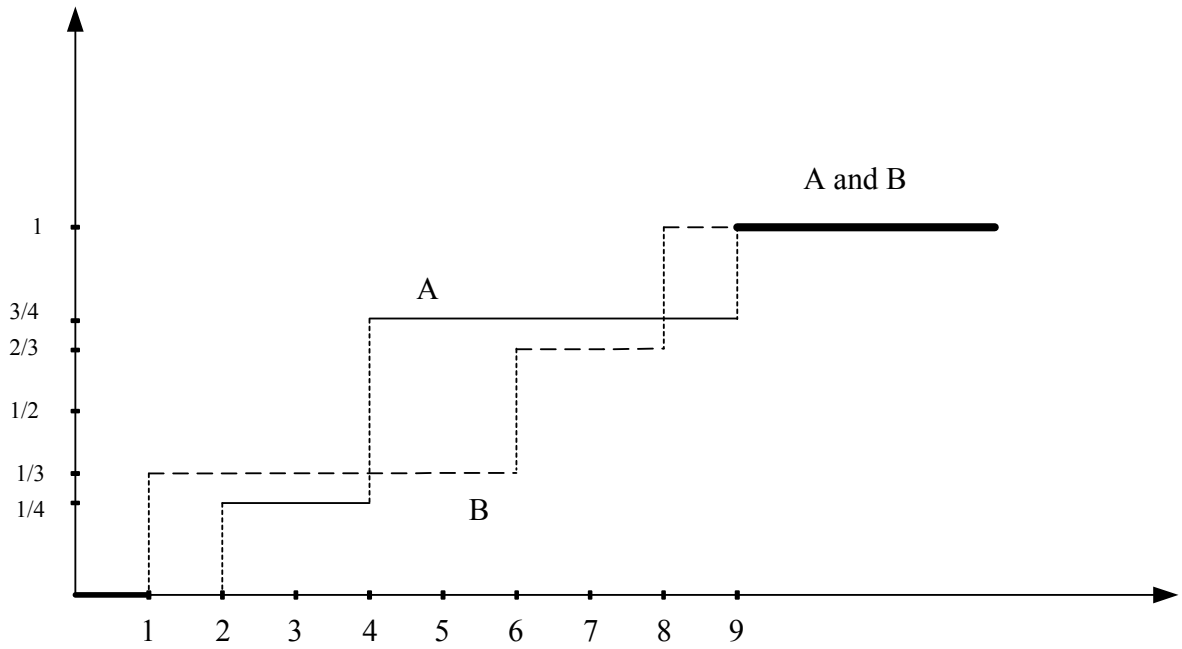
Thus $\tilde{x}_A$ dominates $\tilde{x}_B$ under mean variance.

(ii) Now let's compare them under FSD.

Let $F(\tilde{x}_A)$ be denoted

$F(\tilde{x}_B)$ be denoted

and let us graph $F(\tilde{x}_A)$ and$F(\tilde{x}_B)$



It does not appear that either $\tilde{x}_A$ FSD $\tilde{x}_B$ or $\tilde{x}_B$ FSD $\tilde{x}_A$; either dominates the other in the FSD sense.

Thus, mean-variance dominance does not imply FSD.

(iii)

x	$\int_0^x f_B(t)dt$	$\int_0^x F_B(t)dt$	$\int_0^x f_A(t)dt$	$\int_0^x F_A(t)dt$	$\int_0^x [F_B(t)-F_A(t)]dt$
0	0	0	0	0	0
1	1/3	1/3	0	0	1/3
2	1/3	2/3	1/4	1/4	5/12
3	1/3	1	1/4	1/2	1/2
4	1/3	1 ⅓	3/4	5/4	1/12
5	1/3	1 ⅔	3/4	2	-1/3

There is no SSD as the sign of $\int_0^x [F_B(t)-F_A(t)]dt$ is not monotone.

4.6 a. The certainty equivalent is defined by the equation :

$U(CE_Z) = EU(\tilde{Z})$, since $Y_0 \equiv 0$, and $\tilde{Z} = (16,4;\frac{1}{2})$

$$(CE_Z)^{\frac{1}{2}} = \frac{1}{2}(16)^{\frac{1}{2}} + \frac{1}{2}(4)^{\frac{1}{2}} = \frac{1}{2}(4) + \frac{1}{2}(2) = 3$$

Thus $CE_Z = 9$

b. The insurance policy guarantees the expected payoff :

$$E\tilde{Z} = \frac{1}{2}(16) + \frac{1}{2}(4) = 8 + 2 = 10$$

Π, the premium, satisfies $\Pi = E\tilde{Z} - C\tilde{E}_Z = 1$.

c. The insurance would pay –6 in the high state, +6 in the low state. The most the agent would be willing to pay is 1.

$U(CE_Z) = 10^{\frac{1}{2}} = \pi'(16)^{\frac{1}{2}} + (1-\pi')(4)^{\frac{1}{2}}$, $\pi' = .58$

The probability premium is .08.

d. Now consider the gamble (36, 16, ½) = Z'

$$U(CE_{Z'}) = (CE_{Z'})^{\frac{1}{2}} = \frac{1}{2}(36)^{\frac{1}{2}} + \frac{1}{2}(16)^{\frac{1}{2}}$$

$$(CE_{Z'})^{\frac{1}{2}} = \frac{1}{2}(6) + \frac{1}{2}(4) = 5$$

$CE_{Z'} = 25$

$\Pi_{Z'} = 1$ (as before)

π'' solves : $(26)^{\frac{1}{2}} = \pi''(36)^{\frac{1}{2}} + (1-\pi'')(16)^{\frac{1}{2}}$

$5.10 = \pi''6 + (1-\pi'')4 = 2\pi''+4$

$$2\pi'' = 1.1,\ \pi'' = \frac{1.1}{2} = .55$$

Thus the probability premium is $.55 - .50 = .05$

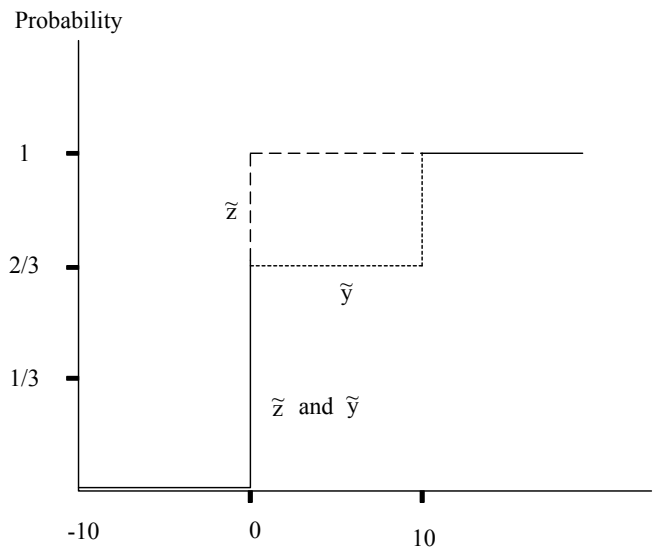
The probability premium has fallen became the agent is wealthier in the case and is operating on a less risk averse portion of his utility curve. As a result, the premium, as measured in probability terms, is less.

4.7 No. Reworking the data of Table 3.3 shows that it is not always the case that $\int_0^x [F_4(t) - F_3(t)]dt > 0$.

Specifically, for x = 5 to 7, $F_3(x) > F_4(x)$. Graphically this corresponds to the fact that in Fig 3.6 area B of the graph is now bigger than area A.

4.8 a. State by state dominance : no.

b. FSD : yes. See graph



These two notions are not equivalent.

Chapter 5

5.1. For full investment in the risky asset the first order condition has to satisfy the following:

$$E[U'(Y_0(1+\tilde{r}))(\tilde{r}-r_f)] \geq 0$$

Now expanding $U'(Y_0(1+\tilde{r}))$ around $Y_0(1+r_f)$, we get, after some manipulations and ignoring higher terms :

$$E[U'(Y_0(1+\tilde{r}))(\tilde{r}-r_f)]$$
$$= U'[Y_0(1+r_f)]E(\tilde{r}-r_f) + U''[Y_0(1+r_f)]E(\tilde{r}-r_f)^2 Y_0 \geq 0$$

Hence,

$E(\tilde{r}-r_f) \geq R_A[Y_0(1+r_f)]E(\tilde{r}-r_f)^2 Y_0$ which is the smallest risk premium required for full investment in the risky asset.

5.2. a.

$$R(\pi_1) = (1-a)2 + a = 2 - a$$
$$R(\pi_2) = (1-a)2 + 2a = 2$$
$$R(\pi_3) = (1-a)2 + 3a = 2 + a$$

b.

$$EU = \pi_1 U(2-a) + \pi_2 U(2) + \pi_3 U(2+a)$$
$$\frac{\partial EU}{\partial a} = -\pi_1 U'(2-a) + \pi_3 U'(2+a) = 0$$
$$a = 0 \Leftrightarrow \underset{>0}{U'(2)}[\pi_3 - \pi_1] = 0 \Leftrightarrow \pi_3 = \pi_1 \Leftrightarrow E(z) = 2$$

c.

Define $W(a) = E(U(Y_0(1+r_f) + a(\tilde{r}-r_f)))$

$$= E(U(2 + a(z-2)))$$
$$W'(a) = E(U'(2+a(z-2))(z-2)) = 0$$

d.

$\bullet\ U(Y) = 1 - \exp(-bY)$

$$W(a) = \pi_1[1-\exp(-b(2-a))] + \pi_2[1-\exp(-b(2))] + \pi_3[1-\exp(-b(2+a))]$$

$$W'(a) = -\pi_1 b\exp(-b(2-a)) + \pi_3 b\exp(-b(2+a)) = 0$$

$$\ln(\pi_3 b)(-b(2+a)) = \ln(\pi_1 b)(-b(2-a))$$

$$a = \frac{\ln(\pi_1) - \ln(\pi_3)}{2b}$$

$\bullet\ U(Y) = \frac{1}{1-\gamma} Y^{1-\gamma}$

$$W(a) = \pi_1\left[\frac{1}{1-\gamma}(2-a)^{1-\gamma}\right] + \pi_2\left[\frac{1}{1-\gamma}(2)^{1-\gamma}\right] + \pi_3\left[\frac{1}{1-\gamma}(2+a)^{1-\gamma}\right]$$

$$W'(a) = -\pi_1(2-a)^{\gamma} + \pi_3(2+a)^{\gamma} = 0$$

$$a = 2\frac{\pi_3^{1/\gamma} - \pi_1^{1/\gamma}}{\pi_1^{1/\gamma} + \pi_3^{1/\gamma}}$$

Assuming $\pi_3 > \pi_1, b > 0, 0 < \gamma < 1$ we have in either cases $\frac{\partial a}{\partial Y} > 0$.

e.

$\bullet\ U(Y) = 1 - \exp(-bY)$

$R_A = b$

$\bullet\ U(Y) = \frac{1}{1-\gamma} Y^{1-\gamma}$

$R_A = \gamma / Y$

5.3 a.
$$\begin{aligned} Y &= (1+\tilde{r})a + (Y_0 - a)(1+r_f) \\ &= Y_0(1+r_f) + a(\tilde{r} - r_f) \end{aligned}$$

b. $\max_a\ EU(Y_0(1+r_f) + a(\tilde{r} - r_f))$

F.O.C.: $E\{U'(Y_0(1+r_f) + a^*(\tilde{r} - r_f))(\tilde{r} - r_f)\} = 0$

$E\{U''(Y_0(1+r_f) + a^*(\tilde{r} - r_f))(\tilde{r} - r_f)^2\} < 0$

Since the second derivative is negative we are at a maximum with respect to a at $a = a^*$, the optimum.

c. We first want to formally obtain $\frac{da^*}{dY_0}$. Take the total differential of the F.O.C. with respect to Y_0.

Since $E\{U'(Y_0(1+r_f)+a(\tilde{r}-r_f))(\tilde{r}-r_f)\}=0$,

$$E\left\{U''(Y_0(1+r_f)+a(\tilde{r}-r_f))(\tilde{r}-r_f)\left[(1+r_f)+\frac{da}{dY_0}(\tilde{r}-r_f)\right]\right\}=0$$

$$-\frac{E\{U''(Y_0(1+r_f)+a(\tilde{r}-r_f))(\tilde{r}-r_f)\}(1+r_f)}{E\{U''(Y_0(1+r_f)+a(\tilde{r}-r_f))(\tilde{r}-r_f)^2\}}=\frac{da}{dY_0}$$

The denominator is <0; in conjunction with the negative preceeding the term we see that the sign of $\frac{da}{dY_0}$ is entirely dependent on $E\{U''(Y_0(1+r_f)+a(\tilde{r}-r_f))(\tilde{r}-r_f)\}(1+r_f)$ the numerator.

Since $(1+r_f)>0$ it is in fact dependent on $EU''(Y_0(1+r_f)+a(\tilde{r}-r_f))(\tilde{r}-r_f)$. We want to show this latter expression is positive and hence $\frac{da}{dY_0}>0$, as our intuition would suggest.

d. $R_A'(Y)<0$ is declining absolute risk aversion: as an investor becomes wealthier he should be willing to place more of his wealth (though not necessarily proportionately more) in the risky asset, if he displays $R_A'(Y)<0$.

e. We will divide the set of risky realizations into two sets: $\tilde{r}\geq r_f$ and $r_f>\tilde{r}$. <u>We maintain the assumption $R_A'(Y)<0$</u>.

Case 1: $\tilde{r}\geq r_f$; Notice that

$$Y_0(1+r_f)+a(\tilde{r}-r_f)>Y_0(1+r_f).$$

Then, $R_A(Y_0(1+r_f)+a(\tilde{r}-r_f))\leq R_A(Y_0(1+r_f))$.

Case 2: if $\tilde{r}<r_f$

$$Y_0(1+r_f)+a(\tilde{r}-r_f)<Y_0(1+r_f).$$

Thus $R_A(Y_0(1+r_f)+a(\tilde{r}-r_f))>R_A(Y_0(1+r_f))$

f. Now we will use these observations.

<u>Case 1</u>: $\tilde{r}\geq r_f$

$$-\frac{U''(Y_0(1+r_f)+a(\tilde{r}-r_f))}{U'(Y_0(1+r_f)+a(\tilde{r}-r_f))}=R_A(Y_0(1+r_f)+a(\tilde{r}-r_f))$$
$$\leq R_A(Y_0(1+r_f))$$

Hence

$$U''(Y_0(1+r_f)+a(\tilde{r}-r_f))\geq -R_A(Y_0(1+r_f))U'(Y_0(1+r_f)+a(\tilde{r}-r_f))$$

Since $\tilde{r}-r_f\geq 0$ for this case,

(i) $\quad U''(Y_0(1+r_f)+a(\tilde{r}-r_f))(\tilde{r}-r_f) \geq -R_A(Y_0(1+r_f))U'(Y_0(1+r_f)+a(\tilde{r}-r_f))(\tilde{r}-r_f)$

<u>Case 2</u>: $\quad \tilde{r} < r_f$

$$-\frac{U''(Y_0(1+r_f)+a(\tilde{r}-r_f))}{U'(Y_0(1+r_f)+a(\tilde{r}-r_f))} = R_A(Y_0(1+r_f)+a(\tilde{r}-r_f))$$
$$\geq R_A(Y_0(1+r_f)).$$

Thus,

$$U''(Y_0(1+r_f)+a(\tilde{r}-r_f)) \leq -R_A(Y_0(1+r_f))U'(Y_0(1+r_f)+a(\tilde{r}-r_f)).$$

Since for this case $\tilde{r}-r_f < 0$,

(ii) $\quad U''(Y_0(1+r_f)+a(\tilde{r}-r_f))(\tilde{r}-r_f) \geq -R_A(Y_0(1+r_f))U'(Y_0(1+r_f)+a(\tilde{r}-r_f))(\tilde{r}-r_f)$

Since (i) and (ii) encompass "all the probability," we can write:

$$E\{U''(Y_0(1+r_f)+a(\tilde{r}-r_f))(\tilde{r}-r_f)\}$$
$$> -R_A(Y_0(1+r_f))\ E\{U'(Y_0(1+r_f)+a(\tilde{r}-r_f))(\tilde{r}-r_f)\}$$

Since by the FOC for a,
$E\{U'(Y_0(1+r_f)+a(\tilde{r}-r_f))(\tilde{r}-r_f)\} = 0$

we know $E\{U''(Y_0(1+r_f)+a(\tilde{r}-r_f))(\tilde{r}-r_f)\} \geq 0$

and $\dfrac{da}{dY_0} \geq 0$.

Key assumption: declining absolute risk aversion.

5.4. Let $\tilde{R}_A$ and $\tilde{R}_B$ represent the gross returns on the two assets ;

$$E(U(a\tilde{R}_A+(1-a)\tilde{R}_B)) > E(aU(\tilde{R}_A)+(1-a)U(\tilde{R}_B)) = aE(U(\tilde{R}_A))+(1-a)E(U(\tilde{R}_B))$$
$$= E(U(\tilde{R}_A)) = E(U(\tilde{R}_B))$$

since the utility function is concave and gross asset returns are identically distributed. This result means that the investor is going to invest in both securities – it is never optimal in this situation to invest only in one of the two assets. He thus chooses to diversify. Moreover if the investor cares only about the first two moments he will invest equal amounts in the assets to minimize variance. To show this note that the expected return on the portfolio is constant independently of the chosen allocation. A mean-variance investor will thus choose a to minimize the variance of

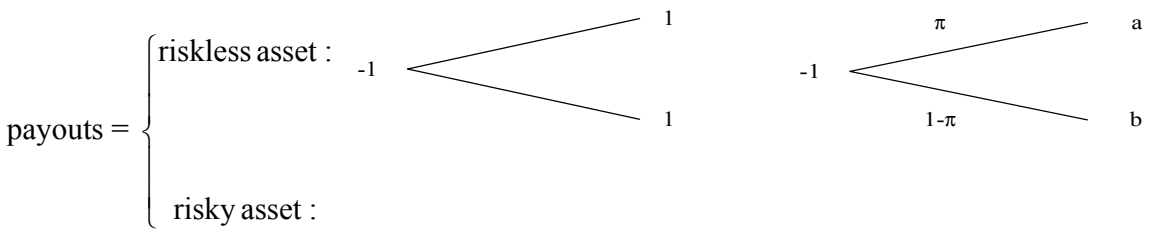
the portfolio. The latter is

$a^2 \operatorname{var}\tilde{R}_A + (1-a)^2 \operatorname{var}\tilde{R}_B = \operatorname{var}\tilde{R}_A(2a^2+1-2a) = \operatorname{var}\tilde{R}_A(1+2(a^2-a))$.

Minimizing the portfolio variance thus amounts to minimizing the quantity (a^2-a) which requires selecting a=1/2.

5.5 $Y_0 \equiv 1$

payouts =



Since the price of both assets is 1, the number of units purchased equals the value of the purchase.

Let $x_1 =$ units (value) of risk free asset demanded

$x_2 =$ units (value) of risky asset demanded

Since $Y_0 \equiv 1$, $x_1 + x_2 = 1$ and $x_2 = 1 - x_1$

a. Necessary condition for $x_1 > 0$:

"$r_f > a$" or, in this case $a < 1$

Necessary condition for the demand for the risky asset to be positive :

$\pi a + (1-\pi)b > 1$.

b. We would expect $\frac{dx_1}{da} < 0$

The optimization problem is

$$\max \pi U(x_1 \cdot 1 + (1-x_1)a) + (1-\pi)U(x_1 \cdot 1 + (1-x_1)b)$$

s.t. $x_1 + x_2 \le 1$ or $0 \le x_1 \le 1$

The F.O.C. for an interior solution is

$\pi U'(x_1 + (1-x_1)a)(1-a) + (1-\pi)U'(x_1 + (1-x_1)b)(1-b) = 0$. This equation implicitly defines $x_1 = x_1(a)$; taking the total differential yields :

$$\pi U'(x_1 + (1-x_1)a)(-1) + \pi U''(x_1 + (1-x_1)a)(1-a)\left[(1-a)\frac{dx_1}{da} + (1-x_1)\right]$$

$$+ (1-\pi)U''(x_1 + (1-x_1)b)(1-b)^2 \frac{dx_1}{da} = 0$$

$$\frac{dx_1}{da} = \frac{\pi U'(x_1 + (1-x_1)a) - \pi(1-a)(1-x_1)U''(x_1 + (1-x_1)a)}{\pi U''(x_1 + (1-x_1)a)(1-a)^2 + (1-\pi)U''(x_1 + (1-x_1)b)(1-b)^2} < 0$$

as the numerator is positive and the denominator negative.

We would expect that an increase in the probability of the unfavorable risky asset state a, would tend to decrease the demand for the risky asset. The "world" is becoming riskier.

Thus $\frac{dx_1}{d\pi} > 0$.

Chapter 6

6.1 A high β does not say anything about the level of diversification. The β speaks about the co-variations between the returns on an asset or a portfolio and the returns on the market portfolio. But it does not say anything about the degree of diversification of a portfolio. A portfolio with a high β may or may not be well diversified and it may or may not be far away from the efficient frontier.

6.2 Let $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$ be the total risk common to every asset. For an equally weighted portfolio:

$$\sigma_P^2 = \sum_{i=1}^{3} w_i^2 \sigma_i^2 = 3(\frac{1}{9})\sigma^2; \ \sigma_P = \sqrt{\frac{1}{3}}\,\sigma$$

The fraction of asset i's risk that it contributes to a portfolio is given by ρ_{iP}.
Without loss of generality consider asset 1.

$$\rho_{1P} = \frac{\mathrm{cov}(r_1, \sum_{i=1}^{3} w_i r_i)}{\sigma_1 \sigma_P}$$

$$= \frac{\sum_{i=1}^{3} w_i \,\mathrm{cov}(r_1, r_i)}{\sigma_1 \sigma_P}$$

$$= \frac{\frac{1}{3}\mathrm{cov}(r_1, r_1)}{\sigma_1 \sigma_P} \qquad (\text{since } \mathrm{cov}(r_i, r_j) = 0 \text{ if } i \neq j)$$

$$= \frac{\frac{1}{3}\sigma^2}{\sigma\sqrt{\frac{1}{3}}\sigma} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}} = 0.57735$$

$$(\text{check: } \sigma_P = \sum_{i=1}^{3}(\rho_{iP}\sigma_i)w_i = \sum_{i=1}^{3}\frac{1}{3}(\frac{1}{\sqrt{3}}\sigma) = \frac{1}{\sqrt{3}}\sigma)$$

Thus, the fraction 0.57735 of the asset's risk is contributed to the portfolio and the fraction 1 - 0.57735 = 0.42265 is diversified away.

6.3 The CML describes the risk/return tradeoff for efficient portfolios, while the SML describes the same tradeoff for arbitrary assets (that are in "M").

For arbitrary assets that have the same statistical characteristics as M (i.e. that are perfectly positively correlated with M), the two should give identical expected return estimates - and they do:

$$\text{SML: } E\tilde{r}_j = r_f + \frac{\rho_{jM}\sigma_j}{\sigma_M}\left[E\tilde{r}_M - r_f\right], \text{ for any asset j.}$$

If "j" is perfectly positively correlated with M,

$$E\tilde{r}_j = r_f + \frac{\sigma_M\sigma_j}{\sigma_M}\left[\frac{Er_M - r_f}{\sigma_M}\right]$$

$$= r_f + \sigma_j\left[\frac{Er_M - r_f}{\sigma_M}\right],$$ which is the equation of the capital market line for an efficient portfolio "j".

6.4

a. No! It is incorrect. The CAPM tells you to equate the expected return on the loan equal to r_f. Since there is some probability of default, you must set the rate higher than r_f in order to insure an expected return equal to r_f.

b. Let r_L = rate on the loans. You want

$$1.1 = 0.95(1+r_L) + 0.05(1)$$
$$r_L = 0.105263 \text{ or } 10.5263\%$$

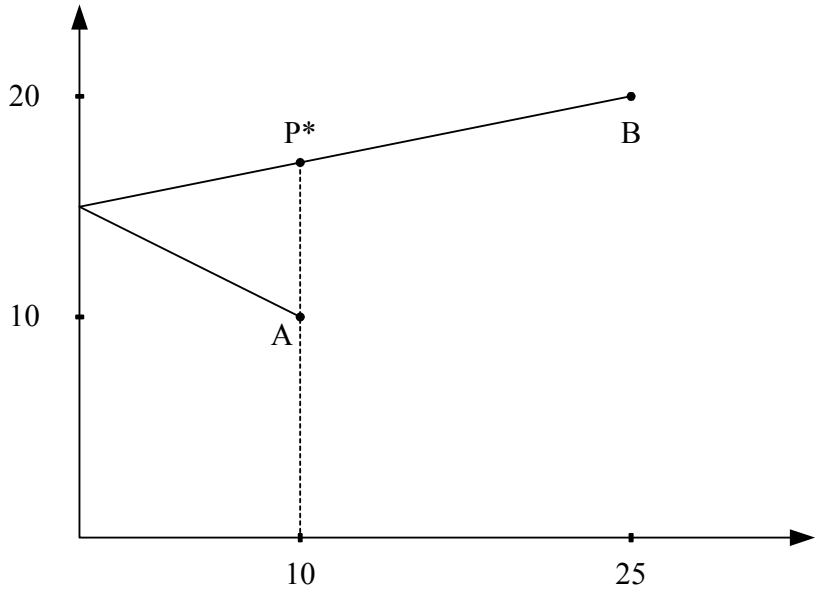
c. Again, let r_L = the rate on the loans:

$1.1 = 0.95(1+r_L) + 0.05(0.8)$ (if there is a default, only 80% of the value of the loan is recoverable).

$$1.1 = 0.95(1+r_L) + 0.04$$
$$1.06 = 0.95 + 0.95\, r_L$$
$$r_L = 0.11/0.95 = 0.1157 \text{ or } 11.57\%$$

6.5 This problem 'starkly' illustrates the gains to diversification. There are two ways to solve it.

a. Method 1 : use the hint



Since $\rho_{AB} = -1$

$$\sigma_P = 10\% = w_B\sigma_B - (1 - w_B)\sigma_A$$
$$= w_B(25\%) - (1 - w_B)(10\%)$$
$$10\% = w_B(25\%) - 10\% + 10\% w_B$$
$$w_B = \frac{20\%}{35\%} = \frac{4}{7}; w_A = \frac{3}{7}.$$

$Er_{P*} = \frac{3}{7}(10\%) + \frac{4}{7}(20\%) = 15.71\%$, a 5.71% gain.

b. Method 2 : first solve for the minimum risk portfolio.

$$\sigma_P = 0 = w_A\sigma_A - (1 - w_A)\sigma_B$$
$$0 = w_A(10\%) - (1 - w_A)(25\%)$$
$$w_A = \frac{5}{7}; w_B = \frac{2}{7}..$$

$$Er_{\substack{\text{min risk portfolio} \\ (\sigma_P = 0)}} = \frac{5}{7}(10\%) + \frac{2}{7}(20\%) = 12.85\%$$

Slope of line joining B + min risk portfolio is $\frac{20\% - 12.85\%}{25\%}$

Thus, $Er_{P*} = 12.85\% + \left(\frac{20\% - 12.85\%}{25\%}\right)(10\%) = 15.71\%$, a 5.71% gain.

6.6 a. The basis for answering this question is the following :
Let C, D be two assets with $\sigma_C < \sigma_D$. Let us consider adding some of 'D' to 'C'. Under what circumstances will the risk go down ? Let $w_C = 1 - w_D$.

$$\sigma_P^2 = (1 - w_D)\sigma_C^2 + w_D^2\sigma_D^2 + 2w_D(1 - w_D)\,\mathrm{cov}(\tilde{r}_C, \tilde{r}_D)$$

Then

$$\frac{\partial\sigma_P^2}{\partial w_D}\Big|_{w_D=0} = 2\left[\text{cov}(\tilde{r}_C, \tilde{r}_D) - \sigma_C^2\right]$$

If $\text{cov}(\tilde{r}_C, \tilde{r}_D) - \sigma_C^2 < 0$, if we add 'D' to 'C' the portfolio's risk will decline below that of σ_C. Equivalently

$$\text{cov}(\tilde{r}_C, \tilde{r}_D) < \sigma_C^2$$

$$\rho_{CD}\sigma_C\sigma_D < \sigma_C^2, \text{ or } \rho_{CD} < \frac{\sigma_C}{\sigma_D}$$

or, for the case at hand,

$$\rho_{\text{your port, Australian index}} < \frac{\sigma_{\text{your port.}}}{\sigma_{\text{Aust. index}}}.$$

b. and c.

So we have to compute these data from the sample statistics that we are given.

$$\hat{E}r_{your} = \frac{1}{6}\{.54 + .24 - .06 + .24 - .06 + .54\} = .24$$

$$\hat{E}r_{Aus.} = \frac{1}{6}\{.50 + .10 - .10 + .60 - .20 + .80\} = .2833$$

$$\hat{\sigma}^2_{your} = \frac{1}{5}\{(.54 - .24)^2 + (.24 - .24)^2 + (-.06 - .24)^2 + (.24 - .24)^2 + (-.06 - .24)^2 + (.54 - .24)^2\}$$

$$= \frac{1}{5}\{.09 + .09 + .09 + .09\} = .072$$

$$\hat{\sigma}^2_{Aus.} = \frac{1}{5}\{(.50 - .2833)^2 + (-.10 - .2833)^2 + (.10 - .2833)^2 + (.60 - .2833)^2 + (.20 - .2833)^2 + (.80 - .2833)^2\}$$

$$= \frac{1}{5}\{.04696 + .14692 + .03360 + .10030 + .23358 + .2670\}$$

$$= \frac{1}{5}\{.82836\}$$

$$= .165672$$

$$\text{cov}(r_{your}, r_{Aus.}) = \frac{1}{5}\{(.54 - .24)(.50 - .2833) + (.24 - .24)(-.10 - .2833) + (-.06 - .24)(.10 - .2833) + (.24 - .24)(.60 - .2833) + (-.06 - .24)(.20 - .2833) + (.54 - .24)(.80 - .2833)\}$$

$$= \frac{1}{5}\{.06501 + .05499 + .14499 + .15501\}$$

$$= .084$$

now check the above inequality :

$$\rho_{your,Aus.} = \frac{\text{cov}(r_{your}, r_{Aus.})}{\sigma_{your}\sigma_{Aus.}} = \frac{.084}{\sqrt{.072}\sqrt{.166}} = .77$$

Clearly $\rho = .77 > \dfrac{\sigma_{your}}{\sigma_{Aus.}} = \dfrac{.2683}{.4070} = .6592$

So the answer are :

b. no

c. no. These are, in fact, two ways of asking the same questions.

d. Whenever you are asked a question like this, it is in reference to a regression ; in this case

$$\tilde{r}_{Aus} = \hat{\alpha}_{Aus} + \hat{\beta}_{Aus} r_{your} + \tilde{\varepsilon}_{Aus}$$

$$\Rightarrow \sigma^2_{Aus} = (\hat{\beta}_{Aus})^2 \sigma^2_{r_{your}} + \sigma^2_{\varepsilon_{Aus}}$$

The fraction of Australian's variation explained by variations in your portfolio's return is

$$R^2 = \frac{(\hat{\beta}_{Aus})^2 \sigma^2_{your}}{\sigma^2_{Aus}} = \frac{\left(\dfrac{\rho_{Aus,your}\sigma_{Aus}\sigma_{your}}{\sigma^2_{your}}\right)^2 \sigma^2_{your}}{\sigma^2_{Aus}}$$

$$= \left(\rho_{Aus,your}\right)^2$$

$$= (.77)^2 = .59$$

e. No ! How could it be ? Adding Australian stocks doesn't reduce risk. Even if it did, you don't know that your portfolio is identical to the true M.

Chapter 7

7.1. Write the SML equation to make the market risk premium appear, then multiply by $\frac{\sigma_M}{\sigma_M}$,

$$E(r_j) = r_f + (E(r_M) - r_f)\beta_j = r_f + (E(r_M) - r_f)\frac{\sigma_{jM}}{\sigma_M^2}$$
$$= r_f + \frac{(E(r_M) - r_f)}{\sigma_M}\frac{\sigma_{jM}\sigma_M}{\sigma_M^2}.$$

Rewrite the last term

$$\frac{\sigma_{jM}\sigma_M}{\sigma_M^2} = \frac{\sigma_j\sigma_M\rho_{jM}\sigma_M}{\sigma_M^2} = \sigma_j\rho_{jM}.$$

Then we get

$$E(r_j) = r_f + \frac{(E(r_M) - r_f)}{\sigma_M}(\sigma_j\rho_{jM})$$

and the conclusion follows since $0 \le |\rho_{jM}| \le 1$.

7.2. Intuitively, the CML in the 'more risk averse economy' should be steeper, in view of its risk/return trade-off interpretation. This is true in particular because one would expect the risk free rate to be lower, as the demand for the risk free asset should be higher, and the return on the optimal risky portfolio to be higher, as the more risk averse investors require a higher compensation for the risk they bear. Note, however, that the Markovitz model is not framed to answer such a question explicitly. It builds on 'given' expected returns that are assumed to be 'equilibrium'. If we imagine, as in this question, a change in the primitives of the economy, we have to turn to our intuition to guess how these given returns would differ in the alternative set of circumstances. The model does not help us with this reasoning. For such (fundamental) questions, a general equilibrium setting will prove superior.

7.3 The frontier of the economy where asset returns are more correlated and where diversification opportunities are thus lower is contained inside the efficient frontier of the economy where assets are less correlated. If the risk free rate was constant, this would guarantee that the slope of the CML would be lower in the economy, and the reward for risk taking lower as well. This appears counter-intuitive – with less diversification opportunities those who take risks should get a higher reward –, an observation which suggests that the risk free rate should be lower in the higher correlation economy. Refer to our remarks in the solution to 6.2 : the CAPM model, by its nature, does not explicitly help us answer such a question.

7.4. If investors hold homogeneous expectations concerning asset returns, mean returns on risky assets -per dollar invested- will be the same. Otherwise they would face different efficient frontiers and most likely would invest different proportions in risky assets. Moreover, the marginal rate of substitution between risk and return would depend on the level of wealth.

7.5. Using standard notations and applying the formulas, we get
A = 3.77, B = 5.85, C = 2.65, D = 1.31

$$g = \begin{bmatrix} 1.529 \\ -0.059 \\ -0.471 \end{bmatrix} \quad h = \begin{bmatrix} -0.618 \\ 0.235 \\ 0.382 \end{bmatrix}$$

$$E(r_{MVP}) = 1.42$$

$$w_{MVP} = \begin{bmatrix} 0.652 \\ 0.275 \\ 0.072 \end{bmatrix}$$

$$E(r_{ZCP_3}) = 1.3028$$

$$w_{ZCP_3} = \begin{bmatrix} 0.725 \\ 0.248 \\ 0.028 \end{bmatrix}$$

7.6 a. The agent's problem is (agent i):

$$\max_{(x_1^i, x_2^i, \ldots, x_j^i)} E\left\{ U_i \left[(Y_0^i - \sum_j x_j^i)(1 + r_f) + \sum_j x_j^i (1 + \tilde{r}_j) \right] \right\}$$

The F.O.C. wrt asset j is :

$$E\{U_i'(\tilde{Y}_i)(-(1 + r_f) + (1 + \tilde{r}_j))\} = 0 \text{, or}$$

$$E\{U_i'(\tilde{Y}_i)(\tilde{r}_j - r_f)\} = 0 \qquad (1)$$

b. We apply the relationship

$Exy = cov(x, y) + ExEy$ to equation (1).

$$0 = E\{U_i'(\tilde{Y}_i)(\tilde{r}_j - r_f)\} = E\{U_i'(\tilde{Y}_i)\}E(\tilde{r}_j - r_f) + cov(U_i'(\tilde{Y}_i), \tilde{r}_j - r_f)$$

$$= E\{U_i'(\tilde{Y}_i)\}E(\tilde{r}_j - r_f) + cov(U_i'(\tilde{Y}_i), \tilde{r}_j)$$

Thus, $E\{U_i'(\tilde{Y}_i)\}E(\tilde{r}_j - r_f) = -cov(U_i'(\tilde{Y}_i), \tilde{r}_j) \qquad (2)$

c. We make use of the relationship

$cov(g(\tilde{x}), \tilde{y}) = E(g'(\tilde{x}))\, cov(\tilde{x}, \tilde{y})$, where we identify $g(\tilde{Y}_i) = U_i'(\tilde{Y}_i)$.

Apply this result to the R.H.S. of equation (2) yields :

$$E\{U_i'(\tilde{Y}_i)\}E(\tilde{r}_j - r_f) = -E(U_i''(\tilde{Y}_i))\, cov(\tilde{Y}_i, \tilde{r}_j). \qquad (3)$$

d. We can rewrite equation (3) as :

$$E(\tilde{r}_j - r_f) = \frac{-E\{U_i''(\tilde{Y}_i)\}}{E\{U_i'(\tilde{Y}_i)\}} \operatorname{cov}(\tilde{Y}_i, \tilde{r}_j)$$

Let us denote $R_{A_i} = \frac{-E\{U_i''(\tilde{Y}_i)\}}{E\{U_i'(\tilde{Y}_i)\}}$ as it is reminiscent (but not equal to) the Absolute Risk Aversion measure.

The above equation can be rewritten as

$$E(\tilde{r}_j - r_f) = \left(\frac{1}{\frac{1}{R_{A_i}}} \right) \operatorname{cov}(\tilde{Y}_i, \tilde{r}_j), \text{ or}$$

$$\left(\frac{1}{R_{A_i}} \right) E(\tilde{r}_j - r_f) = \operatorname{cov}(\tilde{Y}_i, \tilde{r}_j),$$

Summing over all agents i gives

$$\sum_{i=1}^{I} \left(\frac{1}{R_{A_i}} \right) E(\tilde{r}_j - r_f) = \sum_{i=1}^{I} \operatorname{cov}(\tilde{Y}_i, \tilde{r}_j), \text{ or}$$

$$E(\tilde{r}_j - r_f) \left(\sum_{i=1}^{I} \left(\frac{1}{R_{A_i}} \right) \right) = \operatorname{cov}\left(\sum_{i=1}^{I} \tilde{Y}_i, \tilde{r}_j \right)$$

Let us identify $\sum_{i=1}^{I} \tilde{Y}_i \equiv Y_{MO}(1 + \tilde{r}_M)$. Then we have :

$$E(\tilde{r}_j - r_f) \left(\sum_{i=1}^{I} \left(\frac{1}{R_{A_i}} \right) \right) = \operatorname{cov}\left(Y_{MO}(1 + \tilde{r}_M), \tilde{r}_j \right)$$

$$= Y_{MO} \operatorname{cov}(\tilde{r}_M, \tilde{r}_j)$$

Thus,

$$E(\tilde{r}_j - r_f) = \frac{Y_{MO}}{\left(\sum_{i=1}^{I} \left(\frac{1}{R_{A_i}} \right) \right)} \operatorname{cov}\left(\tilde{r}_M, \tilde{r}_j \right) \qquad (4)$$

e. Let w_j be the proportion of economy wide wealth invested in asset j.

Then, for all j

$$w_j E(\tilde{r}_j - r_f) = \frac{Y_{MO}}{\left(\sum_{i=1}^{I} \left(\frac{1}{R_{A_i}} \right) \right)} w_j \operatorname{cov}\left(\tilde{r}_M, \tilde{r}_j \right)$$

Thus,

$$\sum_{j=1}^{J} w_j E(\tilde{r}_j - r_f) = \frac{Y_{MO}}{\left(\sum_{i=1}^{I} \left(\frac{1}{R_{A_i}} \right) \right)} \sum_{j=1}^{J} w_j \operatorname{cov}\left(\tilde{r}_M, \tilde{r}_j \right).$$

It follows that

$$E\left(\sum_{j=1}^{J} w_j \tilde{r}_j - r_f\right) = \frac{Y_{MO}}{\left(\sum_{i=1}^{I}\left(\frac{1}{R_{A_i}}\right)\right)} \operatorname{cov}\left(\tilde{r}_M, \sum_{j=1}^{J} w_j \tilde{r}_j\right)$$

By construction,

$$\sum_{j=1}^{J} w_j \tilde{r}_j = \tilde{r}_M .$$

Then

$$E\tilde{r}_M - r_f = \frac{Y_{MO}}{\left(\sum_{i=1}^{I}\left(\frac{1}{R_{A_i}}\right)\right)} \operatorname{cov}(\tilde{r}_M, \tilde{r}_M)$$

$$E\tilde{r}_M - r_f = \frac{Y_{MO}}{\left(\sum_{i=1}^{I}\left(\frac{1}{R_{A_i}}\right)\right)} \operatorname{var}(\tilde{r}_M). \qquad (5)$$

f. (4) states that

$$E(\tilde{r}_j - r_f) = \frac{Y_{MO}}{\left(\sum_{i=1}^{I}\left(\frac{1}{R_{A_i}}\right)\right)} \operatorname{cov}(\tilde{r}_M, \tilde{r}_j)$$

From (5) $\frac{Y_{MO}}{\left(\sum_{i=1}^{I}\left(\frac{1}{R_{A_i}}\right)\right)} = \frac{E(\tilde{r}_M - r_f)}{\operatorname{var}(\tilde{r}_M)}$; substituting this latter expression into (4) gives :

$$E(\tilde{r}_j - r_f) = \frac{\operatorname{cov}(\tilde{r}_M, \tilde{r}_j)}{\operatorname{var}(\tilde{r}_M)} E(\tilde{r}_M - r_f)$$, the traditional CAPM.

Chapter 8

8.1. a.

$$q_i = \frac{1}{(1+r_i)^i}$$

$$q_1 = 0.91$$

$$q_2 = 0.8224$$

$$q_3 = 0.7424$$

These are in fact the corresponding risk-free discount bond prices.

b. The matrix is the same at each date. The n-period A-D matrix is then $[A-D]^n$. If we are in state i today, we look at line i, written $[A-D]_i^n$, and we sum the corresponding A-D prices to obtain a sure payoff of one unit in each future state. Since it is assumed we are in state 1

$$[A-D]_1 = 0.28 + 0.33 + 0.30 = 0.91 = q_1$$

$$[A-D]_1^2 = 0.8224 = q_2$$

$$[A-D]_1^3 = 0.7424 = q_3$$

8.2 The price of an A-D security is the (subjective) probability weighted MRS in the corresponding state. It is determined by three considerations: the discount factor which is imbedded in the MU of future consumption, the state probability and the relative scarcities reflected in the intertemporal marginal rate of substitution, that is, in the ratio of the future MU to the present MU. The latter is affected by the expected consumption/ endowment in the future state and by the shape of the agents'
utility functions (their rates of risk aversion).

8.3. We determine a term structure for each initial state.

To-day's state is 1:

$$1 + r_1^1 = \frac{1}{0.53 + 0.43} = 1.0417$$

$$(1+r_1^2)^2 = 1.0800 \qquad 1 + r_1^2 = 1.0392$$

$$(1+r_1^3)^3 = 1.1193 \qquad 1 + r_1^3 = 1.0383$$

To-day's state is 2 :

$$1 + r_2^1 = 1.0310$$

$$(1+r_2^2)^2 = 1.0679 \qquad 1 + r_2^2 = 1.0334$$

$$(1+r_2^3)^3 = 1.1067 \qquad 1 + r_2^3 = 1.0344$$

8.4. We determine the price of A-D securities for each date, starting with bond 1 for date 1, q_1 = 96/100 = 0.96. Then we use the method of pricing intermediate cash flows with A-D prices to

price bonds of longer maturity, for example the price of bond 2 is such that

$$900 = 100 \times q_1 + 1100 \times q_2 \equiv \frac{100}{1+r_1} + \frac{1100}{(1+r_2)^2}$$

which gives $q_2 = \frac{900 - 100 \times q_1}{1100} = 0.7309$. Similarly

$q_3 = 0.5331$

$q_4 = 0.3194$

$q_5 = 0.01608$

8.5. a. Given preferences and endowments, it is clear that the allocation {(4, 2, 2) ,(4, 2, 2)} is PO and feasible. In general, there is an infinity of PO allocations.

b. Yes, but only if one of the following securities is traded

$$s_1 = \begin{matrix} -1 \\ 1 \end{matrix} \quad \text{or } s_2 = \begin{matrix} 1 \\ -1 \end{matrix}$$

For example, Agent 1 would sell s_1, and Agent 2 would buy it. In general one security is not sufficient to complete the markets when there are two future states.

c. Agents will be happy to store the commodity for two reasons : consumption smoothing – they are pleased to transfer consumption from period 1 to period 2-, and in addition by shifting to tomorrow some of the current consumption they are able to reduce somewhat (but not fully) the endowment risk they face. For these two reasons, storing will enable them to increase their utility level.

d. Remember aggregate uncertainty means that the total quantity available at date 2 is not the same for all the states. If one agent is risk-neutral, he will however be willing to bear all the risks. Provided enough trading instruments exist, the consumption of the risk-averse agent can thus be completely smoothed out and this constitutes a Pareto Optimum.

8.6 a.

1. Because of the variance term diminishing utility, consumption should be equated across states for each agent.

2. There are many Pareto optima. For example, the allocations below are both Pareto optimal :

	t=0	t=1	
		$\theta = 1$	$\theta = 2$
<u>Allocation 1</u>			
Agent 1	4	3	3
Agent 2	4	3	3
<u>Allocation 2</u>			
Agent 1	5	4	4
Agent 2	3	2	2

The set of Pareto optima satisfies:
$\left\{\left((c_0^1, c_1^1, c_2^1), (c_0^2, c_1^2, c_2^2)\right): c_1^1 = c_2^1 \text{ (and thus } c_2^1 = c_2^2), c_1^2 + c_1^1 = 6; c_0^1 + c_0^2 = 8\right\}$
3. Yes. Given E(c) in the second period, var c is minimized.

b.
1. The Pareto optima satisfy

$$\max_{c_0^1, c_1^1, c_2^1} \left\{ c_0^1 + \frac{1}{4}\ln c_1^1 + \frac{3}{4}\ln c_2^1 + \lambda\left[8 - c_0^1 + \frac{1}{4}\ln(6 - c_1^1) + \frac{3}{4}\ln(6 - c_2^1)\right]\right\}$$

The F.O.C.'s are:

i) $c_1^0 : 1 - \lambda = 0$

ii) $c_1^1 : \frac{1}{4}\left(\frac{1}{c_1^1}\right) + \lambda(\frac{1}{4})\left(\frac{1}{6 - c_1^1}\right)(-1) = 0$

iii) $c_2^1 : \frac{3}{4}\left(\frac{1}{c_2^1}\right) + \lambda(\frac{3}{4})\left(\frac{1}{6 - c_2^1}\right)(-1) = 0$

From (ii) $\frac{1}{c_1^1} = \lambda\left(\frac{1}{6 - c_1^1}\right) \Rightarrow c_1^1 = \frac{6}{1+\lambda}$.

From (iii) $\frac{1}{c_2^1} = \lambda\left(\frac{1}{6 - c_2^1}\right) \Rightarrow c_2^1 = \frac{6}{1+\lambda}$

A Pareto optimum clearly requires $c_1^1 = c_2^1$, and thus $c_1^2 = c_2^2$;

If $\lambda > 1$, $c_0^1 = 0, c_0^2 = 8$

If $\lambda = 1$, $c_0^1 + c_0^2 = 8$

If $\lambda < 1$, $c_0^1 = 8, c_0^2 = 0$

The Pareto optimal allocation here and for the first part of the problem are the same. Both agents are risk averse and we would expect them to try to standardize period 1 consumption.

2. Agents' problems can be written

Agent 1: $\max_{Q_1^1, Q_2^1} (4 - P_1Q_1^1 - P_2Q_2^1) + \frac{1}{4}\ln(1 + Q_1^1) + \frac{3}{4}\ln(5 + Q_2^1)$

Agent 2: $\max_{Q_1^2, Q_2^2} (4 - P_1Q_1^2 - P_2Q_2^2) + \frac{1}{4}\ln(5 + Q_1^2) + \frac{3}{4}\ln(1 + Q_2^2)$

Market clearing conditions:

$$Q_1^1 + Q_1^2 = 0$$
$$Q_2^1 + Q_2^2 = 0$$

(both securities are in zero net supply).

The F.O.C.'s are:

Agent 1: $Q_1^1 : P_1 = \frac{1}{4}\left(\frac{1}{1 + Q_1^1}\right)$

$Q_2^1 : P_2 = \frac{3}{4}\left(\frac{1}{5 + Q_2^1}\right)$

Agent 2: $Q_1^2 : P_1 = \frac{1}{4}\left(\frac{1}{5+Q_1^2}\right)$

$$Q_2^2 : P_2 = \frac{3}{4}\left(\frac{1}{1+Q_2^2}\right)$$

These F.O.C.'s, together with market clearing imply, as expected:

$$\frac{1}{1+Q_1^1} = \frac{1}{5+Q_1^2} \Rightarrow Q_1^1 = 2; Q_1^2 = -2.$$

$$\frac{1}{5+Q_2^1} = \frac{1}{1+Q_2^2} \Rightarrow Q_2^1 = -2; Q_2^2 = +2.$$

Thus, $P_1 = \frac{1}{4}\left(\frac{1}{1+Q_1^1}\right) = \frac{1}{4}\left(\frac{1}{3}\right) = \frac{1}{12}$

$$P_2 = \frac{3}{4}\left(\frac{1}{5+Q_2^1}\right) = \frac{3}{4}\left(\frac{1}{3}\right) = \frac{3}{12}$$

Allocations at Equilibrium:

	t=0	t=1	
		$\theta = 1$	$\theta = 2$
Agent 1:	$4 - \frac{1}{12}(2) - \frac{3}{12}(-2) = 4\frac{1}{3}$	3	3
Agent 2:	$4 - \frac{1}{12}(-2) - \frac{3}{12}(2) = 3\frac{2}{3}$	3	3

This is a Pareto Optima, consumption is stabilized in t=1. However, since agent 1 had more consumption in the more likely state, he is paid in terms of t=0 consumption for agreeing to the exchanges. Agent 2 transfers t=0 wealth to him.

3. Now only (1,0) is traded. The C.E. will not be Pareto optimal as the market is incomplete. The C.E. is as follows:

   Agent 1: $\max_{Q_1^1}(4 - P_1Q_1^1) + \frac{1}{4}\ln(1+Q_1^1) + \frac{3}{4}\ln(5)$

   Agent 2: $\max_{Q_1^2}(4 - P_1Q_1^2) + \frac{1}{4}\ln(5+Q_1^2) + \frac{3}{4}\ln(1)$

   The F.O.C.'s are :

   Agent 1: $P_1 = \frac{1}{4}\left(\frac{1}{1+Q_1^1}\right)$

   Agent 2: $P_1 = \frac{1}{4}\left(\frac{1}{5+Q_1^2}\right)$

   Thus

   $$\frac{1}{1+Q_1^1} = \frac{1}{5+Q_1^2} \Rightarrow Q_1^1 = +2; Q_1^2 = -2. \quad P_1 = \frac{1}{4}\left(\frac{1}{3}\right) = \frac{1}{12}.$$

   Allocation

	t=0	t=1	
		$\theta = 1$	$\theta = 2$
Agent 1:	$4 - \frac{1}{6} = 3\frac{5}{6}$	3	5
Agent 2:	$4 + \frac{1}{6} = 4\frac{1}{6}$	3	1

Consumption is stabilized in state θ_1: effectively agent 1 buys consumption insurance from agent 2.

8.7 The Pareto optima satisfy:

$$\max_{c_0^1, c_1^1, c_2^1} .25 c_0^1 + .5\left[\frac{1}{2}\ln c_1^1 + \frac{1}{2}\ln c_2^1\right] + \lambda\left[6 - c_0^1 + \frac{1}{2}\ln(6 - c_1^1) + \frac{1}{2}\ln(6 - c_2^1)\right]$$

The F.O.C.'s are

$$c_0^1 : .25 - \lambda = 0$$

$$c_1^1 : .5(\frac{1}{2})\left(\frac{1}{c_1^1}\right) + \lambda(\frac{1}{2})\left(\frac{1}{6 - c_1^1}\right)(-1) = 0$$

$$c_2^1 : .5(\frac{1}{2})\left(\frac{1}{c_2^1}\right) + \lambda(\frac{1}{2})\left(\frac{1}{6 - c_2^1}\right)(-1) = 0$$

$$\frac{1}{2}\left(\frac{1}{c_1^1}\right) = \lambda\left(\frac{1}{6 - c_1^1}\right) \Rightarrow 6 - c_1^1 = 2c_1^1\lambda,\ c_1^1 = \frac{6}{1 + 2\lambda}.$$

$$\frac{1}{2}\left(\frac{1}{c_2^1}\right) = \lambda\left(\frac{1}{6 - c_2^1}\right) \Rightarrow 6 - c_2^1 = 2c_2^1\lambda,\ c_2^1 = \frac{6}{1 + 2\lambda}$$

Thus $c_1^1 = c_2^1$, and therefore $c_1^2 = c_2^2$;

If there is no aggregate risk and the agents preferences are the same state by state, then a Pareto optimum will require perfect risk sharing. This example has these features.

The Pareto optimum is clearly not unique. The set of Pareto optima can be described by:

For all $\lambda \geq 0$

	t=0	t=1	
		$\theta = 1$	$\theta = 2$
Agent 1:	$\begin{cases} 6 \text{ if } \lambda < .25 \\ 0 \text{ if } \lambda > .25 \end{cases}$	$\frac{6}{1 + 2\lambda}$	$\frac{6}{1 + 2\lambda}$
Agent 2:	$\begin{cases} 0 \text{ if } \lambda < .25 \\ 6 \text{ if } \lambda > .25 \end{cases}$	$6\left(1 - \frac{1}{1 + 2\lambda}\right)$	$6\left(1 - \frac{1}{1 + 2\lambda}\right)$

$\lambda = .25$, indeterminate

In the second case (state 2 endowment = 5 for agent 1, 3 for agent 2), there will be a Pareto optimum but it will be impossible to achieve perfect risk sharing as there is aggregate risk.

b. The agents' problems are:

Agent 1: $\max_{Q^1,R^1} .25(2-P_Q Q^1 - P_R R^1) + .5\left\{\frac{1}{2}\ln(2+Q^1) + \frac{1}{2}\ln(4+R^1)\right\}$

Agent 2: $\max_{Q^2,R^2} (4-P_Q Q^2 - P_R R^2) + \frac{1}{2}\ln(4+Q^2) + \frac{1}{2}\ln(2+R^2)$

Where, in equilibrium, $\begin{matrix} Q^1+Q^2=0 \\ R^1+R^2=0 \end{matrix}$ (market clearing). Both securities are in zero net supply.

The F.O.C.'s are

<u>Agent 1:</u>

$$Q^1: .25\,P_Q = .5\left(\frac{1}{2}\right)\left(\frac{1}{2+Q_1^1}\right) \Leftrightarrow P_Q = \frac{1}{2+Q_1^1}$$

$$R^1: .25\,P_R = .5\left(\frac{1}{2}\right)\left(\frac{1}{4+R^1}\right) \Leftrightarrow P_R = \frac{1}{4+R^1}$$

<u>Agent 2:</u>

$$Q^2: \quad P_Q = \frac{1}{2}\left(\frac{1}{4+Q^2}\right)$$

$$R^2: \quad P_R = \frac{1}{2}\left(\frac{1}{2+R^2}\right)$$

This implies:

$$\frac{1}{2+Q^1} = \frac{1}{2}\left(\frac{1}{4+Q^2}\right) = \frac{1}{2}\left(\frac{1}{4-Q^1}\right)$$

$$Q^1 = 2;\ Q^2 = -2$$

$$\frac{1}{4+R^1} = \frac{1}{2}\left(\frac{1}{2+R^2}\right) = \frac{1}{2}\left(\frac{1}{2-R^1}\right)$$

$2(2-R^1)=4+R^1$, $R^1=0$, $R^2=0$

As a result, $P_Q = \frac{1}{2+Q^1} = \frac{1}{2+2} = \frac{1}{4}$.

$$P_R = \left(\frac{1}{4+R^1}\right) = \frac{1}{4}$$

The implied allocations are thus:

	t=0	t=1	
		θ_1	θ_2
Agent 1:	2-1/2=1.5	4	4
Agent 2:	4+1/2=4.5	2	2

c. Let us assume the firm can introduce 1 unit of either security. Either way, the problems of the agents and their F.O.C.'s are not affected. What is affected are the market clearing conditions:

If 1 unit of Q is introduced	If 1 unit of R is introduced
$Q^1 + Q^2 = 1$	$Q^1 + Q^2 = 0$
$R^1 + R^2 = 0$	$R^1 + R^2 = 1$

Let's value the securities in either case.

<u>If one unit of Q is introduced:</u>

The F.O.C.'s <u>become</u>

<u>Agent</u> 1:

$$P_Q = \frac{1}{2+Q^1}$$

$$P_R = \frac{1}{4+R^1}$$

<u>Agent</u> 2:

$$P_Q = \left(\frac{1}{4+Q^2}\right)\frac{1}{2} = \frac{1}{2}\left(\frac{1}{4+1-Q^1}\right) = \frac{1}{2(5-Q^2)}$$

$$P_R = \frac{1}{2}\left(\frac{1}{2+R^2}\right) = \frac{1}{2}\left(\frac{1}{2-R^1}\right)$$

The equation involving R are unchanged. Thus P_R =1/4, R^1=0, R^2=0
For the security Q we need to solve:

$$\frac{1}{2+Q^1} = \frac{1}{2(5-Q^2)}, \Rightarrow 10 - 2Q^1 = 2 + Q^1$$

$$Q^1 = \frac{8}{3};\ Q^2 = 1 - Q^1 = 1 - \frac{8}{3} = -\frac{5}{3}$$

Thus $P_Q = \dfrac{1}{2+\frac{8}{3}} = \dfrac{1}{14/3} = \dfrac{3}{14} < .25$.

You know the price had to go down: there is more supply of the security.
The implied allocations are thus:

	t=0	t=1	
		$\theta = 1$	$\theta = 2$
Agent 1:	$2 - \frac{1}{2} = 1.5$	4	4
Agent 2:	$4 + \frac{1}{2} = 4.5$	2	2

<u>If one unit of R is introduced:</u>
The first order conditions become, with market clearing conditions imposed:

Agent 1
$$\begin{cases} P_Q = \dfrac{1}{2+Q^1} \\ P_R = \dfrac{1}{4+R^1} \end{cases}$$

Agent 2 $\begin{cases} P_Q = \frac{1}{2}\left(\frac{1}{4+Q^2}\right) = \frac{1}{2(4-Q^1)} \\ P_R = \frac{1}{2}\left(\frac{1}{2+R^2}\right) = \frac{1}{2}\left(\frac{1}{3-R^1}\right) \end{cases}$

So, P_Q is unchanged, and P_Q=1/4 $Q^1 = 2, Q^2 = -2$.

Solving for P_R:

$$\frac{1}{4+R^1} = \frac{1}{2(3-R^1)}, \quad R^1 = \frac{2}{3}; R^2 = 1 - R^1 = \frac{1}{3}$$

$$P_{R^1} = \frac{1}{4+R^1} = \frac{1}{4+\frac{2}{3}} = \frac{1}{\frac{14}{3}} = \frac{3}{14} < .25$$

	t=0	t=1	
		$\theta = 1$	$\theta = 2$
Agent 1:	$2 - \frac{1}{2} - \frac{2}{3}\frac{3}{14} = 2 - \frac{9}{14}$	4	4+2/3
Agent 2:	$4 + \frac{1}{2} - \frac{1}{3}\frac{3}{14} = 4 + \frac{6}{14}$	2	2+1/3

The firm is indifferent as to which security it sells – either way it receives the same thing. Either way a Pareto optimum is achieved since, with no short sales constraints, the market is complete. Thus a C.E. is Pareto Optimal.
Agent 1 wishes to transfer income to period t=1. The introduction of more securities of either type will reduce the cost to him of doing that. Agent 2, however, will receive lower prices – either way – for the securities he issues. He will be hurt.

8.8 a. At a P.O. allocation there is no waste and there are no possibilities to redistribute goods and make everyone better off. From the viewpoint of social welfare there seems to be no argument not to search for the realization of a Pareto Optimum. Beyond considerations of efficiency, however, considerations of social justice might suggest some non-optimal allocations are in fact socially preferable to some Pareto optimal ones. These issues are at the heart of many political discussions in a world where redistribution across agents is not costless. From a purely financial perspective, we associate the failure to reach a Pareto optimal allocation with the failure to smooth various agents' consumptions across time or states as much as would in fact be feasible. Again there is a loss in welfare and this is socially relevant: we should care.

b. The answer to a indicates we should care since complete markets are required to guarantee that a Pareto optimal allocation is reached. Are markets complete? certainly not! Are we far from complete markets? Would the world be much better with significantly more complete markets? This is a subject of passionate debates that cannot be resolved here. You may want to re-read the concluding comments of Chapter 1 at this stage.

Chapter 9

9.1. a. The CCAPM is an intertemporal model whereas the CAPM is a one-period model. The CCAPM makes a full investors homegeneity assumption but does not require specific utility functions.

b. The key contribution of the CCAPM resides in that the portfolio problem is indeed inherently intertemporal. The link with the real side of the economy is also more apparent in the CCAPM which does provide a better platform to think about many important questions in asset management.

c. The two models are equivalent in a one-period exchange economy since then aggregate consumption and wealth is the same. More generally, the prescriptions of the two models would be very similar in situations where consumption would be expected to be closely correlated with variations in the value of the market portfolio.

9.2. a. $\max(0, S_{t+1}(\theta)-p^*)$

b. $S_{t+1}(\theta) = \sum_{\tau=1}^{\infty} \delta^{\tau} \frac{1/c_{t+1+\tau}}{1/c_{t+1}(\theta)} c_{t+1+\tau} = \frac{\delta}{1-\delta} c_{t+1}(\theta)$

c. $q_{t+1}(\theta) = p(\theta)\delta \frac{c_t}{c_{t+1}(\theta)}$

d. The price of the option is,

$$C_t = \sum_{\theta' \in A} q_{t+1}(\theta')\left(S(\theta') - p^*\right)$$

where A is a set of states θ' for which $(S_{t+1}(\theta')-p^*) \geq 0$.

9.3. a. $(S_{t+1}(\theta)-p^*)$

b. $S_{t+1}(\theta) = \sum_{\tau=1}^{\infty} \delta^{\tau} \frac{1/c_{t+1+\tau}}{1/c_{t+1}(\theta)} c_{t+1+\tau} = \frac{\delta}{1-\delta} c_{t+1}(\theta)$

c. $q_{t+1}(\theta) = p(\theta)\delta \frac{c_t}{c_{t+1}(\theta)}$

d. The price of the forward contract is,

$$F_t(\theta) = \sum_{\theta} q_{t+1}(\theta)\left(S_{t+1}(\theta) - p^*\right).$$

9.4. a. After maximization, the pricing kernel from date 0 to date t takes the form $\frac{\delta^t}{\delta^0}\frac{c_0}{c_t} = m_t$. Now the value of the wealth portfolio is $P_0 = E_0 \sum_{t=0}^{T} m_t e_t$. At equilibrium we have $e_t = c_t$.

Proportionality follows immediately from $E\sum_{t=0}^{T} m_t e_t = E\sum_{t=0}^{T} m_t c_t$. With log utility we even have $P_0 = \frac{1}{1-\delta} c_0$.

b. Let us first define the return on the wealth portfolio as $\tilde{r}_1 = \frac{P_1 + e_1 - P_0}{P_0}$. Inserting prices and rearranging gives $\tilde{r}_1 = \frac{\frac{\delta}{1-\delta} c_1 + c_1 - \frac{1}{1-\delta} c_0}{\frac{1}{1-\delta} c_0}$. Defining consumption growth as $\bar{c}_{t+1} = c_{t+1} - c_t$ we get $\tilde{r}_1 = \frac{1}{c_0} \bar{c}_1$.

c.

$$
\begin{aligned}
P_0 &= 100 E_0[m_1 + m_2] \\
&= 100\delta E_0\left[\frac{c_0}{c_1} + \delta \frac{c_0}{c_2}\right] \\
&= 100\delta E_0\left[\frac{(c_2 - c_1)c_0 + (1+\delta)c_1 c_0}{c_1 c_2}\right] \\
&= 100\delta E_0\left[\{\tilde{r}_2 + (1+\delta)\}\frac{c_0}{c_2}\right]
\end{aligned}
$$

9.5.

$$
\begin{aligned}
P_i &= \sum q_s d_{is} \\
1 &= \sum q_s \frac{d_{is}}{P_i} \\
&= \sum q_s R_{is} \\
&= \sum m_s \pi_s R_{is} \\
&= E(mR_i) \\
&= E(m)E(R_i) + Cov(m, R_i) \\
&= \frac{E(R_i)}{1+r_f} + Cov(m, R_i) \\
R_f &= E(R_i) + Cov(m, R_i) R_f
\end{aligned}
$$

and

$$
R_f = E(R_M) + Cov(m, R_M) R_f
$$

$$
\frac{E(R_i) - R_f}{E(R_M) - R_f} = \frac{Cov(m, R_i)}{Cov(m, R_M)}
$$

$$
E(R_i) - R_f = \frac{Cov(m, R_i)}{Cov(m, R_M)}[E(R_M) - R_f]
$$

Note that if $Cov(m,R_M) = Var(R_M)$ we have exactly the CAPM equation. This relation holds for example with quadratic utility.

Chapter 10

10.1. a. Markets are complete. Find the state prices from

$$\begin{aligned} &5q_1 + 10q_2 + 15q_3 = 8 \\ &q_1 + q_2 + q_3 = \frac{1}{1.1} \\ &3q_3 = 1 \end{aligned} \quad \rightarrow \quad \begin{aligned} q_1 &= 0.55151 \\ q_2 &= 0.02424 \\ q_3 &= 1/3 \end{aligned}$$

b. The put option has a price of $3q_1$. Risk neutral probabilities are derived from

$$\pi_s = 1.1q_S \text{ where } s = 1, 2, 3 \qquad \begin{aligned} \pi_1 &= 0.60667 \\ \pi_2 &= 0.02667 \\ \pi_3 &= 0.36667 \end{aligned}$$

c. Consumption at date 0 is 1. The pricing kernel is given by

$$m_s = \frac{q_S}{p_s} \text{ where } s = 1, 2, 3 \qquad \begin{aligned} m_1 &= 1.83838 \\ m_2 &= 0.06060 \\ m_3 &= 0.11111 \end{aligned}$$

10.2. $m_s = \frac{q_S}{p_s}$ where $s = 1, 2, 3$ Program for agent 1

$$\max_{c_1^1, c_1^2} \left\{ \left(10 + q1 + 5q_2 - c_1^1 q_1 - c_1^2 q_2\right) + \left(\frac{1}{3}\ln c_1^1 + \frac{2}{3}\ln c_1^2\right) \right\}$$

The FOC is

$$-q_1 + \frac{1}{3}\frac{1}{c_1^1} = 0$$

$$-q_2 + \frac{2}{3}\frac{1}{c_1^2} = 0$$

and similarly for agent 2. This yields

$$\frac{1}{c_1^1} = \frac{1}{c_2^1}, \frac{1}{c_1^2} = \frac{1}{c_2^2}$$

$$\Leftrightarrow c_1^1 = c_2^1, c_1^2 = c_2^2$$

Using the market clearing conditions we get

$$c_1^1 = c_2^1 = \frac{5}{2}$$

$$c_1^2 = c_2^2 = \frac{11}{2}$$

so that $q_1 = 2/15$ and $q_2 = 4/33$.

Now construct the risk neutral probabilities as follows:

$$\pi_1 = \frac{q_1}{q_1 + q_2}$$

$$\pi_2 = \frac{q_2}{q_1 + q_2}$$

which satisfy the required conditions to be probabilities.
Computation of the risk-free rate is as usual: $q_1 + q_2 = 1/(1+r_f)$.
The market value of endowments can be computed as follows

$$MV_i = \frac{1}{1+r_f}\left(\pi_1 e_i^1 + \pi_2 e_i^2\right) \equiv q_1 e_i^1 + q_2 e_i^2 .$$

10.3. Options and market completeness.

The put option has payoffs [1,1,1,0]. The payoff matrix is then

$$m = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Of course, the fourth row gives the payoffs of the put option. We have to solve the system

$$mw = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrix on the RHS is the A-D securities payoff matrix. The solution is

$$w = \begin{pmatrix} 1 & -2 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{pmatrix}$$

We could also have checked the determinant condition on matrix m, which states that for a square matrix (number of states = number of assets), if the determinant is not null, then the system has a unique solution. Here Det(m) = -1.

10.4. a. An A-D security is an asset that pays out 1 unit of consumption in a particular state of the world. The concept is very useful since if are able to extract A-D prices from traded assets they enable us to price every complex security. This statement is valid even if no A-D security is traded. To price a complex security from A-D prices, make up the portfolio of A-D securities providing the same state-by-state payoff as the security to be priced and check what is the cost of this portfolio.

b. Markets are not complete : Determinant of the payoff matrix = 0.

c. No: # of assets < # of states.

Completeness can be reached by adding a put on asset one with strike 12 (Det = 126).

A-D security from calls:
long call on B (strike 5), two short calls on B (strike 6), long call on B (strike 7)

$$\begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} -2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

An A-D security with puts:

long put on B (strike 8), two short puts on B (strike 7), long put on B (strike 6)

$$\begin{pmatrix}3\\2\\1\\0\end{pmatrix} -2\begin{pmatrix}2\\1\\0\\0\end{pmatrix} +\begin{pmatrix}1\\0\\0\\0\end{pmatrix} =\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

Chapter 12

12.1. a.

$$EU = 1 + .96(.5\times\ln 1.2 + .5\times\ln .833) + (.96)^2(.25\times\ln 1.44 + .5\times\ln 1 + .25\times\ln .6944) = 0$$

b.

The maximization problem of the representative agent is

max $[\ln(c_0) + \delta(\pi_{11}\ln(c_{11}) + \pi_{12}\ln(c_{12})) + \delta^2(\pi_{21}\ln(c_{21}) + \pi_{22}\ln(c_{22}) + \pi_{23}\ln(c_{23}))]$

s.t. $e_0 + q_{11}e_{11} + q_{12}e_{12} + q_{21}e_{21} + q_{22}e_{22} + q_{23}e_{23} = c_0 + q_{11}c_{11} + q_{12}c_{12} + q_{21}c_{21} + q_{22}c_{22} + q_{23}c_{23}$

(take the consumption at date 0 as a numeraire, its prise is at 1;

q_{ij} is time 0 price of AD security that pays 1 unit of consumption at date i in state j)

The Lagrangian is given by

$$L = EU + \lambda\begin{pmatrix} e_0 + q_{11}e_{11} + q_{12}e_{12} + q_{21}e_{21} + q_{22}e_{22} + q_{23}e_{23} \\ -c_0 + q_{11}c_{11} + q_{12}c_{12} + q_{21}c_{21} + q_{22}c_{22} + q_{23}c_{23} \end{pmatrix}$$

FOC's are:

$$\frac{\partial L}{\partial c_0} = \frac{1}{c_0} - \lambda = 0$$

$$\frac{\partial L}{\partial c_{11}} = \pi_{11}\delta\frac{1}{c_{11}} - \lambda q_{11} = 0$$

.

.

.

$$\frac{\partial L}{\partial c_{23}} = \pi_{23}\delta^2\frac{1}{c_{23}} - \lambda q_{23} = 0$$

A-D prices, risk neutral probabilities, and the pricing kernel can be derived easily from the FOC's. For example, at date t = 0 we have

$$q_{11} = \pi_{11}\delta\frac{1}{c_{11}\lambda} = \pi_{11}\delta\frac{c_0}{c_{11}} = \pi_{11}\frac{MU_{11}}{MU_0} = \pi_{11}m_{11}$$

...

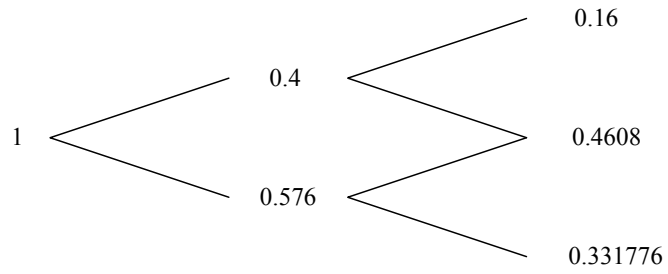
$$q_{23} = \pi_{23}\delta^2\frac{1}{c_{23}\lambda} = \pi_{23}\delta^2\frac{c_0}{c_{23}} = \pi_{23}\frac{MU_{23}}{MU_0} = \pi_{23}m_{23}$$

where m_{ij} is the pricing kernel. Risk neutral probabilities at date one are given by:

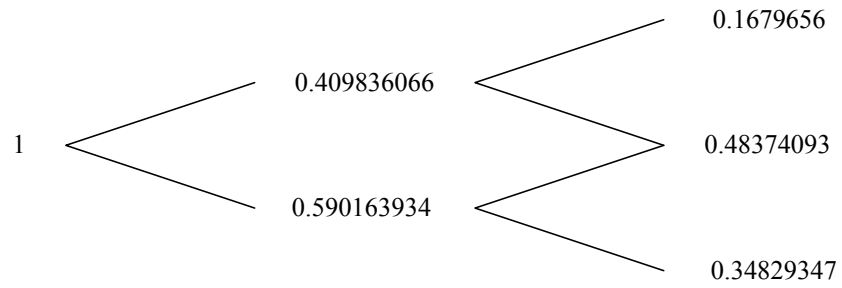
$$\pi_{11}^{RN} = \frac{q_{11}}{q_{11} + q_{12}} \text{ and } \pi_{12}^{RN} = \frac{q_{12}}{q_{11} + q_{12}}$$

$$\pi_{21}^{RN} = \frac{q_{21}}{q_{21} + q_{22} + q_{23}}, \pi_{22}^{RN} = \frac{q_{22}}{q_{21} + q_{22} + q_{23}} \text{ and } \pi_{23}^{RN} = \frac{q_{23}}{q_{21} + q_{22} + q_{23}}$$

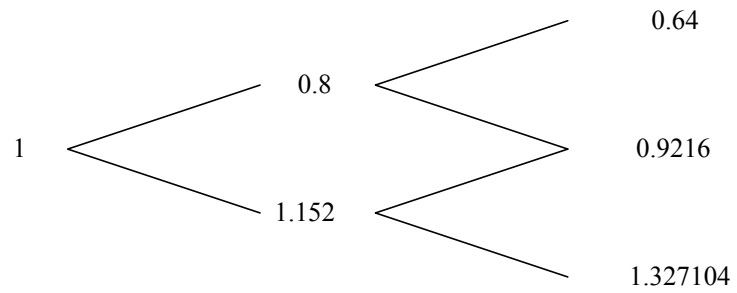
state prices



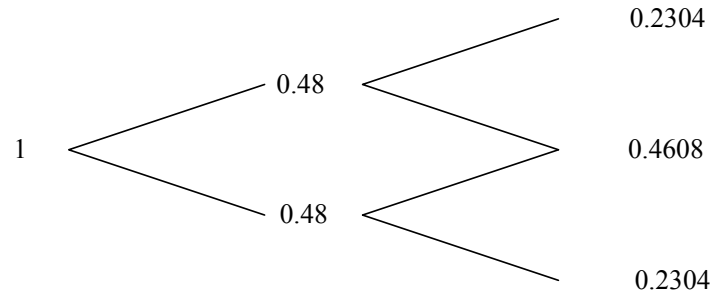
risk neutral probabilities



pricing kernel



c.
valuation



value 2.8816

d.
The one period interest rate at date zero is:

$$r_{0,1} = \frac{1}{(q_{11} + q_{12})} - 1 = 2.459\% .$$

The two period interest rate at date zero is:

$$r_{0,2} = \sqrt{\frac{1}{(q_{21} + q_{22} + q_{23})}} - 1 = 2.459\% .$$

Even though the economy is stochastic with log utility there is no term premia.

The price of a one period bond is $q_b(1) = \frac{1}{1.02459} = .976$ and the price of a two period bond is $q_b(2) = \frac{1}{(1{,}02459)^2} = .953$.

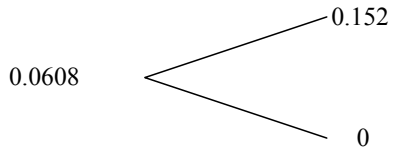
e. The valuation of the endowment stream is

	2.8816	2.352	1.44
price	1.8816	1.152	
space		1.63333333	1
		0.8	
			0.694444444

At date one and two we have one value with payoffs (upper cell) and the value after the cash flow arrived (lower cell).

The value of the option, using either state prices, pricing kernel, or risk neutral valuation, is

option value



f.

The price process is as in e. Now we need to solve for u, d, R, and risk neutral probabilities.

$$u = \frac{2.352}{1.8816} = \frac{1.44}{1.152} = 1.25$$

$$d = \frac{1.6333}{1.8816} = .8681$$

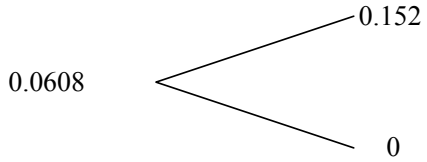
$$R = 1 + r = 1.02459$$

$$q_{11} = \frac{R - d}{u - d} = \frac{1.02459 - .8681}{1.25 - .8681} = .4098$$

(Compare this value with q_{11} in b))

The value of the option is

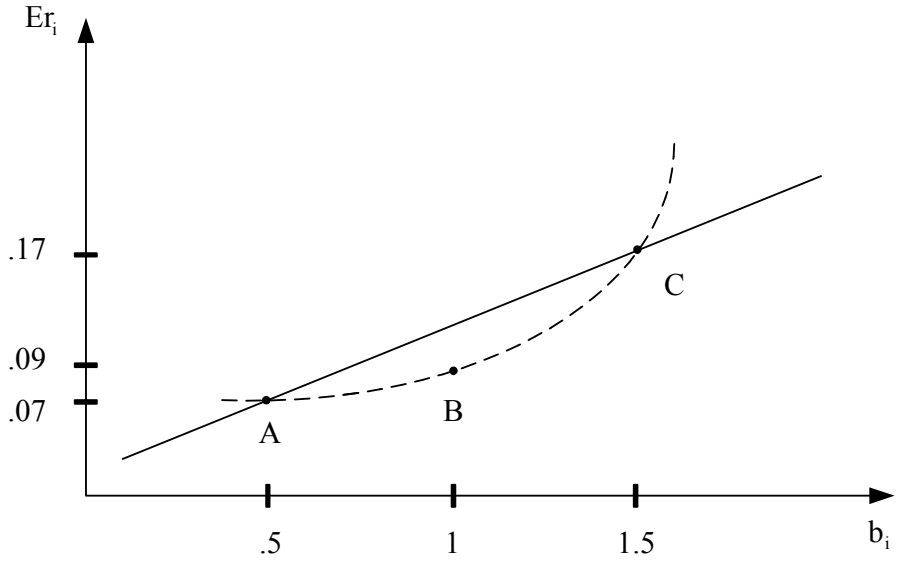
option value



g.
In part b we saw that pricing via A-D prices, risk-neutral probabilities, and pricing kernel are essentially the same. These methods rely on the payoffs of the endowment stream. In contrast to b, risk neutral probabilities are elicited in part f from the price process. Of course, the risk-neutral probabilities are the same as in b. This is not surprising since prices are derived from utility maximization of the relevant cash flows. Thus risk-neutral probabilities of the cash flow stream coincide with the risk neutral probabilities of the price of the asset.

Chapter 13

13.1 a.



b. Using A, B ; we want

$$b_P = 0 = w_A b_A + (1 - w_A) b_B$$
$$0 = w_A(.5) + (1 - w_A)(1)$$
$$.5 w_A = 1; w_A = 2, w_B = -1$$

Using B, C :

$$b_P = 0 = w_B b_B + w_C b_C$$
$$0 = w_B(1) + w_C(1.5)$$
$$0 = w_B + 1.5 - 1.5 w_B$$
$$.5 w_B = 1.5; w_B = 3, w_C = -2$$

c. We need to find the proportions of A and C that give the same b as asset B.

Thus, $b_B = 1 = w_A b_A + (1 - w_A) b_C$

$$1 = w_A(.5) + (1 - w_A)(1.5)$$
$$\Rightarrow w_A = \frac{1}{2}; w_C = \frac{1}{2}$$

With these proportions :

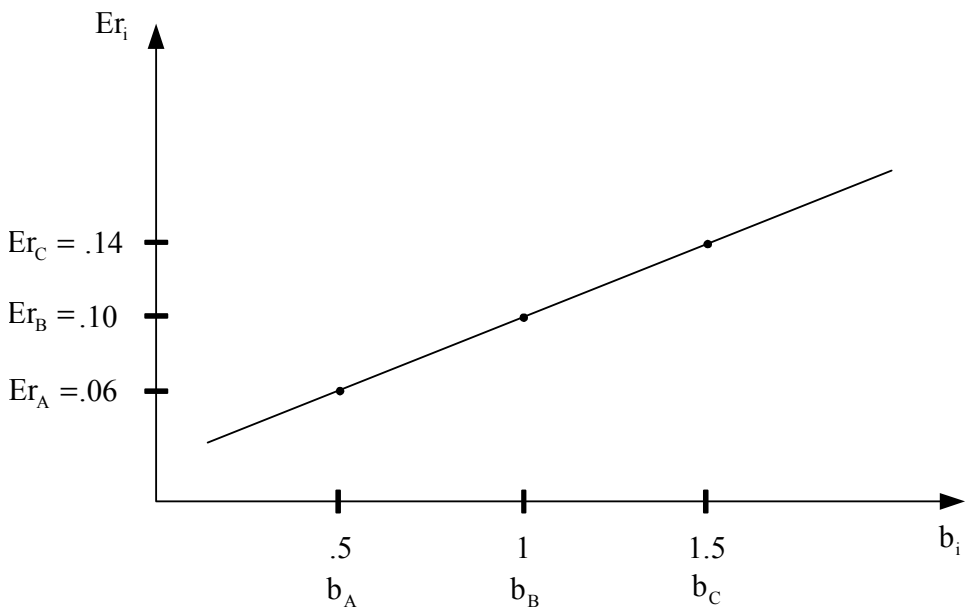
$$Er_P = \frac{1}{2} Er_A + \frac{1}{2} Er_C$$
$$= \frac{1}{2}(.07) + \frac{1}{2}(.17) = .12 > .09 = Er_B$$
$$r_{P_{\substack{w_A = \frac{1}{2} \\ w_C = \frac{1}{2}}}} = .12 + 1F_1 + e_P$$
$$R_{P_{w_B = 1}} = .09 + 1F_1 + e_B, \qquad \text{cov}(e_P, e_B) \equiv 0.$$

Now we assume these assets are each well diversified portfolios so that $e_P = e_B \equiv 0$.

An arbitrage portfolio consist of shorting B and buying the portfolio composed of $w_A = w_C = \frac{1}{2}$ in equal amounts, you will earn 3% riskless.

d. As a result the prices of A, C will rise and their expected returns fall. The opposite will happen to B.

e.



There is no longer an arbitrage opportunity : expected returns are consistent with relative systematic risk.

13.2. a. Since $\text{cov}(\tilde{r}_M, \varepsilon_j) = 0$,

$$\begin{aligned}\sigma_j^2 &= \text{var}(\alpha_j) + \text{var}(\beta_{jM}\tilde{r}_M) + \text{var}(\varepsilon_j)\\ &= 0 + \beta_{jM}^2\sigma_M^2 + \sigma_{\varepsilon j}^2\\ &= \beta_{jM}^2\sigma_M^2 + \sigma_{\varepsilon j}^2\end{aligned}$$

b. $\sigma_{ij} = \text{cov}(\alpha_i + \beta_{iM}\tilde{r}_M + \tilde{\varepsilon}_i, \alpha_j + \beta_{jM}\tilde{r}_M + \tilde{\varepsilon}_j)$

$= \text{cov}(\beta_{iM}\tilde{r}_M + \tilde{\varepsilon}_i, \beta_{jM}\tilde{r}_M + \tilde{\varepsilon}_j)$ (constants do not affect covariances)

$= \text{cov}(\beta_{iM}\tilde{r}_M + \tilde{\varepsilon}_i, \beta_{jM}\tilde{r}_M) + \text{cov}(\beta_{iM}\tilde{r}_M + \tilde{\varepsilon}_i, \tilde{\varepsilon}_j)$ (since $\tilde{r}_M, \tilde{\varepsilon}_j$ are independent)

$= \text{cov}(\beta_{iM}\tilde{r}_M, \beta_{jM}\tilde{r}_M) + \text{cov}(\tilde{\varepsilon}_i, \beta_{jM}\tilde{r}_M) + \text{cov}(\beta_{iM}\tilde{r}_M, \tilde{\varepsilon}_j) + \text{cov}(\tilde{\varepsilon}_i, \tilde{\varepsilon}_j)$ (since $\tilde{r}_M, \tilde{\varepsilon}_i$ are independent)

$= \text{cov}(\beta_{iM}\tilde{r}_M, \beta_{jM}\tilde{r}_M)$; since by contruction of the regression relationship all other covariances are zero.

$= \beta_{iM}\beta_{jM}\,\text{cov}(\tilde{r}_M, \tilde{r}_M) = \beta_i\beta_j\sigma_M^2$

13.3. The CAPM model is an equilibrium model built on structural hypotheses about investors' preferences and expectations and on the condition that asset markets are at equilibrium. The APT observes market prices on a large asset base and derives, under the hypothesis of no arbitrage, the implied relationship between expected returns on individual assets and the expected returns on a small list of fundamental factors.
Both models lead to a linear relationship explaining expected returns on individual assets and portfolios. In the case of the CAPM, the SML depends on a single factor, the expected excess return on the market portfolio. The APT opens up the possibility that more than one factor are priced in the market and are thus necessary to explain returns. The return on the market portfolio could be one of them, however. Both models would be compatible if the market portfolio were simply another way to synthesize the several factors identified by the APT: under the conditions spelled out in section 12.4, the two models are essentially alternative ways to reach the same 'truth'. Empirical results tend to suggest, however, that this is not likely to be the case.
Going from expected returns to current price is straightforward but requires formulating, alongside expectations on future returns, expectations on the future price level and on dividend payments.

13.4. The main distinction is that the A-D theory is a full structural general equilibrium theory while the APT is a no-arbitrage approach to pricing. The former prices all assets from assumed primitives. The latter must start from the observations of quoted prices whose levels are not explained.
The two theories are closer to one another, however, if one realizes that one can as well play the 'no arbitrage' game with A-D pricing. This is what we did in Chapter VIII. There the similarities are great: start from given unexplained market prices for 'complex' securities and extract from them the prices of the fundamental securities. Use the latter for pricing other assets or arbitrary cash flows.
The essential differences are the following: A-D pricing focuses on the concept of states of nature and the pricing of future payoffs conditional on the occurrence of specific future states. The APT replaces the notion of state of nature with the 'transversal' concept of factor. While in the former the key information is the price of one unit of consumption good in a specific future date-state, in the latter the key ingredient extracted from observed prices is the expected excess return obtained for bearing one unit of a specified risk factor.

13.5 True. The APT is agnostic about beliefs. It simply requires that the observed prices and returns, presumably the product of a large number of agents trading on the basis of heterogeneous beliefs, are consistent in the sense that no arbitrage opportunities are left unexploited.

Chapter 15

15.1. a. These utility functions are well known. Agent 1 is risk-neutral, agent 2 is risk-averse.
b. A PO allocation is one such that agent 2 gets smooth consumption.
c. Given that agent 2 is risk-averse, he buys A-D1 and sells AD2, and gets a smooth consumption; Agent 1 is risk-neutral and is willing to buy or sell any quantity of A-D securities. We can say agent 2 determines the quantities, and agent 1 determines the prices of the AD securities.
Solving the program for agent 1 gives the following FOC:

$q_1 = \delta\pi$

$q_2 = \delta(1-\pi)$

The price of AD securities depends only on the probability of each state.
Agent 2's optimal consumption levels are $c^2 = c^2(\theta_1) = c^2(\theta_2) = (2\delta(1-\pi)+1)/(1+\delta)$ which is 1 if $\pi = 0.5$.
d. Note: it is not possible to transfer units of consumption across states. Price of the bond is δ. Allocation will not be PO.
Available security

t=0	t=1	
$-p^b$	θ_1	θ_2
	1	1

Let the desired holdings of this security by agents 1 and 2 be denoted by Q_1 and Q_2 respectively.
Agent maximization problems :

Agent 1 : $\max_{Q_1}(1-Q_1p^b) + \delta\{\pi(1+Q_1) + (1-\pi)(1+Q_1)\}$

Agent 2 : $\max_{Q_2} \ln(1-Q_2p^b) + \delta\{\pi \ln Q_2 + (1-\pi)\ln(2+Q_2)\}$

The F.O.C.s are :

1. Agent 1 : $-p^b + \delta = 0$ or $p^b = \delta$.

Agent 2 : $\dfrac{1}{1-Q_2p^b}(-p^b) + \delta\left\{\dfrac{\pi}{Q_2} + \dfrac{(1-\pi)}{2+Q_2}\right\} = 0$

2. We know also that $Q_1 + Q_2 = 1$ in competitive equilibrium in addition to these equations being satisfied. Substituting $p^b = \delta$ into the 2nd equation yields, after simplification,

$$(1+\delta)(Q_2)^2 + (1+2\pi\delta)Q_2 - 2\pi = 0$$

$$Q_2 = \frac{-(1+2\pi\delta) \pm \sqrt{(1+2\pi\delta)^2 - 4(1+\delta)(-2\pi)}}{2(1+\delta)}$$

$$= \frac{-(1+2\pi\delta) \pm \sqrt{1+4\pi^2\delta^2 + 4\pi\delta + 8\pi + 8\pi\delta}}{2(1+\delta)}$$

$$Q_2 = \frac{-(1+2\pi\delta) \pm \sqrt{(1+2\pi\delta)^2 + 8\pi + 8\pi\delta}}{2(1+\delta)}$$

with $Q_1 = 1 - Q_2$

3. Suppose $\pi = .5$ and $\delta = 1/3$

$$Q_2 = \frac{-(4/3) \pm \sqrt{(4/3)^2 + 8/6 + 4}}{2(4/3)}$$

$$= -\frac{1}{2} \pm \frac{3}{8}(\frac{8}{3})$$

$Q_2 = \frac{1}{2}$ (We want the positive root. Otherwise agent 2 will have no consumption in the $\theta = 1$ state)

$Q_1 = \frac{1}{2}$.

15.2 When markets are incomplete :

i) MM does not hold: the value of the firm may be affected by the financial structure of the firm.

ii) It may not be optimal for the firm's manager to issue the socially preferable set of financial instruments

15.3 a. Write the problem of a risk neutral agent :

$$\text{Max } 30 - p^Q Q^2 + 1/3 \; ½ \; (15 + Q^2) + 2/3 \; ½ \; 15$$

$$\text{FOC: } -p^Q + 1/6 = 0 \text{ thus } p^Q = 1/6 \text{ necessarily!}$$

This is generic: risk neutrality implies no curvature in the utility function. If the equilibrium price differs from 1/6, the agent will want to take infinite positive or negative positions!
At that price, check that the demand for asset Q by agent 1 is zero:

$$\text{Max } 20 - 1/6 \; Q^1 + ½ \; 1/3 \; \ln(1 + Q^1) + ½ \; 2/3 \; \ln(5)$$

$$\text{FOC: } -1/6 + 1/6 \; 1/(1 + Q^1) = 0$$

$$Q^1 = 0$$

Thus there is no risk sharing. The initial allocation is not Pareto – optimal. The risk averse agent is exposed to significant risk at date 1. If the state probabilities were ½, nothing would change, except that the equilibrium price becomes:

$$P^Q = ¼$$

b. $p^Q = 1/6$, $Q^1 = 0$ (the former FOC is not affected), $p^R = 1/3$

FOC of agent 1 wrt R:

$$-1/3 + (½)2/3\ 1/(5 + R^1) = 0$$

$$1 = 5 + R^1$$

$$R^1 = -4$$

So agent 1 sells 4 units of asset R. He reduces his t = 1 risk. At date 1, he consumes 1 unit in either state. He is compensated by increasing his date 0 consumption by $p^R\ R^1 = 1/3(4) = 4/3$. The allocation is Pareto optimal, as expected from the fact that markets are now complete.

Post trade allocation:

	t = 0	t = 1	t = 2
Agent 1	20 4/3	1	1
Agent 2	28 2/3	15	19

15.4

	t=0	t=1	
		$\theta = 1$	$\theta = 2$
Agent 1	4	6	1
Agent 2	6	3	4

$$U^1(c_0, \tilde{c}_1(\theta)) = \tfrac{1}{2}\ln c_0^1 + E\ln c_1^1(\theta)$$
$$U^2(c_0, \tilde{c}_1(\theta)) = \tfrac{1}{2}c_0^2 + E\ln c_1^2(\theta)$$
$$\text{Prob}(\theta_1) = .4 \quad \text{Prob}(\theta_2) = .6$$

a. Initial utilities –

$$\text{Agent 1:} \quad U^1(c_0, \tilde{c}_1(\theta)) = \tfrac{1}{2}\ln(4) + .4\ln(6) + .6\ln(1)$$
$$= \tfrac{1}{2}(1.386) + .4(1.79)$$
$$= .693 + .716 = 1.409$$

$$\text{Agent 2:} \quad U^2(c_0, \tilde{c}_1(\theta)) = \tfrac{1}{2}(6) + .4\ln(3) + .6\ln(4)$$
$$= 3 + .439 + .832$$
$$= 4.271$$

c. Firm's output

	t=1	
	$\theta = 1$	$\theta = 2$
-p	2	3

Agent 1's Problem (presuming only this security is issued).

$$\max_{Q_1} \frac{1}{2}\ln(4 - pQ_1) + .4\ln(6 + 2Q_1) + .6\ln(1 + 3Q_1)$$

Agent 2's Problem

$$\max_{Q_2} \frac{1}{2}(6 - pQ_2) + .4\ln(3 + 2Q_2) + .6\ln(4 + 3Q_2)$$

The F.O.C.'s are:

Agent 1: $$\frac{1}{2}\left(\frac{p}{4 - pQ_1}\right) = .4\left(\frac{1}{6 + 2Q_1}\right)(2) + .6\left(\frac{1}{1 + 3Q_1}\right)(3)$$

Agent 2: $$\frac{1}{2}p = .4\left(\frac{1}{3 + 2Q_2}\right)(2) + .6\left(\frac{1}{4 + 3Q_2}\right)(3)$$

$$Q_2 = 1 - Q_1$$

These can be simplified to

(i) $$\frac{p}{4 - pQ_1} = \frac{1.6}{6 + 2Q_1} + \frac{3.6}{1 + 3Q_1}$$

(ii) $$p = \frac{1.6}{5 - 2Q_1} + \frac{3.6}{7 - 3Q_1}$$

The solution to this set equations via matlab is

$$p = 1.74$$
$$Q_1 = 1.245$$

Thus $Q_2 = 1 - 1.245 = -.245$ (short sale).

Thus $V_F = 1.74$.

The post-trade allocations are

	t=0	t=1	
		$\theta = 1$	$\theta = 2$
Agent 1:	$4-(1.74)(1.245)=1.834$	$6+2(1.245)=8.49$	$1+3(1.245)=4.735$
Agent 2:	$6-(1.74)(-.245)=6.426$	$3-2(.245)=2.51$	$4-3(.245)=3.265$

Post-trade (ex ante) utilities:

$$\text{Agent 1:} \quad \frac{1}{2}\ln(1.834)+.4\ln(8.49)+.6\ln(4.735)$$
$$=.3032+.4(2.14)+.6(1.555)$$
$$=.3032+.856+.933$$
$$=2.0922$$

$$\text{Agent 2:} \quad \frac{1}{2}(6.426)+.4\ln(2.51)+.6\ln(3.265)$$
$$=3.213+.368+.70996$$
$$=4.291$$

Nearly all the benefit goes to agent 1. This is not entirely surprising as the security payoffs are more useful to him for consumption smoothing.
For agent 2, the marginal utility of a unit of consumption in period 1 is less than the marginal utility of a unit in period 0. His consumption pattern across states in t=1 is also relatively smooth, and the security available for sale is not particularly useful in correcting the existing imbalance. Taken together, he is willing to "sell short" the security or, equivalently, to borrow against the future.
The reverse is true for agent 1 especially on the issue of consumption smoothing across t=1 states: he has very little endowment in the more likely state. Furthermore the security pays relatively more in this particular state. Agent 1 thus wishes to save and acquires "most" of the security.
If the two states were of equal probability agent 1 would have a bit less need to smooth, and thus his demand would be relatively smaller. We would expect p to be smaller in this case.

c. The Arrow-Debreu securities would offer greater opportunity for risk sharing among the agents without the presence of the firm. (We would expect V_F to be less than in b).
However, each agent would most likely have a higher utility ex ante (post-trade).

d. Let the foreign government issue 1 unit of the bond paying (2.2); let its price be p.
Agent Problems:

$$\text{Agent 1:} \quad \max_{Q_1} \frac{1}{2}\ln(4-pQ_1)+.4\ln(6+2Q_1)+.6\ln(1+2Q_1)$$

Agent 2: $\max_{Q_2} \frac{1}{2}(4 - pQ_2) + .4\ln(3 + 2Q_2) + .6\ln(4 + 2Q_2)$

Where, in equilibrium $Q_1 + Q_2 = 1$

F.O.C's:

Agent 1: $\frac{1}{2}\left(\frac{p}{4 - pQ_1}\right) = \frac{.4(2)}{6 + 2Q_1} + \frac{.6(2)}{1 + 2Q_1}$

Agent 2: $\frac{1}{2}p = \frac{.4(2)}{3 + 2Q_2} + \frac{.6(2)}{4 + 2Q_2}$

Substituting $Q_2 = 1 - Q_1$, these equations become:

$$\frac{p}{4 - pQ_1} = \frac{1.6}{6 + 2Q_1} + \frac{2.4}{1 + 2Q_1}$$

$$p = \frac{1.6}{5 - 2Q_1} + \frac{2.4}{6 - 2Q_1}$$

Solving these equations using matlab yields

$p = 1.502$
$Q_1 = 1.4215$
and thus $Q_2 = -.4215$

The bond issue will generate $p = 1.502$.

Post Trade allocations:

	t=0	t=1	
		$\theta = 1$	$\theta = 2$
Agent 1:	$4 - (1.502)(1.4215) = 1.865$	$6 + 2(1.425) = 8.843$	$1 + 2(1.4215) = 3.843$
Agent 2:	$6 - (1.502)(-.4215) = 6.633$	$3 + 2(-.4215) = 2.157$	$4 + 2(-.4215) = 3.157$

The utilities are:

$$\text{Agent 1:} \quad \frac{1}{2}\ln(1.865)+.4\ln(8.843)+.6\ln(3.843)$$
$$=.3116+.8719+.8078$$
$$=1.9913$$

$$\text{Agent 2:} \quad \frac{1}{2}(6.633)+.4\ln(2.157)+.6\ln(3.157)$$
$$=3.3165+.3075+.690$$
$$=4.314\ .$$

Once again; both agents are better off after trade. Most of the benefits still go to agent 1; however, the incremental benefit to him is less than in the prior situation because the security is less well situated to his consumption smoothing needs.

e. When the bond is issued by the local government, one should specify i) where the proceeds from the bond issue go, and ii) how the t=1 payments in the bond contracts will be financed. In a simple closed economy, the most natural assumption is that the proceeds from the issue are redistributed to the agents in the economy and similarly that the payments are financed from taxes levied on the same agents. If these redistributive payments and taxes are lump-sum transfers, they will not affect the decisions of individuals, nor the pricing of the security. But the final allocation will be modified and closer (equal ?) to the initial endowments. In more general contexts, these payments may have distortionary effects.

15.5 a.

Agent 1 : $\max\ \frac{1}{2}x_1^1+\frac{1}{2}x_2^1$

s.t.

$$q_1x_1^1+q_2x_2^1 \le q_1e_1^1+q_2e_2^1$$

Agent 2 : $\max\ \frac{1}{2}\ln x_1^2+\frac{1}{2}\ln x_2^2$

s.t.

$$q_1x_1^2+q_2x_2^2 \le q_1e_1^2+q_2e_2^2$$

Substituting the budget constraint into the objective function :

$$\text{Agent 1 :}\ \max_{x_2^1}\ \frac{1}{2}\left(\frac{q_1e_1^1+q_2e_2^1-q_2x_2^1}{q_1}\right)+\frac{1}{2}x_2^1$$

$$\text{Agent 2 :}\ \max_{x_2^2}\ \frac{1}{2}\ln\left(\frac{q_1e_1^2+q_2e_2^2-q_2x_2^2}{q_1}\right)+\frac{1}{2}\ln x_2^2$$

FOC's

Agent 1 : $\frac{1}{2}\left(\frac{-q_2}{q_1}\right)+\frac{1}{2}=0 \Rightarrow q_1 = q_2$

Agent 2 : $\frac{1}{2}\left(\frac{q_1}{q_1 e_1^2 + q_2 e_2^2 - q_2 x_2^2}\right)\left(\frac{q_2}{q_1}\right)=\frac{1}{2}\left(\frac{1}{x_2^2}\right)$ which, taking into account that the two prices are equal, solves for

$x_2^2 = x_1^2 = \frac{e_1^2 + e_2^2}{2}$, i.e.,

agent 2's consumption is fully stabilized.

b. Each agent owns one half of the firm, which can employ simultaneously two technologies :

	t=0	t=1	
		$\theta = 1$	$\theta = 2$
Technology 1	-y	y	y
Technology 2	-y	3y	0

Let x be the portion of input invested in technology 2. Since there are 2 units invested in total, 2-x is invested in technology 1. In total we have :

	t=0	t=1	
		$\theta = 1$	$\theta = 2$
Invested in tech. 1	2-x	2-x	2-x
Invested in tech. 2	x	3x	0
Each firm owner receives		1+x	$1-\frac{1}{2}x$

Considering that the two Arrow-Debreu prices necessarily remain equal, agent 2 solves

$$\max_{x, x_2^2} \tfrac{1}{2}\ln(2+\tfrac{1}{2}x+e_1^2+e_2^2-x_2^2)+\tfrac{1}{2}\ln x_2^2$$

It is clear that he wants x to be as high as possible, that is, x =2.

The FOC wrt x_2^2 solves for

$x_2^2 = x_1^2 = \frac{2+\frac{1}{2}x+e_1^2+e_2^2}{2}$.

Again there is perfect consumption insurance for the risk averse agent (subject to feasibility, that is, an interior solution).

Agent 1 solves

$$\max_{x} \tfrac{1}{2}(2+\tfrac{1}{2}x+e_1^1+_2 e_2^1-x_2^1)+\tfrac{1}{2}x_2^1$$

Clearly he also wants x to be as high as possible. So there is agreement between the two firm owners to invest everything (x = 2) in the second more productive but riskier technology. For agent 1, this is because he is risk neutral. Agent 2 , on the other hand, is fully insured, thanks to complete markets. Given that fact, he also prefers the more productive technology even though it is risky.

c. There cannot be any trade in the second period ; agents will consume their endowments at that time.

Agent 1 solves

$$\max_x \tfrac{1}{2}(1+x+e_1^1)+\tfrac{1}{2}(1-\tfrac{1}{2}x+e_2^1)=$$

$$\max_x 1+\tfrac{1}{4}x+\frac{e_1^1+e_2^1}{2}$$; clearly he still wants to invest as much as possible in technology 2.

Agent 2 solves

$$\max_x \tfrac{1}{2}\ln(1+x+e_1^2)+\tfrac{1}{2}\ln(1-\tfrac{1}{2}x+e_2^2)$$

which, after derivation, yields

$$x=\frac{1+2e_2^2-e_1^2}{2}.$$

That is, the (risk averse) agent 2 in general wants to invest in the risk-free technology. There is thus disagreement among firm owners as to the investment policy of the firm. This is a consequence of the incomplete market situation.

d. The two securities are now

	t=1	
	$\theta = 1$	$\theta = 2$
A bond	1	1
Technology 2	1+x	$1-\tfrac{1}{2}x$

These two securities can replicate (1,0) and (0,1). To replicate (1,0), for instance, invest a in the bond and b in the firm where a and b are such that

$a + (1+x)b = 1$

$a + (1-\tfrac{1}{2}x)b = 0$

This system implies $b = \tfrac{2}{3}x$ and $a = 1-\tfrac{2}{3}x(1+x)$.

Given that the markets are complete, both agents will agree to invest x=2. Thus $b = \tfrac{1}{3}$ and a =0 replicate (1,0).

Chapter 16

16.1. The maximization problem for the speculator's is:

$$\max_f EU\left[c^* + \left(p^f - p\right)f\right]$$

Let us rewrite the program in the spirit of Chapter IV:
$W(f) = E\left\{U\left(c^* + \left(p^f - p\right)f\right)\right\}$. The FOC can then be written
$W'(f) = E\left\{U'\left(c^* + \left(p^f - p\right)f\right)\left(p^f - p\right)\right\} = 0$. From (U"<0) we find
$W''(f) = E\left\{U''\left(c^* + \left(p^f - p\right)f\right)\left(p^f - p\right)^2\right\} < 0$. This means that f>0 iff
$W'(0) = \left\{U'\left(c^*\right)\right\}E\left(p^f - p\right) > 0$. From U'>0 we have f>0 iff $E\left(p^f - p\right) > 0$. The two other cases follow immediately.

16.2. Let us reason with the help of an example. Suppose the current futures price p^f is \$100. The marginal cost of producing the corresponding commodity is \$110. The producer's expectation as to the t=1 spot price for her output is \$120.

The producer could speculate in the sense of deciding to produce an extra unit of output since she expects to be able to sell it tomorrow at a price that covers her marginal cost with a margin (\$10) that we can even assume sufficient to compensate for the risk being taken.

The rule we have derived in this chapter would, however, suggest that this is the wrong decision. Under the spelled out hypotheses, the futures price is the definite signal of production (Equation 16.3).

It is easy to see, indeed, that if the producer wants to take a position on the basis of her expectations on the t=1 spot price, she will make a larger unit profit by speculating on the futures market rather than on the spot market. This is because the cost of establishing a 'spot' speculative position is the cost of producing the commodity, that is, \$110 while the cost of establishing a speculative position on the futures market is \$100 only. If the producers' expectations turn out to be right, she will make \$20 on a unit speculative futures position while she will make only \$10 by speculating on the spot market, that is, by producing the commodity. Note that this reasoning is generic as long as the futures price is below the cost of production. If this was not the case, producing would not involve any speculation in the sense that the output could be sold profitably on the futures market at the known futures price.

16.3 When the object exchanged is a financial asset and investors have heterogeneous information, a price increase may reveal some privately held information leading some buyers and sellers to reevaluate their positions. As a consequence, the increase in price may well lead to a fully rational increase in demand. In that sense the law of demand does not apply in such a context.

16.4 When investors share the same information, the only source of trading are liquidity needs of one or another group of investors. In the CCAPM with homogeneous investors, there simply is no trading. Prices support the endowment allocations. In a world of heterogeneous information, the liquidity-based trades are supplemented by exchanges between investors who disagree (and in

some sense agree to disagree) about the fundamental valuation of the object being exchanged: I sell because I believe at the current price the stock of company A is overvalued knowing that the investor who buys from me is motivated by the opposite assessment. Obviously trading volume is necessarily higher in the second world, which appears to be closer to the one we live in than the former.

16. 5

$$E\tilde{\pi} = p\,y\,\bar{e} - \frac{1}{2}y^2$$

$$\operatorname{var}\tilde{\pi} = p^2\,y^2\,\operatorname{var}\tilde{e} + z^2\,\operatorname{var}\tilde{q} - 2\,z\,p\,y\,\operatorname{cov}(\tilde{e},\tilde{q})$$

FOC:

$$y: \qquad p\,\bar{e} - y - \gamma\left[p^2\,y\,\operatorname{var}\tilde{e} - z\,p\,\operatorname{cov}(\tilde{e},\tilde{q})\right] = 0$$

This is a maginal product = marginal cost condition where the latter includes a risk premium.

$$z: \qquad z\,\operatorname{var}\tilde{q} - p\,y\,\operatorname{cov}(\tilde{e},\tilde{q}) = 0$$

$$z = \frac{p\,y\,\operatorname{cov}(\tilde{e},\tilde{q})}{\operatorname{var}\tilde{q}}$$

p y = value placed at risk by fluctuations of exchange rates.

Optimal position in hedging instrument 2 is the product of py by a coefficient that we may call β, which is the sensitivity of $\tilde{e}$ to changes in the price of the hedging instrument as in

$$\tilde{e} = \alpha + \beta\,\tilde{q} + \tilde{\varepsilon}\,.$$

If the coefficient β is 1, the optimal position in the hedging instrument is exactly equal to the value placed at risk py. Given the value of the optimal *z*, the FOC wrt *y* can be written as:

$$p\,\bar{e} - y - \gamma p^2\,y\,\operatorname{var}\tilde{e}\left[1 - \frac{[\operatorname{cov}(\tilde{e},\tilde{q})]^2}{\operatorname{var}\tilde{q}\,\operatorname{var}\tilde{e}}\right] = 0$$

$$p\,\bar{e} - y - \gamma p^2\,y\,\operatorname{var}\tilde{e}\left[1 - R^2\right] = 0$$

where R^2 is the correlation coefficient in the above regression of $\tilde{q}$ on $\tilde{e}$.