Equilibrium Positive Interest Rates: A Unified View

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This article develops precise connections among two general approaches to building interest rate models: a general equilibrium approach using a pricing kernel and the Heath, Jarrow, and Morton framework based on specifying forward rate volatilities and the market price of risk. The connections exploit the observation that a pricing kernel is uniquely determined by its drift. Through these connections we provide, for any arbitrage-free term structure model, a representative-consumer real production economy supporting that term structure model in equilibrium. We put particular emphasis on models in which interest rates remain positive. By modeling the dynamics of the drift of the pricing kernel, we construct a new family of Markovian-positive interest rate models.

This article develops precise connections among two general approaches to term structure modeling, with particular emphasis on models in which interest rates stay positive: an approach based on direct modeling of the pricing kernel (the marginal utility of optimal consumption), and the Heath, Jarrow, and Morton (1992; hereafter HJM) framework based on specifying forward rate volatilities and the market price of risk. Starting from the primitive data of either perspective, we show how to obtain the primitive data of the other. Our treatment of pricing kernels builds on Rogers' (1997) approach, and our consideration of positive interest rate models includes links with the formulation of Flesaker and Hughston (1996a). In the course of developing connections among these frameworks and as a consequence of them we obtain the following additional results:

- We give an economic interpretation to the observation that a pricing kernel is completely determined by its drift, under the reasonable requirement that the prices of discount bonds vanish as their maturity increases. This observation also implies that once this drift is specified the construction of all other term structure quantities follows.
- For an arbitrary term structure model formulated in either approach we construct a representative-consumer production economy equilibrium

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supporting that term structure. In particular, we show explicitly that every HJM arbitrage-free model is supported by a production economy equilibrium of the type developed by Cox, Ingersoll, and Ross (1985a).

- We develop a criterion under which an HJM model generates positive interest rates, based on a relation between the HJM framework and that of Flesaker and Hughston (1996a).
- We introduce a general class of positive interest rate models based on specifying the drift of the pricing kernel as a function of Markov processes. The models can fit any initial term structure, provide reasonably tractable expressions for bond prices and forward rates, and match elements of the bond return covariance matrix.

We now briefly describe the modeling approaches and then outline the rest of the article. Consider a representative consumer economy in which the representative consumer maximizes expected discounted utility with constant discount factor ρ,

$$\mathsf{E}_t\left(\int_t^\infty e^{-\rho\tau}U(c_\tau)d\tau\right).$$

where $U(\cdot)$, defined on $\mathbb{R}^+$, represents the consumer's Von Neumann–Morgenstern preferences, and c_t is the consumption rate at time t. Let c_t^* denote the optimal consumption process. In equilibrium, the time t price of a contingent claim that pays X units of account at some future date T is $\pi^X(t) = \mathsf{E}_t(\frac{Z_T}{Z_t}X)$ for $0 \le t \le T \le \infty$, where

$$Z_t = e^{-\rho t}\frac{\partial U}{\partial c_t^*}$$

is the marginal utility of optimal consumption at time t, also called the pricing kernel or state-price density. It is well known that in substantial generality [e.g., see Duffie (1996)]

$$dZ_t = Z_t\left(-r(t)dt - \sum_{j=1}^m \phi_j(t)dW_j(t)\right). \tag{1}$$

where $r(t)$ is the short rate in the economy and $\phi_j(t)$ is the market price of risk associated with the jth random factor $W_j(t)$, a standard Brownian motion. Therefore, if one directly models the pricing kernel Z_t as

$$dZ_t = \mu_Z(t)dt + \sum_{j=1}^m Y_j(t)dW_j(t), \tag{2}$$

one can recover the short rate $r(t) = -\mu_Z(t)/Z_t$ and the market price of risk $\phi_j(t) = -Y_j(t)/Z_t$. Assuming that the consumer's utility function is strictly

increasing (the consumer is nonsatiable), we get $Z_t > 0$ for all t. It follows that if $-\mu_Z(t)$ is taken to be positive for all t, then $r(t) > 0$.

Historically the term structure of interest rates was first modeled by specifying the dynamics of the short rates and the market prices of risk. For examples of this type of work, see Vasicek (1977), Cox, Ingersoll, and Ross (1985b), Hull and White (1990), Longstaff and Schwartz (1991). It is clear from Equation (1) that this approach is equivalent to specifying the pricing kernel. More recently, researchers have modeled pricing kernels directly in the sense that they start with explicit expressions for the dynamics of Z_t. Constantinides (1992) develops such a model in which the pricing kernel is driven by Ornstein–Uhlenbeck processes. Flesaker and Hughston (1996a), while not explicitly referring to a pricing kernel, develop a general framework that has a natural interpretation from the pricing kernel, a point observed by Rogers (1997). Rogers (1997) treats the modeling of pricing kernels in a very general setting and observes that they can be modeled as potentials.[1] He also proposes a general class of models in which pricing kernels are obtained from Markov processes. Hagan and Woodward (1997) and Hunt, Kennedy, and Pelsser (1997) construct models in which a numeraire asset is constructed as a function of a Markov process; specifying the dynamics of a numeraire is similar to specifying the dynamics of a pricing kernel.

We will work with Equation (2), taking μ_Z as the most basic modeling element. Once μ_Z is specified, the pricing kernel will be completely determined. In contrast, specifying a process for Z (rather than μ_Z) could lead to a term structure in which the prices of discount bonds do *not* vanish as their maturity increases. A further advantage of modeling μ_Z is that the initial term structure can be easily matched through a restriction on the unconditional mean of μ_Z [see Equation (8)].

Rogers (1997) discusses several routes by which a pricing kernel can be constructed. One approach starts from a positive process $-\mu_Z(t)$, sets

$$A_t = -\int_0^t \mu_Z(\tau)d\tau,$$

and defines

$$Z_t = \mathsf{E}_t(A_\infty) - A_t \tag{3}$$

as a pricing kernel; clearly, Z is thus determined by A. Justification for calling this process[2] a pricing kernel is provided in Section 4, where we construct a production economy equilibrium that indeed generates this process

[1] A *potential*, denoted by Z_t, is a right-continuous nonnegative supermartingale satisfying $\lim_{t\to\infty}\mathsf{E}(Z_t) = 0$. See, for example, Karatzas and Shreve (1991, p. 18.)

[2] Readers familiar with the general theory of stochastic processes will notice that Equation (3) is the Doob–Meyer decomposition of a potential [see, e.g., Karatzas and Shreve (1991, pp. 18–28)].

as its marginal utility of optimal consumption. As a consequence, μ_Z has the meaning of instantaneous expected change in the marginal utility of optimal consumption. We will show that Equation (3) is equivalent to the identity

$$Z_t = \mathsf{E}_t\left(\int_t^\infty \frac{Z_\tau}{B_\tau}\, dB_\tau\right).$$

where B_t represents a money market account. We can interpret this equation [hence also Equation (3)] as follows. The left side is the marginal utility from consuming one additional unit of account at time t. The right side is the expected marginal utility resulting from investing one unit of account in the money market at time t, keeping the size of that investment fixed at one unit in perpetuity, and at each time $\tau > t$ consuming the dividend dB_τ / B_τ paid at time τ. In equilibrium, a representative agent should be indifferent between the additional utility at time t on the left and the expected stream of additional utility on the right. This is the economic interpretation of this identity and hence also of Equation (3) and the statement that the pricing kernel is determined by its drift.

In Section 1 we detail the construction of a term structure model starting from μ_Z and provide technical conditions under which the model admits an equivalent martingale measure. In Section 2 we develop the relation between the pricing kernel approach and that of Heath, Jarrow, and Morton (1992). The HJM framework specifies the evolution of forward rates $f(t, T)$ through a process of the form

$$df(t, T) = \alpha(t, T)dt + \sum_{j=1}^{m} \sigma_j(t, T)dW_j(t).$$

HJM showed that the absence of arbitrage then implies

$$\alpha(t, T) = \sum_{j=1}^{m}\left(\phi_j(t) + \int_t^T \sigma_j(t, \tau)\, d\tau\right)\sigma_j(t, T).$$

The primitive data in this model are therefore the forward rate volatilities σ_j and the risk prices ϕ_j. We show how these determine μ_Z and vice versa.

In Section 3 we use a relation between the Flesaker and Hughston (1996a) and Heath, Jarrow, and Morton (1992) frameworks to obtain a means of verifying positivity of interest rates from the HJM primitive data. Section 4 develops an equilibrium justification for all of the formulations considered in this article. In Section 5, based on the theory developed in previous sections, we build a family of positive interest rate models using nonnegative Markov processes. These models can fit any initial forward rate curve, provide reasonably tractable expressions for bond prices and forward rates, and

match elements of the bond return covariance matrix. As a specific example, we present a positive interest rate model based on a reflected Brownian motion.

1. A Pricing Kernel Approach to Modeling Positive Interest Rates

In this section we build on Rogers (1997) to construct an arbitrage-free positive interest rate model from an arbitrary positive process $-\mu_Z$, which turns out to be the drift of a pricing kernel. In an equilibrium setting, μ_Z can be interpreted as the instantaneous expected change in the marginal utility of optimal consumption.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space that characterizes all the uncertainty in the economy, and $W_1(t), \ldots, W_m(t)$ be m independent standard Brownian motions representing underlying random factors that drive the economy. We choose $\{\mathcal{F}_t\}$ to be the P-augmentation of the natural filtration generated by $W_1(t), \ldots, W_m(t)$, which represents the accumulated information up to time t.

We start with an $\mathcal{F}_t$-adapted positive continuous process $-\mu_Z(t)$ and set

$$A_t = -\int_0^t \mu_Z(\tau)d\tau.$$

Assume the following technical condition holds:

Assumption 1. $\mathsf{E}(A_\infty^2) < \infty$, *where* $A_\infty = \lim_{t\to\infty} A_t$,

and define [as in Equation (1.7) of Rogers (1997)]

$$Z_t = \mathsf{E}\left(A_\infty \mid \mathcal{F}_t\right) - A_t, \quad \text{for every } 0 \le t \le \infty. \tag{4}$$

Observe that Assumption 1 implies $\mathsf{E}(\sup_{0\le t\le\infty} Z^2(t)) < \infty$. Without loss of generality, we normalize A_t so that $Z_0 = \mathsf{E}(A_\infty) = 1$.

We proceed to define the time t price of a zero coupon bond that pays 1 unit of account at maturity T to be

$$p(t, T) = \frac{1}{Z_t}\mathsf{E}\left(Z_T \mid \mathcal{F}_t\right), \quad \text{for } 0 \le t \le T \le \infty. \tag{5}$$

Since A_T is strictly increasing, $\lim_{T\to\infty} p(t, T) = \frac{1}{Z_t}\lim_{T\to\infty}\mathsf{E}(A_\infty - A_T \mid \mathcal{F}_t) = 0$, which is desirable. If one thinks of Z_T / Z_t as the intertemporal marginal rate of substitution between time T consumption and time t consumption, then it is clear that Equation (5) yields the price of a zero coupon bond. The economic intuition of Equation (4) will be given shortly.

It is standard that the instantaneous forward rate $f(t, T)$ is given by

$$f(t,T) = -\frac{\partial \log(p(t,T))}{\partial T} = \frac{1}{\mathsf{E}(Z_T \mid \mathcal{F}_t)} \frac{\partial \mathsf{E}(A_T \mid \mathcal{F}_t)}{\partial T}$$
$$= \frac{1}{\mathsf{E}(Z_T \mid \mathcal{F}_t)} \mathsf{E}\left(\frac{\partial A_T}{\partial T} \Big| \mathcal{F}_t\right).^3 \tag{6}$$

and the short rate is

$$r(t) = f(t,t) = \frac{1}{Z_t} \mathsf{E}\left(\frac{\partial A_T}{\partial T} \Big| \mathcal{F}_t\right)\Big|_{T=t} = \frac{1}{Z_t}\frac{\partial A_t}{\partial t} = -\frac{\mu_Z(t)}{Z_t}. \tag{7}$$

This specializes to Equation (2.4) of Rogers (1997).

Since $p(0,T) = \mathsf{E}(Z_T) = \mathsf{E}(A_\infty) - \mathsf{E}(A_T)$, we have

$$\mathsf{E}\left(\frac{\partial A_T}{\partial T}\right) = -\frac{\partial p(0,T)}{\partial T} = f(0,T)\exp\left(-\int_0^T f(0,\tau)d\tau\right), \tag{8}$$

which can be used to match a given initial forward rate curve.[4] As we will show $\mu_Z(t) = -\frac{\partial A_t}{\partial t}$ is the drift of Z_t, from Equation (8), we conclude that the initial term structure is equivalent to a constraint on the unconditional mean of the drift of a pricing kernel.

By a *positive* interest rate model, we mean one in which the bond prices $p(t,T)$ are strictly decreasing in T, which is equivalent to the requirement that $f(t,T)$ is positive for all $0 \le t \le T < \infty$, since $p(t,T) = \exp(-\int_t^T f(t,\tau)d\tau)$. It follows from Equation (6) that the interest rate model constructed above is a positive one.

To proceed, we introduce the following processes. Let B_t be the money market account induced by $r(t)$, that is,

$$B_t = \exp\left(\int_0^t r(\tau)d\tau\right) \quad \text{for } 0 \le t < \infty.$$

[3] It follows from Fubini's theorem that

$$\int_0^T \mathsf{E}\left(\frac{\partial A_\tau}{\partial \tau}\Big|\mathcal{F}_t\right)d\tau = \mathsf{E}\left(\int_0^T \frac{\partial A_\tau}{\partial \tau}d\tau \Big| \mathcal{F}_t\right) = \mathsf{E}(A_T \mid \mathcal{F}_t).$$

Differentiating both sides of the above equation with respect to T yields

$$\frac{\partial \mathsf{E}(A_T \mid \mathcal{F}_t)}{\partial T} = \mathsf{E}\left(\frac{\partial A_T}{\partial T}\Big|\mathcal{F}_t\right) \quad \text{a.s.}$$

[4] This initialization is consistent with our normalization since

$$Z_0 = \mathsf{E}\left(\int_0^\infty \frac{\partial A_\tau}{\partial \tau}d\tau\right) = \int_0^\infty \mathsf{E}\left(\frac{\partial A_\tau}{\partial \tau}\right)d\tau = \int_0^\infty -\frac{\partial p(0,\tau)}{\partial \tau}d\tau = p(0,0) = 1.$$

We denote by M_t the martingale $\mathsf{E}(A_\infty \mid \mathcal{F}_t)$. By Assumption (1), M_t is square integrable for every $t \ge 0$. It follows from the martingale representation theorem [see, e.g., Karatzas and Shreve (1991, pp. 182–189)] that there exist unique adapted processes $Y_1(t), \ldots, Y_m(t)$ such that $\mathsf{E}\left(\int_0^T Y_j^2(t)dt\right) < \infty$ $(1 \le j \le m)$ for every $0 < T \le \infty$, and

$$M_t = \mathsf{E}(A_\infty) + \sum_{j=1}^m \int_0^t Y_j(\tau)\, dW_j(\tau); \quad 0 \le t \le \infty. \tag{9}$$

As shown later in this section, $-Y_j(t)/Z_t$ turns out to be the market price of risk associated with the jth random factor $W_j(t)$.

Note that $Z_t = M_t - A_t$, so by Equation (7),

$$dZ_t = dM_t - dA_t = dM_t - r_t Z_t\, dt. \tag{10}$$

Because M_t is almost surely continuous, Z_t is too. Since Z_t is strictly positive, Z_t is bounded away from zero (almost surely) on any finite time horizon, and hence $r(t)(= -\mu_Z(t)/Z_t)$ remains finite.

Again by Equation (7) and noticing that $dB_t = r_t B_t dt$, we can rewrite Equation (4) as

$$Z_t = \mathsf{E}\left(-\int_t^\infty \mu_Z(\tau)\,d\tau \Big| \mathcal{F}_t\right) = \mathsf{E}\left(\int_t^\infty Z_\tau r_\tau\, d\tau \Big| \mathcal{F}_t\right)$$
$$= \mathsf{E}\left(\int_t^\infty \frac{Z_\tau}{B_\tau}\, dB_\tau \Big| \mathcal{F}_t\right). \tag{11}$$

We can interpret this equation [hence also Equation (4)] as follows. The left side is the marginal utility from consuming one additional unit of account at time t. The right side is the expected marginal utility resulting from investing one unit of account in the money market at time t, keeping the size of that investment fixed at one unit in perpetuity, and at each time $\tau > t$ consuming the dividend dB_τ/B_τ paid at time τ. In equilibrium, a representative agent should be indifferent between the additional utility at time t on the left and the expected stream of additional utility on the right. This is the economic interpretation of the equality in Equation (11) and hence also of the statement that the pricing kernel is determined by its drift.

The following theorem identifies the equivalent martingale measure in this model.

Theorem 1. *For each fixed $T \in [0,\infty)$, assume Assumption (1) and Assumption 2*

$$\mathsf{E}\left(\exp\left(\sum_{j=1}^m \int_0^T \frac{Y_j(t)}{Z_t}\, dW_j(t) - \frac{1}{2}\sum_{j=1}^m \int_0^T \left(\frac{Y_j(t)}{Z_t}\right)^2 dt\right)\right) = 1 \tag{12}$$

hold.[5] *Then there exists an equivalent probability measure $\widetilde{P}$, given by the likelihood ratio process $\frac{d\widetilde{P}}{dP}\,|_{\mathcal{F}_T} = Z_T B_T$, under which the zero coupon bond price can be expressed as*

$$p(t,T) = B_t \widetilde{\mathsf{E}}\left(\frac{1}{B_T}\,\Big|\,\mathcal{F}_t\right) = \widetilde{\mathsf{E}}\left(\exp\left(-\int_t^T r(\tau)d\tau\right)\,\Big|\,\mathcal{F}_t\right)$$

for $0 \le t \le T < \infty$. That is, $p(t,T)/B_t$ is a martingale under $\widetilde{P}$.

Proof. See Appendix A. ■

This equivalent martingale measure, $\widetilde{P}$, is the so-called *risk-neutral measure*. It is well-known from Harrison and Pliska (1981) that the existence of such a measure implies the absence of arbitrage in the class of self-financing trading strategies. By Theorem 1, the market price of risk associated with the jth random factor, denoted by $\phi_j(t)$, is

$$\phi_j(t) = -\frac{Y_j(t)}{Z_t} \quad \text{for } 1 \le j \le m. \tag{13}$$

Furthermore, the time t fair price of an attainable contingent claim that pays X units of account at time T, denoted by $\pi^X(t)$, is

$$\pi^X(t) = B_t \widetilde{\mathsf{E}}\left(\frac{X}{B_T}\,\Big|\,\mathcal{F}_t\right) = \frac{1}{Z_t}\mathsf{E}(Z_T X \mid \mathcal{F}_t),$$

where X is a nonnegative $\mathcal{F}_T$-measurable random variable with $E(X^2) < \infty$. Hence, Z_t indeed plays the role of pricing kernel under the probability measure P.

To see the generality of this approach, consider *any* interest rate model with a continuous pricing kernel Z_t satisfying $\mathsf{E}\left(\sup_{0\le t\le\infty} Z^2(t)\right) < \infty$ and a continuous short rate process r_t. Since the time t price $p(t,T)$ of a zero coupon bond that pays 1 unit of account at maturity T is less than 1, $\mathsf{E}(Z_T \mid \mathcal{F}_t) < Z_t$, namely, Z_t is a supermartingale. It is reasonable to require $\lim_{T\to\infty} p(t,T) = 0$, which implies that $\lim_{T\to\infty}\mathsf{E}(Z_T) = 0$. As first observed by Rogers (1997), these two properties make Z_t what is known as a potential in the theory of stochastic processes. By the Doob–Meyer decomposition theorem [see, e.g., Karatzas and Shreve (1991, pp. 18–28)], there exists a unique (up to indistinguishability) adapted continuous increasing process A_t with $\mathsf{E}(A_\infty^2) < \infty$ so that $Z_t = \mathsf{E}(A_\infty \mid \mathcal{F}_t) - A_t$ for $0 \le t \le \infty$. This shows the existence of a positive process $-\mu_Z(t) = \frac{\partial A_t}{\partial t}$[6] that gives rise to the pricing kernel we started with.

[5] A sufficient condition for Assumption 2 to hold is that $\left(\frac{Y_1(t)}{Z_t}, \ldots, \frac{Y_m(t)}{Z_t}\right)$ is bounded in R^m for all $0 \le t \le T$.

[6] We can differentiate A_t because its paths are increasing functions and an increasing function is differentiable almost everywhere. The requirement that short rate process is continuous guarantees that $\mu_Z(t)$ is continuous and thus that A_t is the integral of its derivative.

Differentiating Equation (4) and using Equations (9) and (10), we get

$$\begin{aligned} dZ_t &= \mu_Z(t)dt + \sum_{j=1}^m Y_j(t)dW_j(t) \\ &= Z_t\left(-r(t)dt - \sum_{j=1}^m \phi_j(t)dW_j(t)\right), \end{aligned} \tag{14}$$

where the second equality follows from Equations (7) and (13). By the uniqueness of the stochastic integral representation of M_t, the volatilities Y_j, $1 \le j \le m$, of the pricing kernel are uniquely determined by $A_\infty = -\int_0^\infty \mu_Z(\tau)d\tau$,[7] and hence by $\{\mu_Z(t) : 0 \le t \le \infty\}$, the drift of Z_t. It is implicit in Rogers (1997) that Z_t is determined by A_t, at least if one takes his Equation (1.7) as a starting point. We have similarly shown

Theorem 2. *Consider an interest rate model in which the short rate process is continuous and the pricing kernel Z_t is an Itô process satisfying $\mathsf{E}(\sup_{0\le t\le\infty} Z^2(t)) < \infty$. Suppose that the prices of discount bonds go to zero as their maturity approaches infinity. Then the volatility of the pricing kernel is completely determined by its drift.*

This theorem indicates that, to model a pricing kernel, all we need to specify is its drift. In Section 5 we will make use of this observation to construct a family of positive interest rate models that are based on nonnegative Markov processes.

There is a common perception that making parameters time dependent could resolve the problem of fitting initial term structure. Constantinides' model (1992) shows that this does not always work because adding time dependence to parameters will allow one to fit the current term structure, but could result in losing the positivity of interest rates. Motivated by the general construction outlined above, we provide a remedy to this conflict.

Constantinides takes

$$Z_t = \exp\left(-\left(g + \frac{\sigma_0}{2}\right)t + \sigma_0 W_0(t) + \sum_{i=1}^N \left(x_i(t) - \alpha_i\right)^2\right),$$

where $x_i(t)$ $(1 \le i \le N)$ are Ornstein–Uhlenbeck processes defined by

$$dx_i(t) = -\lambda_i x_i(t)dt + \sigma_i dW_i(t),$$

$W_0(t), W_1(t), \ldots, W_N(t)$ are independent standard Brownian motions, and g, $\sigma_0 \ge 0$, $\sigma_i > 0$, α_i, and $\lambda_i > 0$ $(1 \le i \le N)$ are constants. It follows from

[7] More explicitly, applying the Clark formula [e.g., see Ocone (1984)] to Equation (9), we get $Y_j(t) = \mathsf{E}(D_t^j A_\infty \mid \mathcal{F}_t)$, where D is the Malliavin–Fréchet functional derivative.

Itô's rule that the drift of this pricing kernel is

$$\mu_Z(t) = Z_t\Big(-g + \sum_{i=1}^{N}\sigma_i^2 - 2\sum_{i=1}^{N}\lambda_i x_i(t)(x_i(t) - \alpha_i) + 2\sum_{i=1}^{N}\sigma_i^2(x_i(t) - \alpha_i)^2\Big).$$

For Z_t to be a potential, $\mu_Z < 0$ is required; this is ensured by the restrictions in equations (10) and (11) of Constantinides (1992), $\sigma_i^2 < \lambda_i$ for $1 \le i \le N$, and $g - \sum_{i=1}^{N}\big(\sigma_i^2 + (\lambda_i\alpha_i^2/2(1 - \frac{\sigma_i^2}{\lambda_i}))\big) > 0$. The market prices of risk associated with the 0th random factor and the ith random factor are $-\sigma_0$ and $-2\sigma_i(x_i(t) - \alpha_i)$, respectively. By Equation (4), the short rate is $r(t) = -\mu_Z(t)/Z_t$, which, after completing the square for $x_i(t)$, is Equation (19) in Constantinides (1992).[8]

Since $\mathsf{E}(\mu_Z(t)) = -f(0,t)\exp\big(-\int_0^t f(0,\tau)d\tau\big)Z_0$, it is clear that, using the Z_t above, one can *not* match every initial forward rate curve $f(0,t)$. As argued by Constantinides [see Constantinides (1992, p. 537)], adding a deterministic function or making parameters time dependent can result in loosing the positivity of interest rates. Motivated by Equation (8), we redefine the instantaneous expected changes in pricing kernel as

$$\bar{\mu}_Z(t) = -\frac{\mu_Z(t)}{\mathsf{E}(\mu_Z(t))} f(0,t)\exp\Big(-\int_0^t f(0,\tau)d\tau\Big).$$

and set $\bar{A}_t = -\int_0^t \bar{\mu}_Z(\tau)d\tau$. Then, the new pricing kernel is $\bar{Z}_t = \mathsf{E}(\bar{A}_\infty \mid \mathcal{F}_t) - \bar{A}_t$. It follows from the theory developed in this section that this model has positive interest rates and can fit any initial forward rate curve. Since the moments of $x_i(t)$ can be explicitly computed, analytical tractability remains in the revised model. We omit the details of the derivation.

To incorporate interest rate models in which negative rates are allowed, we drop the requirement that $-\mu_Z(t)$ be positive and, instead, we take an adapted (almost surely) continuous process $\mu_Z(t)$ and define $A_t = -\int_0^t \mu_Z(\tau)d\tau$. Assume that $M_t = \lim_{T\to\infty}\mathsf{E}(A_T \mid \mathcal{F}_t)$ exists and $M_t > A_t$ a.s. for all $0 \le t < \infty$. Then $Z_t = M_t - A_t$ is a strictly positive continuous semimartingale. As before, we define the time t price of a zero coupon bond that pays 1 unit of account at maturity T to be $p(t,T) = \frac{1}{Z_t}\mathsf{E}(Z_T \mid \mathcal{F}_t)$ for $0 \le t \le T \le \infty$. Then, using the same argument as given above, we can show that there exists an equivalent martingale measure in the constructed model if M_t is square integrable and Assumption 2 is satisfied.

[8] Notice that here we use a different approach to obtain short rates. Constantinides obtained $r(t)$ by first computing bond yield $y(t,T)$ and then taking the limit of $y(t,T)$ as $T \to t$.

2. Relation to the HJM Framework

In this section we explore the relation between modeling pricing kernels and modeling instantaneous forward rates as proposed by Heath, Jarrow, and Morton (1992) [see also Babbs (1997)]. We show, starting from the primitive data of one approach, how to obtain the primitive data of the other. On one hand, this relation enables us to explicitly construct an equilibrium supporting any HJM model (see Section 4); on the other hand, it allows us to take advantage of the HJM approach when we model pricing kernels (see Section 5). For an equivalence between HJM and short-rate models, see Baxter (1997).

To write down the dynamics of instantaneous forward rates [see Equation (6)] in the framework presented in Section 1, we apply the martingale representation theorem. Since $\mathsf{E}(A_T \mid \mathcal{F}_t)$ is a square integrable martingale, there exist unique adapted processes $Y_j(t,T)$ $(1 \le j \le m)$ such that $\mathsf{E}\big(\int_0^T Y_j^2(t,T)dt\big) < \infty$ for every $0 < T \le \infty$, and

$$\mathsf{E}(A_T \mid \mathcal{F}_t) = \mathsf{E}(A_T) + \sum_{j=1}^{m}\int_0^t Y_j(\tau,T)dW_j(\tau);\ 0 \le t \le \infty. \tag{15}$$

It follows from the uniqueness of the martingale representation that $\lim_{T\to\infty} Y_j(t,T) = Y_j(t)$ a.s. where $Y_j(t)$ is defined in Equation (9). Observe that $Y_j(t,T) = 0$ for $t \ge T \ge 0$ and $1 \le j \le m$.[9]

By Equations (4) and (15), we have

$$\mathsf{E}(Z_T \mid \mathcal{F}_t) = p(0,T) + \sum_{j=1}^{m}\int_0^t \big(Y_j(\tau) - Y_j(\tau,T)\big)dW_j(\tau). \tag{16}$$

and

$$Z_t = p(0,t) + \sum_{j=1}^{m}\int_0^t \big(Y_j(\tau) - Y_j(\tau,t)\big)dW_j(\tau). \tag{17}$$

Hence, modeling a pricing kernel is equivalent to specifying a family of adapted square integrable processes $\{Y_j(t,T),\ 1 \le j \le m,\ 0 \le t \le T :\ Y_j(t,t) = 0,\ \lim_{T\to\infty} Y_j(t,T) < \infty\}$ that determines a supermartingale through Equation (17).

[9] To see this, we apply the Clark formula [see, e.g., Ocone (1984)] to Equation (15) and get $Y_j(t,T) = \mathsf{E}(D_t^j A_T \mid \mathcal{F}_t)$, where D is the Malliavin–Fréchet functional derivative. For $t \ge T \ge 0$, since A_T is adapted to $\mathcal{F}_t$, $D_t^j A_T = 0$, that implies that $Y_j(t,T) = 0$.

Lemma 1. *In the framework developed in Section 1, the dynamics of the instantaneous forward rate are given by*

$$df(t,T) = \sum_{j=1}^{m} \phi_j(t,T) \frac{\partial \phi_j(t,T)}{\partial T} dt + \sum_{j=1}^{m} \frac{\partial \phi_j(t,T)}{\partial T} dW_j(t). \tag{18}$$

where

$$\begin{aligned} \phi_j(t,T) &= -\frac{Y_j(t) - Y_j(t,T)}{\mathsf{E}(Z_T \mid \mathcal{F}_t)} \\ &= -\frac{Y_j(t) - Y_j(t,T)}{p(0,T) + \sum_{j=1}^{m} \int_0^t \left(Y_j(\tau) - Y_j(\tau,T)\right) dW_j(\tau)} \end{aligned} \tag{19}$$

for $0 \le t \le T < \infty$ *and* $1 \le j \le m$.

Proof. By Equations (15) and (16), we have

$$d\left(\frac{\partial \mathsf{E}(A_T \mid \mathcal{F}_t)}{\partial T}\right) = \sum_{j=1}^{m} \frac{\partial Y_j(t,T)}{\partial T} dW_j(t). \tag{20}$$

and

$$d\,\mathsf{E}(Z_T \mid \mathcal{F}_t) = \sum_{j=1}^{m} \left(Y_j(t) - Y_j(t,T)\right) dW_j(t).$$

The conclusion follows from applying Itô's rule to $f(t,T) = \frac{1}{\mathsf{E}(Z_T \mid \mathcal{F}_t)} \frac{\partial \mathsf{E}(A_T \mid \mathcal{F}_t)}{\partial T}$. ■

If we set $\sigma_j(t,T) = \frac{\partial \phi_j(t,T)}{\partial T}$, then the drift of the instantaneous forward rate, denoted by $\alpha(t,T)$, can be rewritten as

$$\alpha(t,T) = \sum_{j=1}^{m} \left(\phi_j(t) + \int_t^T \sigma_j(t,\tau)\, d\tau\right) \sigma_j(t,T), \tag{21}$$

which is HJM forward rate drift condition for the existence of an equivalent martingale measure in an interest rate model. Through Theorem 1, we have provided another proof of this fact.

Now suppose we are given the dynamics of instantaneous forward rate as

$$df(t,T) = \alpha(t,T)\, dt + \sum_{j=1}^{m} \sigma_j(t,T)\, dW_j(t).$$

where $\alpha(t,T)$ satisfies Equation (21). To obtain the pricing kernel in the HJM framework, we define $\phi_j(t,T) = \phi_j(t) + \int_t^T \sigma_j(t,\tau)\, d\tau$. The additional conditions HJM found for the existence of an equivalent martingale

measure are

1. [condition (12.b) in Heath, Jarrow, and Morton (1992)] $\phi_j(t)$ satisfies Assumption 2 (in which $\frac{Y_j(t)}{Z_t}$ is replaced by $-\phi_j(t)$), and
2. [condition (12.c) in Heath, Jarrow, and Morton (1992)] the process

$$\exp\left(\sum_{j=1}^{m} \int_0^t -\phi_j(\tau,T) dW_j(\tau) - \frac{1}{2} \sum_{j=1}^{m} \int_0^t \phi_j^2(\tau,T) d\tau\right)$$

is a martingale.

In Appendix B we show that this exponential martingale is in fact the likelihood ratio process $\frac{d\overline{P}^T}{dP}\big|_{\mathcal{F}_t}$ relating the *forward* measure[10] associated with T to the physical measure P, and, in the framework presented in Section 1, this likelihood ratio process can be written as

$$\left.\frac{d\overline{P}^T}{dP}\right|_{\mathcal{F}_t} = \frac{E(Z_T \mid \mathcal{F}_t)}{p(0,T)} \quad \text{for } 0 \le t \le T.$$

Equating the two expressions for the likelihood ratio process, we find that

$$\begin{aligned} \mathsf{E}(Z_T \mid \mathcal{F}_t) = p(0,T) \exp\Bigg(&\sum_{j=1}^{m} \int_0^t -\phi_j(\tau,T) dW_j(\tau) \\ &- \frac{1}{2} \sum_{j=1}^{m} \int_0^t \phi_j^2(\tau,T) d\tau\Bigg). \end{aligned} \tag{22}$$

It follows that the pricing kernel is

$$Z_t = p(0,t) \exp\left(\sum_{j=1}^{m} \int_0^t -\phi_j(\tau,t) dW_j(\tau) - \frac{1}{2} \sum_{j=1}^{m} \int_0^t \phi_j^2(\tau,t) d\tau\right), \tag{23}$$

and the zero coupon bond price, under P, is given by[11]

$$\begin{aligned} p(t,T) &= \frac{1}{Z_t} \mathsf{E}(Z_T \mid \mathcal{F}_t) \\ &= \frac{p(0,T)}{p(0,t)} \exp\left(\sum_{j=1}^{m} \int_0^t -\left(\phi_j(\tau,T) - \phi_j(\tau,t)\right) dW_j(\tau)\right) \\ &\quad \times \exp\left(-\frac{1}{2} \sum_{j=1}^{m} \int_0^t \left(\phi_j^2(\tau,T) - \phi_j^2(\tau,t)\right) d\tau\right). \end{aligned} \tag{24}$$

[10] A forward measure is one associated with taking a bond maturing at T as numeraire. See El Karoui, Geman, and Rochet (1995) and Jamshidian (1996).

[11] This bond price formula was also obtained through different methods in Carverhill (1995) and Flesaker and Hughston (1996b).

Moreover, by Equation (7), we obtain the drift of the pricing kernel expressed in terms of $\phi_j(t,T)$ $(1 \le j \le m)$:

$$\mu_Z(t) = -p(0,t)\exp\left(\sum_{j=1}^{m}\int_0^t -\phi_j(\tau,t)dW_j(\tau) - \frac{1}{2}\sum_{j=1}^{m}\int_0^t \phi_j^2(\tau,t)d\tau\right)$$
$$\times\left(f(0,t) + \sum_{j=1}^{m}\int_0^t \frac{\partial\phi_j(\tau,t)}{\partial t}dW_j(\tau) + \sum_{j=1}^{m}\int_0^t \phi_j(\tau,t)\frac{\partial\phi_j(\tau,t)}{\partial t}d\tau\right). \quad (25)$$

Summarizing the results in this section, we obtain

Theorem 3. *The following provides the relation between the pricing kernel approach and the Heath-Jarrow-Morton framework:*[12]

1. *Given a pricing kernel Z_t, there exists a unique family of adapted square integrable processes $\{Y_j(t,T),\ 1 \le j \le m,\ 0 \le t \le T\}$, with $Y_j(t,t) = 0$ and $\lim_{T\to\infty} Y_j(t,T) = Y_j(t)$, so that $Z_t = p(0,t) + \sum_{j=1}^m \int_0^t \left(Y_j(\tau) - Y_j(\tau,t)\right)dW_j(\tau)$. For $1 \le j \le m$, define $\phi_j(t,T)$ as in Equation (19). Then the volatility $\sigma_j(t,T)$ of instantaneous forward rate is $\frac{\partial\phi_j(t,T)}{\partial T}$ and the market price of risk associated with the jth random factor is $\phi_j(t) = \phi_j(t,t)$;*
2. *Given the volatility $\sigma_j(t,T)$ of instantaneous forward rate and the market price of risk $\phi_j(t)$, define $\phi_j(t,T) = \phi_j(t) + \int_t^T \sigma_j(t,\tau)d\tau$. Then the pricing kernel Z_t in the HJM framework is given by Equation (23). We also have $Y_j(t) = -\phi_j(t)Z_t$ and $Y_j(t,T) = Y_j(t) + \phi_j(t,T)\mathsf{E}(Z_T \mid \mathcal{F}_t)$, where $\mathsf{E}(Z_T \mid \mathcal{F}_t)$ is given by Equation (22). Moreover, if Z_t is a strict supermartingale, then the interest rates are positive.*

3. Positive Interest Rates in HJM

Flesaker and Hughston (1996a) present a positive interest rate framework in which zero coupon bond prices are modeled as

$$p(t,T) = \frac{\int_T^\infty h(\tau)M(t,\tau)d\tau}{\int_t^\infty h(\tau)M(t,\tau)d\tau} \quad \text{for } 0 \le t \le T < \infty, \quad (26)$$

where $M(t,T)$ is a family of strictly positive continuous martingales indexed by T and h is a deterministic positive function. Clearly, these bond prices decrease with maturity and are thus consistent with positive interest rates. If one models the pricing kernel $Z(t)$ as

$$Z_t = \int_t^\infty h(\tau)M(t,\tau)d\tau, \quad (27)$$

then it is easy to verify that bond prices are of the form of Equation (26) and hence that interest rates remain positive. Musiela and Rutkowski (1997), Rogers (1997, p. 159), and Rutkowski (1997) also make this observation. The converse, however, has not previously been established: can the pricing kernel associated with *any* positive interest rate model be put in the form of Equation (27)? We will show that this is indeed the case and use this to derive a criterion under which interest rates in an HJM model are guaranteed to stay positive.

As before, we take $-\mu_Z(t)$ as an adapted positive (almost surely) continuous process and set $A_t = -\int_0^t \mu_Z(\tau)d\tau$. Suppose Assumptions 1 and 2 hold and define $Z_t = \mathsf{E}(A_\infty \mid \mathcal{F}_t) - A_t$ for every $0 \le t \le \infty$ and normalize A_t so that $Z_0 = \mathsf{E}(A_\infty) = 1$. The key fact is that Z_t can be represented as follows:[13]

Lemma 1. *For any potential (pricing kernel) Z_t, there exists a positive deterministic function $h(T)$ and a family of strictly positive continuous martingales $M(t,T)$ indexed by T, for $0 \le t \le T < \infty$, with $M(0,T) = 1$, such that*

$$Z_t = \int_t^\infty h(\tau)M(t,\tau)d\tau \quad \text{for every } 0 \le t < \infty. \quad (28)$$

Therefore, the Flesaker–Hughston framework consists of all arbitrage-free positive interest rate models.

Proof. Define $N(t,T) = \mathsf{E}(Z_T \mid \mathcal{F}_t)$. We have

$$\frac{\partial N(t,T)}{\partial T} = -\frac{\partial\mathsf{E}(A_T \mid \mathcal{F}_t)}{\partial T} = -\mathsf{E}\left(\frac{\partial A_T}{\partial T}\Big|\mathcal{F}_t\right).$$

Set

$$M(t,T) = \frac{1}{\mathsf{E}\left(\frac{\partial A_T}{\partial T}\right)}\mathsf{E}\left(\frac{\partial A_T}{\partial T}\Big|\mathcal{F}_t\right), \quad (29)$$

then $M(t,T)$ is a strictly positive martingale in t with $M(0,T) = 1$. Writing

$$h(T) = \mathsf{E}\left(\frac{\partial A_T}{\partial T}\right), \quad (30)$$

[12] The requirement that $\lim_{T\to\infty} p(t,T) = 0$ is a standing assumption in all models treated in this paper.

[13] We thank Lane Hughston for discussions that helped motivate our formulation of this result.

we get

$$N(t,T) = \int_T^\infty h(\tau)M(t,\tau)d\tau.$$

Equation (28) follows from $N(t,t) = Z_t$.

Since no-arbitrage is equivalent to the existence of a pricing kernel, by Equations (5) and (28), we have shown that, in an arbitrage-free economy, the time t price of a zero coupon bond that pays 1 unit of account at maturity T can be always expressed as Equation (26). ■

Note that, by Equations (8) and (30), one has

$$h(T) = f(0,T)\exp\left(-\int_0^T f(0,\tau)d\tau\right). \tag{31}$$

which can be used to match any given initial forward rate curve $f(0,T)$.

Now suppose we start with the Flesaker–Huston formulation, that is, a strictly positive function $h(T)$ given by Equation (31) and a family of positive martingales expressed as

$$M(t,T) = \exp\left(\sum_{j=1}^m \int_0^t \eta_j(\tau,T)dW_j(\tau) - \frac{1}{2}\sum_{j=1}^m \int_0^t \eta_j^2(\tau,T)d\tau\right). \tag{32}$$

Then Z_t, defined as in Equation (28), is a strictly positive continuous potential. Since

$$dZ_t = \sum_{j=1}^m \left(\int_t^\infty h(\tau)\eta_j(t,\tau)M(t,\tau)d\tau\right)dW_j(t) - h(t)M(t,t)dt,$$

we get the positive process introduced in Section 1:

$$-\mu_Z(t) = \frac{\partial A_t}{\partial t} = h(t)M(t,t).$$

The following theorem presents a deeper relation between the pricing kernel approach and the Flesaker–Huston formulation:

Theorem 4. *Using the notation of Section 2, we have*

1. *Given a pricing kernel* $Z_t = p(0,t) + \sum_{j=1}^m \int_0^t \left(Y_j(\tau) - Y_j(\tau,t)\right)dW_j(\tau)$, *define*

$$\eta_j(t,T) = \frac{\frac{\partial Y_j(t,T)}{\partial T}}{h(T) + \sum_{j=1}^m \int_0^t \frac{\partial Y_j(\tau,T)}{\partial T}dW_j(\tau)} \quad 0 \le t \le T \le \infty. \tag{33}$$

for each $1 \le j \le m$. *Then* $M(t,T)$ *defined by* $\eta_j(t,T)$ $(1 \le j \le m)$ *through Equation* (32) *is the family of positive martingales used in the Flesaker–Hughston framework;*

2. *Given a family of positive martingales* $M(t,T)$ *expressed as in Equation* (32), *define*

$$Y_j(t,T) = \int_t^T h(\tau)\eta_j(t,\tau)M(t,\tau)d\tau \quad 0 \le t \le T \le \infty, \tag{34}$$

for each $1 \le j \le m$. *Then* $\{Y_j(t,T)\}$ *is the family of square integrable processes that yields the pricing kernel* Z_t.

Proof. See Appendix C. ■

To see the dynamics of instantaneous forward rates in the Flesaker–Hughston framework, we apply Equations (19) and (34), and get

$$\begin{aligned}\phi_j(t,T) &= -\frac{\int_t^\infty h(\tau)\eta_j(t,\tau)M(t,\tau)d\tau - \int_t^T h(\tau)\eta_j(t,\tau)M(t,\tau)d\tau}{\mathrm{E}\left(\int_T^\infty h(\tau)M(T,\tau)d\tau \mid \mathcal{F}_t\right)} \\ &= -\frac{\int_T^\infty h(\tau)\eta_j(t,\tau)M(t,\tau)d\tau}{\int_T^\infty h(\tau)M(t,\tau)d\tau},\end{aligned}$$

for $0 \le t \le T$ and $(1 \le j \le m)$. At $t = T$, this reduces to the representation of the market price of risk in Flesaker and Hughston (1997). Together with Equation (18), it puts the Flesaker–Hughston models into the HJM framework. Flesaker and Hughston (1996b) make a similar connection via bond prices. It is illuminating to find the converse of this relation because it provides a way of verifying positivity of interest rates directly from the HJM primitive data and because it confirms that every HJM model with positive interest rates admits the Flesaker–Hughston formulation.

Theorem 5. *In the HJM framework, suppose the instantaneous forward rates evolve according to*

$$df(t,T) = \sum_{j=1}^m \left(\phi_j(t) + \int_t^T \sigma_j(t,\tau)d\tau\right)\sigma_j(t,T)dt + \sum_{j=1}^m \sigma_j(t,T)dW_j(t).$$

Define

$$\eta_j(t,T) = -\phi_j(t) - \int_t^T \sigma_j(t,\tau)d\tau + \frac{\sigma_j(t,T)}{f(t,T)}. \tag{35}$$

Then, the zero coupon bond price $p(t,T)$ *is strictly decreasing in* T *if and only if* $\eta_j(t,T)$ $(1 \le j \le m)$ *define a family of exponential martingales* $\{M(t,T)\}$ *through Equation* (32).[14] *Moreover, they are the martingales used in the Flesaker–Hughston framework.*

Proof. See Appendix C. ■

[14] A sufficient condition for $M(t,T)$ to be a martingale in t is the Novikov condition [see, e.g., Karatzas and Shreve (1991, p. 199)] or simply that $(\eta_1(t,T),\ldots,\eta_m(t,T))$ is bounded in $\mathbb{R}^m$ for all $0 \le t \le T$.

4. A Supporting Equilibrium

In this section, through explicit construction we demonstrate that every HJM model (including the Flesaker–Hughston positive interest rate models) arises as the equilibrium term structure in a Cox–Ingersoll–Ross production economy [Cox, Ingersoll, and Ross (1985a,b), and Longstaff and Schwartz (1991)]. Although this type of equivalence between equilibrium and arbitrage-free formulations is widely understood in general terms and in a variety of specific settings [see, e.g., Ross (1977), Dybvig and Ross (1989), and Duffie (1996)], we know of no previous result specifically supporting term structure models in the generality of the HJM framework.

Suppose we are given the primitive data of an HJM model, $\phi_j(t, T)$ ($1 \le j \le m$), satisfying the technical conditions that guarantee the existence of an equivalent martingale measure. We exogenously specify $\mu_Z(t)$ through Equation (25). Assume that all physical investment is performed through a single stochastic constant-returns-to-scale technology that produces a good that is either consumed or reinvested in production. The realized returns on physical investment are governed by the following stochastic differential equation:

$$\frac{dQ_t}{Q_t} = \left(-\frac{\mu_Z(t)}{Z_t} + \sum_{j=1}^{m} \left(\frac{Y_j(t)}{Z_t} \right)^2 \right) dt - \sum_{j=1}^{m} \frac{Y_j(t)}{Z_t} dW_j(t). \tag{36}$$

where Z_t and $Y_j(t)$ are determined by Equations (4) and (9),[15] and $W_j(t)$ ($1 \le j \le m$) are m independent standard Brownian motions. In other words, we are taking $\mu_Z(t)$ as a "shock" process that affects the productivity of investment.

Further assume that a representative consumer seeks to maximize

$$\mathsf{E}\left(\int_0^\infty e^{-\rho\tau} \ln(c_\tau)\, d\tau \right),$$

subject to the budget constraint

$$dX_t = X_t \frac{dQ_t}{Q_t} - c_t dt, \quad \text{and} \quad X_t \ge 0.$$

[15] More precisely, we have

$$Z_t = \int_0^t \mu_Z(\tau) d\tau - \mathsf{E}\left(\int_0^\infty \mu_Z(\tau) d\tau \,\middle|\, \mathcal{F}_t \right)$$

and

$$Y_j(t) = \mathsf{E}\left(D_t^j \left(-\int_0^\infty \mu_Z(\tau) d\tau \right) \middle| \mathcal{F}_t \right).$$

where D is the Malliavin–Fréchet functional derivative, and $\mathcal{F}_t$ is the information set generated by independent random factors $W_j(t)$ ($1 \le j \le m$).

where c_t represents time t consumption rate, $\rho > 0$ is the utility discount factor, and X_t denotes wealth at time t with $X_0 = \frac{1}{\rho}$.

Theorem 6. *The optimal consumption rate is $c_t^* = e^{-\rho t} \frac{1}{Z_t}$, which is uniquely determined by the "shock" process μ_Z, and the optimal wealth process is $X_t^* = \frac{1}{\rho} c_t^*$. Therefore, Z_t is the marginal utility of optimal consumption.*

Proof. See Appendix D. ■

Theorem 6 is an extension of Cox, Ingersoll, and Ross' results in that Cox, Ingersoll, and Ross (1985a,b) constructed a Markovian economy, which allows them to utilize the Hamilton–Jacobi–Bellman equation for solving the representative consumer's maximization problem. In our case, the "shock" process $\mu_Z(t)$ itself may not be Markovian. Therefore, we have to rely upon convex duality and martingale techniques [see, e.g., Cvitanić and Karatzas (1992)] to find the optimal consumption rate. Moreover, by Equation (8), we see that the current term structure of interest rates imposes a condition on the shock process through its unconditional mean.

5. A General Class of Positive Interest Rate Models

We will use nonnegative Markov processes to model instantaneous expected changes in pricing kernels.[16] This yields a family of positive interest rate models that can fit any given initial forward rate curve and have easy-to-evaluate expressions for bond prices and forward rates. More specifically, we are going to take the Markov process as a reflected Brownian motion and show how to use the link between the pricing kernel approach and the HJM framework to match elements of the bond return covariance matrix.

Let X_t be a nonnegative Markov process, driven by m independent standard Brownian motions $W_1(t), \ldots, W_m(t)$, with $\mathsf{E}(X_t) = l(t) > 0$ for all $t \ge 0$. We set

$$-\mu_Z(t) = j(t) X_t + g(t),$$

where $g(t)$ is a deterministic function that satisfies $0 < g(t) < -\frac{\partial p(0,t)}{\partial t}$, $p(0, t)$ ($t \ge 0$) are the initial bond prices and $j(t) = \left(-\frac{\partial p(0,t)}{\partial t} - g(t)\right) \frac{1}{l(t)}$. Note that

$$\mathsf{E}(\mu_Z(t)) = \frac{\partial p(0, t)}{\partial t} = -f(0, t) \exp\left(-\int_0^t f(0, \tau) d\tau\right),$$

[16] Rogers (1997) also gives examples of pricing kernels constructed from Markov processes, but the approach and resulting models are different. Hunt, Kennedy, and Pelsser (1997) directly model the value of zero coupon bonds or other assets as functions of a Markov process. Hagan and Woodward (1997) model an abstract numeraire as a function of a Markov process.

where $\{f(0,t)\}_{t\geq 0}$ denotes the initial forward rate curve. We define

$$A_t = \int_0^t -\mu_Z(\tau)d\tau = \int_0^t \big(j(\tau)X_\tau + g(\tau)\big)\,d\tau.$$

Assume that there exist two positive constants c_1 and c_2 so that $c_1 < f(0,t) < c_2$ for all $t \geq 0$. It is immediate that, if $X_t/I(t)$ and $\mathrm{E}(X_t^2)$ are at most of polynomial growth as t goes to infinity, then $A_\infty = \lim_{t\to\infty} A_t$ exists, and $\mathrm{E}(A_\infty^2) < \infty$.

Let $\{\mathcal{F}_t\}$ be the filtration representing the information available up to time t. As before, we set $Z_t = \mathrm{E}(A_\infty \mid \mathcal{F}_t) - A_t$, and define the time t price of a zero coupon bond that pays 1 unit of account at maturity T to be $p(t,T) = \frac{1}{Z_t}\mathrm{E}(Z_T \mid \mathcal{F}_t)$. For $x \in \mathrm{R}$ and $T \geq t \geq 0$, we write

$$L(t,T;x) = \mathrm{E}\left(X_T \mid X_t = x\right).$$

Then

$$\begin{aligned}\mathrm{E}\left(Z_T \mid \mathcal{F}_t\right) &= \mathrm{E}\left(\int_T^\infty \big(j(\tau)X_\tau + g(\tau)\big)\,d\tau \,\Big|\, X_t\right)\\ &= L(t,T;X_t)J(T) + I(t,T;X_t) + G(T),\end{aligned}$$

where $J(T) = \int_T^\infty j(\tau)d\tau$, $I(t,T;X_t) = \int_T^\infty J(\tau)\frac{\partial L(t,\tau;X_t)}{\partial \tau}d\tau$ and $G(T) = \int_T^\infty g(\tau)d\tau$.

Since $L(t,t;X_t) = X_t$, we have

$$Z_t = X_t J(t) + I(t,t;X_t) + G(t).$$

Proposition 1. *The zero coupon bond price can be expressed as*

$$p(t,T) = \frac{L(t,T;X_t)J(T) + I(t,T;X_t) + G(T)}{X_t J(t) + I(t,t;X_t) + G(t)}.$$

Remark. $p(t,T)$ only depends on the current state of X_t, independent of the past, which makes the evaluation easy to carry out.

To obtain an expression for the instantaneous forward rate, we need

$$\begin{aligned}\mathrm{E}\left(\frac{\partial A_T}{\partial T}\Big|\mathcal{F}_t\right) &= \mathrm{E}\left(j(T)X_T + g(T) \mid X_t\right) = j(T)\mathrm{E}\left(X_T \mid X_t\right) + g(T)\\ &= j(T)L(t,T;X_t) + g(T).\end{aligned}$$

By Equations (6) and (7), we get

Proposition 2. *The instantaneous forward rate is*

$$f(t,T) = \frac{L(t,T;X_t)j(T) + g(T)}{L(t,T;X_t)J(T) + I(t,T;X_t) + G(T)},$$

and the short rate is

$$r(t) = \frac{X_t j(t) + g(t)}{X_t J(t) + I(t,t;X_t) + G(t)}.$$

We notice that, as far as computation is concerned, the key quantity in the above model is $L(t,T;X_t)$. To get more explicit formulas, from now on, we take

$$X_t = |W(t)|,$$

where $W(t)$ is a standard Brownian motion. First, the law of the iterated logarithm implies that $\frac{|W(t)|}{\sqrt{t}}$ is at most of the order $\sqrt{2\log\log t}$ as t goes to infinity. Hence A_∞ exists. Since $\mathrm{E}(|W(t)|^2) = t^2$, $\mathrm{E}(A_\infty^2) < \infty$. Moreover, for $x \in \mathrm{R}$ and $t \geq 0$, we have

$$\begin{aligned}L(t,T;x) &= \mathrm{E}\left(|W(T)| \,\big|\, W(t) = x\right)\\ &= \sqrt{\frac{2(T-t)}{\pi}}e^{-\frac{x^2}{2(T-t)}} + x\left(2\Phi\left(\frac{x}{\sqrt{T-t}}\right) - 1\right),\end{aligned}$$

where $\Phi(\cdot)$ denotes the cumulative standard normal distribution. It follows that

$$\frac{\partial L(t,T;W(t))}{\partial T} = \frac{1}{\sqrt{2\pi(T-t)}}e^{-\frac{W^2(t)}{2(T-t)}}.$$

Applying Itô's rule, we obtain

$$d\,\mathrm{E}\left(\frac{\partial A_T}{\partial T}\Big|\, W(t)\right) = j(T)\left(2\Phi\left(\frac{W(t)}{\sqrt{T-t}}\right) - 1\right)dW(t).$$

By Equation (20),

$$\frac{\partial Y(t,T)}{\partial T} = j(T)\left(2\Phi\left(\frac{W(t)}{\sqrt{T-t}}\right) - 1\right).$$

That is,

$$\begin{aligned}Y(t,T) &= \int_t^T j(\tau)\left(2\Phi\left(\frac{W(t)}{\sqrt{\tau-t}}\right) - 1\right)d\tau\\ &= \mathrm{sgn}(W(t))J(t) - J(T)\left(2\Phi\left(\frac{W(t)}{\sqrt{T-t}}\right) - 1\right)\\ &\quad - \frac{W(t)}{\sqrt{2\pi}}\int_t^T J(\tau)e^{-\frac{W^2(t)}{2(\tau-t)}}(\tau-t)^{-\frac{3}{2}}\,d\tau.\end{aligned}$$

Recall that $\{Y(t,T): 0 \le t \le T\}$ gives rise to the martingale representation of $\mathsf{E}(Z_T \mid \mathcal{F}_t)$ through Equation (16). Hence, we have

$$Y(t) = \lim_{T\to\infty} Y(t,T) = \operatorname{sgn}(W(t))J(t) - \frac{W(t)}{\sqrt{2\pi}} \int_t^{\infty} J(\tau) e^{-\frac{W^2(t)}{2(\tau-t)}} (\tau - t)^{-\frac{3}{2}} d\tau.$$

To see the representation of this model in the Heath–Jarrow–Morton framework, we note that

$$\begin{aligned}\phi(t,T) &= -\frac{Y(t) - Y(t,T)}{\mathsf{E}(Z_T \mid \mathcal{F}_t)}\\ &= \frac{-J(T)\left(2\Phi\left(\frac{W(t)}{\sqrt{T-t}}\right) - 1\right) + \frac{W(t)}{\sqrt{2\pi}} \int_T^{\infty} J(\tau) e^{-\frac{W^2(t)}{2(\tau-t)}} (\tau-t)^{-\frac{3}{2}} d\tau}{\left(W(t)\left(2\Phi\left(\frac{W(t)}{\sqrt{T-t}}\right) - 1\right) + \sqrt{\frac{2(T-t)}{\pi}} e^{-\frac{W^2(t)}{2(T-t)}}\right) J(T) + I(t,T; W(t)) + G(T)}.\end{aligned}$$

Since $\lim_{T\to t} 2\Phi\left(\frac{W(t)}{\sqrt{T-t}}\right) - 1 = \operatorname{sgn}(W(t))$, the market price of risk is

$$\begin{aligned}\phi(t) &= \lim_{T\to t} \phi(t,T)\\ &= \frac{-\operatorname{sgn}(W(t))\, J(t) + \frac{W(t)}{\sqrt{2\pi}} \int_t^{\infty} J(\tau) e^{-\frac{W^2(t)}{2(\tau-t)}} (\tau-t)^{-\frac{3}{2}} d\tau}{|W(t)| J(t) + I(t,t; W(t)) + G(t)}.\end{aligned}$$

It follows from Equations (18) and (24) that the volatilities of instantaneous forward rates are

$$\frac{\partial \phi(t,T)}{\partial T}, \quad t \le T \tag{37}$$

and the volatilities of zero coupon bond prices are

$$\phi(t) - \phi(t,T), \quad t \le T. \tag{38}$$

Therefore, one could use either Equation (37) to match estimated market volatilities of forward rates or Equation (38) to match elements of the bond return covariance matrix.

The formulas above involve only deterministic integrals and are therefore amenable to numerical evaluation. Also, this model could be easily generalized to a multifactor model by taking

$$-\mu_Z(t) = j_1(t)\left|W_1(t)\right| + \cdots + j_m(t)\left|W_m(t)\right| + g(t),$$

where $W_1(t), \ldots, W_m(t)$ are independent standard Brownian motions and $j_i(t)$ $(1 \le i \le m)$ and $g(t)$ are positive deterministic functions.

6. Conclusion

In this article we develop relationships between interest rate models formulated through pricing kernels and through the Heath, Jarrow, and Morton (1992) framework. We demonstrate that HJM models can be supported by Cox–Ingersoll–Ross production economies and show how to verify the positivity of interest rates directly from HJM primitive data. By modeling the dynamics of the drift of the pricing kernel, we generate a new class of positive interest rate models that can fit any type of initial yield curve, match elements of the bond return covariance matrix, and yet have reasonably tractable expressions for bond prices and forward rates.

Appendix A: Proof of Theorem 1

Proof. The key fact that needs to be shown is that $Z_t B_t$ is an exponential martingale. For any $0 < T < \infty$, we set $\epsilon(T) = \min_{0\le t\le T} Z_t$. Then $\epsilon(T) > 0$ a.s. since Z_t is strictly positive and almost surely continuous. Since $Y_j(t)$ $(1 \le j \le m)$ are square integrable, we have

$$\int_0^T \left(\frac{Y_j(t)}{Z_t}\right)^2 dt \le \frac{1}{\epsilon(T)^2} \int_0^T Y_j^2(t) dt < \infty \quad \text{a.s.} \quad \text{for } 1 \le j \le m.$$

Therefore, the expression in the expectation in Equation (12) is well defined.

Applying Itô's rule, we get

$$\begin{aligned} d(Z_t B_t) &= B_t dZ_t + Z_t B_t r(t) dt\\ &= B_t \left(dM_t - dA_t + Z_t r(t) dt\right)\\ &= B_t \left(dM_t - dA_t + dA_t\right) \quad \text{[by Equation (7)]}\\ &= Z_t B_t \sum_{j=1}^{m} \frac{Y_j(t)}{Z(t)} dW_j(t) \quad \text{[by Equation (9)]}.\end{aligned}$$

It follows that $Z_t B_t$ is an exponential local martingale. By Assumption 2, it is an exponential martingale.[17]

Now, we can define an equivalent probability measure $\tilde{P}$ by the following Girsanov transformation

$$\left.\frac{d\tilde{P}}{dP}\right|_{\mathcal{F}_T} = Z_T B_T \quad \text{for } 0 \le T < \infty.$$

It follows from Bayes' rule that

$$p(t,T) = \frac{1}{Z_t} \mathsf{E}(Z_T \mid \mathcal{F}_t) = \frac{1}{Z_t} Z_t B_t \tilde{\mathsf{E}}\left(Z_T \frac{1}{Z_T B_T} \middle| \mathcal{F}_t\right) = B_t \tilde{\mathsf{E}}\left(\frac{1}{B_T} \middle| \mathcal{F}_t\right).$$

■

[17] This follows from the facts that a local martingale bounded from below is a supermartingale and a supermartingale of constant expectation is a martingale.

Appendix B: Forward Measure

Recall that one way to define the *forward measure* associated with time $T > 0$, denoted by $\overline{P}^T$, is that it is the probability measure on $(\Omega, \mathcal{F}, P)$, equivalent to P, under which the instantaneous forward rate $f(t, T)$ is a martingale for $0 \le t \le T$ [see El Karoui, Geman, and Rochet (1995) and Jamshidian (1996)].

Proposition 3. *In the framework presented in Section 1, the forward measure associated with $T > 0$ is given by the following likelihood ratio process*

$$\left.\frac{d\overline{P}^T}{dP}\right|_{\mathcal{F}_t} = \frac{\mathsf{E}(Z_T \mid \mathcal{F}_t)}{p(0,T)} \quad \textit{for } 0 \le t \le T. \tag{39}$$

Proof. Since $\mathsf{E}(Z_T \mid \mathcal{F}_t)$ is a strictly positive martingale and $\mathsf{E}(Z_T) = p(0,T)$, Equation (39) is well defined. By Equation (6),

$$\begin{aligned} f(t,T) &= \frac{1}{\mathsf{E}(Z_T \mid \mathcal{F}_t)} \mathsf{E}\left(\frac{\partial A_T}{\partial T} \Big| \mathcal{F}_t\right) \\ &= \frac{1}{\mathsf{E}(Z_T \mid \mathcal{F}_t)} \frac{\mathsf{E}(Z_T \mid \mathcal{F}_t)}{\mathsf{E}(Z_T)} \overline{\mathsf{E}}^T \left(\frac{\partial A_T}{\partial T} \frac{\mathsf{E}(Z_T)}{\mathsf{E}(Z_T \mid \mathcal{F}_t)} \Big| \mathcal{F}_t\right) \\ &= \overline{\mathsf{E}}^T\left(\frac{1}{Z_T}\frac{\partial A_T}{\partial T} \Big| \mathcal{F}_t\right) = \overline{\mathsf{E}}^T(r(T) \mid \mathcal{F}_t). \end{aligned}$$

■

Proposition 4. *In the Heath–Jarrow–Morton framework, the forward measure associated with $T > 0$ is given by*

$$\left.\frac{d\overline{P}^T}{dP}\right|_{\mathcal{F}_t} = \exp\left(\sum_{j=1}^m \int_0^t -\phi_j(\tau, T) dW_j(\tau) - \frac{1}{2}\sum_{j=1}^m \int_0^t \phi_j^2(\tau, T) d\tau\right) \quad \textit{for } 0 \le t \le T.$$

Proof. Using the notation of Section 2, we can write the dynamics of instantaneous forward rates under the physical measure P as

$$df(t,T) = \sum_{j=1}^m \phi_j(t,T)\frac{\partial \phi_j(t,T)}{\partial T} dt + \sum_{j=1}^m \frac{\partial \phi_j(t,T)}{\partial T} dW_j(t).$$

and under the forward measure $\overline{P}^T$ as

$$df(t,T) = \sum_{j=1}^m \frac{\partial \phi_j(t,T)}{\partial T} d\overline{W}_j^T(t).$$

Therefore,

$$\overline{W}_j^T(t) = W_j(t) + \int_0^t \phi_j(\tau, T) d\tau.$$

Since condition (12.c) in Heath, Jarrow, and Morton (1992) guarantees that $\phi_j(t,T)$ $(1 \le j \le m)$ define an exponential martingale, the conclusion follows from the Girsanov theorem. ■

Appendix C: Proof of Results in Section 3

Proof of Theorem 4. By Equation (29), we have

$$\begin{aligned} dM(t,T) &= \frac{1}{\mathsf{E}\left(\frac{\partial A_T}{\partial T}\right)} d\mathsf{E}\left(\frac{\partial A_T}{\partial T} \Big| \mathcal{F}_t\right) \\ &= \frac{1}{\mathsf{E}\left(\frac{\partial A_T}{\partial T}\right)} \sum_{j=1}^m \frac{\partial Y_j(t,T)}{\partial T} dW_j(t) \quad \text{(by (20))} \\ &= \sum_{j=1}^m \frac{\frac{\partial Y_j(t,T)}{\partial T}}{\frac{\partial \mathsf{E}(A_T \mid \mathcal{F}_t)}{\partial T}} M(t,T)\, dW_j(t). \end{aligned}$$

The first part of theorem follows from the fact that $dM(t,T)$ can also be written as

$$dM(t,T) = \sum_{j=1}^m \eta_j(t,T) M(t,T)\, dW_j(t) \quad \text{[by Equation (32)]}.$$

Conversely, suppose we are given the family of martingales $M(t,T)$. Let $\{Y_j(t,T)\}$ be the family of square integrable processes that yields the pricing kernel. Then we have

$$\begin{aligned} Y_j(t,T) &= \int_t^T \frac{\partial Y_j(t,\tau)}{\partial \tau} d\tau \\ &= \int_t^T \eta_j(t,T) \frac{\partial \mathsf{E}(A_\tau \mid \mathcal{F}_t)}{\partial \tau} d\tau \quad \text{[by Equation (33)]} \\ &= \int_t^T h(\tau)\eta_j(t,\tau) M(t,\tau) d\tau \quad \text{[by Equations (29) and (30)]}. \end{aligned}$$

■

Proof of Theorem 5. The fact that the zero coupon bond price is strictly decreasing in T is equivalent to $f(t,T) > 0$. By Lemma 1, there exits a family of positive martingales $M(t,T)$ $(0 \le t \le T)$ so that the zero coupon bond price can be expressed as Equation (26). Theorem 4 tells us $\eta_j(t,T)$ $(1 \le j \le m)$ defined in Equation (33) give rise to $M(t,T)$ through Equation (32).

On the other hand, in Lemma 1, we show that $\phi_j(t,T)$ $(1 \le j \le m)$ defined in Equation (19) are the primitive data for the HJM framework. Differentiating Equation (19) with respect to T, we get

$$\eta_j(t,T) = -\phi_j(t,T) + \frac{1}{f(t,T)}\frac{\partial \phi_j(t,T)}{\partial T}.$$

The conclusion follows. ■

Appendix D

Proof of Theorem 6. We use the martingale method and the convex duality argument [see Cvitanić and Karatzas (1992)] to show $c_t^* = e^{-\rho t}\frac{1}{Z_t}$ is the optimal consumption rate. We first define the following convex decreasing function:

$$\widetilde{U}(y) = \max_{0<x<\infty} \{\ln(x) - xy\} = \ln\left(\frac{1}{y}\right) - 1, \qquad 0 < y < \infty. \tag{40}$$

Using Equation (36), the budget constraint reads

$$dX_t = X_t\left(-\frac{\mu_Z(t)}{Z_t} + \sum_{j=1}^m \left(-\frac{Y_j(t)}{Z_t}\right)^2\right)dt - \sum_{j=1}^m X_t \frac{Y_j(t)}{Z_t} dW_j(t) - c_t dt$$
$$= X_t\left(r(t) + \sum_{j=1}^m \phi_j(t)^2\right)dt + \sum_{j=1}^m X_t\phi_j(t)dW_j(t) - c_t dt.$$

where $r(t)$ and $\phi_j(t)$ are taken as two stochastic processes that are induced by $\mu_Z(t)$. Set $B_t = e^{\int_0^t r(\tau)d\tau}$. By Itô's rule, we get

$$d\left(\frac{X_t}{B_t}\right) = \left(\frac{1}{B_t}\sum_{j=1}^m \phi_j(t)^2 - \frac{c_t}{B_t}\right)dt + \frac{1}{B_t}\sum_{j=1}^m \phi_j(t)dW_j(t)$$
$$= -\frac{c_t}{B_t}dt + \frac{1}{B_t}\sum_{j=1}^m \phi_j(t)d\widetilde{W}_j(t).$$

where $\widetilde{W}_j(t) = W_j(t) + \int_0^t \phi_j(\tau)d\tau$ is a standard Brownian motion under the equivalent probability measure $\widetilde{P}$ defined by $\frac{d\widetilde{P}}{dP}\mid_{\mathcal{F}_t} = Z_t B_t$. (Note that the existence of an equivalent martingale measure in the HJM model guarantees this change of measure is well defined.)

Therefore,

$$\frac{X_t}{B_t} = \frac{1}{\rho} - \int_0^t \frac{c_\tau}{B_\tau}d\tau + \sum_{j=1}^m \int_0^t \frac{\phi_j(\tau)}{B_\tau}d\widetilde{W}_j(\tau). \tag{41}$$

Taking expectations, using the nonnegativity of wealth and letting $t \to \infty$, we have

$$0 \le \frac{1}{\rho} - \widetilde{\mathsf{E}}\left(\int_0^\infty \frac{c_\tau}{B_\tau}d\tau\right) = \frac{1}{\rho} - \mathsf{E}\left(\int_0^\infty c_\tau Z_\tau d\tau\right).$$

It follows that, for any $\chi > 0$,

$$\mathsf{E}\left(\int_0^\infty e^{-\rho t}\ln(c_t)dt\right) \le \mathsf{E}\left(\int_0^\infty e^{-\rho t}\ln(c_t)dt\right) + \chi\left(\frac{1}{\rho} - \mathsf{E}\left(\int_0^\infty c_t Z_t dt\right)\right)$$
$$\le \frac{\chi}{\rho} + \mathsf{E}\left(\int_0^\infty e^{-\rho t}\widetilde{U}\left(e^{\rho t}\chi Z_t\right)\right)dt \quad \text{[by Equation (40)]}$$
$$= \frac{\chi}{\rho} + \mathsf{E}\left(\int_0^\infty e^{-\rho t}\left(\ln\left(\frac{1}{e^{\rho t}\chi Z_t}\right) - 1\right)dt\right).$$

Since the value of χ that makes Equation (41) hold as an equality when c_t is substituted by $\frac{e^{-\rho t}}{\chi Z_t}$ is 1, we obtain

$$\mathsf{E}\left(\int_0^\infty e^{-\rho t}\ln(c_t)\,dt\right) \le \mathsf{E}\left(\int_0^\infty e^{-\rho t}\ln(c_t^*)\,dt\right).$$

Applying Itô's rule to $X_t^* = \frac{1}{\rho Z_t}e^{-\rho t}$ and making use of Equation (14), we get

$$dX_t^* = X_t^*\left(r(t) + \sum_{j=1}^m \phi_j(t)^2\right)dt + \sum_{j=1}^m X_t^*\phi_j(t)dW_j(t) - e^{-\rho t}\frac{1}{Z_t}dt.$$

Therefore $X_t^* = \frac{1}{\rho}c_t^*$ is the optimal wealth process. ∎

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International Competition and Exchange Rate Shocks: A Cross-Country Industry Analysis of Stock Returns

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This article systematically examines the importance of exchange rate movements and industry competition for stock returns. Common shocks to industries across countries are more important than competitive shocks due to changes in exchange rates. Weekly exchange rate shocks explain almost nothing of the relative performance of industries. Using returns measured over longer horizons, the importance of exchange rate shocks increases slightly and the importance of industry common shocks increases more substantially. Both industry and exchange rate shocks are more important for industries that produce internationally traded goods, but the importance of these shocks is economically small for these industries as well.

Economists, journalists, and politicians around the world argue that some of the industries in their country compete vigorously with the same industries in other countries and that exchange rate shocks affect their competitiveness. In the United States it is routinely stated that some U.S. industries compete with Japanese industries and that a depreciation of the yen is bad for these U.S. industries and good for the rival Japanese industries: "If the yen falls, trade tensions could intensify between the U.S. and Japan as autos and machinery from Japan gain a competitive edge."[1] Further, the exchange rate literature shows that exchange rate shocks lead to persistent deviations from purchasing power parity.[2] Froot and Klemperer (1989) and Knetter (1989, 1993), among others, demonstrate that deviations from purchasing power parity lead to sharp changes in price markups and profit margins for exporters.

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[1] "If the yen is any guide, Asia's crisis isn't over," by Bernard Wysocki, Jr., *Wall Street Journal*, June 15, 1998, A1.

[2] See Froot and Rogoff (1995) for a review of this literature.

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