

# Randomly Recurring Investment Opportunities: Real Options with Illiquid Projects\*

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## Abstract

Although the vast majority of real options models assume that investment opportunities are permanently available, many real-world investment opportunities are sporadic. Options to invest can suddenly become blacked-out, only to be followed by a re-opening in the future. For example, the option to develop real estate may open or close depending upon zoning and growth control decisions, competitive entry, or input (labor or material) availability. Similar factors influence complex R&D options. We term such randomly recurring investment opportunities as options on illiquid projects. We derive and analyze a model of such illiquid options, and find that the potential for exercise opportunities to be shut down provides an incentive for early exercise while the investment opportunity is open. We examine the conditions that impact the liquidity discount. We also consider the case in which the degree of illiquidity is endogenously determined, for instance by a regulator. The model is extended to an equilibrium setting, and the notion of illiquidity is generalized to allow for costly exercise during times of illiquidity.

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# 1 Introduction

The real options approach to analyzing investment under uncertainty has become part of the mainstream literature of financial economics. The subject of real options now typically comprises an entire chapter in corporate finance textbooks.<sup>1</sup> Essentially, the real options approach posits that the opportunity to invest in a project is analogous to an American call option on the investment opportunity. Once that analogy is made, the vast and rigorous machinery of financial options theory is at the disposal of real investment analysis. The real options approach is well summarized in Dixit and Pindyck (1994).

A fundamental assumption underlying the nature of a real option is the time over which the option may be exercised. In much of the real options literature, it is assumed that the investment opportunity is permanently available for exercise (analogous to a perpetual call option), and the firm simply chooses the optimal time at which to invest.<sup>2</sup> However, there is also a more recent literature that assumes the opposite extreme: when the investment opportunities present themselves, they are now-or-never propositions.<sup>3</sup> While both of these extremes may make sense for characterizing some real-world investment opportunities, we argue that many realistic investment opportunities are better modeled as randomly recurring, where the ability to invest is itself stochastic and opportunities disappear and reappear over time.

In this paper we provide a model of randomly recurring real options, where investment opportunities are subject to shocks that limit a firm's ability to exercise potentially valuable growth options. This notion that there are times at which options can be exercised and times at which options cannot be exercised is analogous to the notion of liquidity in asset markets. Perhaps the closest analogy to liquidity comes from Longstaff (2005) in which he models illiquid assets as ones that are subject to an open period (in which assets can be freely bought and sold) and a blackout period (in which assets cannot be bought or sold). We shall often use this illiquidity analogy in this paper to model the opening and closing of investment opportunities, where instead of having illiquidity refer to the inability to purchase a financial asset (at least without significant cost), here illiquidity refers to the inability to invest in a real asset. In the model in this paper, the investment opportunity will move stochastically back and forth through open and blackout periods, making the forward-looking investment decision more complicated than either the perpetual or now-or-never variants in the literature.<sup>4</sup>

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<sup>1</sup>For example, see Berk and DeMarzo (2007) and Brealey, Myers and Allen (2006).

<sup>2</sup>This perpetual investment opportunity assumption is seen in the seminal papers such as Brennan and Schwartz (1985) and McDonald and Siegel (1986).

<sup>3</sup>For example, see Berk, Green and Naik (1999) and Bernardo, Chowdhry and Goyal (2006). Note that while such an assumption makes the timing decision trivial (invest if and only if the opportunity has a positive net present value), this literature is primarily focused on explaining the time-series and cross-sectional properties of security returns.

<sup>4</sup>Dupuis and Wang (2002) provide a model of optimal stopping with random intervention times.

In order to highlight the importance of modeling the intermittent nature of real options, consider the following two examples of well-known applications.

*Example 1: Real Estate Development*

A classic case of a real option is real estate investment. As in Titman (1985) and Williams (1991), the development of real estate is analogous to a perpetual American call option on a building, where the exercise price is equal to the construction cost. However, it is easy to identify several features of real estate markets that make development opportunities illiquid. First, real estate development is a highly regulated industry subject to the vagaries of local bureaucratic forces. Growth controls can be tightened or loosened, the permitting process can become attenuated or more costly, and building codes and zoning laws are subject to change. As the regulatory environment moves from lax to strict, a promising development opportunity can be taken away, forcing the developer to wait for the investment opportunity to reopen in the future. Second, many real estate markets are quite competitive, and promising investment opportunities can be suddenly preempted by rival developers. Models of game theoretic competitive interplay in real options have been formally developed by Williams (1993), Grenadier (1995, 2002) and others. However, in markets with incomplete information and model uncertainty, the impact of the competitive environment on exercise viability might best be modeled in a probabilistic sense, as in the present model. Third, the act of development involves technical and operational complexities. Thus, such factors as input availability (labor, materials, capital), weather or natural disasters can make the investment opportunity subject to temporary blackout periods.

*Example 2: Pharmaceutical Research and Development*

Another well-known case of a real option is drug research and development (R&D). Schwartz (2004) provides a careful description of the option-like characteristics of R&D in the pharmaceutical industry. The development of a drug is a multi-stage option, where the successful development of a drug may take ten or more years to complete. The nature of the industry makes it clear that many of the development opportunities (at various stages from initial R&D to product roll-out) are prone to the uncertain arrival of blackout and reopening periods. First, the initial stages of investment are characterized by significant technical uncertainty. Early research on a potentially highly profitable drug can reach a technical bottleneck that precludes the ability to invest in the next stage of research. However, the very nature of technological progress can mean that such technical bottlenecks may be removed in the future. Second, before a potentially profitable drug can be marketed, it must receive Federal Drug Administration approval. Such approval is uncertain in both outcome and duration. Finally, since drugs receive patent protection which ensure

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In their model, the periods at which the option can be exercised are of infinitesimal dimension, occurring at random points in time. This will be seen to be a special case of our more general model.

monopoly rights to market the drug until the patent expires, the outcome of a patent race can determine whether the firm can exercise its option to market the drug.<sup>5</sup>

Just as illiquidity in financial asset markets significantly impacts economic outcomes, so to does illiquidity in investing in real assets. In our basic model, a firm holds an irreversible opportunity to invest in an asset with stochastic payoff  $X$ , by paying the investment cost  $I$ . In this model, we make the additional assumption that the project is illiquid, in that the investment option is not always exercisable. In particular, we assume that there are two possible periods: an open period and a black-out period. During the open period, the firm can freely exercise the option. During the black-out period, the firm cannot exercise the option. We allow the open and black-out periods to last for random durations of time. That is, when one is in an open period, at a random point in time the market enters a black-out period. Similarly, when one is in a black-out period, at a random point in time the market enters an open period.<sup>6</sup> The interconnectedness of the option in these two states makes the investment decisions more complicated than in the standard fully-liquid case, as the valuation equations become interlinked. Nevertheless, closed-form solutions are obtained.

As intuition would suggest, the intermittent nature of the investment opportunity motivates firms to invest earlier than they would if the asset was fully liquid. This result is not unlike the result of Grenadier (2002) who demonstrates that industry competition compels firms to exercise their options sooner. However, here it is not the fear of preemption by competitors that motivates early exercise, but fear of losing the investment opportunity itself.

We find that the cost of illiquidity depends on the relative timing of the opening and closing of the investment opportunity. We see that in some reasonable parameterizations illiquid options would be subject to a 20% discount relative to liquid options, while in others they may be subject to only a trivial discount. Similarly, the discount for illiquidity is greater in more volatile environments.

While in the basic model's formulation we assume that the degree of illiquidity is exogenous, in some cases a regulator or industry group might have some control over the market's liquidity. We extend the model to a setting where the timing of the closing and reopening of investment projects is set to maximize a planner's objective function. We look at a particular case in which the investment project has a positive externality. Thus, by allowing for some amount of illiquidity, the planner can accelerate investment in light of the positive externality.

The model is also extended to an equilibrium setting. In the basic model, a single firm chooses the optimal time to invest under stochastic illiquidity. In an extension to a perfectly competitive industry equilibrium setting, multiple firms contemplate

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<sup>5</sup>For models of patent races, see Fudenberg, Gilbert, Stiglitz and Tirole (1983) and Lambrecht (2000).

<sup>6</sup>In a later section we generalize the model to allow for exercise during the black-out period, but at a potentially higher cost (although not infinite, as in the basic model).

entry into an industry, when entry is prone to periodic closings and reopenings. The equilibrium entry strategies are derived and compared with that of the monopolist setting. In equilibrium, firms will enter in incremental amounts when the market is open, but may enter in discrete clusters when a profitable blacked-out market suddenly opens.

To some extent, our model of investment illiquidity is somewhat extreme in that a closed market provides no ability to invest at any price. Thus, we loosen this restriction by providing a more general model of illiquidity. In this generalized setting, rather than ruling out exercise during the illiquid period, we allow the option to be exercised, but at a potentially higher exercise price. As the illiquid exercise price approaches infinity, the resulting problem mimics the basic model of the paper. When compared with the basic model, the generalized model produces the following implications. In an open market, with generalized liquidity, the investment is undertaken later than in the basic model. The intuition is clear: since the generalized model provides an opportunity to invest in the illiquid period (albeit at a higher cost), there is less pressure to exercise early in fear of being entirely denied the investment opportunity. Second, in the generalized setting, the option may be exercised during the illiquid period, but at a point later than the fully liquid trigger. This stands to reason since in order to motive a firm to invest at a high cost and not wait until the market becomes once again liquid, the opportunity must have substantial profitability.

The remainder of the paper is organized as follows. Section 2 describes the underlying model and provides the solutions for the valuation and exercise strategy for randomly recurring investment opportunities.. Section 3 considers four special cases of the model in order to further intuition. Section 4 analyzes the implied discount for illiquidity. Section 5 permits an endogenous specification of illiquidity, and considers a planner's solution for a market with positive investment externalities. Section 6 considers the implications of our model in an equilibrium setting. Section 7 generalizes the model to permit investment during illiquid periods, and Section 8 concludes.

## 2 The Basic Model

### 2.1 Assumptions and Setup

Consider the setting for a standard irreversible investment problem. (See Brennan and Schwartz, 1985; McDonald and Siegel, 1986; and Dixit and Pindyck, 1994.) A risk-neutral firm possesses an opportunity to invest in a project. Let  $X$  denote the payoff value process of the underlying project. Assume that the project payoff value evolves as a geometric Brownian motion process:

$$dX(t) = \alpha X(t)dt + \sigma X(t)dz_t, \tag{1}$$

where  $\alpha$  is the instantaneous conditional expected percentage change in  $X$  per unit time,  $\sigma$  is the instantaneous conditional standard deviation per unit time, and  $dz$  is the increment of a standard Wiener process. Investment at any time costs  $I$ . Thus, the payoff from investment at time  $t$  is then given by  $X(t) - I$ .

In this model, we make the additional assumption that the project is illiquid, in that the investment option is not always exercisable. In particular, we assume that there are two possible periods: an open period and a black-out period. During the open period, the firm can freely exercise the option. During the black-out period, the firm cannot exercise the option. In a later section we generalize the model to allow for exercise during the black-out period, but at a potentially higher cost.

We allow the open and black-out periods to last for random durations of time. That is, when one is in an open period, at a random point in time the market enters a black-out period. Similarly, when one is in a black-out period, at a random point in time the market enters an open period. For simplicity, we assume that the length of the open and black-out periods are exponentially distributed with parameters  $\lambda_1$  and  $\lambda_2$ , respectively. Stated in another way, the conclusion of the open period is modeled as a Poisson process with intensity  $\lambda_1$ , while the conclusion of the black-out period is modeled as a Poisson process with intensity  $\lambda_2$ . By appropriately selecting the parameterization of  $\lambda_1$  and  $\lambda_2$ , we can account for a wide variety of illiquidity settings.

## 2.2 The Fully Liquid Benchmark

In this subsection, we briefly consider the standard real options model in which the underlying project is liquid and can always be freely exercised. Let  $F(X)$  denote the value of the investment option, and let  $X^*$  denote the fully liquid exercise trigger. Using standard arguments (i.e., Dixit and Pindyck, 1994), over the region prior to exercise,  $F(X)$  solves the differential equation:

$$0 = \frac{1}{2}\sigma^2 X^2 F'' + \alpha X F' - rF, \quad X < X^*, \quad (2)$$

subject to the boundary conditions:

$$F(X^*) = X^* - I, \quad (3)$$

$$F'(X^*) = 1, \quad (4)$$

$$F(0) = 0. \quad (5)$$

The first boundary condition is the value-matching condition. It simply states that, at the moment the option is exercised, the payoff is  $X^* - I$ . The second boundary condition is the smooth-pasting or high-contact condition. (See Merton, 1973, for a discussion of the high-contact condition.) This condition ensures that the exercise trigger is chosen so as to maximize the value of the option. The third boundary

condition reflects the fact that  $X = 0$  is an absorbing barrier for the underlying project value process.

The solutions for the option value and optimal exercise triggers are:

$$F(X) = \begin{cases} \left(\frac{X}{X^*}\right)^\beta (X^* - I) & \text{for } X < X^*, \\ X - I & \text{for } X \geq X^*. \end{cases} \quad (6)$$

$$X^* = \frac{\beta}{\beta - 1} I, \quad (7)$$

where  $\beta = \frac{-(\alpha - \frac{1}{2}\sigma^2) + \sqrt{(\alpha - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}}{\sigma^2} > 1$ .

### 2.3 The Illiquid Case

Let  $V(X)$  denote the value of the investment option during an open period, and  $W(X)$  denote the value of the investment option during a black-out period. Exercise may only occur during the open period, and let  $\bar{X}$  be the optimal investment trigger at which the option is exercised. During the open period,  $V(X)$  solves the differential equation

$$0 = \frac{1}{2}\sigma^2 X^2 V'' + \alpha X V' - rV + \lambda_2(W - V), \quad X \leq \bar{X} \quad (8)$$

Notice that upon the arrival of the black-out period, the value of the open option switches to the value of the blacked-out option.<sup>7</sup>

Eq. (8) is solved subject to appropriate boundary conditions. These boundary conditions are identical to those of the fully liquid benchmark in the previous section:

$$V(\bar{X}) = \bar{X} - I, \quad (9)$$

$$V'(\bar{X}) = 1, \quad (10)$$

$$V(0) = 0. \quad (11)$$

During the black-out period, there are two regions to consider. When the current level of  $X$  is greater than or equal to the exercise trigger  $\bar{X}$ , a jump to the open period would result in the immediate exercise of the option. Conversely, when the current level of  $X$  is less than the exercise trigger  $\bar{X}$ , a jump to the open period would not result in the immediate exercise of the option. We thus decompose the value function  $W(X)$  into two parts:

$$W(X) = \begin{cases} W_H(X) & \text{for } X \geq \bar{X}, \\ W_L(X) & \text{for } X < \bar{X}. \end{cases} \quad (12)$$

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<sup>7</sup>See Dixit and Pindyck (1994), Chapter 4, Section 1.1 for a derivation of the equilibrium differential equation for mixed processes with both Poisson and diffusion components.

$W_H(X)$  and  $W_L(X)$  satisfy the following coupled differential equations:

$$0 = \frac{1}{2}\sigma^2 X^2 W_H'' + \alpha X W_H' - r W_H + \lambda_1 (X - I - W_H), \quad X \geq \bar{X}, \quad (13)$$

$$0 = \frac{1}{2}\sigma^2 X^2 W_L'' + \alpha X W_L' - r W_L + \lambda_1 (V - W_L), \quad X < \bar{X}, \quad (14)$$

subject to

$$W_H(\bar{X}) = W_L(\bar{X}), \quad (15)$$

$$W_H'(\bar{X}) = W_L'(\bar{X}), \quad (16)$$

$$W_L(0) = 0, \quad (17)$$

$$\frac{W_H(X)}{X} \text{ is bounded as } X \rightarrow \infty. \quad (18)$$

The first two boundary conditions ensure that the value function  $W(X)$  is continuously differentiable at  $\bar{X}$  (see Dixit, 1993, Section 3.8). The third boundary condition reflects the fact that zero is an absorbing barrier for  $X$ . The fourth boundary condition is a “no bubbles” condition (see Dixit and Pindyck, 1994, Chapter 6).

Note that the cost of illiquidity comes from the fact that during the black-out period, when  $X > \bar{X}$ , the firm cannot exercise the option, no matter how high the payoff becomes. In anticipation of this investment inefficiency during the black-out period, the firm will distort its investment trigger during the open period.

## 2.4 Model Solution

We now solve the system of differential equations for  $V(X)$  and  $W(X)$ , along with the optimal exercise trigger  $\bar{X}$ . Note that the solution will be considerably more complicated than that for the standard real options case, since their associated differential equations will be interlinked.

The general solution for  $W_H$  is:

$$W_H(X) = A_1 X^{\beta_2} + \frac{\lambda_1}{r + \lambda_1 - \alpha} X - \frac{\lambda_1}{r + \lambda_1} I, \quad X \geq \bar{X}, \quad (19)$$

where

$$\beta_2 = \frac{-(\alpha - \frac{1}{2}\sigma^2) - \sqrt{(\alpha - \frac{1}{2}\sigma^2)^2 + 2(r + \lambda_1)\sigma^2}}{\sigma^2} < 0.$$

Note that we have eliminated the positive root as an exponent of  $X$  to satisfy boundary condition (18).

When  $X < \bar{X}$ , we have the coupled system of differential equations that jointly determine  $V(X)$  and  $W_L(X)$ :

$$0 = \frac{1}{2}\sigma^2 X^2 V'' + \alpha X V' - (r + \lambda_2)V + \lambda_2 W_L, \quad (20)$$

$$0 = \frac{1}{2}\sigma^2 X^2 W_L'' + \alpha X W_L' - (r + \lambda_1)W_L + \lambda_1 V. \quad (21)$$

Conjecture that the value functions for  $V(X)$  and  $W_L(X)$  are given by

$$V(X) = B_1X^\gamma + C_1X^\eta, \quad (22)$$

$$W_L(X) = B_2X^\gamma + C_2X^\eta, \quad (23)$$

where the coefficients  $B_1, B_2, C_1, C_2$ , and  $\gamma > 0$  and  $\eta > 0$  are to be determined. Note that we can safely drop any negative power terms because of the absorbing barrier conditions, in that  $V(0) = W_L(0) = 0$ . Without loss of generality, let  $\gamma \geq \eta$ .

Substituting the conjectured value functions (22) and (23) into (20) and (21) gives the following system of four equations:

$$K_1(\gamma)B_2 + \lambda_1B_1 = 0, \quad (24)$$

$$K_1(\eta)C_2 + \lambda_1C_1 = 0. \quad (25)$$

$$K_2(\gamma)B_1 + \lambda_2B_2 = 0, \quad (26)$$

$$K_2(\eta)C_1 + \lambda_2C_2 = 0, \quad (27)$$

where

$$K_1(z) = \frac{\sigma^2}{2}z(z-1) + \alpha z - (r + \lambda_1), \quad (28)$$

$$K_2(z) = \frac{\sigma^2}{2}z(z-1) + \alpha z - (r + \lambda_2). \quad (29)$$

Conditions (24) through (27) jointly imply that  $\gamma$  and  $\eta$  are the two positive roots to the following polynomial equation:<sup>8</sup>

$$K_1(z)K_2(z) = \lambda_1\lambda_2. \quad (30)$$

We thus have

$$\eta = \frac{-(\alpha - \frac{1}{2}\sigma^2) + \sqrt{(\alpha - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}}{\sigma^2} = \beta > 1, \quad (31)$$

$$\gamma = \frac{-(\alpha - \frac{1}{2}\sigma^2) + \sqrt{(\alpha - \frac{1}{2}\sigma^2)^2 + 2(r + \lambda_1 + \lambda_2)\sigma^2}}{\sigma^2} > \eta. \quad (32)$$

Note that  $K_1(\eta) = -\lambda_1$  and  $K_1(\gamma) = \lambda_2$ .

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<sup>8</sup>It is simple to show that there are two positive roots and two negative roots to the polynomial equation  $K_1(z)K_2(z) = \lambda_1\lambda_2$ .  $K_1(z) = 0$  at one negative value and one positive value. The same is true for  $K_2(z) = 0$ . Thus, the LHS equals zero at two negative values and two positive values. Also, the LHS goes to positive infinity as  $z$  goes to both negative infinity and positive infinity. Finally, the LHS, when evaluated at  $z = 0$ , equals  $(r + \lambda_1)(r + \lambda_2)$ , which is greater than  $\lambda_1\lambda_2$ . Therefore, there are two positive values and two negative values that satisfy this polynomial equation.

Therefore, after solving for the roots  $\eta$  and  $\gamma$ , we are then able to express  $B_2$  in terms of  $B_1$ , and express  $C_2$  in terms of  $C_1$  as follows:

$$B_2 = -\frac{\lambda_1}{K_1(\gamma)}B_1 = -\frac{\lambda_1}{\lambda_2}B_1, \quad (33)$$

$$C_2 = -\frac{\lambda_1}{K_1(\eta)}C_1 = C_1. \quad (34)$$

We now have four parameters to determine:  $A_1, B_1, C_1$ , and  $\bar{X}$ . To do so, we use the 4 boundary conditions (9), (10), (15), and (16).

$$B_1\bar{X}^\gamma + C_1\bar{X}^\eta = \bar{X} - I, \quad (35)$$

$$\gamma B_1\bar{X}^\gamma + \eta C_1\bar{X}^\eta = \bar{X}, \quad (36)$$

$$-\frac{\lambda_1}{\lambda_2}B_1\bar{X}^\gamma + C_1\bar{X}^\eta = A_1\bar{X}^{\beta_2} + \frac{\lambda_1}{r + \lambda_1 - \alpha}\bar{X} - \frac{\lambda_1}{r + \lambda_1}I, \quad (37)$$

$$-\gamma\frac{\lambda_1}{\lambda_2}B_1\bar{X}^\gamma + \eta C_1\bar{X}^\eta = \beta_2 A_1\bar{X}^{\beta_2} + \frac{\lambda_1}{r + \lambda_1 - \alpha}\bar{X}. \quad (38)$$

Solving yields:

$$\bar{X} = I \left[ \frac{\beta_2}{r + \lambda_1} - \frac{\eta}{\lambda_2} \frac{\beta_2 - \gamma}{\gamma - \eta} - \frac{\gamma}{\lambda_1} \frac{\beta_2 - \eta}{\gamma - \eta} \right] \quad (39)$$

$$\times \left[ \frac{1 - \eta}{\lambda_2} \frac{\beta_2 - \gamma}{\gamma - \eta} - \frac{\gamma - 1}{\lambda_1} \frac{\beta_2 - \eta}{\gamma - \eta} + \frac{\beta_2 - 1}{r + \lambda_1 - \alpha} \right]^{-1}, \quad (40)$$

$$A_1 = \bar{X}^{-\beta_2} \left[ -\frac{\lambda_1 \bar{X} (1 - \eta) + \eta I}{\lambda_2} \frac{1}{\gamma - \eta} + \frac{\bar{X} (\gamma - 1) - \gamma I}{\gamma - \eta} - \frac{\lambda_1 \bar{X}}{r + \lambda_1 - \alpha} + \frac{\lambda_1 I}{r + \lambda_1} \right] \quad (41)$$

$$B_1 = \frac{\bar{X}(1 - \eta) + I\eta}{(\gamma - \eta)\bar{X}^\gamma}, \quad (42)$$

$$C_1 = \frac{\bar{X}(\gamma - 1) - I\gamma}{(\gamma - \eta)\bar{X}^\eta}.$$

Intuition suggests that illiquidity induces earlier exercise. While the market is open, fear of being locked out during a potential upcoming black-out period causes the firm to invest earlier than it would if the investment project was fully liquid. The following proposition states this more formally.

**Proposition 1** For any  $\lambda_2 > 0$ ,  $\bar{X} < X^*$ .

**Proof.** Consider equation (8) subject to boundary conditions (9) through (11). Note that in this range, the value of the open option  $V$  is greater than the value of the blacked-out option  $W$ . Thus for positive  $\lambda_2$ , differential equation (8) is like the standard real options valuation equation (as denoted in equations (2) through (5)), but with the addition of the fact that the option pays a “negative dividend” of  $\lambda_2(W - V)$ . Thus, while holding the option, one is paying a negative dividend (relative to the fully liquid case), creating an incentive to exercise sooner. Thus,  $\bar{X} < X^*$ . ■

### 3 Special cases

In this section we look at four special cases, where we consider the corner solutions with respect to the probability of the opening and closing of the market. This should prove helpful in providing intuition for the model's results. Specifically, here we see the solutions for the limiting cases as the jump intensities approach zero or infinity.

#### Case I: Black-Out Period is Permanent ( $\lambda_1 = 0$ )

Consider an environment that is subject to a shock that would lead to the permanent closure of the opportunity to invest in the project. One might think of being preempted by a major competitor or the loss of a patent fight (which isn't quite permanent, but similar in its economic ramifications). In such cases, once an open option is transformed into a blacked-out one, there is no possibility that the option will ever again become open. Clearly, this threat of closure will exert a significant pressure on the firm to exercise the option while it remains open.

While the current period is open, the agent shall essentially discount at a rate with  $r + \lambda_2$ . Taking limits of  $\bar{X}$  in equation (39) gives us:

$$\lim_{\lambda_1 \rightarrow 0} \bar{X} = \lim_{\lambda_1 \rightarrow 0} I \left[ -\frac{\gamma}{\lambda_1} \frac{\beta_2 - \eta}{\gamma - \eta} \right] \left[ -\frac{\gamma - 1}{\lambda_1} \frac{\beta_2 - \eta}{\gamma - \eta} \right]^{-1} \quad (43)$$

$$= \frac{\gamma}{\gamma - 1} I, \quad (44)$$

where  $\gamma$  is the positive root with discount rate  $r + \lambda_2$  (as  $\lambda_1 \rightarrow 0$ ).

This result is analogous to the well-known bond pricing result in which for a bond subject to total default with a jump parameter of  $\lambda_2$ , the pricing formula substitutes the term " $r + \lambda_2$ " for the term " $r$ " in the riskless formula.

#### Case II: Open Period is Permanent ( $\lambda_2 = 0$ )

This case is basically the reverse of the previous one. Case 2 is one in which the investment opportunity is currently closed, but awaits a shock that permanently opens the investment opportunity. Such a setting might include a pharmaceutical firm awaiting FDA approval to manufacture and market a drug, or a firm awaiting a technological breakthrough that permits the introduction of a new product.

If a black-out period ever exists, then once the period becomes open it remains open forever after. Thus, once one is in an open period (where the option may be freely exercised), there is no possibility of the option ever becoming illiquid. Thus trivially, the open option fits the standard real option model solved in Subsection 2.2, where the optimal exercise trigger is  $X^*$ .

Verifying our solution, we can take limits of  $\bar{X}$  in equation (39) to give us:

$$\lim_{\lambda_2 \rightarrow 0} \bar{X} = \lim_{\lambda_2 \rightarrow 0} I \left[ -\frac{\eta}{\lambda_2} \frac{\beta_2 - \gamma}{\gamma - \eta} \right] \left[ \frac{1 - \eta}{\lambda_2} \frac{\beta_2 - \gamma}{\gamma - \eta} \right]^{-1} \quad (45)$$

$$= \frac{\eta}{\eta - 1} I \equiv \frac{\beta}{\beta - 1} I \equiv X^*. \quad (46)$$

### Case III: Black-Out Period is Infinitesimal ( $\lambda_1 = \infty$ )

Here we consider a setting in which the market may be blacked-out (in fact, for any finite jump amplitude), but only for an infinitesimal interval. Thus we are thinking here about markets in which closings are only for very short durations. For example, one might think of real estate development where weather and scheduling issues temporarily intervene. Obviously, as the duration of a black-out becomes infinitesimal, with continuous sample paths of  $X$ , the option exercise setting converges to that of the benchmark fully-liquid model. The solution should therefore conform to the standard real options case.

To see this, note that  $\gamma \rightarrow \infty$  as  $\lambda_1 \rightarrow \infty$ , and

$$\lim_{\lambda_1 \rightarrow \infty} \bar{X} = \lim_{\lambda_1 \rightarrow \infty} I \left[ -\frac{\eta}{\lambda_2} \frac{\beta_2 - \gamma}{\gamma - \eta} - \frac{\gamma}{\lambda_1} \frac{\beta_2 - \eta}{\gamma - \eta} \right] \left[ \frac{1 - \eta}{\lambda_2} \frac{\beta_2 - \gamma}{\gamma - \eta} - \frac{\gamma - 1}{\lambda_1} \frac{\beta_2 - \eta}{\gamma - \eta} \right]^{-1} \quad (47)$$

$$= \lim_{\lambda_1 \rightarrow \infty} I \left[ -\frac{\eta}{\lambda_2} \frac{\beta_2 - \gamma}{\gamma - \eta} \right] \left[ \frac{1 - \eta}{\lambda_2} \frac{\beta_2 - \gamma}{\gamma - \eta} \right]^{-1} \quad (48)$$

$$= \frac{\eta}{\eta - 1} I \equiv \frac{\beta}{\beta - 1} I \equiv X^*. \quad (49)$$

Thus, Cases II and III have equivalent exercise strategies.

### Case IV: Open Period is Infinitesimal ( $\lambda_2 = \infty$ )

In this setting, opportunities to invest are fleeting. Thus at times that investment opportunities present themselves, they are quickly lost. One might imagine a market made up of many relatively similar competitors, who quickly take advantage of investment opportunities, preempting others and denying them the benefits of investment. In this case, we consider the limiting case in which the open period becomes a singleton (for each realized arrival of the Poisson shock). This is the special case treated in Dupuis and Wang (2002). In this case, the only instances at which the option may be exercised are those moments at which the jump occurs to briefly open the market. The firm may either exercise immediately, or wait until future instants at which the market briefly opens.

Note that with  $\lambda_2 \rightarrow \infty$ ,  $\gamma \rightarrow \infty$ . Hence, we have

$$\begin{aligned} \lim_{\lambda_2 \rightarrow \infty} \bar{X} &= \lim_{\lambda_2 \rightarrow \infty} \left( \frac{\beta_2}{r + \lambda_1} - \frac{\beta_2 - \eta}{\lambda_1} \right) \left( -\frac{\beta_2 - \eta}{\lambda_1} + \frac{\beta_2 - 1}{r + \lambda_1 - \alpha} \right)^{-1} I, \\ &= \lim_{\lambda_2 \rightarrow \infty} \left( \eta - \frac{r}{r + \lambda_1} \beta_2 \right) \left( \eta - \frac{r - \alpha}{r + \lambda_1 - \alpha} \beta_2 - \frac{\lambda_1}{r + \lambda_1 - \alpha} \right)^{-1} I \\ &= \left( \eta - \frac{r}{r + \lambda_1} \beta_2 \right) \left( \eta - 1 - \frac{r - \alpha}{r + \lambda_1 - \alpha} (\beta_2 - 1) \right)^{-1} I. \end{aligned} \quad (50)$$

## 4 The Discount for Illiquidity

Consider the value discount due to the potential inability to exercise the option. An exercisable option on an illiquid project (during the open period) has a value of  $V(X)$ , while an option on a fully liquid project has a value of  $F(X)$ .  $V(X)$  and  $F(X)$  are summarized in equations (6) and Section 2.4, respectively:

$$F(X) = \begin{cases} \left(\frac{X}{X^*}\right)^\beta (X^* - I) & \text{for } X < X^*, \\ X - I & \text{for } X \geq X^*. \end{cases} \quad (51)$$

$$V(X) = \begin{cases} B_1 X^\gamma + C_1 X^\eta & \text{for } X < \bar{X}, \\ \bar{X} - I & \text{for } X \geq \bar{X}. \end{cases} \quad (52)$$

We recall from Proposition 1 that  $\bar{X} < X^*$ .

In this section we will focus on the discount for illiquidity for an option that is open over the region  $X < \bar{X}$ . That is, if one holds a currently exercisable option on an illiquid project, how much is its price lowered relative to that of freely exercisable option with all other terms the same? We define this discount for liquidity by:

$$D(X) = \frac{F(X) - V(X)}{F(X)}, \quad X < \bar{X}, \quad (53)$$

where  $D(X) \in [0, 1]$ . Similarly,  $\bar{D}(X)$  would be the price (as a percentage of the fully liquid option price) that one would be willing to pay in order to make one's illiquid option fully liquid.

Figure 1 depicts the impact on the illiquidity discount  $D(X)$  as the jump intensities  $\lambda_1$  and  $\lambda_2$  vary, over the range of  $X$ . The top curve depicts the most illiquid environment, where  $\lambda_1 = 0.05$  and  $\lambda_2 = 2.0$ . This means that on average, an open market is blacked-out every six months, while a blacked-out market opens up every 20 years. For this illiquid environment, the illiquidity discount is around 20% for most of the range of  $X$ , declining slightly as the option approaches its exercise trigger. The middle curve depicts an intermediate illiquid environment, where  $\lambda_1 = 0.1$  and  $\lambda_2 = 1.0$ . This means that on average, an open market is blacked-out every twelve months, while a blacked-out market opens up every 10 years. For this intermediate environment, the illiquidity discount is around 10%. The bottom curve depicts the most liquid environment, where  $\lambda_1 = 1.0$  and  $\lambda_2 = 0.50$ . This means that on average, an open market is blacked-out every two years, while a blacked-out market opens up every twelve months. For this most liquid environment, there is virtually no discount for illiquidity. Note that in all environments, the illiquidity discount falls as  $X$  increases, since the illiquid option will be exercised immediately once  $X$  reaches the exercise trigger.<sup>9</sup>

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<sup>9</sup>Default parameter values for Figure 1 are  $\alpha = 0.02$ ,  $r = 0.04$ ,  $\sigma = 0.3$ , and  $I = 1.0$ .

Figure 2 depicts the impact on the illiquidity discount  $D(X)$  as the market volatility varies. The top curve depicts the most volatile environment where  $\sigma = 0.35$ , the middle curve depicts the intermediate volatile environment where  $\sigma = 0.20$ , and the bottom curve depicts the least volatile environment where  $\sigma = 0.05$ . Note that greater volatility increases the discount to illiquidity. The intuition is relatively straightforward. As is well-known in the real options literature, there is a valuable option to wait before exercising. This option to wait is more valuable, the greater the underlying volatility. However, with an illiquid project, the option to wait is curtailed, as one is compelled to exercise early so as to avoid being shut out from a blacked-out market. Thus, in markets with greater volatility, the discount for illiquidity is larger as it detracts from fundamentally more valuable options to wait.<sup>10</sup>

## 5 Endogenizing Liquidity

We now consider a setting in which one or both of the liquidity parameters  $\lambda_1$  and  $\lambda_2$  are set endogenously.<sup>11</sup> Consider the setting of a regulatory entity that sets these parameters so as to maximize some objective function. In particular, consider the case in which investment provides a positive externality. Specifically, let the private cost of investment be  $I$ , but the public cost of investment be  $I - \theta$ , with  $\theta \in (0, I)$ . Note that by introducing illiquidity, we have learned that investment is triggered earlier. Thus, this is what the regulator will attempt to do by optimally choosing the  $\lambda_1$  and/or  $\lambda_2$ .

For any pair of parameters  $\lambda_1$  and  $\lambda_2$ , the firm will exercise at the trigger  $\bar{X}$  denoted in (39), where we make explicit the parameter dependence by writing it as  $\bar{X}(\lambda_1, \lambda_2)$ . We assume that the value to the regulatory entity is equal to the payoff of  $\bar{X}(\lambda_1, \lambda_2) - (I - \theta)$ , at the time of exercise.

Similar to the setup of the firm's problem in Section 2, let the value functions for the regulator be denoted by  $\hat{V}(X; \lambda_1, \lambda_2)$ ,  $\hat{W}_H(X; \lambda_1, \lambda_2)$ , and  $\hat{W}_L(X; \lambda_1, \lambda_2)$ , for the values to the regulator when the market is open, closed with  $X \geq \bar{X}$ , and closed with  $X < \bar{X}$ , respectively. These can be expressed as:

$$\hat{V}(X; \lambda_1, \lambda_2) = \hat{B}_1(\lambda_1, \lambda_2)X^{\gamma(\lambda_1, \lambda_2)} + \hat{C}_1(\lambda_1, \lambda_2)X^\eta, \quad (54)$$

$$\hat{W}_H(X; \lambda_1, \lambda_2) = -\frac{\lambda_1}{\lambda_2}\hat{B}_1(\lambda_1, \lambda_2)X^{\gamma(\lambda_1, \lambda_2)} + \hat{C}_1(\lambda_1, \lambda_2)X^\eta, \quad (55)$$

$$\hat{W}_L(X; \lambda_1, \lambda_2) = \hat{A}_1(\lambda_1, \lambda_2)X^{\beta_2(\lambda_1)} + \frac{\lambda_1}{r + \lambda_1 - \alpha}X - \frac{\lambda_1}{r + \lambda_1}(I - \theta). \quad (56)$$

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<sup>10</sup>Default parameter values for Figure 2 are  $\alpha = 0.02$ ,  $r = 0.04$ ,  $\lambda_1 = 0.05$ ,  $\lambda_2 = 2.0$ , and  $I = 1.0$ .

<sup>11</sup>It is also possible that regulation could take a non-random form, in which the opening and closing of the market are chosen deterministically. However, in keeping with the stochastic setting of our model, we assume that regulators only control the parameters governing the stochastic processes that determine the timing of such events.

The boundary conditions are:

$$\hat{V}(\bar{X}(\lambda_1, \lambda_2); \lambda_1, \lambda_2) = \bar{X}(\lambda_1, \lambda_2) - (I - \theta), \quad (57)$$

$$\hat{W}_H(\bar{X}(\lambda_1, \lambda_2); \lambda_1, \lambda_2) = \hat{W}_L(\bar{X}(\lambda_1, \lambda_2); \lambda_1, \lambda_2), \quad (58)$$

$$\frac{\partial \hat{W}_H}{\partial \bar{X}}(\bar{X}(\lambda_1, \lambda_2); \lambda_1, \lambda_2) = \frac{\partial \hat{W}_L}{\partial \bar{X}}(\bar{X}(\lambda_1, \lambda_2); \lambda_1, \lambda_2). \quad (59)$$

Note these boundary conditions are identical to those in the basic model, with the exception that the trigger is determined by the regulated firm, and thus does not require a smooth-pasting condition for the regulator.

The solutions are:

$$\hat{B}_1(\lambda_1, \lambda_2) = \frac{\bar{X}(\lambda_1, \lambda_2)^{-\gamma(\lambda_1, \lambda_2)}}{[\beta_2(\lambda_1) - \gamma(\lambda_1, \lambda_2)] \frac{\lambda_1}{\lambda_2} + \beta_2(\lambda_1) - \eta} \times \quad (60)$$

$$\left\{ \frac{\bar{X}(\lambda_1, \lambda_2)}{r + \lambda_1 - \alpha} [(1 - \beta_2(\lambda_1)) \lambda_1 - (\eta - \beta_2(\lambda_1))(r + \lambda_1 - \alpha)] + \quad (61)$$

$$\frac{I - \theta}{r + \lambda_1} [\eta(r + \lambda_1) - \beta_2(\lambda_1)r] \right\}, \quad (62)$$

$$\hat{C}_1(\lambda_1, \lambda_2) = \bar{X}(\lambda_1, \lambda_2)^{-\eta} \left[ \bar{X}(\lambda_1, \lambda_2) - (I - \theta) - \hat{B}_1(\lambda_1, \lambda_2) \bar{X}(\lambda_1, \lambda_2)^{\gamma(\lambda_1, \lambda_2)} \right] \quad (63)$$

$$\hat{A}_1(\lambda_1, \lambda_2) = \frac{1}{\beta_2(\lambda_1) \bar{X}(\lambda_1, \lambda_2)^{\beta_2(\lambda_1)}} \left\{ -\frac{\lambda_1}{\lambda_2} \hat{B}_1(\lambda_1, \lambda_2) \gamma(\lambda_1, \lambda_2) \bar{X}(\lambda_1, \lambda_2)^{\gamma(\lambda_1, \lambda_2)} \right. \quad (64)$$

$$\left. \hat{C}_1(\lambda_1, \lambda_2) \eta \bar{X}(\lambda_1, \lambda_2)^\eta - \frac{\lambda_1}{r + \lambda_1 - \alpha} \bar{X}(\lambda_1, \lambda_2) \right\}. \quad (65)$$

Let us consider a particular example to solve. Consider special case *IV* from Section 3 in which the open period is always infinitesimal ( $\lambda_2 = \infty$ ). Thus, the regulator must choose  $\lambda_1$  (the intensity of market openings) so as to maximize its current value function. Assume that the market is currently closed, and that  $X$  is currently below the chosen trigger  $\bar{X}(\lambda_1, \infty)$ .

Thus, the regulator's problem is:

$$\max_{\lambda_1 \geq 0} \hat{W}_L(X; \lambda_1, \infty). \quad (66)$$

Figure 3 displays the optimal choice of  $\lambda_1$  for varying degrees of positive externalities,  $\theta$ . For a relatively small positive externality,  $\theta = 0.10$ , the optimal  $\lambda_1$  is 2.65, implying that on average the market will be opened every 4.5 months. However, for a larger positive externality,  $\theta = 0.50$ , the optimal  $\lambda_1$  is 0.16, implying that on average the market will be opened every 6.25 years, or 75 months. This is in keeping with our intuition, as the regulator will choose to make the market more illiquid in order to motivate earlier investment with greater positive externalities.<sup>12</sup>

<sup>12</sup>Default parameter values are  $r = 0.04$ ,  $\alpha = 0.02$ ,  $\sigma = 0.3$ ,  $I = 1.0$ , and  $X = 2.0$ .

## 6 Perfectly Competitive Industry Equilibrium with Illiquid Projects

Thus far in this article, firms have been modeled essentially as monopolists, ignoring the impact of potential competitive investment on their own payoffs. In this section, we extend to model of illiquid investment opportunities to the case of a perfectly competitive equilibrium outcome for an industry with free entry. The competitive equilibrium framework that we use is similar to that of Leahy (1993) and Dixit and Pindyck (1994). The key contribution of this section is the extension of the equilibrium to the case with illiquid investment projects.

Consider an industry comprised of a large number of identical firms. Each firm has the option to irreversibly undertake a single investment by paying an up-front investment cost of  $I$  at chosen time  $\tau$ , at moments when the market is open and not blacked-out. Investment is irreversible, in that exit from the industry is not permitted. Upon investment, the project yields a stream of stochastic (profit) flow of  $\{p(s) : s \geq \tau\}$  forever.<sup>13</sup> The industry is perfectly competitive, in that each unit of output is small in comparison with industry supply,  $Q(t)$ . Thus, each entrepreneur acts as a price taker. The equilibrium price is determined by the condition equating industry supply and demand. Each entrepreneur takes as given the stochastic process of price  $p$ . In the rational expectations equilibrium, this conjectured price process will indeed be the market clearing price.

The price of a unit of output is given by the industry's inverse demand curve

$$p(t) = \theta(t) \cdot D [Q(t)], \quad (67)$$

where  $D'(Q) < 0$  and  $\theta(t)$  is a multiplicative shock and is given by following the geometric Brownian motion process:

$$d\theta(t) = \alpha\theta(t)dt + \sigma\theta(t)dz_t. \quad (68)$$

Over an interval of time in which no entry takes place,  $Q(\cdot)$  is fixed, and thus the price process  $p$  evolves as follows:

$$dp(t) = \alpha p(t)dt + \sigma p(t)dz_t. \quad (69)$$

Given the multiplicative shock specification of the demand curve in (67), entry by new firms causes the price process to have an upper reflecting barrier. Thus, in this simple setting, each price taking entrepreneur will take the process (69) with an upper reflecting barrier as given. In the rational expectations equilibrium, the entry response by entrepreneurs who assume such a process will lead precisely to the supply process that equates supply and demand.

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<sup>13</sup>We assume no variable costs of production, and thus the process  $p$  represents cash flow process.

Conjecture that the equilibrium entry will be at the trigger  $p_{eq}$ , and thus in equilibrium the price process will have an upper reflecting barrier at  $p_{eq}$ . We will begin by considering the value of an active firm (one that has already paid the entry cost and is producing output) in an open market,  $\Phi(p)$ , and in a blacked-out market,  $\Psi(p)$ . By the standard argument,  $\Phi(p)$  satisfies the equilibrium differential equation:

$$\frac{1}{2}\sigma^2 p^2 \Phi'' + \alpha p \Phi' - r\Phi + p + \lambda_2(\Psi - \Phi) = 0, \quad p \leq p_{eq}. \quad (70)$$

The impact of the reflecting barrier necessitates the boundary condition<sup>14</sup>:

$$\Phi'(p_{eq}) = 0. \quad (71)$$

Now, consider the active firm when the market is blacked-out. There will be two ranges of interest: one where the current price is above the entry threshold  $p_{eq}$  (and thus there are firms that would like to enter but cannot), and one where the current price is below the entry threshold  $p_{eq}$  (and thus there is no current desire for new entry). Let the value of the active firm in the black-out period be denoted by  $\Psi_H(p)$ , and  $\Psi_L(p)$ , for the two regions, respectively.

In the range in which  $p > p_{eq}$ , if the market opens a discrete number of new firms will enter causing the price to fall sufficiently to ensure that entrants have a zero net present value from entry.

$$0 = \frac{1}{2}\sigma^2 p^2 \Psi_H'' + \alpha p \Psi_H' - r\Psi_H + p + \lambda_1(I - \Psi_H), \quad p > p_{eq}. \quad (72)$$

Conversely, when  $p \leq p_{eq}$ , if the market opens, no new entry will occur.

$$0 = \frac{1}{2}\sigma^2 p^2 \Psi_L'' + \alpha p \Psi_L' - r\Psi_L + p + \lambda_1(\Phi - \Psi_L), \quad p \leq p_{eq}. \quad (73)$$

The additional boundary conditions are:

$$\Psi_H(p_{eq}) = \Psi_L(p_{eq}), \quad (74)$$

$$\Psi_H'(p_{eq}) = \Psi_L'(p_{eq}), \quad (75)$$

$$\Phi(p_{eq}) = I. \quad (76)$$

The first two boundary conditions ensure that the value of the active firm in a blacked-out market is continuously differentiable at  $p_{eq}$ . The third boundary condition ensures the free-entry condition of new entry obtaining a zero net present value.

Note that we ensure that free entry causes zero profits to entrants in both entry cases: when the market is open and  $p = p_{eq}$ , and when the blacked-out market suddenly opens when  $p > p_{eq}$ . In an open market when  $p = p_{eq}$ , the marginal entrant receives the net present value of  $\Phi(p_{eq}) - I = 0$ . In a blacked-out market

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<sup>14</sup>See Malliaris and Brock (1982, p. 200).

that suddenly opens when  $p > p_{eq}$ , the discrete cluster of entrants causes  $p$  to fall to  $p_{eq}$ , where again the net present value is zero.

The solution is:

$$\Phi(p) = b_1 p^\gamma + c_1 p^\eta + \frac{p}{r - \alpha}, \quad (77)$$

$$\Psi_H(p) = -\frac{\lambda_1}{\lambda_2} b_1 p^\gamma + c_1 p^\eta + \frac{p}{r - \alpha}, \quad (78)$$

$$\Psi_L(p) = a_1 p^{\beta_2} + \frac{p}{r + \lambda_1 - \alpha} + \frac{\lambda_1}{r + \lambda_1} I, \quad (79)$$

where:

$$p_{eq} = (r - \alpha) \bar{X}, \quad (80)$$

$$b_1 = \frac{p_{eq}^{-\gamma}}{\eta - \gamma} \left[ \eta I - (\eta - 1) \frac{p_{eq}}{r - \alpha} \right], \quad (81)$$

$$c_1 = \frac{p_{eq}^{-\eta}}{\gamma - \eta} \left[ \gamma I - (\gamma - 1) \frac{p_{eq}}{r - \alpha} \right], \quad (82)$$

$$a_1 = p_{eq}^{-\beta_2} \left[ -\frac{\lambda_1}{\lambda_2} B_1 p_{eq}^\gamma + C_1 p_{eq}^\eta + \frac{\lambda_1}{(r + \lambda_1 - \alpha)(r - \alpha)} p_{eq} - \frac{\lambda_1}{r + \lambda_1} I \right]. \quad (83)$$

Importantly, as has been demonstrated by Leahy (1993) and others, the exercise trigger for a perfectly competitive industry equals the monopolist trigger. This is clear as the value of the project at the equilibrium trigger is equal to  $\frac{p_{eq}}{r - \alpha} = \bar{X}$ . The intuition is that the reflecting barrier has two exactly opposing effects: it lowers the value of the payoff from exercise (since the future cash flow is capped at the barrier), while it also lowers the option value of waiting. Interestingly, this result survives the extension to illiquid projects. However, unlikely the Leahy setting of perfect liquidity, in this illiquid setting new entry may occur (in discrete jumps) at prices above the equilibrium trigger level  $p_{eq}$ , at those times when the market opens with pent-up entry demand.

## 7 A More General Model of Illiquidity

In the basic model, when the market is blacked-out, the option cannot be exercised at any price. One might view this as a fairly extreme form of illiquidity. In this section, rather than ruling out exercise during the illiquid period, we allow the option to be exercised, but at a potentially higher exercise price. Specifically, the option can be exercised in the liquid period for  $I$ , and in the illiquid period for  $I + K$ , with  $K > 0$ . Note that if we let  $K \rightarrow \infty$ , we will be back to our initial model.

There will be two exercise triggers. In the liquid market, exercise will occur at the trigger  $\bar{X}_1$ , while in the illiquid market exercise will occur at the trigger  $\bar{X}_2$ , where

$\bar{X}_1 < \bar{X}_2$ . Let  $\Omega(X)$  denote the option value during the liquid period, and  $\Pi(X)$  denote the option value during the illiquid period.  $\Omega(X)$  must satisfy the following differential equation:

$$0 = \frac{1}{2}\sigma^2 X^2 \Omega'' + \alpha X \Omega' - r\Omega + \lambda_2 (\Pi - \Omega), \quad X \leq \bar{X}_1 \quad (84)$$

subject to

$$\Omega(\bar{X}_1) = \bar{X}_1 - I, \quad (85)$$

$$\Omega'(\bar{X}) = 1. \quad (86)$$

The first boundary condition is the value-matching condition, while the second condition is the smooth-pasting condition. In addition, we will ensure that  $\Omega(0) = 0$ . Notice that a jump with intensity  $\lambda_2$  shifts the value function from  $\Omega(X)$  to  $\Pi(X)$ .

As usual, the value function during the illiquid period,  $\Pi(X)$ , can be decomposed into two regions:  $\Pi_H(X)$  when  $X > \bar{X}_1$ , and  $\Pi_L(X)$  when  $X \leq \bar{X}_1$ . During the illiquid period,  $\Pi_H(X)$  and  $\Pi_L(X)$  must satisfy the following differential equations:

$$0 = \frac{1}{2}\sigma^2 X^2 \Pi_H'' + \alpha X \Pi_H' - r\Pi_H + \lambda_1 (X - I - \Pi_H), \quad \bar{X}_1 < X < \bar{X}_2, \quad (87)$$

$$0 = \frac{1}{2}\sigma^2 X^2 \Pi_L'' + \alpha X \Pi_L' - r\Pi_L + \lambda_1 (\Pi_H - \Pi_L), \quad X \leq \bar{X}_1, \quad (88)$$

subject to

$$\Pi_H(\bar{X}_2) = \bar{X}_2 - I - K, \quad (89)$$

$$\Pi_H'(\bar{X}_2) = 1, \quad (90)$$

$$\Pi_L(\bar{X}_1) = \Pi_H(\bar{X}_1), \quad (91)$$

$$\Pi_L'(\bar{X}_1) = \Pi_H'(\bar{X}_1). \quad (92)$$

The first two boundary conditions are the value-matching and smooth-pasting conditions. The second two boundary conditions ensure that  $\Pi(X)$  is continuously differentiable at  $\bar{X}_1$ .

The general solutions are:

$$\Omega(X) = \bar{B}_1 X^\gamma + \bar{C}_1 X^\eta, \quad (93)$$

$$\Pi_L(X) = -\frac{\lambda_1}{\lambda_2} \bar{B}_1 X^\gamma + \bar{C}_1 X^\eta, \quad (94)$$

$$\Pi_H(X) = \bar{A}_1 X^{\beta_2} + \bar{A}_2 X^{\nu_2} + \frac{\lambda_1}{r + \lambda_1 - \alpha} X - \frac{\lambda_1}{r + \lambda_1} I, \quad (95)$$

where

$$\beta_2 = \frac{-(\alpha - \frac{1}{2}\sigma^2) - \sqrt{(\alpha - \frac{1}{2}\sigma^2)^2 + 2(r + \lambda_1)\sigma^2}}{\sigma^2} < 0, \quad (96)$$

$$\nu_2 = \frac{-(\alpha - \frac{1}{2}\sigma^2) + \sqrt{(\alpha - \frac{1}{2}\sigma^2)^2 + 2(r + \lambda_1)\sigma^2}}{\sigma^2} > 1. \quad (97)$$

We now have six equations and six unknowns ( $\bar{A}_1, \bar{A}_2, \bar{B}_1, \bar{C}_1, \bar{X}_1, \bar{X}_2$ ):

$$\bar{B}_1 \bar{X}_1^\gamma + \bar{C}_1 \bar{X}_1^\eta = \bar{X}_1 - I, \quad (98)$$

$$\gamma \bar{B}_1 \bar{X}_1^\gamma + \eta \bar{C}_1 \bar{X}_1^\eta = \bar{X}_1, \quad (99)$$

$$\bar{A}_1 \bar{X}_2^{\beta_2} + \bar{A}_2 \bar{X}_2^{\nu_2} + \frac{\lambda_1}{r + \lambda_1 - \alpha} \bar{X}_2 - \frac{\lambda_1}{r + \lambda_1} I = \bar{X}_2 - I - K, \quad (100)$$

$$\beta_2 \bar{A}_1 \bar{X}_2^{\beta_2} + \nu_2 \bar{A}_2 \bar{X}_2^{\nu_2} + \frac{\lambda_1}{r + \lambda_1 - \alpha} \bar{X}_2 = \bar{X}_2, \quad (101)$$

$$\bar{A}_1 \bar{X}_1^{\beta_2} + \bar{A}_2 \bar{X}_1^{\nu_2} + \frac{\lambda_1}{r + \lambda_1 - \alpha} \bar{X}_1 - \frac{\lambda_1}{r + \lambda_1} I = -\frac{\lambda_1}{\lambda_2} \bar{B}_1 \bar{X}_1^\gamma + \bar{C}_1 \bar{X}_1^\eta, \quad (102)$$

$$\beta_2 \bar{A}_1 \bar{X}_1^{\beta_2} + \nu_2 \bar{A}_2 \bar{X}_1^{\nu_2} + \frac{\lambda_1}{r + \lambda_1 - \alpha} \bar{X}_1 = -\gamma \frac{\lambda_1}{\lambda_2} \bar{B}_1 \bar{X}_1^\gamma + \eta \bar{C}_1 \bar{X}_1^\eta. \quad (103)$$

$\bar{X}_1$  and  $\bar{X}_2$  may be solved for via the following two equation system:

$$\begin{aligned} & \bar{X}_2^{-\beta_2} \left[ (\nu_2 - 1) \left( \frac{r - \alpha}{r + \lambda_1 - \alpha} \right) \bar{X}_2 + \nu_2 \left( \frac{\lambda_1 I_1 - (r + \lambda_1)(I + K)}{r + \lambda_1} \right) \right] \\ = & \bar{X}_1^{1-\beta_2} \left[ \frac{\nu_2 \left[ (\gamma - 1) + \frac{\lambda_1}{\lambda_2} (\eta - 1) \right] - \left( \eta(\gamma - 1) + \frac{\lambda_1}{\lambda_2} \gamma (\eta - 1) \right)}{\gamma - \eta} - \frac{\lambda_1 (\nu_2 - 1)}{r + \lambda_1 - \alpha} \right] \\ & + \bar{X}_1^{-\beta_2} I \left[ \nu_2 \left( \frac{\lambda_1}{r + \lambda_1} - \frac{\frac{\lambda_1}{\lambda_2} \eta + \gamma}{\gamma - \eta} \right) + \frac{\eta \gamma}{\gamma - \eta} \frac{\lambda_1 + \lambda_2}{\lambda_2} \right] \end{aligned} \quad (104)$$

$$\begin{aligned} & \bar{X}_2^{-\nu_2} \left[ (\beta_2 - 1) \left( \frac{r - \alpha}{r + \lambda_1 - \alpha} \right) \bar{X}_2 + \beta_2 \left( \frac{\lambda_1 I_1 - (r + \lambda_1)(I + K)}{r + \lambda_1} \right) \right] \\ = & \bar{X}_1^{1-\nu_2} \left[ \frac{\beta_2 \left[ (\gamma - 1) + \frac{\lambda_1}{\lambda_2} (\eta - 1) \right] - \left( \eta(\gamma - 1) + \frac{\lambda_1}{\lambda_2} \gamma (\eta - 1) \right)}{\gamma - \eta} - \frac{\lambda_1 (\beta_2 - 1)}{r + \lambda_1 - \alpha} \right] \\ & + \bar{X}_1^{-\nu_2} I \left[ \beta_2 \left( \frac{\lambda_1}{r + \lambda_1} - \frac{\frac{\lambda_1}{\lambda_2} \eta + \gamma}{\gamma - \eta} \right) + \frac{\eta \gamma}{\gamma - \eta} \frac{\lambda_1 + \lambda_2}{\lambda_2} \right] \end{aligned} \quad (105)$$

The remaining parameters then satisfy:

$$\bar{A}_1 = \frac{\bar{X}_2^{-\beta_2}}{\nu_2 - \beta_2} \left[ (\nu_2 - 1) \left( \frac{r - \alpha}{r + \lambda_1 - \alpha} \right) \bar{X}_2 + \nu_2 \left( \frac{\lambda_1 I - (r + \lambda_1)(I + K)}{r + \lambda_1} \right) \right] \quad (106)$$

$$\bar{A}_2 = \frac{\bar{X}_2^{-\nu_2}}{\beta_2 - \nu_2} \left[ (\beta_2 - 1) \left( \frac{r - \alpha}{r + \lambda_1 - \alpha} \right) \bar{X}_2 + \beta_2 \left( \frac{\lambda_1 I - (r + \lambda_1)(I + K)}{r + \lambda_1} \right) \right] \quad (107)$$

$$\bar{B}_1 = \bar{X}_1^{-\gamma} \frac{1}{\gamma - \eta} [ -(\eta - 1) \bar{X}_1 + \eta I ], \quad (108)$$

$$\bar{C}_1 = \bar{X}_1^{-\eta} \frac{1}{\gamma - \eta} [ (\gamma - 1) \bar{X}_1 - \gamma I ]. \quad (109)$$

The following proposition established the impact that the generalized notion of illiquidity has on the timing of exercise.

**Proposition 2** *The generalized model of illiquidity implies that the liquid option is exercised later than it is in the basic illiquidity model of Section 2, with  $\bar{X}_1 > \bar{X}$ . The liquid option, however, is still exercised earlier than in the case of full liquidity, with  $\bar{X}_1 < X^*$ . Finally, during the illiquid period, the option may now be exercised, but at a trigger above the full-liquidity trigger, with  $\bar{X}_2 > X^*$ . Thus,  $\bar{X} < \bar{X}_1 < X^* < \bar{X}_2$ .*

**Proof.** Consider equation (84) subject to boundary conditions (85) and (86). Note that in this range, the value of the liquid option  $\Omega$  is greater than the value of the illiquid option  $\Pi$ . Thus for positive  $\lambda_2$ , differential equation (84) is like the standard real options valuation equation (as denoted in equation (2) and its associated boundary conditions), but with the addition of the fact that the option pays a “negative dividend” of  $\lambda_2(\Pi - \Omega)$ . Thus, while holding the option, one is paying a negative dividend (relative to the fully liquid case), creating an incentive to exercise sooner. Thus,  $\bar{X}_1 < X^*$ . Note however that this negative dividend of  $\lambda_2(\Pi - \Omega)$  is smaller than the negative dividend of  $\lambda_2(W - V)$  embodied in differential equation (8) of the basic illiquidity model, since the basic illiquidity model allows no exercise during the black-out period, and thus the value of the blacked-out option is dominated by the value of the illiquid option in this current extension. Thus, with a now smaller negative dividend in the extended illiquidity model, there is less reason to defer exercise during the liquid period, leading to  $\bar{X} < \bar{X}_1$ . Finally, it is clear that as  $K \rightarrow 0$ ,  $\bar{X}_1$  and  $\bar{X}_2$  both converge to  $X^*$ . For any  $K > 0$ , all else held constant, the terms of the illiquid option become more onerous, leading to later exercise. Hence,  $\bar{X}_2 > X^*$ . ■

## 8 Conclusion

While the real options approach to modeling investment under uncertainty has proven valuable in understanding the timing of investment decisions, the standard approach is not suitable for analyzing situations in which investment opportunities are randomly recurring. While it is commonly accepted that many financial securities are

illiquid, it is even more clear that many real assets themselves are illiquid. Regulatory changes, technological bottlenecks, and competitive pressures make many real world investment options sporadic . Thus, this article's model should have many potential applications.

The model demonstrates that the potential threat of investment opportunity black-outs can have an impact on investment decisions while the option is open; most notably, firms will find it optimal to speed up their investment in anticipation of being locked out of the market in the future. As the article makes clear, the magnitude of the illiquidity effect depends upon the nature of the relative timing of the opening and closing of the investment opportunities, and the degree of the underlying uncertainty. Clearly, formulating an investment strategy that ignores potential illiquidity in a randomly recurring investment environment will lead to suboptimal investment timing.

The basic model of real options with illiquid projects is flexible enough to be extended in several directions in this article. We allow for the endogenous setting of the degree of illiquidity, as would be the case with a regulator. The model is also extended to the case of equilibrium, where a mass of competitive firms choose when to enter an industry, where entry opportunities are randomly recurring. The notion of illiquidity is also generalized to allow for firms to invest during illiquid periods, albeit at a higher cost.

## References

- [1] Berk, J., and P. DeMarzo, 2007, "Corporate Finance," Pearson Education, Inc., Boston, Massachusetts.
- [2] Berk, J., R. C. Green, and V. Naik, 1999, "Optimal Investment, Growth Options and Security Returns," *Journal of Finance* 54, 1153-1608.
- [3] Bernardo, A. E., B. Chowdhry, and A. Goyal, 2006, "Growth Options, Beta, and the Cost of Capital," *Financial Management*, forthcoming.
- [4] Brealey, R. A., S. C. Myers, and F. Allen, 2006, "Principles of Corporate Finance," McGraw Hill/Irwin, New York.
- [5] Brennan, M.J., and E. Schwartz, 1985, "Evaluating Natural Resource Investments," *Journal of Business* 58, 135-157.
- [6] Dixit, A. K., and R. S. Pindyck, 1994, "Investment Under Uncertainty," Princeton University Press, Princeton, New Jersey.
- [7] Dupuis, P., and H. Wang, 2002, "Optimal stopping with Random Intervention Times," *Advances in Applied Probability* 34, 141-157.
- [8] Fudenberg, D., R. Gilbert, J. Stiglitz, and J. Tirole, 1983, "Preemption, Leapfrogging and Competition in Patent Races," *European Economic Review* 22, 3-31.
- [9] Grenadier, S. R., 1995, "The Persistence of Real Estate Cycles," *Journal of Real Estate Finance and Economics* 10, 95-119.
- [10] Grenadier, S. R., 2002, "Option Exercise Games: An Application to the Equilibrium Investment Strategies of Firms," *Review of Financial Studies* 15, 691-721.
- [11] Lambrecht, B. M., 2000, "Strategic Sequential Investments and Sleeping Patents," in *Project Flexibility, Agency and Product Market Competition: New Developments in the Theory and Application of Real Options Analysis*, (eds) Brennan M J, Trigeorgis L, Oxford University Press, Oxford, 297-323.
- [12] Leahy, J. V., 1993, "Investment in Competitive Equilibrium: The Optimality of Myopic Behavior," *Quarterly Journal of Economics* 108, 1105-1133.
- [13] Longstaff, F. A., 2005, "Asset Pricing in Markets with Illiquid Assets," working paper, UCLA.
- [14] Malliaris, A. G., and W. A. Brock, 1982. "Stochastic Methods in Economics and Finance," North-Holland, New York.

- [15] McDonald, R., and D. Siegel, 1986, "The Value of Waiting to Invest," *Quarterly Journal of Economics* 101, 707-727.
- [16] Merton, R. C., 1973, "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science* 4, 141-183.
- [17] Schwartz, E. S., 2004, "Patents and R&D as Real Options," *Economic Notes* 33, 23-54.
- [18] Titman, S., 1985, "Urban Land Prices Under Uncertainty," *American Economic Review* 75, 505-514.
- [19] Williams, J. T., 1991, "Real Estate Development as an Option," *Journal of Real Estate Finance and Economics* 4, 191-208.

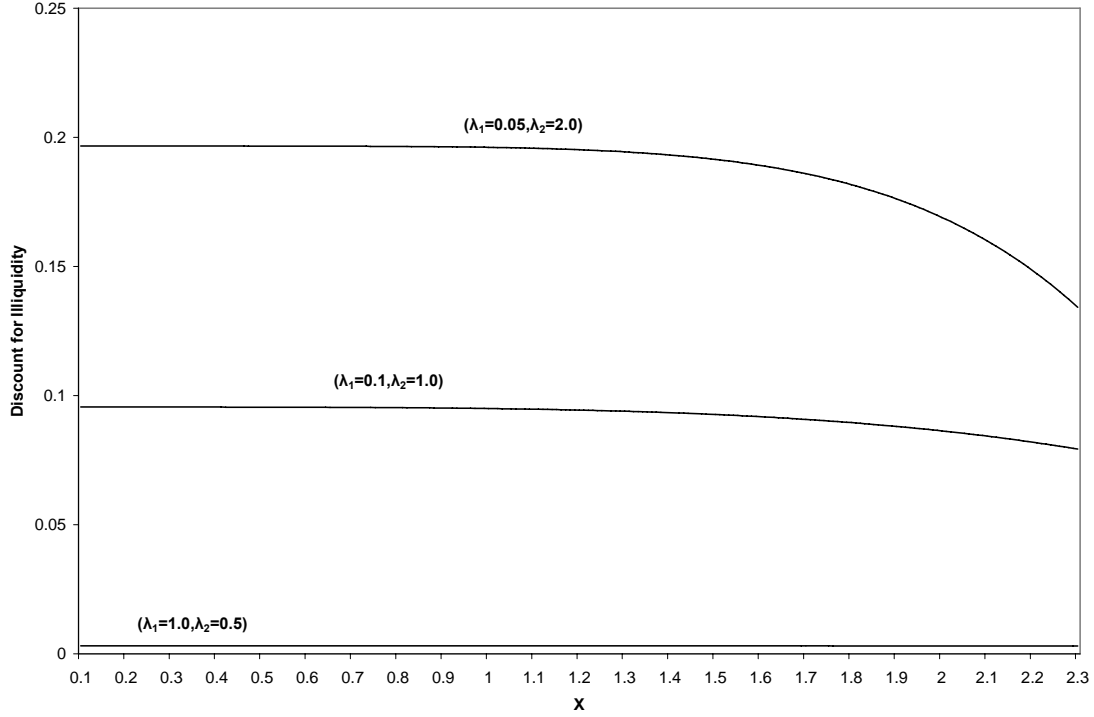


Figure 1: The Discount for Illiquidity in Different Liquidity Environments. This graph plots the impact on the option discount for illiquidity as the jump intensities  $(\lambda_1, \lambda_2)$  that determine the relative timing of the reopening and closing of the investment opportunity, respectively, vary. The top curve depicts the most illiquid environment, the middle curve depicts an intermediate illiquid environment, and the bottom curve depicts the least illiquid environment. We find that in these three settings, the discount varies from about 20% to around 1%. The parameter values are  $\alpha = 0.02$ ,  $r = 0.04$ ,  $\sigma = 0.3$ , and  $I = 1.0$ .

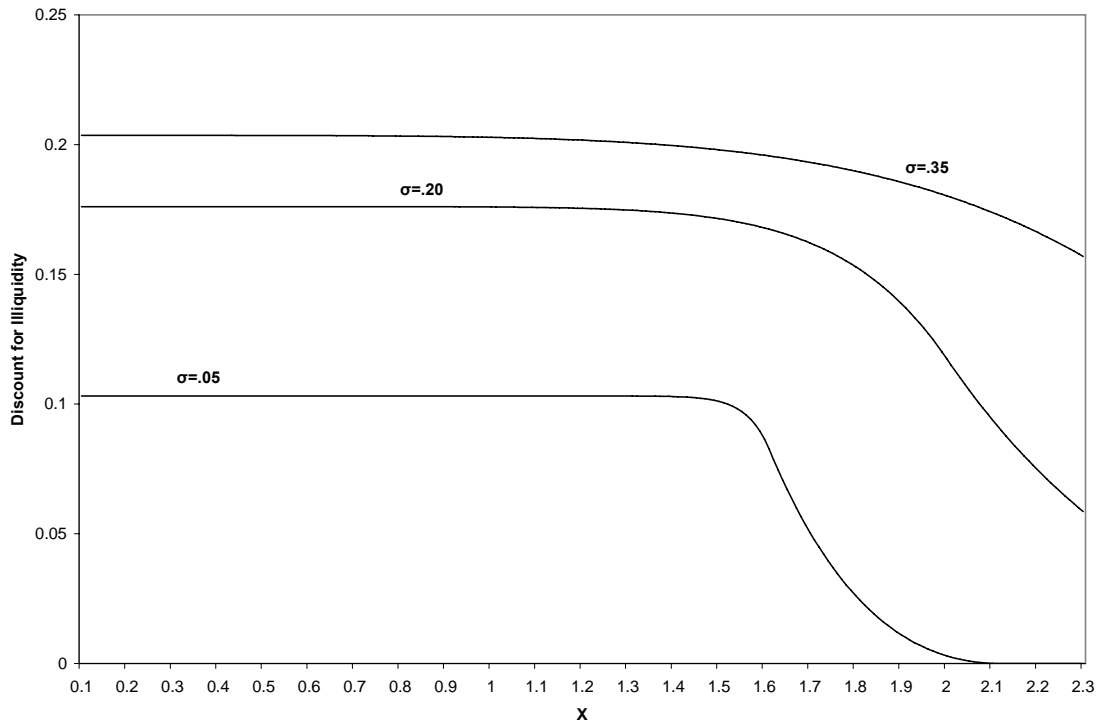


Figure 2: The Discount for Illiquidity in Different Volatility Environments. This graph plots the impact on the option discount as the market volatility varies, over the range of  $X$ . The top curve depicts the most volatile environment, the middle curve depicts an intermediate volatile environment, and the bottom curve depicts the least volatile environment. We find that greater volatility increases the discount for illiquidity. The parameter values are  $\alpha = 0.02$ ,  $r = 0.04$ ,  $\lambda_1 = 0.05$ ,  $\lambda_2 = 2.0$ , and  $I = 1.0$ .

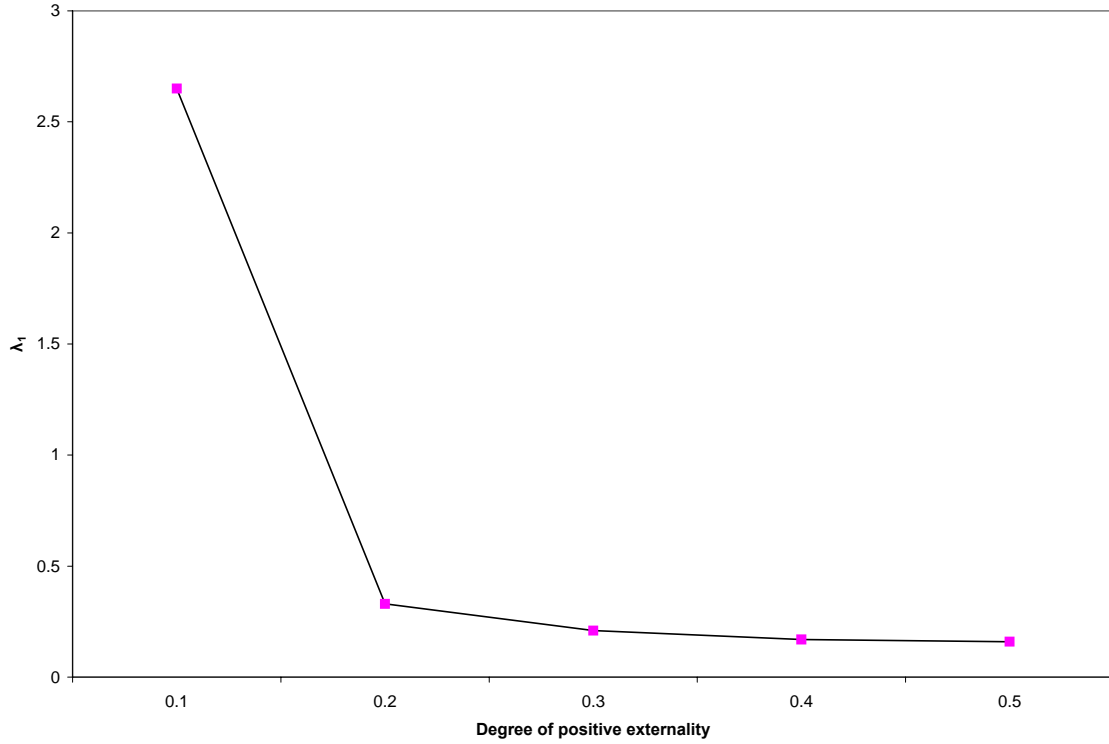


Figure 3: The Endogenous Choice of Liquidity. This figure plots the optimal choice of  $\lambda_1$ , the intensity of market reopenings, chosen by a central planner. Here, the central planner chooses to increase market illiquidity in order to inspire earlier investment to take advantage of a positive externality from investment. For a relatively small positive externality,  $\theta = 0.10$ , the optimal  $\lambda_1$  is 2.65, implying that on average the market will be opened every 4.5 months. However, for a larger positive externality,  $\theta = 0.50$ , the optimal  $\lambda_1$  is 0.16, implying that on average the market will be opened every 6.25 years. Parameter values are  $r = 0.04$ ,  $\alpha = 0.02$ ,  $\sigma = 0.3$ ,  $I = 1.0$ , and  $X = 2.0$ .