Market-Triggered Contingent Capital: Equilibrium Price Dynamics

Paul Glasserman* and Behzad Nouri†

October 2012

Abstract

Using the market price of a firm's equity to trigger a change in the firm's capital structure creates a question of internal consistency because the value of the equity itself depends on the firm's capital structure. Of particular interest is the case of contingent capital for banks, in the form of debt that converts to equity, when conversion is triggered by a decline in the bank's stock price. We analyze the problem of existence and uniqueness of equilibrium values for a firm's liabilities in this context, meaning values consistent with a market-price trigger. The liquidity of the triggering security has important implications for this problem. Limited liquidity in the form of intermittent trading allows multiple equilibria. In contrast, we show that the possibility of multiple equilibria can largely be ruled out if the triggering security is sufficiently liquid for its price to adjust continuously in anticipation of reaching a trigger. Within our general framework, existence of an equilibrium is ensured through appropriate positioning of the trigger level; in the case of contingent capital with a stock price trigger, we need the trigger to be sufficiently high. We formulate our results by comparing prices of claims on post-conversion and no-conversion variants of a firm. If the conversion is to be triggered by a decline in the market price of a claim, then the key condition we need is that the no-conversion price be higher than the post-conversion price when either is above the trigger. Put differently, we require that the trigger be sufficiently high to ensure that this holds. For the design of contingent capital with a stock price trigger, this condition may be interpreted to mean that conversion should be disadvantageous to shareholders. Our results apply as well to other types of changes in capital structure and triggers based on debt values as well as equity values.

1 Introduction

This paper investigates the feasibility of using the market price of a firm's equity to trigger a contractual change in the firm's capital structure. Because the equity value itself depends on the firm's capital structure, such a mechanism creates a question of internal consistency in the design of the trigger.

Our analysis is motivated by proposals to enhance financial stability by having banks issue contingent capital in the form of debt that converts to equity if the bank's financial condition

*403 Uris Hall, Columbia Business School, New York, NY 10027, pg20@columbia.edu

†IEOR Department, Columbia University, New York, NY 10027, bn2164@columbia.edu

deteriorates.[1] This type of contingent convertible debt offers the potential of a private sector alternative to government intervention in a crisis. Should a bank's assets suffer a sharp decline in value, the bank would ordinarily need to raise new equity to remain solvent; as raising equity near financial distress may be difficult, regulators may feel compelled to inject government support if a bankruptcy is deemed sufficiently disruptive. Contingent capital would instead provide a built-in mechanism for increasing a bank's equity through conversion of a portion of its debt. This conversion could eliminate a perceived need for government support and should be relatively immune to regulatory forbearance.

One of the main challenges to the effective use of contingent capital is the design of the trigger for conversion from debt to equity. In the major issuances to date (most notably by Lloyd's Banking Group and Credit Suisse), conversion is triggered by a regulatory capital ratio falling below a threshold. Flannery [11, 12] introduced contingent capital with a market-based trigger (a bank's stock price), based on the view that the market value of equity provides a better indicator of a bank's capital adequacy. Regulatory capital relies on accounting-based measures that are slow to respond to new information and are subject to some discretion in their implementation. In contrast, market signals in the form of stock prices, bond yields, and spreads on credit default swaps are credited with incorporating new information quickly and providing a forward-looking view of a firm's financial condition.[2]

Sundaresan and Wang [30] make the key observation that using a stock price to trigger a conversion of debt to equity is potentially problematic because the stock price is itself affected by the possibility of conversion. Their examples show that a conversion rule may admit multiple equilibria — that is, different stock prices consistent with the same mechanism — or no equilibrium. They caution that indeterminacy or inconsistency in the stock price could create an opening for market manipulation to prompt or prevent hitting of the trigger. They relate the possibility of multiple equilibria to a value transfer between shareholders and contingent capital investors at the point of conversion; an equilibrium with greater transfer of value away from shareholders is associated with a lower stock price and an earlier conversion. They propose a modification to the terms of conversion to prevent a value transfer. Prescott [27] proposes

[1] The Basel Committee [2] and the Financial Stability Oversight Council [10] provide regulatory overviews.

[2] McDonald [23] and Squam Lake Working Group [29] propose dual triggers measuring the condition of the banking sector generally as well as an individual institution, taking the view that contingent capital should convert only in the event of a broad threat to the financial system and not simply to protect an individual institution. Most analyses to date (including Albul et al. [1], Chen et al. [8], Hilscher and Raviv [17], Himmelberg and Tsyplakov [18], and Pennacchi [26]) have modeled conversion based on the value of a bank's assets; asset value is not directly observable but can be approximated using information about the value of bank liabilities (as in Calomiris and Herring [7] or Chen et al. [8]). Koziol and Lawrenz [20] use an earnings-based trigger, and the model in Chen et al. [8] accommodates that mechanism as well. Glasserman and Nouri [14] model an accounting-based capital ratio, allowing the book value and market value of assets to be imperfectly correlated. Hart and Zingales [16] consider a credit default swap trigger.

contract modifications for similar reasons in a static formulation of the problem.

We analyze the existence and uniqueness of equilibrium for contingent capital with a stock price trigger and a broad class of related problems involving a change in capital structure determined by the market price of a firm's liability. An essential feature of our analysis, absent from previous work, is the role of liquidity of the triggering security in the form of dynamic price adjustments that allow the market price to reflect the anticipated effect of conversion. Within the general framework of problems we study, once we have at least one equilibrium, continuous updating of prices rules out the possibility of multiple equilibria under modest additional conditions; indeed, exceptions to these conditions and examples allowing multiple equilibria are then rather contrived. Existence of an equilibrium in our framework typically depends on the level of the conversion trigger. In the basic case of contingent capital with a stock price trigger, we get one (and only one) equilibrium as long as the trigger is sufficiently high.

Our uniqueness results are easiest to describe when information arrives to the market continuously and prices evolve diffusively (although our analysis allows jumps as well). In this setting, consistency with rational expectations requires that an equilibrium stock price change continuously upon hitting the trigger, as only the sudden release of unanticipated information could cause the price to jump. Put differently, the ability of prices to adjust continuously is sufficient to preclude any transfer of value between holders of different claims when the trigger is reached; no contract modification is needed to enforce this property. Where continuous trading admits just one equilibrium, restricting transactions to discrete time points can give rise to multiple equilibria. This restriction on trading introduces a friction that prevents market prices from fully incorporating all relevant information, and it is this friction that creates a transfer of value at the trigger and that allows for multiple prices consistent with the terms of conversion.

This friction reflects illiquidity and, at the other extreme, continuous trading describes a perfectly liquid market. Our results thus imply that the potential pitfalls in contingent capital with a market trigger lie primarily in the liquidity of the price that triggers conversion rather than in the design of the contract itself. In the case of the stock price of a major financial institution, the speed with which information is incorporated into the price makes continuous trading a sensible approximation. Corporate bonds are much less liquid, and indeed Bianchi, Hancock, and Kawano [3] find that trading frequency affects the yields on subordinated debt and impairs the value of these yields as measures of a bank's financial condition. Credit default swaps, with liquidity intermediate to that of stocks and corporate bonds, have been proposed as triggers by Hart and Zingales [16] and others. How introducing a trigger based on a security might affect the volume of trading in that security is an open question, though it seems plausible that trading frequency would increase near the trigger, where the distinction between intermittent and continuous price updating is most important.

For our existence results and indeed our entire framework, we introduce two hypothetical firms — a post-conversion firm for which the trigger has in effect been tripped prior to time zero, and a no-conversion firm not subject to a trigger. In the specific case of contingent capital, the first firm has already had debt converted to equity, and the second firm's debt is not convertible at all. More generally, we use "conversion" to refer to whatever change results from the trigger, which could also be a debt write-down, a forced deleveraging, or an automatic consolidation of a subsidiary. Because neither the post-conversion firm nor the no-conversion firm is subject to an endogenous trigger, the prices of claims on these firms are automatically well-defined. The question of existence is whether one can define a price process (a stock price, for example) for the original firm that coincides with the post-conversion value after crossing the trigger and coincides with the no-conversion value if it never reaches the trigger. The key property we require is that the no-conversion price be higher than the post-conversion price when either is above the trigger. In the setting of contingent capital, one may take this to mean that conversion is disadvantageous to shareholders, but this statement needs to be interpreted with care because, as we have already stressed, there is no cost or benefit at the moment the trigger is reached.

With this structure in place, we construct an equilibrium in which conversion occurs the first time the post-conversion price reaches the trigger; under modest additional conditions, this is the only equilibrium. With trading restricted to discrete time points, this becomes the *minimal* equilibrium, giving the earliest possible conversion date and the lowest stock price among all possible stock prices. We do not model investor decisions, but this phenomenon can be interpreted through behavior implied by a stock price process. With illiquid intermittent trading, the first potential conversion date may occur at an asset level that makes conversion particularly disadvantageous to shareholders; speculation that a preferred conversion opportunity may occur later (or that conversion may be avoided entirely) can keep the stock price high, fulfilling the expectation of a postponed conversion. With continuous trading, prices adjust to reflect a declining asset value, so keeping the stock price high as asset value declines becomes impossible. The first conversion opportunity occurs when conversion can no longer be avoided. The potential conversion opportunities in this interpretation are the dates at which the post-conversion price is at or below the trigger, an assertion we make precise as part of our analysis.

As a contrast to the dynamic models on which we mainly focus and to help introduce ideas, we begin with a static setting. Prices and the triggering event are determined simultaneously, and there is no opportunity for prices to anticipate the effect of conversion because there is no evolution of time. Here we may indeed have no equilibrium, multiple equilibria, or just one equilibrium, and we characterize each case.

This static setting shares some features with Bond, Goldstein, and Prescott [6], but there are also important differences in the problems considered. Bond et al. [6] consider a possible intervention in a firm based on the market price of the firm's equity and a private signal. The intervention affects the firm's fundamentals and thus the value of its equity, giving rise to questions about equilibrium. In our setting, the intervention has no effect on fundamentals — converting debt to equity has no effect on the value of the underlying assets. Bond et al. [6] have an unambiguous mapping from fundamentals to equity value; the possibility of multiple equilibria arises from uncertainty about fundamentals. In contrast, in our setting the source of the problem is that a market trigger can make contingent capital and equity incomplete contracts: even with perfect information about the value of a firm's assets, there are states in which the apportionment of firm value between contingent capital investors and equity holders may not be fully specified.

For the dynamic setting on which we mainly focus, one can draw a partial analogy with the question of market completeness. A market that is dynamically complete with continuous trading typically becomes incomplete when trading is restricted to fixed dates. This incompleteness allows a range of values for the price of a contingent claim, each consistent with the absence of arbitrage (and thus with a market equilibrium). Discrete-time hedging of an option with the underlying assets leaves some residual risk at maturity, and the discrete-time version of our setting leaves some residual risk at conversion in the form of a possible value transfer between claimants. Under appropriate conditions, each of these ranges shrinks to a point as trading frequency increases. For derivatives on liquid underlying assets, the complete market paradigm of continuous trading dominates practice as well as theory. But the analogy between the two settings should not be overstated. In the case of market incompleteness, the range of possible prices corresponds to a range of possible risk preferences; in our setting, specifying preferences for a representative agent does not pin down a single price. Instead, multiple equilibria arise in our setting because, as already noted, restricting trading impairs the ability of the market price to incorporate information and leaves ambiguity about the payouts to investors.

The rest of this paper is organized as follows. Section 2 introduces contingent capital, first detailing the static case and then expanding it to a dynamic capital structure model of the type introduced by Merton [24]. Section 3 abstracts from the specifics of contingent capital to consider more general changes in a firm triggered by the market value of a claim on the firm. The added generality clarifies the essential features leading to existence and uniqueness of equilibria. Section 4 works through applications of these results, first revisiting contingent capital and then examining other capital structure changes including debt write-downs, forced deleveraging, capital access bonds, automatic share repurchases, and a triggered consolidation of a subsidiary. Section 5 extends our results to cover dividends and coupons, debt covenants and bankruptcy costs, and infinite-horizon problems. Section 6 allows jumps in fundamentals. Most technical details are deferred to an appendix.

2 Contingent Capital

2.1 The Static Case

We begin by considering a static setting. This setting illustrates the possibility of having a unique equilibrium, multiple equilibria, or no equilibrium, depending on the level of the conversion trigger; it thus serves as a useful introductory case, and it also describes the situation at maturity in a dynamic formulation we take up later.

We consider a firm with asset value A. The firm's liabilities consist of senior debt with face value B, contingent convertible debt with face value C, and equity. The value of the equity depends on whether the contingent capital converts, and conversion is triggered by the value of the equity, so equity value and the outcome of the conversion trigger are determined simultaneously. In the absence of conversion, the contingent capital remains debt and the value of the equity is

$$S = v(A) = (A - B - C)^+, \tag{1}$$

which is the residual value of the firm after the debt has been repaid, with $x^+ = \max\{x, 0\}$. If we adopt the normalization that there is a single share outstanding, then (1) is the price per share.

Upon conversion, the contingent capital is replaced with $m > 0$ newly issued shares of stock, bringing the total number of shares to $1 + m$. The total value of equity is now $(A - B)^+$, and the price per share is

$$S = u(A) = (A - B)^+/(1 + m). \tag{2}$$

If we posit that the contingent capital converts precisely if $S \leq L$, for some trigger level L, the question arises of whether these specifications are internally consistent and, if they are, whether they admit more than one solution. To make this precise, we introduce the following definition.

Definition 2.1 *A mapping* $S : [0, \infty) \mapsto [0, \infty)$ *is an* equilibrium stock price *if it satisfies*

$$S(A) = \begin{cases} u(A), & \text{if } S(A) \leq L; \\ v(A), & \text{if } S(A) > L. \end{cases} \tag{3}$$

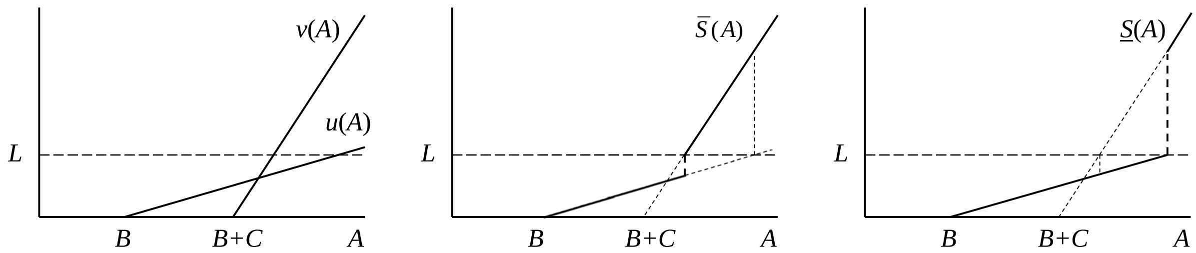

Figure 1: The equity values $u(A)$ and $v(A)$ cross at a height of C/m. With $L > C/m$, (3) admits an infinite number of solutions, all lying between $\underline{S}(A)$ and $\overline{S}(A)$.

for all $A \in [0, \infty)$.

Equation (3) selects one of the two stock price formulas in (1) and (2), depending on whether the stock price is above the trigger L or not. Importantly, the stock price $S(A)$ appears on both sides of the equation.

The situation is illustrated in Figure 1. The left panel shows the functions $u(A)$ and $v(A)$. The lines cross at a height of C/m, so the figure has $L > C/m$. An equilibrium stock price is a mapping from A on the horizontal axis to $S(A)$ on the vertical axis consistent with (3). With $L > C/m$, there are multiple solutions. For example, we may take

$$\underline{S}(A) = \begin{cases} u(A), & \text{if } u(A) \le L; \\ v(A), & \text{otherwise,} \end{cases} \tag{4}$$

as illustrated in the right panel of the figure. We may also take

$$\overline{S}(A) = \begin{cases} u(A), & \text{if } v(A) \le L; \\ v(A), & \text{otherwise,} \end{cases} \tag{5}$$

as illustrated in the center panel. Both $\underline{S}$ and $\overline{S}$ satisfy the conditions in (3) to qualify as equilibrium stock prices. It is not hard to see (as shown more generally in the appendix) that any solution to (3) must lie between $\underline{S}$ and $\overline{S}$; conversely, any mapping S that lies between $\underline{S}$ and $\overline{S}$ and for which $S(A) \in \{u(A), v(A)\}$ for every A satisfies (3), even if it not monotone in the asset level A.

We can interpret the extremal solutions in (4) and (5) as follows. The equity price $\underline{S}(A)$ arises from conversion occurring at the highest asset level consistent with conversion (and all lower asset values); $\overline{S}$ arises from avoiding conversion at the lowest asset value consistent with non-conversion (and all higher asset values). If we imagine sliding A down from some large value, moving from right to left in Figure 1, $\underline{S}$ results from converting at the first opportunity, and $\overline{S}$ results from converting at the last possible point.

When $C = mL$, the trigger L is exactly at the height at which $u(A)$ and $v(A)$ cross, $\overline{S} = \underline{S}$, and there is no other solution to (3) as any solution must lie between these two. If $C > mL$,

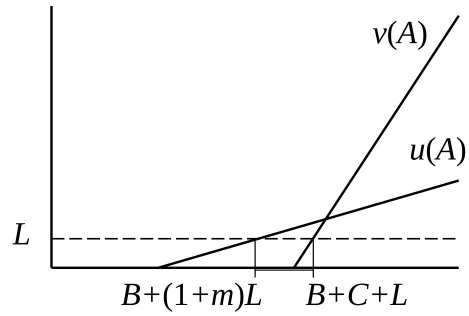

Figure 2: With $L < C/m$, there is no way to define $S(A)$ consistent with (3) for A between $B + (1 + m)L$ and $B + C + L$. Setting $S(A) = v(A) < L$ in this interval violates the first condition in (3), and setting $S(A) = u(A) > L$ violates the second condition.

there is no choice of mapping $S(A)$ consistent with (3). This case is illustrated in Figure 2. In particular, at any value of A for which $B + (1 + m)L < A < B + C + L$, we have

$$v(A) < L < u(A). \tag{6}$$

Thus, there is no way of defining $S(A)$ consistent with (3): if we set $S(A) = v(A) < L$ we violate the first condition in (3), and if we set $S(A) = u(A) > L$ we violate the second condition in (3). We summarize the situation as follows.

Proposition 2.2 *In the static problem (3), (i) if $C < mL$, the problem admits an infinite number of equilibrium stock prices, all lying between $\underline{S}$ and $\overline{S}$; (ii) if $C = mL$, the problem admits just one equilibrium stock price $S = \underline{S} = \overline{S}$; (iii) if $C > mL$, the problem admits no equilibrium stock price.*

In case (i) of the proposition, we get multiple equilibria because the seemingly simple rule defining a stock price in (3) — convert if the price is at or below the trigger — makes equity an incomplete contract. For asset levels between $B + C + L$ and $B + (1 + m)L$, marked by the vertical dashed lines in the center and right panels of Figure 1, the rule in (3) does not fully specify how asset value is apportioned between equity holders and contingent capital investors. An equity payout of either $u(A)$ or $v(A)$ is consistent with the conversion rule.

Generalizing beyond (1) and (2), the essential property of the functions v and u underpinning part (i) of the proposition is that

$$v(A) \geq u(A) \text{ whenever } u(A) \geq L. \tag{7}$$

This says that equity holders prefer non-conversion at high assets levels, and the trigger should be high enough so that this preference holds above the trigger. Condition (7) is equivalent to

$$v(A) \geq u(A) \text{ whenever } \max\{u(A), v(A)\} \geq L; \tag{8}$$

we do not require monotonicity of u or v. When (7) holds, (4) and (5) define equilibria, and they define lower and upper bounds on any other equilibria; if the sets where $v(A) \leq L$ and $u(A) \leq L$ coincide, there is just one equilibrium; and if (6) holds at some A, then no equilibrium is possible. (See the appendix for details.)

2.2 Dynamic Structural Model

There is a natural extension of this static case to a dynamic model of a firm on a time interval $[0, T]$ in the spirit of Merton [24]. In this formulation, A_t, $t \in [0, T]$, describes the evolution of the value of the assets held by the firm; to be concrete, we take A_t to be geometric Brownian motion. The value of these assets — think of loans and securities in the case of a bank — does not depend on how the assets are financed. The firm has issued straight debt and contingent convertible debt, just as before, both maturing at T, with respective par values of B and C. We now seek a stock price *process* S_t, $t \in [0, T]$ satisfying

$$S_T = \begin{cases} u(A_T), & \text{if } S_t \leq L \text{ for some } t \in [0, T]; \\ v(A_T), & \text{otherwise.} \end{cases} \tag{9}$$

In this formulation, the trigger is in effect throughout $[0, T]$, and not just at the terminal date T. This fundamentally changes the problem because the stock price can adjust to anticipate the effect of the trigger — no such adjustment is possible in the static case.

The analysis of (9) depends on contractual terms like coupons on debt or covenants as in Black and Cox [4]. We return to these specific types of features after analyzing a setting that is both simpler and more general. As we develop the general setting, it will be useful to keep this example in mind.

3 Dynamic Setting

3.1 Formulation

In this section, we abstract from the details of contingent convertible debt and consider a more general problem of existence and uniqueness of an equilibrium with a market trigger. We postpone technical details as much as possible to the appendix. We do, however, need to lay out some assumptions. We assume (but see Section 3.4) that investors are risk-neutral and interest rates are zero. This makes the price of any asset equal to the expected value of its future payouts, and the price of an asset without dividends is a martingale. Information arrives to the economy diffusively and prices adjust continuously, though later we allow jumps. More precisely, we work on a probability space $(\Omega, \{\mathcal{F}_t, t \in [0, T]\}, P)$ in which the information flow $\{\mathcal{F}_t, t \in [0, T]\}$ is generated by Brownian motion, possibly multidimensional. All processes we define are adapted to this information flow. The price of any asset that pays no dividends evolves continuously because any martingale adapted to a Brownian filtration is continuous.

We introduce two processes U_t and V_t which may be interpreted as, for example, stock price processes for two hypothetical firms investing in identical assets but with different liabilities and possibly different overall firm size. To connect this setting with contingent capital, think of U_t as the stock price for a firm in which the convertible debt has already been converted to equity, and think of V_t as the stock price for an otherwise identical firm in which the convertible debt is replaced with straight debt that can never convert. By construction, these prices do not depend on an endogenous trigger and are well-defined. Our interest is in the existence and uniqueness of a stock price for a third firm with the same asset process but for which the contingent capital is genuinely convertible. More generally, we are interested in a process S_t satisfying

$$S_T = \begin{cases} U_T, & \text{if } S_t \leq L \text{ for some } t \in [0,T]; \\ V_T, & \text{otherwise,} \end{cases} \tag{10}$$

with U_T, V_T having finite expectations. Assuming this firm pays no dividends, we also require

$$S_t = E_t[S_T], \tag{11}$$

where E_t denotes conditional expectation given $\mathcal{F}_t$. Put abstractly, our problem is the existence and uniqueness of a process S_t satisfying (10) and (11), given U_T and V_T. We will call such a process an equilibrium stock price.

3.2 Existence

A natural counterpart of the condition (7) we used in the static setting is

$$U_T \leq V_T \text{ whenever } U_T \geq L. \tag{12}$$

This requires that the no-conversion price be higher than the post-conversion price above the trigger. It will often be easier to work with the simpler but stronger condition that the no-conversion price is always higher,

$$U_T \leq V_T. \tag{13}$$

The two hypothetical firms pay no dividends (until Section 5), so

$$U_t = E_t[U_T] \text{ and } V_t = E_t[V_T], \tag{14}$$

and (13) implies

$$U_t \leq V_t, \quad \text{for all } t \in [0,T]. \tag{15}$$

This is already more than enough to ensure existence:

Theorem 3.1 *(i) If (13) holds, then there exists at least one equilibrium stock price. (ii) The same holds if (13) is relaxed to (12).*

Proof of part (i). We construct a solution to (10)–(11). Define

$$S_T^* = \begin{cases} U_T, & \text{if } U_t \leq L \text{ for some } t \in [0,T]; \\ V_T, & \text{otherwise,} \end{cases} \tag{16}$$

noting that the trigger event is here defined by U_t, so S_T^* is automatically well-defined. Further define $S_t^* = E_t[S_T^*]$, $t \in [0,T)$, so (11) holds. To show that S^* satisfies (10), we need to show that S_t^* reaches L if and only if U_t does, since then (10) follows from (16).

Because $S_T^* \in \{U_T, V_T\}$, we have $U_T \leq S_T^* \leq V_T$, and taking conditional expectations then yields $U_t \leq S_t^* \leq V_t$ for all $t \in [0,T]$. Consequently, if $S_t^* \leq L$, then $U_t \leq L$.

It remains to show that if $U_t \leq L$ for some $t \in [0,T]$, then the same is true of S^*. Define the stopping time

$$\tau_U = \inf\{t \in [0,T] : U_t \leq L\}, \tag{17}$$

the first time U_t is at or below the trigger, with the convention that $\tau_U = \infty$ if $U_t > L$ for all $t \in [0,T]$. On the event $\{\tau_U \leq T\}$, we have $S_T^* = U_T$ by definition, and then

$$S_{\tau_U}^* = E[S_T^* | \mathcal{F}_{\tau_U}] = E[U_T | \mathcal{F}_{\tau_U}] = U_{\tau_U} \leq L.$$

We have shown that S_t^* and U_t reach the trigger simultaneously or not at all, so we may replace U_t with S_t^* in the first condition in (16), and this yields (10). The continuity of S_t^* confirms that $S_t^* \to L$ as $t \uparrow \tau_U$. $\square$

The construction is illustrated in Figure 3. The equilibrium stock price starts above the post-conversion value U_0 and coincides with the post-conversion value at the instant that both reach the trigger. Prior to conversion, there is some chance the trigger will not be reached and thus some chance the stock price will terminate at the higher value V_T; this possibility is reflected in the stock price being greater than U_t prior to the trigger being reached. We stress that the process U_t is merely used for the proof of existence and need not be observable in the market: the triggering event is ultimately determined solely by the equilibrium stock price process S_t^*.

The following lemma, proved in the appendix, reduces part (ii) of Theorem 3.1 to part (i):

Lemma 3.2 *Suppose we relax (13) to (12). Then S_t is an equilibrium stock price with U_T and V_T in (10) if and only if it is an equilibrium stock price with $\tilde{U}_T = U_T$ and $\tilde{V}_T = \max\{U_T, V_T\}$.*

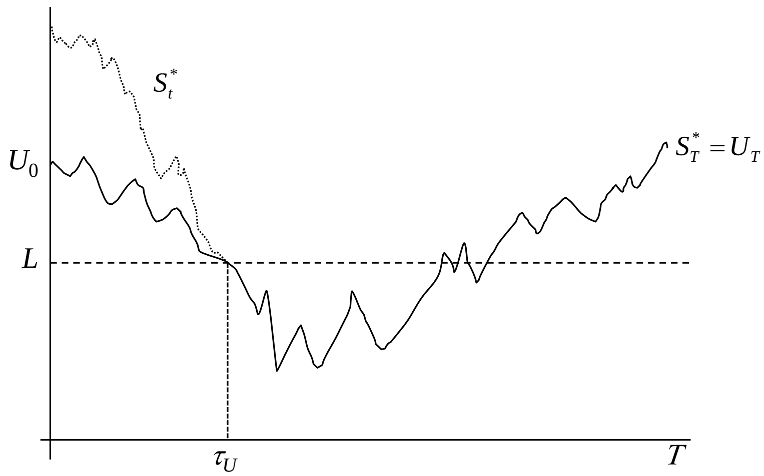

Figure 3: Illustration of the existence argument: By construction S_t^* reaches the trigger L together with U_t or not at all. In this equilibrium, conversion occurs the first time the post-conversion stock price reaches the trigger.

The simple ordering condition $U_T \leq V_T$ in (13) does not hold in the dynamic structural model of contingent capital in Section 2.2 because we do not have $u(A) \leq v(A)$ for all A; the lines describing u and v in Figure 1 cross. The weaker condition (12) holds if the trigger is high enough that $C \leq mL$. To put it another way, the value of the no-conversion payoff $v(A)$ below the trigger is irrelevant, so in the lemma we are free to change $v(A)$ below the trigger to make it greater than or equal to $u(A)$.

When the simple ordering condition (13) holds, the post-conversion price is always lower than the no-conversion price and existence of an equilibrium is automatic; this case is particularly convenient for analysis. Condition (12) has broader scope and is more directly applicable. In our examples, the interpretation of (12) will be that we can ensure the existence of an equilibrium by setting the trigger sufficiently high.

Theorem 3.1 provides sufficient conditions for existence of an equilibrium, so it is natural to ask to what extent conditions of this type are necessary. Dropping the ordering of U_t and V_t generally requires imposing other restrictions on the range of values they can take, as we illustrate through examples, starting with one that ensures existence without any condition on U_t by constraining V_t:

Proposition 3.3 *Suppose V_t can never reach the trigger L; that is, $P(V_t > L \text{ for all } t \in [0,T]) = 1$. Then $S_t = V_t$ is an equilibrium stock price for (10)–(11).*

This no-conversion equilibrium is of limited practical interest because it entails a strong restriction on the range of V_t, but it serves as a useful contrast as we consider what happens without restrictions on the range of U_t and V_t.[3] The following is a partial converse to Theorem 3.1, precluding the possibility of an equilibrium if the no-conversion price can terminate below the trigger without the post-conversion price ever reaching the trigger:

Proposition 3.4 *There is no equilibrium stock price that reaches the trigger at a time when $U_t > L$. In particular, if*

$$P(V_T \leq L \text{ and } U_t > L \text{ for all } t \in [0,T]) > 0 \tag{18}$$

then there is no equilibrium.

Proof. Let S be an equilibrium stock price and define τ_S

$$\tau_S = \inf\{t \in [0,T] : S_t \leq L\}.$$

On the event $\{\tau_S \leq T\}$, $S_T = U_T$, so

$$S_{\tau_S} = E[S_T|\mathcal{F}_{\tau_S}] = E[U_T|\mathcal{F}_{\tau_S}] = U_{\tau_S}. \tag{19}$$

Thus, it is not possible to have $S_{\tau_S} \leq L < U_{\tau_S}$. For the second assertion in the proposition, if an equilibrium S_t exists and U_t does not reach L, then, by the first part of the proposition, S_t never reaches the trigger. But then $S_T = V_T \leq L$ yields a contradiction. □

Viewed from the perspective of this result, the ordering condition (12) that underpins Theorem 3.1 eliminates the problematic possibility in (18) that V_t ends up below the trigger without U_t ever having reached the trigger, which creates a logical inconsistency in trying to find a stock price that satisfies (10), just as (6) does in the static case. The practical implication of these results for contingent capital with a stock price trigger is that the trigger level L and the dilution factor m should be set so that the post-conversion stock price U_t is lower than the no-conversion stock price V_t when both are above the trigger — at high valuation levels, conversion should be disadvantageous to shareholders. We discuss other types of triggers and conversion mechanisms in Section 4.

Equation (19), which we will invoke at several places, reflects the assumption of perfect liquidity embodied in continuous trading. It implies that at conversion the stock price is precisely at the trigger L and exactly equal to the post-conversion price.[4]

[3] Hart and Zingales [16] construct a more interesting no-conversion equilibrium in a simple two-period model with a credit default swap trigger. In their setting, conversion is prevented through actions taken by a firm's managers and not through restrictions on a price process.

[4] This phenomenon is analogous to an observation in Flood and Garber [13, p.4] on their deterministic model of exchange-rate regimes. They argue that the collapse of a fixed-rate regime occurs with the exchange rate equal to a shadow floating value to which the exchange rate switches at the instant the fixed rate can no longer be sustained.

3.3 Uniqueness

We turn next to uniqueness. We describe the argument informally through reference to Figure 4, deferring technical details to the appendix. In the figure, S_t is a candidate equilibrium stock price. It is bounded above by the no-conversion stock price V_t, either because the ordering relation (13) holds or because of the weaker requirement (22) introduced below. In light of the bound, S_t must reach the trigger no later than V_t. However, we know from Proposition 3.4 (and (19) in particular) that S_t can reach the trigger only when the post-conversion stock price is at the trigger — that is, when $U_t = L$. To avoid a contradiction, we must therefore have S_t reach the trigger at τ_U, which then implies that S_t must coincide with the equilibrium S_t^* constructed in (16), establishing uniqueness. More precisely, we will argue that S_t must reach L before U_t reaches $L-\epsilon$, and we will then let ϵ shrink to zero.

Making this argument work relies on three types of conditions. First and most important is the type of ordering relation that pervades all of our results, requiring, in various ways, that the no-conversion stock price be higher than the post-conversion stock price when either is above the trigger. Second, we impose a *free-range condition* ensuring that the evolution of U_t and V_t is not overly constrained. We require, in effect, that the situation depicted in Figure 4 cannot be ruled out: after U_t reaches $L-\epsilon$, there is always a chance that V_t will reach the trigger before U_t does. To complete the argument, we need a *right-continuity* condition stating that the first time U_t reaches $L-\epsilon$ approaches the first time it reaches L as ϵ decreases to zero. Models that violate these conditions are rather contrived, as we will see in two examples below.

To formulate these conditions precisely, let τ_U^ϵ denote the first time U_t is at or below $L-\epsilon$, as in (17) but with L replaced by $L-\epsilon$. The free-range condition requires $P(\tau_U^\epsilon < T) > 0$ for sufficiently small $\epsilon > 0$, and conditional on $\tau_U^\epsilon < T$, we have, almost surely,

$$P(U_t < L \text{ for all } t \in [\tau_U^\epsilon, T] \text{ and } V_t \leq L \text{ for some } t \in [\tau_U^\epsilon, T] | \mathcal{F}_{\tau_U^\epsilon}) > 0, \tag{20}$$

The right-continuity condition requires that, almost surely,

$$\tau_U^\epsilon \to \tau_U \quad \text{as} \quad \epsilon \downarrow 0, \tag{21}$$

Uniqueness means that any two equilibria agree with probability one at each time t.

Theorem 3.5 *Suppose the conditions of Theorem 3.1(i) hold, and suppose the free-range condition (20) and the right-continuity condition (21) hold. Then the equilibrium stock price is unique. The same holds under the conditions of Theorem 3.1(ii) with (12) replaced by*

$$U_t \geq L \Rightarrow U_t \leq V_t, \quad \textit{for all } t \in [0,T]. \tag{22}$$

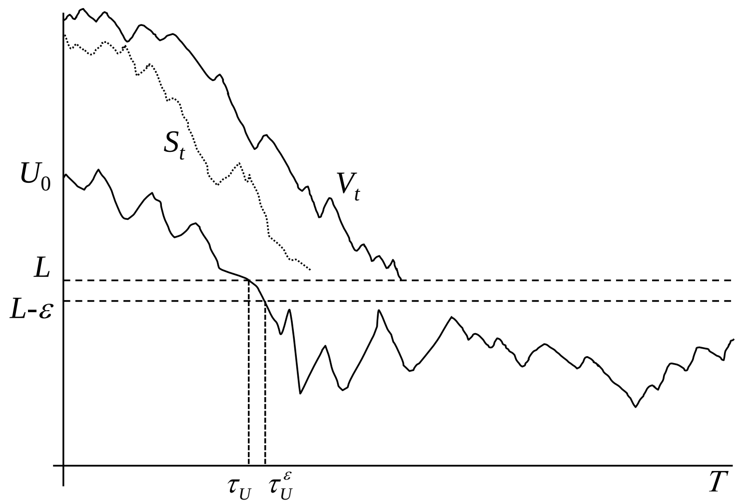

Figure 4: Illustration of the uniqueness argument. A candidate equilibrium S_t must reach the trigger no later than the no-conversion stock price V_t. However, it can reach the trigger only when the post-conversion stock price is at the trigger, $U_t = L$. To avoid a contradiction, S_t must reach the trigger the first time U_t does (or, more precisely, before U_t reaches $L - \epsilon$).

In the dynamic structural model of Section 2.2, with the asset process A_t modeled by geometric Brownian motion, $U_t = E_t[u(A_T)]$ and $V_t = E_t[v(A_T)]$ are both given explicitly as functions of A_t by variants of the Black-Scholes formula. Our free-range condition is then satisfied because there is positive probability that A_t will drop sufficiently low to drive V_t below L before U_t returns to L. To further illustrate the argument, it is helpful to consider an example in which the free-range condition is violated and an example in which the right-continuity condition is violated:

Example 3.6 Suppose the no-conversion price stock price V_t is guaranteed to stay above a level $L_2 > L$. For example, this firm's underlying assets could include more cash than the total value of the firm's debt, putting a lower bound on equity value. Suppose the post-conversion stock price U_t is guaranteed to stay below some level L_1, with $L_1 > L$. For example, this firm's assets might consist entirely of loans that can never be worth more than their par values, putting an upper bound on equity value. With $L < L_1 < L_2$, the ordering condition (15) is automatically satisfied, and (14) holds in the absence of dividend payments. Thus, applying the construction used in Theorem 3.1, we get one equilibrium S_t^* that converts the first time U_t reaches the trigger L (which is then also the first time S_t^* reaches L). However, we get a second equilibrium by setting $S_t = V_t$; this is a no-conversion equilibrium because V_t can never reach the trigger. The free-range condition is violated by this example, and there is nothing to force the two equilibria to coincide.

Example 3.7 Consider, again, the setting of Example 3.6, but suppose that the first time U_t reaches the trigger L it remains pegged at L for an exponentially distributed period of time ξ, violating (21). As before, we have an equilibrium S_t^* that reaches L at τ_U, the first time U_t reaches L. We can construct a second equilibrium $\tilde{S}_t$ that converts at $\tau_U + a$ if the exponential clock ξ runs for at least $a > 0$ time units and if $\tau_U + a \leq T$. More precisely, define $\tilde{S}$ the way we defined S^* in (16), but with the condition $\tau_U \leq T$ replaced by $\tau_U + a \leq \min\{\tau_U + \xi, T\}$. The argument in the proof of Theorem 3.1 showing that S_t^* is an equilibrium applies as well to $\tilde{S}_t$. Once S_t^* reaches the trigger, there is still a strictly positive probability that $\tilde{S}_t$ will never reach the trigger and thus that $\tilde{S}_T$ will terminate at V_T. This implies that $\tilde{S}_t$ is strictly greater that S_t^* during at least the interval in which only S_t^* has converted. Thus, these are two distinct equilibria. In fact, we get a separate equilibrium for each value of a, so this problem admits an infinite number of equilibrium stock prices.

The somewhat pathological situations described in Examples 3.6 and 3.7 are ruled out by the conditions we impose — the free-range condition and the right-continuity condition. We consider these two conditions very natural — indeed, examples that violate them seem rather contrived — so once we have the conditions of Theorem 3.1 in place for existence, uniqueness should be considered "typical."

3.4 Stochastic Discounting

We have assumed that investors are risk-neutral and interest rates are zero largely for notational convenience: under these assumptions, the prices U_t, V_t, and S_t are simply conditional expectations of future payouts. Here we briefly show that our results hold without these assumptions.

We assume the existence of a stochastic discount factor $Z_t > 0$ such that all price processes satisfy relations of the form $U_t = E_t[Z_T U_T]/Z_t$, $V_t = E_t[Z_T V_T]/Z_t$. In a general equilibrium asset pricing model, the stochastic discount factor encodes the time and risk preferences of a representative agent, with Z_t measuring the marginal utility of optimal consumption at time t. Even without such a framework, the absence of arbitrage is essentially equivalent to the existence of a stochastic discount factor.

In modifying our results for this setting, we leave the definition of an equilibrium in (10) unchanged but replace (11) with the pricing relation $S_t = E_t[Z_T S_T]/Z_t$. Observe that $U_T \leq V_T$ continues to imply $U_t \leq V_t$, as in (15), with stochastic discounting, and Lemma 3.2 continues to hold as well. For Proposition 3.4 and the crucial property $S_{\tau_S} = L = U_{\tau_S}$ in (19), we add the condition that Z_t is continuous. Theorems 3.1 and 3.5 hold in this setting with the price processes U_t and V_t now defined through the stochastic discounting relation.

4 Applications and Extensions

This section applies the theoretical results of the previous section to various examples. We start by revisiting contingent capital in the form of debt that converts to equity and then examine other types of changes in capital structure and other types of market triggers.

4.1 Contingent Capital Revisited

We return to the dynamic structural model of Section 2.2. An equilibrium stock price is now a solution to (9) with v and u as defined in (1) and (2). The asset value process A_t is geometric Brownian motion, and $U_t = E_t[U_T]$ and $V_t = E_t[V_T]$.

Corollary 4.1 *If $C \leq mL$, there is a unique equilibrium stock price. If $C > mL$ and $U_0 > L > 0$, there is no equilibrium.*

This result reflects a sharp contrast with its static counterpart, Proposition 2.2: *in this continuous-time setting there is no possibility of multiple equilibria*. Moreover, the possibility of having no equilibrium is easily ruled out by setting the trigger L (or the dilution factor m) to be sufficiently high. In light of the construction of S^* in Theorem 3.1, the unique equilibrium is easy to describe: conversion occurs the first time the post-conversion stock price reaches the trigger. (In the trivial case $U_0 \leq L$ excluded from the statement of the corollary, we always get a unique equilibrium by setting $S_t = U_t$, and if $L = 0$ then the only equilibrium is $S_t = V_t$.)

Multiple equilibria will exist in the case $C < mL$ if the market evolves in discrete time steps, a case considered in Sundaresan and Wang [30]. The key difference is that the first part of Proposition 3.3 does not hold in discrete time: with intermittent trading, it is possible for the post-conversion price U_t to be below the trigger when the trigger is breached. In the terminology of Sundaresan and Wang [30], this would be a value transfer from equity investors to contingent capital investors because the total value of equity at conversion is less than the anticipated amount L, and the difference becomes an increase in the value of contingent capital. Sundaresan and Wang [30] propose modifying the terms of the contingent capital contract to eliminate the possibility of a value transfer and thus to ensure a unique equilibrium.

In the continuous-time case, equation (19) shows that the no-value-transfer condition suggested by Sundaresan and Wang [30] holds *automatically*, even without a contract modification. With perfect liquidity, the stock price is able to adjust continuously and to fully anticipate the effect of conversion; at conversion, any equilibrium stock price must exactly equal L. Just after conversion, the stock price must still equal L (as in (19)). No information is released at the instant of conversion, so there is no cause for a discontinuity in the stock price, and a predictable drop would be an arbitrage opportunity. In contrast, a lack of liquidity prevents stock prices from fully reflecting information about conversion, and it is this friction that allows multiple equilibrium stock prices in the discrete-time case.[5]

We comment briefly on the range of possible equilibria in discrete time, under the condition $C < mL$. The construction of S^* in the existence proof of Theorem 3.1 works in discrete time as well. Indeed, by the ordering argument used there, no equilibrium stock price can reach or cross the trigger before U_t does. If we have $U_T \leq V_T$ (which we may assume in light of Lemma 3.2), then converting later translates to a higher stock price because it entails a higher likelihood of terminating at V_T rather U_T; and a higher stock price confirms that conversion occurs later. Thus, the discrete-time counterpart of the unique continuous-time equilibrium gives the lowest stock price of all possible discrete-time equilibria. Any other discrete-time equilibrium may be interpreted as self-fulfilling speculation (lifting the stock price) that conversion can be delayed. Moreover, because the continuous-time equilibrium reaches L no later than any discrete-time equilibrium (the continuous-time process can reach the trigger between ticks of the discrete clock), it yields a lower stock price. Thus, for any discrete time-step, the full swath of discrete-time equilibrium stock prices lies above the full-liquidity continuous-time equilibrium. The market friction created by discrete-time trading keeps the stock price higher, benefiting equity holders by delaying conversion

Figure 5 illustrates the convergence from discrete to continuous time. The figure plots the smallest and largest possible equity prices and contingent capital prices against the trading frequency, calculated in a binomial lattice. Consistent with the explanation just given, both the upper and lower bounds for equity prices appear to decrease toward a common continuous-time limit.[6] This example uses parameters from Sundaresan and Wang [30]: $A_0 = B = 100$, $C = 6$, $L = 1$, $T = 5$, an asset volatility of 6%, and a risk-free interest rate of 2%.

Sundaresan and Wang [30] compare the effect of varying different parameters; asset volatility appears to have the most pronounced effect, with the range of equity prices at a daily trading frequency collapsing to a single value with asset volatility at 2%. We have found a similar pattern in our numerical experiments and interpret this to mean that the liquidity required to make continuous-time trading a good approximation depends in part on the level of volatility of the underlying assets.

[5] What matters is the frequency of trading, not the frequency with which the trigger is monitored. Even if the trigger is monitored only periodically (e.g., daily), continuous trading allows the price to adjust in anticipation, precluding multiple equilibria.

[6] Numerical results suggest that the range shrinks in proportion to the square root of the time step. Taking the time step as a reflection of liquidity then suggests a relation between the price range and the degree of liquidity. The binomial calculation entails a discretization of price levels as well as time, and this may contribute to the magnitude of the range of prices without necessarily changing the convergence rate.

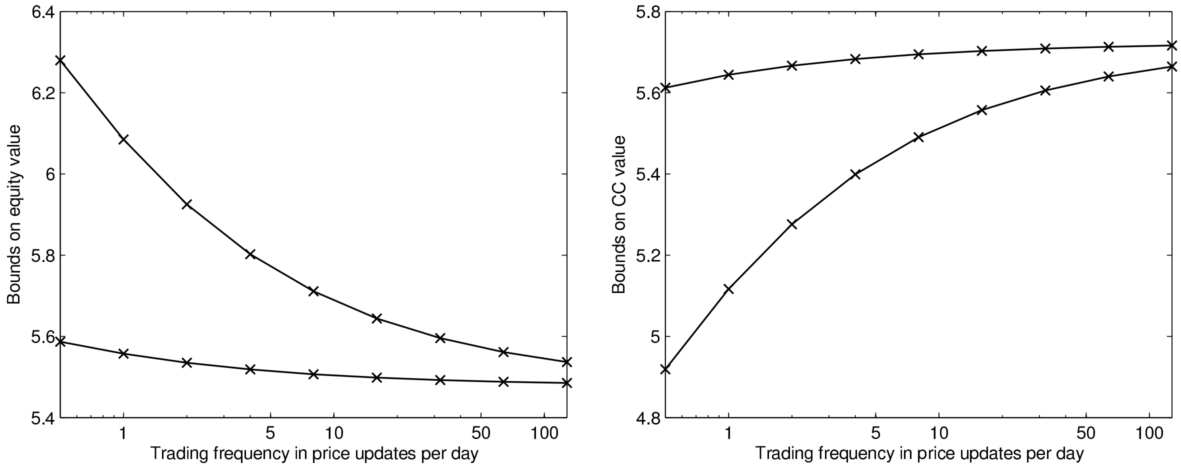

Figure 5: Convergence of discrete-time prices to continuous-time prices as the trading frequency increases in a binomial lattice. The left panel shows the range of stock prices and the right panel shows the range of prices for contingent capital. The example has $A_0 = B = 100$, $C = 6$, trigger level $L = 1$, maturity $T = 5$, an asset volatility of 6%, and a risk-free interest rate of 2%.

We end this section with a final observation on the difference between the dynamic and static settings. It may seem paradoxical that multiple equilibria are possible in the static formulation of Proposition 2.2, which coincides with $T = 0$, but not for any $T > 0$. This apparent paradox is easily resolved. In the static problem with $C < mL$, the bounds in Proposition 2.2 imply that multiple equilibria can differ only at values of A where $u(A) \leq L < v(A)$; every equilibrium stock price has $S(A) = v(A)$ at larger values of A and $S(A) = u(A)$ at lower values of A. In the dynamic setting, we would therefore seem to face the possibility of multiple equilibria if $U_T \leq L < V_T$. However, if $U_T < L$, then $U_t < L$ for some $t < T$ (by continuity) and any equilibrium must have converted before T and satisfies $S_T = U_T$. If $U_T = L$, then we have two potential candidates for an equilibrium stock price, $S_T = U_T$ and $S_T = V_T$, but $P(U_T = L) = P(A_T = B + (1 + m)L) = 0$. In short, the possibility of A_t reaching the problematic interval where $u(A_T) \leq L < v(A_T)$ (where the static problem admits multiple equilibria) without conversion having already occurred has probability zero.

4.2 Debt Triggers and Principal Write-Downs

An automatic debt write-down is often categorized as another form of contingent capital; see, for example, FSOC [10] and Sidley Austin [28]. Rabobank issued debt of this type in March 2010, designed to have the principal reduced to 25% of its original value if a regulatory capital ratio fell below a 7% trigger. Here we apply our results to the case of a market trigger.

To this point, we have taken the market trigger to be a stock price and formulated our equilibrium conditions in terms of stock prices. This interpretation however is not essential to our results, and the example of this section is convenient for illustrating the alternative of a debt trigger.

We adapt our basic dynamic structural model to the case of a principal write down. As before, let B denote the face value of the firm's debt, and now let αB, $0 < \alpha < 1$, be the level to which principal is reduced at the occurrence of a triggering event. We drop the contingent convertible debt from the capital structure (taking $C = 0$) to focus on the write-down. Consider the possibility of using the market value of the firm's debt as the trigger. Bonds are not as liquid as stocks, but credit default swap spreads have been proposed as triggers for contingent capital, and, in the setting of our simple model a CDS trigger is equivalent to a bond price trigger. Writing Δ_t for the market price of the debt, the condition we need is

$$\Delta_T = \begin{cases} \alpha B, & \text{if } \Delta_t \leq L \text{ for some } t \in [0, T]; \\ B, & \text{otherwise,} \end{cases} \tag{23}$$

and then $\Delta_t = E_t[\Delta_T]$ for $t \in [0, T)$. Here we have implicitly assumed that the debt matures at T and pays no coupons; we consider coupons in Section 5.

If we now set

$$U_t = E_t[\min(A_T, \alpha B)] \quad \text{and} \quad V_t = E_t[\min(A_T, B)],$$

then these are the post-write-down and no-write-down bond prices in the basic dynamic structural model. It is immediate that $U_t \leq V_t$ for all t, so Theorem 3.1 ensures existence of an equilibrium — one that triggers a write-down the first time the post-write-down bond value is at or below the trigger. With A_t geometric Brownian motion, the free-range and right-continuity conditions are also satisfied and we have uniqueness.

A stock price trigger for a principal write-down leads to a very different outcome. The post-write-down and no-write-down stock prices are given by

$$U_t = E_t[(A_T - \alpha B)^+] \quad \text{and} \quad V_t = E_t[(A_T - B)^+].$$

Thus, we have $U_t \geq V_t$, which is the opposite of what we need. In fact, with A_t geometric Brownian motion and $U_0 > L$, (18) holds *and there is no equilibrium stock price.* By replacing A_t with a suitably constrained process, we could ensure that V_t is always above L, and then taking $S_t = V_t$ would yield an equilibrium in which the trigger is never reached and the write-down never occurs, but this exception is rather contrived. The overall conclusion of this example is that an equity trigger is incompatible with a principal write-down.

Taken together, this example and the contingent capital bond of Section 4.1 provide valuable insight into our results. The stock price trigger works for the contingent capital bond because the stock price decreases to the trigger and the post-conversion stock price is lower than the no-conversion price. Similarly, in (23), the bond price decreases to the trigger and the post-write-down bond price is lower than the no-write-down bond price. When we try to use a stock price trigger with a principal write-down, we create a situation in which as the stock price decreases toward the trigger the anticipated *increase* in the stock price resulting from the principal write-down pushes the stock price up, creating an inconsistency that precludes any equilibrium that can actually reach the trigger.

This difficulty could be circumvented by setting L above U_0 and having the stock price reach the trigger from below. This is the mirror image of the setting of Section 3, and the results given there hold here with the directions of the inequalities reversed; see the appendix. However, the result is an unpalatable outcome in which a rising stock price triggers a principal write-down.[7]

One case remains to complete this discussion: a senior bond price trigger for contingent capital in the form of debt that converts to equity, as in Section 4.1. This case is easiest of all because the price of the senior debt is $\Delta_t = E_t[\min(A_T, B)]$ regardless of whether the contingent capital converts. There is no ambiguity or potential inconsistency in defining the conversion trigger to be the first time the value of the senior debt falls to some threshold.

4.3 Forced Deleveraging

Consider, next, a firm that issues equity and senior debt with a covenant that requires deleveraging in the event of a sufficiently large decline in the firm's stock price. More specifically, the covenant specifies that if the stock price reaches L, then a fraction $1 - \beta$ of the firm's assets are transferred to the bond holders to retire a fraction $1 - \alpha$ of the firm's debt, $0 < \alpha, \beta < 1$. Equivalently, we may think of a fraction $1 - \alpha$ of the debt as being secured by a fraction $1 - \beta$ of the assets and the bond holders seizing this collateral when the trigger is reached.

The post-deleveraging and no-deleveraging stock prices are given by

$$U_t = E_t[(\beta A_T - \alpha B)^+] \quad \text{and} \quad V_t = E_t[(A_T - B)^+].$$

Here we need to distinguish two cases. If $\alpha \geq \beta$, then $U_t \leq V_t$ and we get existence of an equilibrium stock price, with uniqueness under the free-range and right-continuity conditions

[7] A better example qualitatively consistent with a trigger in this direction is provided by performance-sensitive debt for which the coupon decreases as the stock price increases. Manso et al. [22] value contracts of this type, but with the coupon tied to a firm's asset value rather than its stock price, in which case equilibrium is not a concern.

of Section 3.3. If $\alpha < \beta$, we do not necessarily have $U_t \leq V_t$, but this ordering does hold at all sufficiently high asset levels. In particular, if we set the trigger so that $L \geq (\beta - \alpha)B/(1 - \beta)$, then we can apply Lemma 3.2, modify V_T below the trigger (where its value is irrelevant), and get $U_t \leq V_t$ with this modification. (The appendix on linear contracts provides the details.) Indeed, from the perspective of equity holders this contract is nearly equivalent to the contingent capital setting, and Corollary 4.1 applies with only minor modification: If $L \geq (\beta-\alpha)B/(1-\beta)$, there is a unique equilibrium stock price; if L is below this level and $U_0 > L > 0$, no equilibrium exists.

4.4 Convertible Bonds

The holder of a conventional convertible bond has the option to convert the bond to equity and will do so when the share price rises. This contrasts with contingent capital that converts when the share price falls. To sharpen the parallel between the two cases, we consider convertible bonds that convert automatically the first time the stock price is greater than or equal to L. The no-conversion and post-conversion terminal equity claims are $(A_T - B - C)^+$ and $(A_T - B)^+/(1 + m)$, just as in the case of contingent capital, with C now denoting the face value of the convertible bonds and m denoting the number of shares to which the bonds convert. The only difference is that the trigger is now approached from below rather than from above. In this case, if $C \geq mL$ there is a unique equilibrium and if $C < mL$ and $U_0 < L$ there is no equilibrium. The symmetry of this example is rather particular to this case — the roles of U_t and V_t are not symmetric so we cannot automatically reverse the direction from which the trigger is approached and expect the same conclusions to apply. See Proposition A.2 in the appendix for details.

Bolton and Samama [5] propose a *capital access bond* with an embedded option for the issuing bank to repay the bond with new equity. In their proposal, the conversion option is exercised by the bank when the stock price is sufficiently low. If the conversion threshold were a fixed level of the stock price, this security would function like our contingent capital example, with a unique equilibrium with a sufficiently high trigger. With the bank holding the option to set the trigger, it has an incentive to lower the trigger. At maturity the optimal level is clearly $L = C/m$, as Bolton and Samama [5] note. In other words, in the static case, the optionality moves the trigger to the only level that results in a unique equilibrium stock price.

4.5 Automatic Share Repurchase

A firm that has issued only straight debt and equity commits to buying back shares of its equity when its stock price reaches a trigger level L. The terms of the buyback agreement have the firm reduce the number of shares to a fraction $1/\beta$, $\beta > 1$, of the previous float by increasing its debt B by a fraction $\alpha > 0$.

The post-repurchase and no-repurchase share prices are given by

$$U_t = E_t[\beta(A_T - (1+\alpha)B)^+] \quad \text{and} \quad V_t = E_t[(A_T - B)^+].$$

Despite the apparent similarity to some previous examples, this case behaves very differently. At asset levels greater than $L^* = \alpha\beta B/(\beta - 1)$, we have $U_T \geq V_T$. With the post-repurchase price greater than the no-repurchase price when both are high, we cannot expect to construct an equilibrium based on the stock price decreasing toward the trigger. Indeed, if $L > L^*$, there is no equilibrium with an initial stock price above the trigger; this is a consequence of Proposition A.2(iv) in the appendix. The same result implies that redesigning the trigger to be approached from below does not eliminate the difficulty: in this case, there is no equilibrium if $L < L^*$. The problem in both cases is that the share repurchase can move the stock price in the direction opposite to that in which the stock price approaches the trigger.

4.6 Triggered Consolidation: A Composite Trigger

Banks often use preferred securities issued by a subsidiary trust or similar entity to meet capital requirements. In order that these securities be loss-absorbing for the bank, they are made exchangeable into liabilities of the bank, with the trigger for conversion typically based on the bank's capital ratio. Here we consider a market trigger.

We abstract from the many complex details of hybrid capital instruments and consider a simple structure. We denote by A_t^1 the value of the bank's assets and by B^1 the par value of its debt. The trust's assets (which often include part of the bank's debt) we denote by A_t^2, and we let P^2 denote the par value of the trust's preferred shares. Without the possibility of conversion, the equity values for the bank and the trust would be

$$V_t^1 = E_t[(A_T^1 - B^1)^+] \quad \text{and} \quad V_t^2 = E_t[(A_T^2 - P^2)^+],$$

where we have assumed for simplicity that the preferred shares are the trust's most senior claim. (The assumption that the trust has issued traded equity would apply, for example, to a real estate investment trust, and REIT preferred shares are indeed a source of bank capital.) At conversion, the trust's assets are consolidated with those of the bank, the preferred shares become bank preferred shares, and the trust's shareholders receive m shares of bank stock for each share in the trust. The post-conversion share price is thus

$$U_t = E_t[(A_T^1 + A_T^2 - B^1 - P^2)^+]/(1+m).$$

Without additional constraints on asset values, there is no ordering between the post-conversion and no-conversion share prices. For reasonable choices of asset value processes, (18) holds (with V_T^1 in place of V_T), so using the bank's stock price to trigger conversion will not yield an equilibrium. We can, however, use a composite market trigger based on the sum of the two stock prices — the price of a portfolio of the two shares. To be consistent with a conversion trigger at L, the portfolio price needs to satisfy

$$S_T = \begin{cases} (1+m)U_T, & \text{if } S_t \leq L \text{ for some } t \in [0,T]; \\ V_T^1 + V_T^2, & \text{otherwise,} \end{cases}$$

and $S_t = E_t[S_T]$. We now have $(1+m)U_t \leq V_t^1 + V_t^2$ for all $t \in [0,T]$, so we have existence of an equilibrium, and we get uniqueness under the modest additional conditions discussed in Section 3.3. Once the conversion time is defined (the first time S_t reaches L), the stock prices for the bank and trust are unambiguous. Using (S_t^1, S_t^2) to denote the two stock prices, we have

$$(S_T^1, S_T^2) = \begin{cases} (U_T, mU_T), & \text{if } S_t \leq L \text{ for some } t \in [0,T]; \\ (V_T^1, V_T^2), & \text{otherwise,} \end{cases}$$

and $S_t^i = E_t[S_T^i]$ for $t \leq T$, $i = 1, 2$. The preferred shares are worth

$$P_T^2 = \begin{cases} \min\{[A_T^1 + A_T^2 - B^1]^+, P^2\}, & \text{if } S_t \leq L \text{ for some } t \in [0,T]; \\ \min\{A_T^2, P\}, & \text{otherwise,} \end{cases}$$

and $P_t^2 = E_t[P_T^2]$ for $t \leq T$.

The inconsistency created by trying to use the bank's stock price alone as the trigger arises from the possibility that the trust's stock price might be high when the bank's stock price is low: if consolidation increases the value of a share of bank stock, then the bank's stock price should be increasing as it decreases toward the trigger. Using the combined trigger eliminates this difficulty.

5 Dividends, Coupons, and Infinite-Horizon Settings

To this point, we have excluded payments of any dividends or coupons, and this has allowed us to express all prices as conditional expectations of terminal payoffs. We now extend the analysis to allow intermediate cash flows. This also allows us to consider infinite-horizon problems in which there are no terminal payouts.

5.1 Dividends

We denote by D_t^U and D_t^V the cumulative dividends paid in $[0,t]$ by the post-conversion and no-conversion firms. We assume that these are continuous processes and defer discrete dividends until our discussion of jumps. The corresponding stock prices are now given by

$$U_t = E_t[U_T + D_T^U - D_t^U] \quad \text{and} \quad V_t = E_t[V_T + D_T^V - D_t^V]; \tag{24}$$

in other words, the current price U_t is the conditional expectation of the sum of the terminal payment U_T and the future dividends $D_T^U - D_t^U$, and similarly for V_t.

We get a simple extension of previous results if we impose the ordering relation (13) and add

$$D_T^U - D_t^U \leq D_T^V - D_t^V, \quad \text{for all } 0 \leq t < T. \tag{25}$$

In light of (24), these conditions imply $U_t \leq V_t$, a property we have used before. When the dividends are given by payout rates δ_t^U, δ_t^V in the sense that

$$D_t^U = \int_0^t \delta_s^U \, ds \quad \text{and} \quad D_t^V = \int_0^t \delta_s^V \, ds, \tag{26}$$

then (25) is implied by $\delta_t^U \leq \delta_t^V$. These conditions on dividends are broadly consistent with the original ordering relationship (13): the condition $U_T \leq V_T$ indicates that the post-conversion equity claim is always less than the no-conversion equity claim, suggesting that the conversion is, in a sense, punitive to shareholders. Reducing dividend payouts after conversion is then consistent with this interpretation. As we did in Theorem 3.1, we will use somewhat weaker and more broadly applicable conditions that impose the ordering when the stock prices are sufficiently high.

We need to modify our definition of an equilibrium stock price to incorporate dividends. The requirement on S_T in (10) continues to hold, but we also need to define an equilibrium dividend process associated with an equilibrium stock price. Writing τ_S for the first time $S_t \leq L$, we require

$$D_t^S = \begin{cases} D_{\tau_S}^V + D_t^U - D_{\tau_S}^U, & \text{if } \tau_S \leq t; \\ D_t^V, & \text{otherwise.} \end{cases} \tag{27}$$

In other words, the cumulative dividends paid by an equilibrium stock price coincide with those of the no-conversion process V_t before the stock price reaches the trigger; once the trigger is reached, the accumulation of subsequent dividend payments coincides with that of the post-conversion process U_t. To complete the definition of an equilibrium pair (S_t, D_t^S) of a price process and dividend process, we need to replace (11) with

$$S_t = E_t[S_T + D_T^S - D_t^S]. \tag{28}$$

We can now extend our previous results on existence and uniqueness.

Theorem 5.1 *(i) If (13) and (25) hold, there is an equilibrium stock price and dividend process. (ii) The same holds if*

$$U_{\tau_U \wedge T} \leq V_{\tau_U \wedge T} \quad \textit{and} \quad E_t[D^U_{\tau_U \wedge T} - D^U_t] \leq E_t[D^V_{\tau_U \wedge T} - D^V_t], \tag{29}$$

for all $t \leq \tau_U$. (iii) In (i), if the right-continuity and free-range conditions hold, the equilibrium is unique. (iv) The same holds in (ii) with the addition of condition (22).

The first condition in (29) is implied by (22). We get a simpler formulation of the second condition in (29) when the cumulative dividends are given by dividend rates in the sense of (26). In the following corollary, we require that the post-conversion price and the post-conversion dividend rate should both be lower than their no-conversion counterparts when the post-conversion price is above the trigger.[8] Put differently, we need the trigger to be sufficiently high to imply these orderings.

Corollary 5.2 *If $U_t \leq V_t$ whenever $U_t \geq L$ and also*

$$\delta^U_t \leq \delta^V_t \ \textit{whenever}\ U_t \geq L \tag{30}$$

then an equilibrium stock price and dividend process exist. If the right-continuity and free range condition hold, the equilibrium is unique.

5.2 Dynamic Structural Model and Debt Covenants

To ground the discussion, we revisit the basic dynamic structural model of Section 2.2. The firm's assets A_t are modeled as geometric Brownian motion, and we posit, as in for example Leland [21], that the assets have a constant payout rate δ and thus generate cash at rate δA_t at time t. The firm's senior debt pays a continuous coupon at rate b_t on a principal of B, and the contingent convertible debt pays a continuous coupon at rate c_t on a principal of C. The coupon rates could depend on the current asset level. The difference $\delta A_t - b_t B - c_t C$ is the rate at which dividends are paid to equity. Here, as in Leland [21] and much of the dynamic capital structure literature, this rate may become negative, in which case it should be interpreted as the issuance of small amounts of additional equity to existing shareholders.

After conversion of the contingent capital, the net dividend per share is $(\delta A_t - b_t B)/(1+m)$. Condition (25) is satisfied if the no-conversion dividend rate is higher than the post-conversion dividend rate,

$$\delta A_t - b_t B - c_t C \geq (\delta A_t - b_t B)/(1+m), \tag{31}$$

[8]This dividend ordering is consistent with the example of performance-sensitive debt mentioned in Section 4.2 if a drop in the stock price triggers an increase in the debt coupon. With the total payout rate held fixed, increasing the debt coupon decreases the dividends to shareholders.

or, equivalently, if $m(\delta A_t - b_t B)/(1+m) \geq c_t C$.

We can relax (31) to the weaker condition in (30), which requires only that (31) hold when $U_t \geq L$. Suppose the coupon rates $b_t \equiv b$ and $c_t \equiv c$ are constant. By the Markov property of geometric Brownian motion, V_t is given by a deterministic function $v_t(A_t)$ of time and the underlying asset level, and similarly for U_t. Assuming a drift of $-\delta$ for A_t (consistent with the assumed payout rate of δ), we get

$$u_t(A_t) = E_t[(A_T - B)^+/(1+m)|A_t] + \frac{A_t}{1+m}(1 - e^{-\delta(T-t)}) - \frac{bB(T-t)}{1+m}$$

and

$$v_t(A_t) = E_t[(A_T - B - C)^+|A_t] + A_t(1 - e^{-\delta(T-t)}) - bB(T-t) - cC(T-t)$$

by evaluating the conditional expectations of the terminal payoffs and the expected dividends. Both $u_t(\cdot)$ and $v_t(\cdot)$ are unbounded and strictly increasing in $A_t > 0$ for every $t < T$. It is easy to see that (31) holds for all sufficiently large A_t, and thus for all sufficiently large U_t, as required by (30). In particular, (30) holds if

$$L \geq \frac{cC}{m\delta} + \frac{(b-\delta)B}{(1+m)\delta}.$$

Some algebra shows that $u_t(A_t) \leq v_t(A_t)$ whenever

$$A_t \geq a_t = B(1 + b(T-t)) + (1+m)C(1 + c(T-t))/m,$$

so we can ensure that (22) holds if the trigger is sufficiently high — high enough that $u_0(a_0) \leq L$. Moreover, with geometric Brownian motion and constant coupon rates, the right-continuity and free-range conditions hold; thus, we get existence and uniqueness of an equilibrium stock price by setting the trigger sufficiently high.

A similar analysis applies with a debt covenant of the type used in Black and Cox [4]. In their model, a debt covenant requires that the firm's assets be liquidated to pay the firm's debt if the asset level falls below a specified boundary. If the boundary is lower than the face value of the debt, then the liquidation results in a loss to bond holders, which could be interpreted as a bankruptcy cost.

Consider, then, a liquidation boundary at an asset level of kB, for some k. With $k < 1$, bond holders suffer a loss at liquidation and there is no residual payment to equity holders. Let τ_A denote the first time A_t reaches the level kB. With constant coupon rates, the no-conversion stock price is given by

$$V_t = E_t\left[(A_T - B - C)^+ \mathbf{1}\{\tau_A > T\} + \int_t^{\tau_A \wedge T} (\delta A_s - bB - cC)\, ds\right],$$

and the post-conversion stock price by

$$U_t = E_t\left[\frac{1}{1+m}(A_T - B)^+ \mathbf{1}\{\tau_A > T\} + \frac{1}{1+m}\int_t^{\tau_A \wedge T} (\delta A_s - bB)\, ds\right].$$

If in addition to $k < 1$ we have

$$k \geq \frac{b}{\delta} + \frac{(1+m)}{m}\frac{c}{\delta}\frac{C}{B},$$

then the no-conversion dividend rate is always at least as large as the post-conversion dividend rate, because this lower bound on k implies that (31) holds whenever $A_t \geq kB$. If $L \geq C/m$, then (22) holds. The values of U_t and V_t are now given by barrier option formulas, rather than the Black-Scholes formula. The unique equilibrium stock price with $C \leq mL$ triggers conversion the first time $U_t = L$; and because $U_t = 0$ at τ_A, the contingent capital is necessarily converted to equity before a liquidation is triggered by the debt covenant.

5.3 Infinite-Horizon Formulation

A small step takes us from a model with dividends to an infinite-horizon formulation. Without a terminal date T, equity value is given by the present value of all future dividends. This entails some technical considerations (discussed in the appendix) but the basic framework and results are largely unchanged.

To illustrate, we consider the basic dynamic structural model with payout rate δ and constant coupon rates b and c on perpetual senior debt with face value B and contingent convertible debt with face value C. To get finite stock prices, we introduce a constant discount rate $r > 0$. The no-conversion equity value is

$$V_t = E_t\left[\int_t^\infty e^{-r(s-t)}(\delta A_s - bB - cC)\, ds | A_t\right] = A_t - \frac{bB + cC}{r},$$

and the post-conversion equity value is

$$U_t = E_t\left[\int_t^\infty e^{-r(s-t)}(\delta A_s - bB)/(1+m)\, ds | A_t\right] = \frac{A_t}{1+m} - \frac{bB}{r(1+m)}.$$

We have implicitly assumed that the drift of A_t is $(r - \delta)$ so that A_t is itself the present value of its future payouts δA_s, $s \geq t$. Algebraic simplification now shows that we can ensure that $U_t \leq V_t$ whenever $U_t \geq L$ by setting $cC/r \leq mL$. This is essentially the same condition as in the finite-horizon setting of Corollary 4.1, except that the principal value C has been replaced with the capitalized value of the coupon payments cC/r.

6 Jumps and Discrete Dividends

Our analysis thus far has relied on the assumption introduced in Section 3 that the information flow to which all processes are adapted is generated by an underlying (possibly multidimensional) Brownian motion. As a consequence of this assumption, all of our price processes have been continuous because all Brownian martingales are continuous. We now introduce jumps in the form of a compound Poisson process

$$J_t = \sum_{n=1}^{N_t} Y_n,$$

where N_t is a Poisson process, $\{Y_1, Y_2, \dots\}$ is an i.i.d. sequence of random vectors, and the Y_n and N_t are independent of each other and of the underlying Brownian motion W_t. For example, this allows us to model the asset value process using a Merton [25] jump-diffusion model

$$\frac{dA_t}{A_{t-}} = \mu\, dt + \sigma\, dW_t + dJ_t^1, \tag{32}$$

where J_t^1 denotes the first coordinate of J_t, with jump sizes greater than -1, and μ and σ are constants. We take all price processes to be continuous from the right and have limits from the left. The size of a jump in A at t is $A_t - A_{t-}$, with the left limit A_{t-} giving the value just before the jump.

The information set $\mathcal{F}_t$ is now the one generated by the joint history of the underlying Brownian motion and the jump process J in $[0, t]$. We require that all processes be adapted to this information. In previous sections, all martingales (all prices in the absence of dividends) were necessarily continuous. In this section, the martingale representation theorem (as in Lemma 4.24 of Jacod and Shiryaev [19]) implies that a martingale can jump only when the Poisson process N_t jumps. In other words, the arrival of new information, such as a jump in fundamentals, can cause a price (in the absence of dividends) to jump, but a conversion from debt to equity cannot by itself cause a price to jump. A jump in a stock price may cause a trigger to be breached, but reaching the trigger cannot generate a jump. This is the key property that allows us to extend our earlier results.

We first establish the extension without dividends. We need to make two modifications to the free-range condition (20). Jumps make it possible for V_t to jump below U_t, so for the upper bound in Figure 4 we introduce

$$\tilde{V}_t = E_t[\max\{U_T, V_T\}].$$

Let T_ϵ denote the first time after τ_U^ϵ that the Poisson process N_t jumps, taking $T_\epsilon = \infty$ if either $\tau_U^\epsilon = \infty$ or if there is no jump in $(\tau_U^\epsilon, T]$. We replace the endpoint T in (20) with T_ϵ and require

$$P(U_t < L \text{ for all } t \in [\tau_U^\epsilon, T_\epsilon) \text{ and } \tilde{V}_t \le L \text{ for some } t \in [\tau_U^\epsilon, T_\epsilon) | \mathcal{F}_{\tau_U^\epsilon}) > 0, \tag{33}$$

almost surely. In other words, after U_t is at or below $L - \epsilon$, $\tilde{V}_t$ reaches the trigger without a jump.

Theorem 6.1 *(i) If $U_T \leq V_T$ whenever $U_T \geq L$ then there exists at least one equilibrium stock price. (ii) If in addition the free-range condition (33) and the right-continuity condition (21) hold, then the equilibrium is unique.*

In the case of contingent capital with asset value described by (32), we have $U_t = u_t(A_t)$ and $V_t = v_t(A_t)$, with u_t and v_t given explicitly by Merton's [25] formula for the price of an option in a jump-diffusion model. The terminal value of the upper bound $\tilde{V}_T = \max\{U_T, V_T\}$ can be replicated with a portfolio of options, so $\tilde{V}_t = \tilde{v}_t(A_t)$ is given by a linear combination of variants of the same formula. It is easy to see that once U_t reaches $L - \epsilon$, the asset process A_t can decline to any positive level in an arbitrarily small time interval without the occurrence of a jump, ensuring that $\tilde{V}_t$ will reach or cross L as required by (33). In short, the only restriction imposed by the theorem on this example is in part (i), the same condition we had in the continuous case. As before, the key requirement for existence and uniqueness is simply $L \geq C/m$.

We now introduce discrete dividends. We assume that dividend payments are restricted to fixed dates $0 < t_1 < \cdots < t_n < T$. For both practical and technical reasons, we also want to avoid the possibility that conversion is triggered by a dividend payment: such an event runs counter to the idea of a market price providing an objective trigger and might reasonably be viewed as market manipulation by the firm.[9] A simple and sensible way to avoid this possibility is to suppose that the post-conversion firm pays no dividends. We will further suppose that the probability of the post-conversion firm reaching or crossing the trigger at a dividend date is zero, $P(\tau_U \in \{t_1, \ldots, t_n\}) = 0$.

We define a cumulative dividend D_t^V for V_t by writing

$$D_t^V = \sum_{i:t_i \leq t} \delta_i^V,$$

for some payments δ_i^V, which, for simplicity we take to be positive. We augment $\mathcal{F}_t$ to include payments made in $[0, t]$. We define an equilibrium stock price and dividend process as a pair (S_t, D_t^S) with terminal value S_T as in (10) and satisfying the pricing relation $S_t = E_t[S_T + D_T^S - D_t^S]$, $t \in [0, T]$, with

$$D_t^S = \sum_{i:t_i \leq t,\, t_i < \tau_S} \delta_i^V; \tag{34}$$

in other words, S_t pays the same dividends as V_t prior to conversion at τ_S.

To establish uniqueness, we need a process $\tilde{V}_t$ that upper bounds any possible equilibrium

[9] The mandatory rights offering proposed by Duffie [9] could be interpreted as a negative dividend triggered by conversion.

stock price. For example, we can take

$$\tilde{V}_t = E_t[\max\{U_T, V_T\} + D_T^V - D_t^V].$$

We modify the definition of T_ϵ to be the minimum of the next dividend date and the time of the next Poisson jump after τ_U^ϵ. Thus, there are no jumps between τ_U^ϵ and T_ϵ either form dividends or from the Poisson process. We use the free-range condition (33) with these two modifications.

Theorem 6.2 *Under the foregoing conditions, (i) if* $U_t \leq V_t$ *whenever* $U_t \geq L$ *then there exists at least one equilibrium stock price and dividend process. (ii) If in addition the free-range condition and the right-continuity condition hold, then the equilibrium is unique.*

7 Concluding Remarks

We have analyzed the problem of internal consistency that arises when a firm's stock price (or the market price of any other liability of the firm) is used to trigger a change in the firm's capital structure. Within a general framework, we have shown that continuous trading largely rules out the possibility of multiple equilibria by allowing the market price to adjust in anticipation of reaching the trigger. In other words, we require the triggering security to be highly liquid. A lack of liquidity prevents the market price from fully incorporating the effect of the trigger, and this allows multiple equilibria. We establish conditions for existence of an equilibrium by comparing no-conversion and post-conversion variants of a firm. In the case of contingent capital, we need the conversion trigger or the dilution ratio at conversion to be sufficiently high.

The cases we have analyzed involve a contractual change in capital structure, as proposed in the design of contingent capital for banks. In practice, most changes in capital structure involve managerial discretion. Investors seek to anticipate a firm's decisions, and these decisions are in part informed by market prices, so similar questions about equilibria arise. These are settings in which the trigger itself is not well-defined; the approach developed here may nevertheless be relevant to these settings as well.

Acknowledgments. The authors thank Mark Flannery, Suresh Sundaresan, and Zhenyu Wang for many valuable discussions on the topic of this paper.

A Appendix

A.1 The Static Case

We abstract from the specific payoff functions in (1) and (2) and let u and v be arbitrary functions from $\mathbb{R}$ to $\mathbb{R}$. We continue to use the definitions in (3)–(5).

Proposition A.1 *Suppose the functions u and v satisfy (7). (i) Then $\underline{S}$ and $\overline{S}$ are equilibrium stock prices. (ii) Every equilibrium stock price satisfies*

$$S(A) \in \{u(A), v(A)\} \quad \textit{and} \quad \underline{S}(A) \leq S(A) \leq \overline{S}(A),$$

for all A, and, conversely, any function satisfying this property is an equilibrium stock price. (iii) If the sets $\{A : u(A) \geq L\}$ and $\{A : v(A) \geq L\}$ coincide there is just one equilibrium stock price, $S = \underline{S} = \overline{S}$. (iv) If (7) does not hold and if $\overline{S}(A) < \underline{S}(A)$ at some A, then no equilibrium exists.

Proof. (i) If $\underline{S}(A) \leq L$ and $\underline{S}(A) = v(A)$ then $v(A) \leq L$, but then (7) implies $u(A) \leq L$ and then $\underline{S}(A) = u(A)$ in light of (4). If $\underline{S}(A) > L$ and $\underline{S}(A) = u(A)$ then $u(A) > L$ so $\underline{S}(A) = v(A)$. Thus, $\underline{S}$ satisfies (3). If $\overline{S}(A) \leq L$ and $\overline{S}(A) = v(A)$ then $v(A) \leq L$ and $\overline{S}(A) = u(A)$. If $\overline{S}(A) > L$ and $\overline{S}(A) = u(A)$ then $u(A) > L$ so $v(A) > L$ and then $\overline{S}(A) = v(A)$. Thus, $\overline{S}$ satisfies (3).

(ii) Suppose S is an equilibrium stock price. If $S(A) \leq L$ then $S(A) = u(A) = \underline{S}(A)$; also, either $\overline{S}(A) = u(A)$ or $\overline{S}(A) = v(A) > L$. If $S(A) > L$ then $S(A) = v(A) \geq u(A)$, so $S(A) \geq \underline{S}(A)$; also, $\overline{S}(A) = v(A)$. Thus $\underline{S} \leq S \leq \overline{S}$. Conversely, let S be any function with the property in (ii). If $S(A) \leq L$ and $S(A) = v(A)$ then $\overline{S}(A) = u(A)$ and $\underline{S}(A) \leq L$ so $\underline{S}(A) = u(A)$; thus $S(A) = u(A)$. If $S(A) > L$ and $S(A) = u(A)$ then $\underline{S}(A) = v(A)$ and $v(A) \geq u(A)$ so $S(A) = v(A)$. Thus, S is an equilibrium stock price.

(iii) If the sets coincide then $\underline{S} = \overline{S}$ so this is the only equilibrium. (iv) Of the four possible combinations that can be constructed by pairing one of the outcomes $u(A) \leq L$ and $u(A) > L$ with either $v(A) \leq L$ or $v(A) > L$, the only one consistent with $\overline{S}(A) < \underline{S}(A)$ is $u(A) > L$ and $v(A) \leq L$. As in the discussion of (6), this rules out the possibility of an equilibrium. □

A.2 Existence and Uniqueness

To complete the proof of Theorem 3.1(ii), we just need to prove Lemma 3.2.

Proof of Lemma 3.2. Suppose S_t is an equilibrium stock price in the sense of (10)–(11). If $S_t > L$ for all $t \in [0, T]$, then $S_T = V_T$ so $V_T > L$, in which case (12) implies $V_T \geq U_T$. Thus, (10) continues to hold if we replace V_T with $\tilde{V}_T = \max\{U_T, V_T\}$. Conversely, suppose (10) holds with V_T replaced by $\tilde{V}_T$. If $V_T \geq U_T$, then $\tilde{V}_T = V_T$, so suppose $U_T > V_T$. In light of (12), this requires $U_T \leq L$ and thus $V_T \leq L$. But then $S_T \leq L$ and $S_T = U_T$. In other words, if $U_T > V_T$ then the value of V_T is irrelevant, so if (10) holds with $\tilde{V}_T$ it holds with V_T. □

Proof of Theorem 3.5. We prove the second case in the theorem. In the first case, any equilibrium must satisfy $U_t \leq S_t \leq V_t$ so a simpler version of the same argument applies.

Let τ_V denote the first time V_t reaches L; condition (22) implies $\tau_U \leq \tau_V$. Let S_t be an equilibrium stock price and let τ_S be the first time it reaches L. If $t \leq \tau_S$ then

$$\begin{aligned}
S_t &= E_t[S_{\tau_S \wedge T}] \\
&= E_t[S_{\tau_S}\mathbf{1}\{\tau_S \leq T\} + S_T\mathbf{1}\{\tau_S > T\}] \\
&= E_t[U_{\tau_S}\mathbf{1}\{\tau_S \leq T\} + V_T\mathbf{1}\{\tau_S > T\}],
\end{aligned}$$

and with (22), $U_T > V_T$ would require $S_T \leq \max\{U_T, V_T\} < L$, so

$$U_t = E_t[U_{\tau_S}\mathbf{1}\{\tau_S \leq T\} + U_T\mathbf{1}\{\tau_S > T\}] \leq S_t \leq E_t[V_{\tau_S}\mathbf{1}\{\tau_S \leq T\} + V_T\mathbf{1}\{\tau_S > T\}] = V_t. \quad (35)$$

This in turn implies that $\tau_S \leq \tau_V$.

Write the event in the free-range condition (20) as

$$\mathcal{A}_\epsilon = \{U_t < L \text{ for all } t \in [\tau_U^\epsilon, T] \text{ and } V_t \leq L \text{ for some } t \in [\tau_U^\epsilon, T]\}.$$

On the event $\mathcal{A}_\epsilon$, we have $\tau_V \leq T$ and therefore $\tau_S \leq T$. However, we know from Proposition 3.4 (and (19) in particular) that $U_{\tau_S} = L$, whereas $U_t < L$ for all $t \in [\tau_U^\epsilon, T]$ on $\mathcal{A}_\epsilon$. Thus, almost surely,

$$P(\mathcal{A}_\epsilon \cap \{\tau_S > \tau_U^\epsilon\} | \mathcal{F}_{\tau_U^\epsilon}, \tau_U^\epsilon < T) = 0. \quad (36)$$

Because the event $\{\tau_S > \tau_U^\epsilon\}$ is contained in $\mathcal{F}_{\tau_U^\epsilon}$, its conditional probability is 0 or 1, so in view of the free-range condition, we must have

$$P(\tau_S > \tau_U^\epsilon | \mathcal{F}_{\tau_U^\epsilon}, \tau_U^\epsilon < T) = 0,$$

almost surely. This implies $P(\tau_S \leq \tau_U^\epsilon | \tau_U^\epsilon < T) = 1$. The right-continuity condition implies $P(\tau_S \leq \tau_U^\epsilon < T) \to P(\tau_S \leq \tau_U < T)$ and $P(\tau_U^\epsilon < T) \to P(\tau_U < T)$. The right-continuity condition also implies $P(\tau_U = T) = 0$; otherwise, there would be positive probability that $\tau_U^\epsilon = \infty$ for all $\epsilon > 0$ while $\tau_U = T$. Thus, we have $P(\tau_S \leq \tau_U | \tau_U \leq T) = 1$. From the proof of Theorem 3.1, we know that $\tau_S \geq \tau_U$. We conclude that $P(\tau_S = \tau_U) = 1$, and thus $S = S^*$, the equilibrium constructed in the proof of Theorem 3.1. □

It is evident from the proof that in the free-range condition we could replace T in the interval $[\tau_U^\epsilon, T]$ with, for example, $[\tau_U^\epsilon, T \wedge \tau_U^\epsilon + h]$, for any $h > 0$, or with the interval $[\tau_U^\epsilon, \tau_V]$.

We turn next to the special case of contingent capital in the case of geometric Brownian motion.

Proof of Corollary 4.1. If $C \leq mL$, then the terminal payments (1) and (2) satisfy (7), so (12) holds and we get existence from Theorem 3.1(ii). By the Markov property, $U_t = u_t(A_t)$ for some deterministic function of time and asset level; $u_t(A_t)$ is essentially the Black-Scholes formula for a call option (monotone and continuous in t and A), and it inherits the right-continuity property (21) from A_t. Let $\tilde{V}_T = \max\{U_T, V_T\}$ and $\tilde{V}_t = E_t[\tilde{V}_T] = E_t[\max\{u(A_T), v(A_T)\}]$. By Lemma 3.2, it suffices to show that the free-range condition holds with $\tilde{V}_t$ in place of V_t. But $\tilde{V}_t \leq u_t(A_t) + v_t(A_t)$, where $v_t(A_t) = E_t[v(A_T)]$ is again given by a variant of the Black-Scholes formula. It is clear that after A_t has fallen far enough that $u_t(A_t) = L - \epsilon$, there is positive probability that it will subsequently fall to the point where $u_t(A_t) + v_t(A_t) \leq L$ before returning to a level at which $u_t(A_t) = L$.

If $C > mL$, then there is strictly positive probability of having A_T end up in the interval $(u_T^{-1}(L), v_T^{-1}(L))$, which is $(B + (1+m)L, B + C + L)$, while having $u_t(A_t) > L$ for all t. In other words, (18) holds and there is no equilibrium. □

A.3 Linear Contracts

In this section, we provide a result that covers several of the examples in Section 4 and others. The no-conversion and post-conversion stock prices are given by

$$U_t = E_t[(aA_T - bB_t)^+] \quad \text{and} \quad V_t = E_t[(A_T - B)^+], \tag{37}$$

for some positive constants $a \neq 1$ and b. (For the case $a = 1$, see Section 4.2.) The existence and uniqueness of an equilibrium stock price with a conversion trigger at L depend on these coefficients. We will refer to (10) as a *lower-trigger* equilibrium because the trigger condition is $S_t \leq L$; we will also consider the case of an *upper-trigger* equilibrium in which the condition is $S_t \geq L$. To rule out uninteresting cases, we assume $U_0 > L$ in the lower-trigger case and $U_0 < L$ in the upper-trigger case. Set $L^* = (a-b)B/(1-a)$; this is the level at which the lines $A \mapsto aA - bB$ and $A \mapsto A - B$ intersect.

Proposition A.2 *In the setting of (14) and (37) with A_t geometric Brownian motion, we have the following:*

(i) *If $a > 1$ and $b \leq a$, there is no lower-trigger equilibrium and there is a unique upper-trigger equilibrium for any L.*

(ii) *If $b < a < 1$, there is a unique lower-trigger equilibrium for any $L \geq L^*$ and no lower-trigger equilibrium otherwise. There is a unique upper-trigger equilibrium for any $L \leq L^*$ and no upper-trigger equilibrium otherwise.*

(iii) *If $a < 1$ and $a \leq b$, there is a unique lower-trigger equilibrium and no upper-trigger equilibrium for any L.*

(iv) *If $1 < a < b$, there is no lower-trigger equilibrium if $L > L^*$ and no upper-trigger equilibrium if $L < L^*$.*

Proof. In case (i), $U_t \geq V_t$ for all $t \in [0, T]$, so existence and uniqueness in the upper-trigger case follows from the arguments used in Theorems 3.1 and 3.5. In the lower-trigger case, (18) holds. The lower-trigger case of (ii) becomes equivalent to the setting of Corollary 4.1 if we replace B with $\tilde{B} = B + C$ and set $a = 1/(1+m)$ and $b = aB/(B+C)$. A symmetric argument works for the upper-trigger case. In (iii), $V_t \geq U_t$ for all $t \in [0, T]$, so existence and uniqueness in the lower-trigger case follows from Theorems 3.1 and 3.5. For the upper-trigger case, (18) holds with the inequalities reversed, ruling out an equilibrium.

For case (iv), observe that the Markov property of A_t allows us to write $U_t = u_t(A_t)$ and $V_t = v_t(A_t)$, where u_t and v_t are deterministic functions of t and A_t (both given by variants of the Black-Scholes formula). Let $u_t^{-1}(L)$ be the asset level that makes $U_t = L$; i.e., that solves $u_t(u_t^{-1}(L)) = L$, and define $v_t^{-1}(L)$ accordingly. If $L > L^*$, then $u_T^{-1}(L) < v_T^{-1}(L)$, and then by continuity of these inverses $u_t^{-1}(L) < v_t^{-1}(L)$ for all $t \in [T-\epsilon, T]$ for some $\epsilon > 0$. Now there is strictly positive probability that A_t will be above $u_t^{-1}(L)$ for all $0 \leq t \leq T$ (recall that $U_0 < L$ so $A_0 > u_0^{-1}(L)$) and then $A_t \in (u_T^{-1}(L), v_T^{-1}(L))$ for all $t \in [T-\epsilon, T]$. But then $V_T = v_T(A_T) < L$ whereas $U_t = u_t(A_t) > L$ for all $t \in [0, T]$; in other words, (18) holds. The second assertion of (iv) follows by a symmetric argument. □

There are two possibilities not covered by Proposition A.2, both arising in case (iv). If $L \leq L^*$, we cannot rule out the possibility of a lower-trigger equilibrium; however, we cannot ensure that $S_t^* \geq U_t$, so the argument in Theorem 3.1 does not apply. Similar comments apply for an upper-trigger equilibrium with $L \geq L^*$.

A.4 Dividends

Proof of Theorem 5.1. Part (i) follows from (ii). For (ii), define S_T^* as in (16) and define a dividend process D^* as in (27) but with τ_S replaced by τ_U. For $t < T$, define

$$S_t^* = E_t[S_T^* + D_T^* - D_t^*].$$

The process D_t^* is continuous because D_t^V and D_t^U are continuous; the process $S_t^* + D_t^*$ is continuous because it is a Brownian martingale; it follows that S_t^* is continuous. For $t \leq \tau_U$,

$$\begin{aligned} S_t^* &= E_t[S^*_{\tau_U \wedge T}] + E_t[D^V_{T\wedge\tau_U} - D_t^V] \\ &= E_t[U_{\tau_U}\mathbf{1}\{\tau_U \leq T\} + V_T\mathbf{1}\{\tau_U > T\}] + E_t[D^V_{T\wedge\tau_U} - D_t^V] \\ &\geq E_t[U_{\tau_U}\mathbf{1}\{\tau_U \leq T\} + U_T\mathbf{1}\{\tau_U > T\}] + E_t[D^U_{T\wedge\tau_U} - D_t^U] \qquad (38) \\ &= U_t. \end{aligned}$$

It follows that S_t^* first reaches L at τ_U, and thus that S_t^* is an equilibrium, as in the proof of Theorem 3.1.

For uniqueness, first observe that any equilibrium stock price S_t is continuous by the argument used for S_t^*, so $S_{\tau_S} = U_{\tau_S} = L$. For (iii), once we have $U_t \le V_t$ for all $t \in [0, T]$, uniqueness follows as in Theorem 3.5. For (iv), if S_t is an equilibrium stock price process then

$$\begin{aligned} S_{\tau_S \wedge T} &= S_{\tau_S}\mathbf{1}\{\tau_S \le T\} + S_T\mathbf{1}\{\tau_S > T\} \\ &= U_{\tau_S}\mathbf{1}\{\tau_S \le T\} + V_T\mathbf{1}\{\tau_S > T\} \le V_{\tau_S \wedge T}, \end{aligned}$$

using $U_{\tau_S} = L$ and (22). It follows that if $t \le \tau_S \wedge T$, then

$$S_t = E_t[D^V_{\tau_S \wedge T} - D^V_t + S_{\tau_S \wedge T}] \le V_t;$$

and if $\tau_S < t \le T$, then $S_t = U_t$, so $S_t \le V_t$ if $S_t \ge L$. Combining these two cases, we conclude that $S_t \le V_t$ for all $t \le \tau_V$. The proof of Theorem 3.5 now applies. □

A.5 Infinite Horizon

For the infinite-horizon formulation, we set $U_t = E_t[D^U_\infty - D^U_t]$ and $V_t = E_t[D^V_\infty - D^V_t]$, where D^U_t and D^V_t are continuous dividend processes, their limits D^U_∞ and D^V_∞ are almost surely finite, and U_t and V_t uniformly integrable. An equilibrium pair (S_t, D^S_t) satisfies $S_t = E_t[D^S_\infty - D^S_t]$, where

$$D^S_t = \begin{cases} D^V_{\tau_S} + D^U_t - D^U_{\tau_S}, & t \ge \tau_S; \\ D^V_t, & t < \tau_S. \end{cases} \tag{39}$$

We modify the free-range condition by fixing a $T > 0$ and requiring $P(\tau^\epsilon_U < \infty) > 0$ and, conditional on $\tau^\epsilon_U < \infty$, we have

$$P(U_t < L \text{ for all } t \in [\tau^\epsilon_U, \tau^\epsilon_U + T] \text{ and } V_t \le L \text{ for some } t \in [\tau^\epsilon_U, \tau^\epsilon_U + T] | \mathcal{F}_{\tau^\epsilon_U}) > 0, \tag{40}$$

almost surely.

Proposition A.3 *(i) If $E_t[D^V_{\tau_U} - D^V_t] \ge E_t[D^U_{\tau_U} - D^U_t]$ for all $t < \tau_U$, then there exists at least one equilibrium stock price and corresponding dividend process. (ii) If $E_t[D^V_{\tau_V} - D^V_t] \ge E_t[D^U_{\tau_V} - D^U_t]$ for all $t < \tau_V$, and the free-range condition (40) and the right-continuity condition (21) hold, then the equilibrium is unique.*

Proof. Define D^*_t as in (39) but with τ_U in place of τ_S, and then set $S^*_t = E_t[D^*_\infty - D^*_t]$. The condition in (i) ensures that $S^*_t \ge U_t$ and thus that S^*_t is an equilibrium by the argument used previously. The condition in (ii) ensures that any equilibrium satisfies $S_t \le V_t$, and then uniqueness follows by the argument used previously. □

A.6 Jumps and Discrete Dividends

Proof of Theorem 6.1. (i) In the proofs of Lemma 3.2 and Theorem 3.1, we need to replace "reaches the trigger" or "reaches L" with "reaches $(-\infty, L]$," because conversion may now be triggered by a jump that crosses L. With this modification, the proofs hold exactly as before because the arguments there do not use continuity.

(ii) Even with jumps, we have $S_{\tau_S} = U_{\tau_S}$, by the definition (10)–(11) of an equilibrium S and the fact $S_{\tau_S} = S_{\tau_S+}$ and $U_{\tau_S} = U_{\tau_S+}$ record values after any jump at τ_S. The argument in the proof of Theorem 3.5 showing in (35) that $U_t \leq S_t$ for all $t \leq \tau_S$ therefore applies. We do not necessarily have $U_{\tau_S} \leq V_{\tau_S}$ because a jump at τ_S could result in $V_{\tau_S} < U_{\tau_S} < L$, so the upper bound in (35) may no longer hold, but we clearly have $S_t \leq \tilde{V}_t$ for all $t \in [0, T]$.

Let $\mathcal{A}_\epsilon$ denote the event in the modified free-range condition (33). On this event, we have $\tau_S \in [\tau_U, T_\epsilon)$ because the upper bound $\tilde{V}_t$ reaches or crosses the trigger before T_ϵ and the lower bound U_t does so at τ_U. However, the path of S_t is continuous throughout the interval $[\tau_U^\epsilon, T_\epsilon)$ because there are no Poisson jumps in this interval. Thus, for S_t to reach the trigger in this interval we would need to have $S_{\tau_S} = L$ and then $U_{\tau_S} = S_{\tau_S} = L$, which is impossible on $\mathcal{A}_\epsilon$. Thus, (36) holds and the rest of the proof goes through exactly as before. □

Proof of Theorem 6.2. If $t < \tau_U$, then throughout the interval $[t, \tau_U \wedge T)$ we have $U_t > L$ and then $U_t \leq V_t$. Define S^* as in the proof of Theorem 5.1, and define D^* as in (34) but with τ_S replaced by τ_U. On the event $\tau_U < T$, we have $S^*_{\tau_U} = U_{\tau_U}$ by construction, and then $S^*_t \geq U_t$ for all $t \leq \tau_U$, as in (38). We conclude that (S^*_t, D^*_t) is an equilibrium as in Theorems 3.1 and 5.1. Uniqueness follows from the free-range and right-continuity conditions, together with the bound $S_t \leq \tilde{V}_t$, for all $t \in [0, T]$, which clearly holds by construction. □

References

[1] Albul, B., Jaffee, D.M., and Tchistyi, A. (2010) Contingent Convertible Bonds and Capital Structure Decisions, working paper, University of California, Berkeley.

[2] Basel Committee on Banking Supervision (2011) *Global systemically important banks: assessment methodology and the additional loss absorbency requirement.* Publication BCBS207, Bank for International Settlements, Basel, Switzerland.

[3] Bianchi, C., Hancock, D., and Kawano, L. (2005) Does trading frequency affect subordinated debt spreads? Finance and Economics Discussion Series 2008-05, Federal Reserve Board, Washington, D.C.

[4] Black, F., and J. C. Cox. (1976). Valuing corporate securities: some effects of bond indenture provisions. *Journal of Finance* **31**, 351–367.

[5] Bolton, P., and Samama, F., (2012) Capital access bonds: contingent capital with an option to convert. *Economic Policy* **27**, 275–317.

[6] Bond, P., Goldstein, I., and Prescott, E.S. (2010). Market-based corrective action, *Review of Financial Studies* **23**, 781–820.

[7] Calomiris, C.W., and Herring, R.J. (2011) Why and How to Design a Contingent Convertible Debt Requirement. Available at SSRN: http://ssrn.com/abstract=1815406

[8] Chen, N., Glasserman, P., Nouri, B., and Pelger, M. (2012) CoCos, bail-in, and tail risk, Working paper.

[9] Duffie, D. (2011) *How Big Banks Fail And What to Do About It.* Princeton University Press, Princeton, New Jersey.

[10] Financial Stability Oversight Council (2012) *Report to Congress on Study of a Contingent Capital Requirement for Certain Nonbank Financial Companies and Bank Holding Companies.* Available at http://www.treasury.gov/initiatives/fsoc/Pages/studies-and-reports.aspx.

[11] Flannery, M.J. (2005) "No Pain, No Gain?" Effecting Market Discipline via "Reverse Convertible Debentures," in H.S. Scott (ed.), *Capital Adequacy Beyond Basel: Banking, Securities, and Insurance*, Oxford University Press.

[12] Flannery, M.J. (2009) Stabilizing Large Financial Institutions with Contingent Capital Certificates. Available at SSRN: http://ssrn.com/abstract=1485689.

[13] Flood, R.P., and Garber, P.M. (1984) Collapsing exchange-rate regimes: some linear examples. *Journal of International Economics* **17**, 1–13.

[14] Glasserman, P., and Nouri, B. (2011) Contingent capital with a capital-ratio trigger. *Management Science*, to appear.

[15] Hancock, D., and Kwast, M.L. (2001) Using Subordinated Debt to Monitor Bank Holding Companies: Is it Feasible? *Journal of Financial Services Research* 20, 147–87.

[16] Hart, O., and Zingales, L. (2011) A new capital regulation for large financial institutions. *American Law and Economics Review* **13**, 453-490.

[17] Hilscher, J., and Raviv, A. (2011). Bank stability and market discipline: debt-for-equity swap versus subordinated notes. Working paper, International Business School, Brandeis University, Waltham, Mass.

[18] Himmelberg, C.P., and Tsyplakov, S. (2011) Pricing contingent capital bonds: incentives matter. Working paper, University of South Carolina.

[19] Jacod, J., and Shiryaev, A. (2003) *Limit Theorems for Stochastic Processes*, Springer, Berlin.

[20] Koziol, C., and Lawrenz, J. (2012) Contingent Convertibles: Solving or Seeding the Next Banking Crisis? *Journal of Banking and Finance* **36**, 1:90–104.

[21] Leland, H. E. (1994). Corporate Debt Value, Bond Covenants, and Optimal Capital Structure. *Journal of Finance* **49**, 1213–1252.

[22] Manso, G., Strulovici, B., and Tchistyi, A. (2010) Performance-sensitive debt. *Review of Financial Studies* **23**, 1819–1854.

[23] McDonald, R. (2010) Contingent Capital with a Dual Price Trigger, working paper, Finance Department, Kellogg School of Management, Northwestern University.

[24] Merton, R. C. (1974) On the pricing of corporate debt: the risk structure of interest rates. *Journal of Finance* **29**, 449–470.

[25] Merton, R.C. (1976) Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics* **3**, 125–144.

[26] Pennacchi, G. (2010) A Structural Model of Contingent Bank Capital, Federal Reserve Bank of Cleveland Working Paper 10-04.

[27] Prescott, E.S. (2011) Contingent capital: the trigger problem. Working paper 11-07, Federal Reserve Bank of Richmond, Richmond, Virginia.

[28] Sidley Austin, LLP (2010) *Hybrid Capital Product Development: Focusing on Tier 1 and Other Hybrid Capital Products.* Available at www.sidleyhybridcapital.com.

[29] Squam Lake Working Group (2009) An Expedited Resolution Mechanism for Distressed Financial Firms: Regulatory Hybrid Securities, Council on Foreign Relations.

[30] Sundaresan, S., and Z. Wang. (2010). Design of contingent capital with a stock price trigger for conversion. Working paper, Federal Reserve Bank of New York.