Optimal Public Debt Management under Credit Frictions*

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Abstract

This paper develops a theory of public debt management in which some households cannot borrow. We consider a constant aggregate endowment economy in which the government finances exogenous stochastic public spending shocks with uniform lump sum taxes and state-contingent public debt. Households are heterogeneous with respect to their constant endowment. Ricardian Equivalence holds if public debt is high but not if it is low since some households are borrowing constrained and public debt is traded at a premium. For low levels of debt, tax as opposed to debt financing raises current inequality but reduces future inequality since returns to savers decline. We show that optimal policy exploits this tradeoff by limiting the supply of public debt. Consequently, policies and allocations respond persistently to spending shocks along the equilibrium path even though state-contingent debt is available and taxes are lump sum. This is because households respond to government policy by saving into the future to protect themselves against borrowing constraints. Nonetheless, in the long run, borrowing constrained households accumulate enough wealth that allocations do not respond persistently to spending shocks and Ricardian Equivalence holds.

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1 Introduction

How should a government structure the timing of taxes and debt under aggregate shocks? Though the Ricardian Equivalence proposition due to Barro (1974) implies an irrelevance to this decision, current theories of optimal debt management break Ricardian Equivalence by assuming distortionary taxes (e.g., Barro, 1979 and Lucas and Stokey, 1983).[1] These theories generate a unique optimal policy in which governments smooth tax distortions by running surpluses and deficits so as to insure against aggregate shocks.

Despite the prevalence of this framework, many economists have suggested that the level of debt not only affects tax distortions but the credit constraints faced by individuals. This view has inspired theoretical work addressing the mechanism by which the issuance of government debt can alleviate credit constraints and a number of empirical and quantitative analyses addressing the size of this effect.[2] Nevertheless, normative theories have ignored government debt's impact on credit frictions, and the implication of these frictions for optimal dynamic debt management in the presence of aggregate shocks are unknown.[3]

The purpose of this paper is to characterize the dynamic optimal structure of taxes and debt in an economy facing public spending shocks and credit constraints.[4] Specifically, we consider a setting in which some households cannot borrow. In order to focus on these credit frictions, we abstract entirely from distortionary taxes and consider a constant endowment economy. All households can purchase state-contingent bonds, and in order for credit constraints to have an impact, we let households be heterogeneous in their constant endowment and we constrain taxes to be uniform and lump sum. Together, these two features imply a differential motive for households to hold bonds which means that credit constraints may bind for a subset of households.

In this environment, the government's choice of public debt affects consumption allocations for low levels of debt but stops affecting them if debt is sufficiently high. The

[1] This is the case for a large body of work which includes–but is by no means limited to–Aiyagari, Marcet, Sargent, and Seppala (2002), Angeletos (2002), Bohn (1988), Buera and Nicolini (2004), Chari, Christiano, and Kehoe (1994), Lustig, Sleet, and Yeltekin (2008), and Marcet and Scott (2008).

[2] See for example Altig and Davis (1989), Hayashi (1985), Heathcoate (2005), Holmstrom and Tirole (1998), Hubbard and Judd (1986), Kocherlakota (2007), Krishnamurthy and Vissing-Jorgensen (2008), Shapiro and Slemrod (2003), Woodford (1990), and Yotsuzuka (1987).

[3] Our work is complementary to Shin (2002), though we focus on the role of credit frictions alone by ignoring the presence of uninsurable risk, allowing for state-contingent bonds, and allowing for lump sum taxes. Aiyagari and McGrattan (1998) also consider the optimal supply of debt in an economy with credit constraints though they ignore aggregate shocks.

[4] Though we discuss our model in terms of public spending shocks, all of our results apply to an economy with productivity shocks.

irrelevance of public debt once its level becomes sufficiently high is a consequence of Ricardian Equivalence. For example, consider an economy equally populated by rich households and poor credit-constrained households in which the government must finance a temporary war today. If the government finances the war primarily with debt as opposed to current taxes, then households will anticipate higher taxes in the future to service this debt, and both rich and poor households will buy government bonds to save in anticipation of this future tax increase. Any further tax increase today with a reduction in debt will be matched by a one to one uniform decrease in household savings and will have no effect on the interest rate or inequality.

In contrast, if the level of public debt is low, then government policy can affect consumption allocations since only a subset of households purchase public debt. In the example of the temporary war, this is because the implied initial taxes are so high that poor households do not want to save and would instead prefer to borrow, though they are unable to do so. Thus, public debt is sold at a premium to the rich, and any decrease in the supply of government bonds requires an decrease in the return to these bonds in order to deter rich savers. By choosing the supply of debt, the government effectively chooses a policy which optimizes on the tradeoff between current and future inequality. If the war is primarily financed via taxes, then current inequality is high since the poor pay for a bigger share of the war, but future inequality is low because debt is low and interest rate payments to the rich are low. The opposite is true if the war is primarily financed via debt.

Our first main result is that a utilitarian government should exploit this tradeoff by limiting the supply of public debt. Government policy should let poor households be borrowing constrained in order to provide the government with an advantage in borrowing since debt is then sold at a premium to the rich. Imagine if such a premium were absent and consider the simple example of the temporary war. In this circumstance, every household's participation in the financial market implies that every household is indifferent on the margin between receiving $\epsilon > 0$ units of consumption today versus $\epsilon R > 0$ units of consumption tomorrow where R is the interest rate. Imagine if the government were to reduce the level of debt and raise the level of initial taxes sufficiently so that poor households are eventually saving nothing and cannot offset further tax increases with borrowing. Then in this circumstance, a further increase in initial taxes by some small amount $\epsilon > 0$ reduces the poor households' wartime consumption by ϵ. Importantly, because the implied reduction in public debt reduces interest rates, it increases poor households' post-war consumption by *more* than ϵR, so that this reduction in debt represents a welfare improvement for the poor households. Therefore, the increase in social inequality during the war is offset by the implied reduction in social inequality after the war.

Our second main result is that optimal policies and allocations respond persistently to spending shocks along the equilibrium path even though state-contingent bonds are available and taxes are lump sum. This occurs because the government limits the supply of public debt, and some households are credit constrained in some states of the world. In states in which these constraints are relaxed, households save into the future so as to reduce the effect of future credit constraints, and this raises their consumption going forward in a persistent fashion. As an example, consider the simple case of a temporary war in a dynamic setting, so that war can recur even after peace has been established. As long as the initial war persists, poor households are constrained and choose a constant consumption. Once the war ends, however, these households will save in anticipation of the possibility of future war, so that if the war occurs a second time, these households have a higher level of wealth than they did the first time, and their consumption is thus higher. Consequently, their consumption allocation during the war depends on whether war is occurring for the first or the second time. The general equilibrium implication is that government policies must also reflect whether war is occurring for the first or the second time and must therefore be history dependent.

This result stands in contrast to the standard result in the literature due to Lucas and Stokey (1983) which states that optimal debt management in the presence of state-contingent bonds and distortionary taxes yields allocations which are independent of history. In their environment, distortions emerge because taxes are linear. In our environment, taxes are not distortionary, though government financing decisions affect the intertemporal allocation of inequality. Whereas the extent to which taxes are distortionary is constant in their environment, in our environment, the extent to which government financing decisions can affect the intertemporal allocation of inequality is endogenous to the savings behavior of households. Thus, despite the availability of contingent debt, public spending shocks have a persistent effect on allocations and policies.

Our third main result is that in the long run, allocations do not respond persistently to spending shocks and Ricardian Equivalence holds. The intuition for this result is that, along the equilibrium path, households which are borrowing constrained respond to these constraints by saving into the future, and eventually, they accumulate enough wealth that these constraints stop binding. Why does the government not tighten these constraints even further in the long run in order to borrow even more cheaply in response? Because borrowing constrained households would anticipate this tightening of credit and would save even more, and eventually borrowing constraints would stop binding. Moreover, such a policy would reduce long run interest rates and reduce savings by the rich in the short run, undermining the government's ability to raise funds in the short run. Therefore, even though allocations respond persistently to shocks along the equilibrium path, they do not do so in the long run and Ricardian Equivalence holds.

This paper is related to a number of literatures. As previously mentioned, this paper is related to the literature on optimal debt management in the presence of aggregate shocks, and it is also related to the literature on the role of debt in alleviating credit constraints. In addition to these two literatures, this paper is related to the literature on optimal dynamic taxes in heterogeneous agent settings such as in Golosov, Kocherlakota, and Tsyvinski (2003) and Albanesi and Sleet (2006). As in this literature, we explicitly consider the government's role in affecting inequality, though in contrast to this literature, we limit the number of fiscal instruments available to the government by imposing that it can only use uniform lump sum taxes and debt.[5] This constraint creates a role for government debt which would otherwise be irrelevant in the presence of a rich set of tax instruments (e.g., Bassetto and Kocherlakota, 2004). In this regard, our study of government debt is related to the work of Bassetto (1999) and Werning (2007) who also allow for heterogeneity and limit fiscal instruments, though they do not consider the role of credit market frictions which is our focus.

The paper proceeds as follows. Section 2 describes an example. Section 3 describes the model. Section 4 characterizes competitive equilibria. Section 5 provides our main results. Section 6 describes extensions of our model. Section 7 concludes and the Appendix provides all of the proofs.

2 Example

We provide intuition for the mechanics of our general dynamic model in a simple two period deterministic example. This example highlights how the government's choice of public debt affects the intertemporal allocation of inequality and how the government chooses to optimally exploit this tradeoff.

2.1 Environment

The economy is populated by two types of households indexed by $i = \{L, H\}$ each of size 1/2. Each household has a constant endowment y^i where $y^H > y^L$ and faces the following

[5] This is equivalent to assuming hidden endowments, anonymous asset trading, and a memoryless government in this framework.

budget constraints at $t = 0$ and $t = 1$, respectively:

$$c_0^i + q_1 b_1^i = y^i - \tau_0, \text{ and} \tag{1}$$
$$c_1^i = y^i - \tau_1 + b_1^i. \tag{2}$$

At date 0, households pay lump sum taxes τ_0 and they use their income net of taxes to purchase consumption c_0^i and public debt b_1^i at a price q_1. At date 1, they receive b_1^i and use their income net of taxes to finance consumption. We assume that poor households are not credit worthy and cannot short public debt, so we impose that $b_1^L \geq 0$. Households choose levels of c_0^i, c_1^i, and b_1^i to maximize their utility $\sum_{t=0,1} \beta^t \log(c_t^i)$ for discount factor $\beta \in (0,1)$ subject to their budget constraints and borrowing limits. This yields the following Euler equations:

$$q_1 = \frac{\beta c_0^H}{c_1^H} \geq \frac{\beta c_0^L}{c_1^L}, \tag{3}$$

where the last inequality is slack only if $b_1^L = 0$.

The government must finance a deterministic stream of public spending $\{G_0, G_1\} = \left\{\overline{G}, 0\right\}$ for $0 < \overline{G} < y^L$, so that there is a war in the initial period. It can raises taxes $\tau_t \gtreqless 0$ and it can also issue public debt $B_1 \gtreqless 0$. Its budget constraints at $t = 0$ and $t = 1$, respectively are:

$$\overline{G} = \tau_0 + q_1 B_1, \text{ and} \tag{4}$$
$$B_1 = \tau_1 \tag{5}$$
$$\text{for } B_1 = \frac{1}{2} b_1^L + \frac{1}{2} b_1^H. \tag{6}$$

The government is utilitarian and maximizes $\frac{1}{2} \sum_{i=1,2} \sum_{t=0,1} \beta^t \log(c_t^i)$.

2.2 Competitive Equilibria

Since the economy lasts only for two periods, the government effectively chooses a level B_1 which maximizes social welfare and which induces a state price q_1 and therefore requires taxes τ_0 and τ_1 so as to satisfy (4) and (5). In the next two sections, we begin by characterizing competitive equilibria under different levels of B_1 and then proceed to characterize optimal policy.

As a benchmark, it is useful to first consider the equilibrium in the absence of credit constraints so that (3) is an equality. In this situation, consumption allocations and interest rates are independent of government policy, and this result is a consequence of the Ricardian Equivalence proposition discussed in Barro (1974). More specifically, for (3) to hold, $c_1^H > y^H$ so that rich agents consume more than their endowment, while $c_1^L < y^L$ so that poor agents consume less than their endowment at date 1.[6] In contrast, at date 0, $c_0^H < y^H - \overline{G}$ so that rich agents consume less than their endowment net of public spending, while $c_0^L > y^L - \overline{G}$ so that poor agents consume more than their endowment net of public spending. The public spending shock at date 0 therefore serves to raise consumption inequality at date 1 above and beyond that which would be implied by endowment differences alone. This occurs because households are taxed uniformly, and poor households at date 0 save less than rich households since they are more burdened by taxation. Rich households effectively finance a bigger portion of government spending at date 0, and this provides them with greater savings at date 1 with which to finance their consumption.

Note that this allocation is independent of the level of debt issued by the government, and this is true because all households have access to credit. For instance, imagine if the government were to balance its budget by choosing $\tau_0 = \overline{G}$ and $B_1 = \tau_1 = 0$. In this situation, poor households would borrow from rich households at date 0 and they would repay their debt to the rich households at date 1. We now compare this result to the equilibria in our economy under credit constraints.

2.2.1 High Public Debt

Let $Y = \frac{1}{2}y^L + \frac{1}{2}y^H$ and define

$$B_1^* = \frac{\overline{G}}{Y - \overline{G}} \frac{y^H - Y}{1 + \beta}.$$

Lemma 1 ***(high public debt)*** *If $B_1 > B_1^*$, then $\left\{ \{c_t^i\}_{i=L,H} \right\}_{t=0}^{1}$ and q_1 are uniquely defined and independent of the value of B_1.*

This lemma states that if the government supplies enough debt at date 0, then fiscal policy has no effect on household allocations and social welfare. The intuition for this

[6] Mathematically, this is because

$$\begin{aligned} \left(c_0^L, c_1^L\right) &= \left(\psi\left(Y - \overline{G}\right), \psi Y\right) \\ \left(c_0^H, c_1^H\right) &= \left((1-\psi)\left(Y - \overline{G}\right), (1-\psi) Y\right) \end{aligned}$$

for $\psi = \left(\left(y^L - \overline{G}\right) / \left(Y - \overline{G}\right) + \beta y^L / Y\right) / (1+\beta)$.

result is as follows. If public debt is sufficiently high, then the level of taxation in the initial period during the public spending increase is very low. Both rich as well as poor households rationally anticipate that taxes must rise in the future in order to finance this spending spree, so that both types of households own positive levels of public debt in order to save in anticipation of this tax increase. Consequently, if the government were to increase current taxes τ_0 by some amount $\epsilon > 0$, then households of all types would anticipate a decrease in future taxes τ_1 by ϵ/q_1 (since government debt is reduced) and they would therefore decrease their savings $q_1 b_1^i$ *uniformly* by ϵ. Thus, the change in government policy has no impact on household allocations and interest rates.

This result is in the spirit of the Ricardian Equivalence result, though it relies on the level of debt being sufficiently high that all households participate in the savings market. This allocation and equilibrium interest rate are identical to those in an economy absent credit constraints. Thus, a high enough supply of public debt allows the government to effectively replicate private markets in economies in which such markets are non-existent. Interestingly, this implies that in contrast to economies in which debts are financed via distortionary taxes (e.g., Barro, 1979 and Lucas and Stokey, 1983), excessively high levels of public debt do not actually reduce social welfare on the margin.

2.2.2 Low Public Debt

We characterize competitive equilibria when the government issues low levels of public debt.

Lemma 2 ***(low public debt)*** *If* $B_1 < B_1^*$, *then* $\left\{ \{c_t^i\}_{i=L,H} \right\}_{t=0}^{1}$ *and* q_1 *are uniquely defined for every* B_1.

Corollary 1 ***(effect of debt)*** *As the government increases* B_1 *from below* B_1^*, *(i)* q_1 *decreases, (ii)* b_1^L *is constant and* b_1^H *increases, (iii)* c_0^L *increases and* c_0^H *decreases, and (iv)* c_1^L *decreases and* c_1^H *increases.*

The lemma states that if the supply of public debt is not too high, then fiscal policy affects equilibrium allocations and the interest rate. The corollary to the lemma states that as the government increases debt and decreases initial taxes, the interest rates increases, the holdings of savings by the rich increases, social inequality at date 0 decreases, and social inequality at date 1 increases. Since $b_1^L = 0$ in this region, substitution of (4) and

(5) into (1) and (2) yields the following sequence of consumption:

$$\left(c_0^L, c_1^L\right) = \left(y^L - \overline{G} + q_1 B_1, y^L - B_1\right), \quad (7)$$

$$\left(c_0^H, c_1^H\right) = \left(y^H - \overline{G} - q_1 B_1, y^H + B_1\right), \quad (8)$$

and substitution into (3) yields an inverse interest rate:

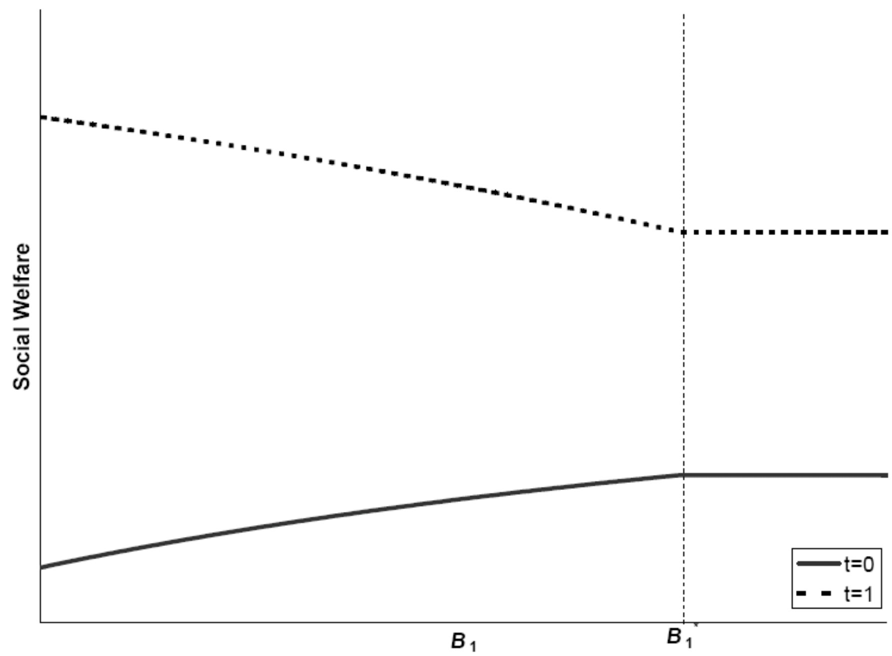
$$q_1 = \beta \frac{y^H - \overline{G}}{y^H + B_1(1+\beta)}. \quad (9)$$

The reason why the supply of public debt affects equilibrium outcomes is because the implied initial taxes are so high that poor households do not want to save and would instead prefer to borrow. However, these households cannot borrow since credit is unavailable to them. This means that the price of this debt is determined by the rich households' demand for it, and the inability of the poor households to short this debt implies that government debt will be sold at a premium to the rich. This additional premium commanded by government debt does not exist if $B_1 > B_1^*$, since if such a premium existed, poor households could easily take advantage of it by simply purchasing fewer government bonds. Consequently, the supply of public debt has a real impact on the riskless interest rate, as any decrease in the supply of government bonds requires a decrease in the return to these bonds in order to convince the rich savers to put fewer resources in public markets.

More specifically, if the government increases initial taxes τ_0 by $\epsilon > 0$, then τ_1 declines by *more* than ϵ/q_1 since q_1 rises (interest rates decline) as debt declines. If τ_0 rises and τ_1 falls, then poor households would ideally borrow more. However, because they are on their credit constraint ($b_1^L = 0$), it is the case that c_0^L falls and c_1^L rises, which must imply from the resource constraint that c_0^H rises and c_1^H falls. Since rich households are not credit constrained, (3) holds, and consequently q_1 falls to justify the reduction in the future consumption of the rich. Therefore, in contrast to the case for which $B_1 > B_1^*$, the supply of debt affects equilibrium consumption and interest rates.

A government choosing a level of debt below B_1^* faces a tradeoffs which is depicted in Figure 1 which displays social welfare at date 0 and at date 1 as a function of B_1. The government can finance most of the spending via initial taxes as opposed to debt by choosing low levels of B_1. The benefit is that social inequality at date 1 is minimized because the rich do not receive much return on savings since public debt and the interest rate is low. Nevertheless, high initial taxes are very burdensome on the poor households so that social inequality at date 0 is high. By raising public debt, the government can reduce these initial taxes by making the rich bear a bigger initial burden of the public spending shock. The cost of this alternative is that it raises social inequality at date 1 when the rich are repaid the interest on their savings. Therefore, as in an economy with distortionary taxes, the government faces an intertemporal tradeoff. However, the objective of the government is not to determine the optimal intertemporal allocation of *economic distortions*, but the optimal intertemporal allocation of *economic inequality*.

Figure 1: Inequality and Debt



2.3 Optimal Government Policy

As a consequence of Lemma 1, one can write the objective of the government as choosing a level of public debt $B_1 \leq B_1^*$ which maximizes

$$\frac{1}{2} \sum_{i=L,H} \sum_{t=0,1} \beta^t \log \left(c_t^i\right) \text{ s.t. } (7), \ (8), \text{ and } (9). \tag{10}$$

The first order condition for the planner is

$$\left(\frac{1}{c_0^L} q_1 - \beta \frac{1}{c_1^L}\right) + \left(B_1 \frac{dq_1}{dB_1}\right) \left(\frac{1}{c_0^L} - \frac{1}{c_0^H}\right) \geq 0, \tag{11}$$

which is slack only if $B_1 = B_1^*$. Note that dq_1/dB_1 is negative from (9). Moreover, at

$B_1 = B_1^*$, the first term on the left hand side of (11) equals zero since the poor agents are not borrowing constrained and (3) is an equality. Moreover, the second term must be negative since $c_0^L < c_0^H$ so that the poor consume less than the rich. Therefore, (11) cannot hold at $B_1 = B_1^*$.

Proposition 1 ***(Optimal Policy)*** *Optimal government policy sets $B_1 < B_1^*$.*

This result states that the government chooses low levels of debt so that the poor feel borrowing constrained. The intuition behind this result is related to our discussion in the previous section. The government effectively wants poor households to be borrowing constrained in order to provide the government with an advantage in borrowing since debt is then sold at a premium to the rich. Imagine if such a premium were absent so that $B_1 = B_1^*$. In this circumstance, every household's participation in the financial market implies that every household is indifferent on the margin between receiving $\epsilon > 0$ units of consumption today versus $\epsilon/q_1 > 0$ units of consumption tomorrow. Consider the effect of an increase in initial taxes τ_0 by ϵ for $\epsilon > 0$ sufficiently small. This reduces poor households' (raises rich households') date 0 consumption by ϵ. Importantly, because the perturbation raises q_1, it raises poor households' (reduces rich households') date 1 consumption by *more* than ϵ/q_1. Therefore, the implied increase in social inequality during the war is offset by the implied reduction in social inequality after the war, and the government's optimal choice of debt makes poor households borrowing constrained.[7]

3 Model

Our simple example shows that the government's decision to choose tax versus debt financing affects the intertemporal allocation of social inequality. In the example, the government optimally chooses a fiscal policy which limits the supply of debt and leaves some households credit constrained. We now generalize this result in a fully dynamic environment with stochastic public expenditure shocks and state-contingent public debt. In contrast to previous models of optimal debt and taxes (e.g., Barro, 1979 and Lucas and Stokey, 1983) we abstract away from distortionary taxes and consider an economy in which households are heterogeneous in their endowments and in their access to credit, as in our simple example. This allows us to generalize Proposition 1 and to consider the

[7] Note that this result does not change if it is instead the rich households which have no access to credit. In this circumstance, B_1^* is actually negative and is associated with a τ_0 which is high enough that rich households balance their budget at $t = 0$.

dynamic response of policy to public spending shocks both in the short run and in the long run.

3.1 Time and Uncertainty

There are discrete time periods $t = \{0, ..., \infty\}$ and a stochastic state $s_t \in S \equiv \{1, ..., M\}$ which follows a first order Markov process with full support for $M \geq 2$. s_0 is given. Let $s^t = \{s_0, ..., s_t\} \in S^t$ represent a history, and let $\pi\left(s^k | s^t\right)$ represent the probability of s^k conditional on s^t for $k \geq t$. The only source of uncertainty is the size of exogenous public spending $G(s_t) \geq 0$ which must be financed by the government.

3.2 Households

There is a continuum of mass 1 of heterogeneous households where each household is permanently of type $i = \{1, ...N\}$ for $N \geq 2$. The household's type specifies its constant endowment $y^i > \max_s G(s)$ and its access to credit $a^i = \{0, 1\}$ where $a^i = 1$ denotes access. Households of type i are of size $\lambda^i > 0$ where $\sum_{i=1}^{N} \lambda^i = 1$. Every household i has a utility function

$$U\left(\left\{c^i\left(s^t\right)\right\}_{t=0}^{\infty}\right) = \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi\left(s^t | s^0\right) u\left(c^i\left(s^t\right)\right) \text{ for } \beta \in (0, 1) \tag{12}$$

where $c^i\left(s^t\right) \geq 0$ represents the consumption of household i at history s^t, $u(\cdot)$ is the household's flow utility function where $u'(\cdot) > 0$, $u''(\cdot) < 0$, $u'(0) = \infty$, and β is the discount factor. We ignore labor decisions for simplicity. In the Appendix, we expand the two period economy of Section 2 to include a labor decision and a linear labor tax and we show that Proposition 1 continues to hold.

The household's dynamic budget constraint at s^t is

$$c^i\left(s^t\right) + \sum_{s^{t+1} \in S^{t+1}} q\left(s^{t+1} | s^t\right) b^i\left(s^{t+1} | s^t\right) = y^i - \tau\left(s^t\right) + b^i\left(s^t | s^{t-1}\right), \tag{13}$$

$$\text{where } b^i\left(s^{t+1} | s^t\right) \geq 0 \text{ if } a^i = 0, \text{ and} \tag{14}$$

$$b^i\left(s^{t+1} | s^t\right) \gtreqless 0 \text{ if } a^i = 1. \tag{15}$$

$b^i\left(s^{t+1} | s^t\right)$ corresponds to the level of public debt purchased by household i at history s^t which repays 1 unit of consumption at $t + 1$ conditional on the realization of history

s^{t+1}.[8] The price of such a claim is $q\left(s^{t+1}|s^t\right)$. $\tau\left(s^t\right) \gtreqless 0$ corresponds to lump sum taxes levied by the government at s^t. Households effectively purchase consumption and contingent government debt with their income net of taxes plus current holdings of debt.[9] Equation (14) implies that households without access to credit cannot short government debt, and (15) implies that households with access to credit can purchase negative public debt.[10] Only for expositional simplicity, we let $b^i\left(s^0|s^{-1}\right) = 0$ for all i.[11] We impose that $\lim_{T\to\infty} q\left(s^{T+1}|s^T\right) b^i\left(s^{T+1}|s^T\right) = 0$ to rule out Ponzi schemes.

The individual household i's problem is to choose a stochastic sequence of consumption $\left\{c^i\left(s^t\right)\right\}_{t=0}^{\infty}$ and debt holdings $\left\{b^i\left(s^{t+1}|s^t\right)\right\}_{t=0}^{\infty}$ which maximize (12) subject to (13) – (15). This leads to the following first order condition for household i:

$$u'\left(c^i\left(s^t\right)\right) \geq \frac{\beta\pi\left(s^{t+1}|s^t\right)}{q\left(s^{t+1}|s^t\right)} u'\left(c^i\left(s^{t+1}\right)\right) \tag{16}$$

where the inequality is strict only if $a^i = 0$ and $b^i\left(s^{t+1}|s^t\right) = 0$. We make the following assumptions.

Assumption 1 (heterogeneity) $y^i \neq y^j$ for some $i \neq j$ and $a^i \neq a^j$ for some $i \neq j$.

Assumption 2 (utility) $U(\cdot)$ is homothetic.

Assumption 1 states that endowments are heterogeneous, some households are credit constrained, and some households are not credit constrained. The existence of credit constrained households guarantees that Ricardian Equivalence does not always hold. The existence of households with credit access is important since it implies that in a dynamic economy, state prices will co-move with at least one household's consumption, and this allows us to consider the short run and long response of fiscal policy to shocks. Note that all of our results hold in an economy where the size of households with credit access is arbitrarily close to zero. Assumption 2 states that the utility function is homothetic. This

[8] We abstract away from longer maturity debt and private bonds only to simplify notation. Our economy is isomorphic to one with non-shortable public bonds and private bonds which are shortable by a subset of households. Note that in the absence of contingent debt, the existence of non-contingent debt of sufficiently long maturity would allow the government to replicate allocations under contingent bonds by analogous arguments as Angeletos (2002).

[9] We exclude capital for simplicity. One can show that an analogous proposition to Proposition 1 holds in the two period example of Section 2 if capital is introduced and all households have access to an identical Cobb-Douglas production technology. Details available upon request.

[10] Note that in contrast to some environments, for example Attanasio and Rios-Rull (2000), government policy does not have a direct effect on constraints on private trades.

[11] This does not alter our main results, though it simplifies discussion since it implies that the only motive for fiscal intervention is the inequality induced by public spending shocks and not by the initial distribution of wealth.

assumption simplifies our proofs as it guarantees that continuation equilibria when (14) does not bind yield state prices which are independent of the wealth distribution.[12]

Note that a critical feature of this environment is not that taxes are lump sum but rather that the tax system leaves households with a differential motive to hold government bonds. As an example, one can easily introduce proportional taxes on the endowment, and our results do not change as long as the maximum tax rate is below 100 percent.

3.3 Government

The government is managed by a utilitarian social planner with full commitment power who has period 0 welfare:

$$\sum_{i=1}^{N} \lambda^i U\left(\left\{c^i\left(s^t\right)\right\}_{t=0}^{\infty}\right). \tag{17}$$

The government's dynamic budget constraint at s^t is:

$$G\left(s_t\right) + B\left(s^t|s^{t-1}\right) = \tau\left(s^t\right) + \sum_{s^{t+1}\in S^{t+1}} q\left(s^{t+1}|s^t\right) B\left(s^{t+1}|s^t\right) \tag{18}$$

$$\text{for } B\left(s^t|s^{t-1}\right) = \sum_{i=1}^{N} \lambda^i b^i\left(s^t|s^{t-1}\right). \tag{19}$$

The government must finance an exogenous stochastic stream of public spending. It pays for this public spending and repays debt holders with current taxes and current borrowing of contingent debt. Equation (19) guarantees that the bond market clears since this is a closed economy. Together, the household's and the government's dynamic budget constraints imply the following resource constraint for the economy:

$$C\left(s^t\right) + G\left(s_t\right) = Y \tag{20}$$

for $C\left(s^t\right) = \sum_{i=1}^{N} \lambda^i c^i\left(s^t\right)$ and Y defined analogously.

In this environment, a feasible allocation is a stochastic sequence $\left\{\left\{c^i\left(s^t\right)\right\}_{i=1}^{N}, G\left(s_t\right)\right\}_{t=0}^{\infty}$ that satisfies (20) and $c^i\left(s^t\right) \geq 0\ \forall i, s^t$. A government policy is a stochastic sequence $\left\{B\left(s^{t+1}|s^t\right), \tau\left(s^t\right), G\left(s_t\right)\right\}_{t=0}^{\infty}$. A state price sequence is $\left\{q\left(s^{t+1}|s^t\right)\right\}_{t=0}^{\infty}$.

Definition 1 *A competitive equilibrium is a feasible allocation, a government policy, and a state price sequence such that (i) given the state price sequence and government policy,*

[12] In particular, it allows us to consider the response of the economy to a localized debt reduction at some history s^{t+1} where (14) does not bind at any other history.

the allocation solves every household's problem, (ii) given the state price sequence, the government policy satisfies the sequence of government budget constraints, and (iii) given the allocation and government policy, the state price sequence clears the bond market.

In this environment, we refer to optimal government policy as the policy which corresponds to the competitive equilibrium which maximizes the utilitarian objective.

4 Competitive Equilibria

Before proceeding to our main results, it is useful to characterize the entire set of competitive equilibria over which the government optimizes. We proceed by first showing that our economy is analogous to an economy with private borrowing and lending of contingent debt subject to history dependent borrowing limits. Second, we use this observation to characterize the entire set of competitive equilibria using the primal approach.

4.1 Analogy to Private Lending Equilibrium

Definition 2 *A private lending equilibrium is feasible allocation, a government policy, and a state price sequence such that (i) given the state price sequence and government policy, the allocation solves every household's problem subject to* $b^i\left(s^{t+1}|s^t\right) \geq \underline{b}\left(s^{t+1}|s^t\right)$ *replacing* (14) *for some exogenous sequence* $\{\underline{b}\left(s^{t+1}|s^t\right)\}_{t=0}^{\infty}$*, (ii) the government policy satisfies* $\tau\left(s^t\right) = G\left(s_t\right)$ *and* $B\left(s^{t+1}|s^t\right) = 0\ \forall s^{t+1}$*, and (iii) given the allocation and government policy, the state price sequence clears the bond market.*

A private lending equilibrium is one in which the government balances its budget in every period so that $G\left(s_t\right)$ is deducted uniformly from every household's budget. At every history, households trade state-contingent debt where households with limited access to private credit are subject to history dependent borrowing limits. Note that in such an equilibrium, (16) is slack only if $a^i = 0$ and $b^i\left(s^{t+1}|s^t\right) = \underline{b}\left(s^{t+1}|s^t\right)$ so that the borrowing limit binds.

Proposition 2 ***(equivalence)*** *An allocation and state price sequence is a competitive equilibrium under a debt sequence* $\{B\left(s^{t+1}|s^t\right)\}_{t=0}^{\infty}$ *if and only if it is a private lending equilibrium under a debt limit sequence* $\{\underline{b}\left(s^{t+1}|s^t\right)\}_{t=0}^{\infty}$ *for* $\underline{b}\left(s^{t+1}|s^t\right) = -B\left(s^{t+1}|s^t\right)$*.*

This proposition states that for any allocation and state price sequence in our economy there exists an equivalent allocation and state price sequence in a private lending economy with debt limits and vice versa. Moreover, the proposition states that the debt limits in the private lending economy correspond to the level of contingent government debt of our economy so that in choosing public debt the government is implicitly choosing the maximal amount of private borrowing. This proposition is analogous to Theorem 1 of Kocherlakota (2007) who shows the equivalence between a private and public bond economy.

The intuition behind this proposition is as follows. If private credit markets are available and the government runs a balanced budget, then households can offset their tax burdens by borrowing and lending in private markets while satisfying their borrowing limit. If private markets are unavailable, however, then the government can stop running a balanced budget, it can issue public debt, and this provides households with the exact same stream of consumption as in the private lending economy. The government effectively borrows on behalf of the households since it has the unique ability to repay debts contracts with tax revenue. Thus the government implicitly determines how much it is allowing credit constrained households to borrow in choosing its level of debt. Mathematically, one can gain an insight into this proposition by substituting (18) into (13) to achieve:

$$c^i\left(s^t\right)+G\left(s_t\right)+\sum_{s^{t+1}\in S^{t+1}} q\left(s^{t+1}|s^t\right)\left(b^i\left(s^{t+1}|s^t\right)-B\left(s^{t+1}|s^t\right)\right)=y^i+\left(b^i\left(s^t|s^{t-1}\right)-B\left(s^t|s^{t-1}\right)\right), \tag{21}$$

which is a dynamic budget constraint which corresponds to that of a private lending economy in which every household holds $\left(b^i\left(s^t|s^{t-1}\right)-B\left(s^t|s^{t-1}\right)\right)$ levels of private debt at s^t. Constraint (14) implies that $b^i\left(s^{t+1}|s^t\right)-B\left(s^{t+1}|s^t\right)\geq -B\left(s^{t+1}|s^t\right)$ if $a^i=0$, so that the household is effectively constrained in its private borrowing. This leads to the below proposition which is analogous to Lemmas 1 and 2 in the simple two period example. For any variable x, let $\mathbf{B}_\epsilon\left(x\right)$ represent an open ball of radius ϵ around x.

Proposition 3 ***(debt relevance)*** *$\exists\left\{B^*\left(s\right)\right\}_{s\in S}$ s.t.*

1. *If $B\left(s^{t+1}|s^t\right)>B^*\left(s_{t+1}\right)\ \forall s^{t+1}$, then $\left\{\left\{c^i\left(s^t\right)\right\}_{i=1}^N\right\}_{t=0}^\infty$ and $\left\{q\left(s^{t+1}|s^t\right)\right\}_{t=0}^\infty$ are uniquely defined and independent of the value of $\left\{B\left(s^{t+1}|s^t\right)\right\}_{t=0}^\infty$.*

2. *If $B\left(s^{t+1}|s^t\right)<B^*\left(s_{t+1}\right)$ for some s^{t+1}, then $\left\{\left\{c^i\left(s^t\right)\right\}_{i=1}^N\right\}_{t=0}^\infty$ and $\left\{q\left(s^{t+1}|s^t\right)\right\}_{t=0}^\infty$ do not correspond to equilibrium allocations for all $\left\{B\left(s^{t+1}|s^t\right)\right\}_{t=0}^\infty\in\mathbf{B}_\epsilon\left(\left\{B\left(s^{t+1}|s^t\right)\right\}_{t=0}^\infty\right)$.*

The proposition states that if the stream of government debt is sufficiently high, then small changes to fiscal policy have no effect on the sequence of consumption and on the sequence of state prices. Intuitively, no household is borrowing constrained, so that the equilibrium corresponds to the unique equilibrium which emerges with full participation in financial markets. Ricardian Equivalence holds since any debt financed reduction in current taxes leads to a one for one uniform increase in savings by households, leaving consumption allocations and interest rates unchanged.

This picture is drastically different if the stream of government debt is not sufficiently high so that at some history, some households are borrowing constrained. In this circumstance, small changes to fiscal policy at some date at which some households are constrained will have real effects on the economy. Intuitively, some households would like to be borrowing from the government and thus do not fully offset a reduction in current taxes with an increase in savings. More generally, a low level of public debt implies that households who save must pay a premium for doing so since some households are unable to short public debt. This is stated formally in the below corollary, where we note that given Assumption 2, $q\left(s^{t+1}|s^t\right) = \beta\pi\left(s^{t+1}|s^t\right) u'\left(Y - G\left(s_{t+1}\right)\right)/u'\left(Y - G\left(s_t\right)\right) \forall s^{t+1}$ if $B\left(s^{t+1}|s^t\right) > B^*\left(s_{t+1}\right) \forall s^{t+1}$ since state prices are determined by the representative consumer.

Corollary 2 *If $B\left(s^{t+1}|s^t\right) < B^*\left(s_{t+1}\right)$ for some s^{t+1}, then*

$$q\left(s^{t+1}|s^t\right) \geq \beta\pi\left(s^{t+1}|s^t\right) u'\left(Y - G\left(s_{t+1}\right)\right)/u'\left(Y - G\left(s_t\right)\right) \ \forall s^{t+1}, \tag{22}$$

and (22) *is slack for at least one s^{t+1}.*

4.2 Competitive Allocations

Proposition 2 implies we can consider the entire set of private lending equilibria and this set will be equivalent to the set of competitive equilibria in our setting. We use this observation to abstract away from prices and government policy and describe necessary and sufficient conditions for an allocation to be a competitive equilibrium. Let i^* correspond to some household with credit access so that $a^{i^*} = 1$ and so that (16) implies that the price of s^t consumption in terms of s^0 consumption is:

$$q\left(s^t|s^0\right) = \beta^t\pi\left(s^t|s^0\right)\frac{u'\left(c^{i^*}\left(s^t\right)\right)}{u'\left(c^{i^*}\left(s^0\right)\right)}.[13] \tag{23}$$

Proposition 4 ***(implementability)*** *An allocation is a competitive equilibrium if and*

[13] This is derived from $q\left(s^t|s^0\right) = q\left(s^t|s^{t-1}\right) \times ... \times q\left(s^1|s^0\right)$.

only if it satisfies (20) $\forall s^t$,

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi\left(s^t|s^0\right) u'\left(c^{i^*}\left(s^t\right)\right)\left(y^i - G\left(s_t\right) - c^i\left(s^t\right)\right) = 0 \ \forall i, \tag{24}$$

$$\frac{u'\left(c^{i^*}\left(s^{t+1}\right)\right)}{u'\left(c^{i^*}\left(s^t\right)\right)} = \frac{u'\left(c^i\left(s^{t+1}\right)\right)}{u'\left(c^i\left(s^t\right)\right)} \ \forall i, s^{t+1} \ \textit{if } a^i = 1, \ \textit{and} \tag{25}$$

$$\min\left\{\begin{array}{c} \frac{u'\left(c^{i^*}\left(s^{t+1}\right)\right)}{u'\left(c^{i^*}\left(s^t\right)\right)} - \frac{u'\left(c^i\left(s^{t+1}\right)\right)}{u'\left(c^i\left(s^t\right)\right)}, \\ \Gamma\left(s^{t+1}|s^t\right) - \\ \sum_{l=t+1}^{\infty}\sum_{s^l \in S^l} \beta^{l-t-1}\pi\left(s^l|s^{t+1}\right)\frac{u'\left(c^{i^*}\left(s^l\right)\right)}{u'\left(c^{i^*}\left(s^{t+1}\right)\right)}\left(y^i - G\left(s_l\right) - c^i\left(s^l\right)\right) \end{array}\right\} = 0 \ \forall i, s^{t+1} \ \textit{if } a^i = 0 \tag{26}$$

for some stochastic sequence $\left\{\Gamma\left(s^{t+1}|s^t\right)\right\}_{t=0}^{\infty}$.

This proposition states that if a stochastic sequence of consumption satisfies a set of conditions, then it can be implemented as a competitive equilibrium, and any competitive equilibrium must satisfy this set of conditions. Condition (24) states that the present discounted value of every household's surplus equals its initial debt of zero, where household i^*'s consumption is used to discount the surpluses. Together with (20), it guarantees that the dynamic budget constraints of all households and the government are satisfied for some sequence of contingent debt. Condition (25) guarantees that (16) binds if $a^i = 1$ since these households have perfect access to credit. Condition (26) guarantees that if (16) is slack for some households with $a^i = 0$, then these households are necessarily choosing levels of debt $b^i\left(s^{t+1}|s^t\right) = 0$ so that the present discounted value of their surpluses must all be equal to some common amount $\Gamma\left(s^{t+1}|s^t\right)$. Importantly, (26) implies that

$$\frac{u'\left(c^{i^*}\left(s^{t+1}\right)\right)}{u'\left(c^{i^*}\left(s^t\right)\right)} \geq \frac{u'\left(c^i\left(s^{t+1}\right)\right)}{u'\left(c^i\left(s^t\right)\right)} \ \forall i, s^{t+1} \text{ if } a^i = 0, \tag{27}$$

so that household i^*'s marginal utility grows weakly faster than household i's marginal utility if $a^i = 0$. Note that in a simple economy in which only one type of household does not have access to credit, (26) reduces to (27).

Consider how a government implements an allocation which satisfies the conditions of Proposition 4. If (27) is slack for some i at s^{t+1}, then the government chooses $B\left(s^{t+1}|s^t\right) = \Gamma\left(s^{t+1}|s^t\right)$ so as to ensure that $b^i\left(s^{t+1}|s^t\right) = 0$ for borrowing constrained households. If instead (27) is an equality for all i at s^{t+1}, then the government can issue any level of debt $B\left(s^{t+1}|s^t\right)$ which exceeds

$$\max_{i \text{ s.t. } a^i=0} \left\{ \sum_{l=t+1}^{\infty} \sum_{s^l \in S^l} \beta^{l-t-1} \pi\left(s^l|s^{t+1}\right) \frac{u'\left(c^{i^*}\left(s^l\right)\right)}{u'\left(c^{i^*}\left(s^{t+1}\right)\right)} \left(y^i - G\left(s_l\right) - c^i\left(s^l\right)\right) \right\}, \tag{28}$$

so that no household is borrowing constrained. For any sequence of debt which satisfies these conditions, the government then chooses a sequence of lump sum taxes which satisfies its dynamic budget constraints given the implied sequence of state prices. Given this characterization of competitive equilibria, we now move to consider the welfare maximizing competitive sequence of allocations.

5 Optimal Policy

In this section we focus on the characterization of optimal policy. First, we show that the government chooses $B\left(s^{t+1}|s^t\right) < B^*\left(s_{t+1}\right)$ for some s^{t+1} so that some households are borrowing constrained in the optimal equilibrium. Second, we show that this implies that allocations and policies respond in a persistent fashion to public spending shocks even though state-contingent debt is available. This result is important since it contrasts with that of Lucas and Stokey (1983) who show that optimal debt management in the presence of distortionary taxes and complete markets involves allocations and policies which are independent of history. Finally, we show that all competitive equilibrium allocations–and by consequence the optimal equilibrium allocation–must converge to an equilibrium with Ricardian Equivalence in the long run. Thus, policies and allocations do not respond in persistent fashion to shocks in the long run.

5.1 Short Run

Given Proposition 4, the objective of the government is:

$$\max_{\left\{\left\{c^i\left(s^t\right)\right\}_{i=1}^N\right\}_{t=0}^{\infty}} (17) \text{ s.t. } (20), (24), (25), \text{and } (26). \tag{29}$$

Our first result generalizes Proposition 1.

Theorem 1 ***(optimal policy)*** *$\exists s_0 \in S$ s.t. the solution to (29) admits $B\left(s^{t+1}|s^t\right) < B^*\left(s_{t+1}\right)$ for some s^{t+1}.*

This theorem states that there exists an initial state such that optimal government policy lets some households be borrowing constrained at some history. The intuition for this proposition is that the government effectively wants some households to be borrowing constrained in order to provide the government with an advantage in borrowing since debt is then sold at a premium to rich savers. This serves to reduce social inequality and therefore maximizes social welfare.

The intuition for this result is analogous to that of Proposition 1, though the proof is somewhat more complicated because of the dynamic and stochastic nature of the problem. The proof relies on the existence of a history s^t at which $b^i\left(s^{t+1}|s^t\right)$ is monotonically increasing in the endowment y^i in an economy in which $B\left(s^{t+1}|s^t\right) = B^*\left(s_{t+1}\right)\ \forall s^{t+1}$. This is guaranteed by the homotheticity of the utility function–which implies that savings are monotonic in the endowment–and the existence a history s^t at which public spending is expected to decline into the future so that poor households are more likely to borrow today relative to the rich since aggregate consumption is expected to rise in the future. At such a point, the government can reduce government debt and therefore penalize the rich savers going forward by charging them a premium for their savings.

More specifically, at such a point s^t where $B\left(s^{t+1}|s^t\right) = B^*\left(s_{t+1}\right)$, no household is borrowing constrained and all households are indifferent on the margin between receiving $\epsilon > 0$ additional consumption at s^t and receiving $\epsilon/q\left(s^{t+1}|s^t\right)$ units of consumption a s^{t+1}. Imagine if the government were to reduce the level of debt $B\left(s^{t+1}|s^t\right)$ by $\epsilon/q\left(s^{t+1}|s^t\right) > 0$ arbitrarily small, where this would cause some household to become borrowing constrained and $q\left(s^{t+1}|s^t\right)$ to increase. Operationally, this change reduces the value of future taxes to all households at s^{t+1} by *more* than $\epsilon/q\left(s^{t+1}|s^t\right)$ (since $q\left(s^{t+1}|s^t\right)$ rises), and it raises the value of taxes prior to s^{t+1} by ϵ. The reduction in interest rate benefits households who are saving less than average from s^t to s^{t+1} to the detriment of households who are saving more than average from s^t to s^{t+1}. Nonetheless, the social benefit outweighs the social cost since the households who are saving less have lower equilibrium consumption and higher marginal utility than those who are saving more.

As an aside, note that Theorem 1 does not depend on the assumption that some agents are able to borrow. This is because the argument involves a local perturbation around a point at which borrowing constraints do not bind, and it does not require us to appeal to Proposition 4 which characterizes the entire set of competitive equilibrium allocations.

Corollary 3 *Theorem 1 applies in economies with $a^i = 0\ \forall i$.*

An important implication of the Theorem 1 is that optimal allocations respond persistently to public spending shocks.

Theorem 2 ***(persistence)*** *$\exists s_0 \in S$ s.t. the solution to* (29) *admits $\{c^i(s^t)\}_{i=1}^N \neq \{c^i(s^l)\}_{i=1}^N$ for some $s^t \neq s^l$ and $s_l = s_t$.*

This theorem states that there are two paths from s_0 which lead to the same state which are associated with different allocations. In other words, the allocations in state s at date t are a function of history. The intuition for this result is that households facing borrowing constraints take advantage of better economic conditions to save in order to lessen the burden of future borrowing constraints. Therefore, allocations at a given date will depend not only on the current shock but on whether constrained households have been able to build a buffer stock to protect against borrowing limits. Therefore, allocations are a function of history, and by Propositions 3 and 4, this means that policies are also a function of history.

As an example, consider an economy oscillating stochastically between war and peace which is populated with rich households and credit constrained poor households as in the example of Section 2. As long as the initial war persists, poor households are constrained and choose a constant consumption. Once the war ends, however, these households will decide to save in anticipation of the possibility of future war (i.e., they will lend to the government or to the rich during peace), so that if the war occurs a second time, these households have a higher level of savings than they did the first time, and their consumption is thus higher. Consequently, the consumption allocation during the war depends on whether war is occurring for the first or the second time.

This result stands in contrast to the standard result in the literature due to Lucas and Stokey (1983) which states that optimal debt management in the presence of state-contingent debt and distortionary taxes yields allocations which are independent of history. In their environment, distortions emerge because taxes are linear. In our environment, taxes are not distortionary, though government financing decisions affect the intertemporal allocation of inequality. Whereas the extent to which taxes are distortionary is constant in their environment, in our environment, the extent to which government financing decisions can affect the intertemporal allocation of inequality is endogenous to the savings behavior of households. Thus, despite the availability of contingent debt, public spending shocks have long term effects on allocations and policies.

Mathematically, one can gain an insight into Theorem 2 by appealing to Proposition 2 and considering the individual household's maximization problem subject to a sequence of exogenous debt limits $\{-B(s^{t+1}|s^t)\}_{t=0}^{\infty}$. Household i's problem given that $a^i = 0$ is to effectively maximize (12) subject to

$$\sum_{t=0}^{\infty}\sum_{s^t\in S^t} q\left(s^t|s^0\right)\left(y^i - G\left(s_t\right) - c^i\left(s^t\right)\right) = 0, \text{ and} \tag{30}$$

$$\sum_{l=t}^{\infty}\sum_{s^l\in S^l} q\left(s^l|s^0\right)\left(y^i - G\left(s_l\right) - c^i\left(s^l\right)\right) \leq q\left(s^t|s^0\right) B\left(s^t|s^{t-1}\right) \ \forall s^t, \tag{31}$$

where we have multiplied both sides of (21) by $q\left(s^t|s^0\right)$ and taken the sum of these constraints so that given (14) they reduce to (30) and (31). Let $\mu^i\left(s^0\right)$ and $\mu^i\left(s^t\right)$ represent the Lagrange multipliers for (30) and (31), respectively, and let $\Psi^i\left(s^t\right) = \Psi^i\left(s^{t-1}\right) + \mu^i\left(s^t\right)$ for $\Psi^i\left(s^0\right) = \mu^i\left(s^0\right)$. If allocations are independent of history, then from (23), this implies that $q\left(s^t|s^0\right) = \beta^t\pi\left(s^t|s^0\right)$ if $s_t = s_0$ since consumption for all households–including those with access to credit–at s^t and s^0 is the same. First order conditions would yield

$$\beta^t\pi\left(s^t|s^0\right) u'\left(c^i\left(s^t\right)\right) = q\left(s^t|s^0\right)\Psi^i\left(s^t\right) = \beta^t\pi\left(s^t|s^0\right)\Psi^i\left(s^t\right), \tag{32}$$

which implies that if $c^i\left(s^t\right) = c^i\left(s^0\right)$, then it must be that $\mu^i\left(s^l\right) = 0$ for all s^l for $l \geq 1$ leading up to s^t. However, this contradicts Theorem 1 which states that (31) must bind somewhere for some i. In other words, it must eventually be the case that $c^i\left(s^t\right)$ exceeds $c^i\left(s^0\right)$, since household i's optimal way of relaxing (31) whenever it binds is to raise future consumption at all future dates since this smooths consumption into the future.

5.2 Long Run

We have argued that optimal policy involves some households being borrowing constrained along the equilibrium path, and this induces allocations to respond persistently to shocks. A natural question concerns whether this characterization holds in the long run. The next theorem shows that *any* competitive allocation converges in the long run to an allocation in which Ricardian Equivalence holds.

Theorem 3 ***(long run Ricardican Equivalence)*** *If* $\left\{\left\{c^i\left(s^t\right)\right\}_{i=1}^N\right\}_{t=0}^{\infty}$ *is a competitive equilibrium allocation, then*

$$\lim_{t\to\infty}\left(\frac{u'\left(c^{i^*}\left(s^{t+1}\right)\right)}{u'\left(c^i\left(s^{t+1}\right)\right)} - \frac{u'\left(c^{i^*}\left(s^t\right)\right)}{u'\left(c^i\left(s^t\right)\right)}\right) =^{a.s.} 0 \ \forall i. \tag{33}$$

Theorem 3 states that in the long run, marginal utility growth is equated across households. This implies that consumption allocations co-move with the state and are independent of history. Such an economy effectively corresponds to one in which Ricardian Equivalence holds since no household is borrowing constrained and since such an equilibrium can be implemented by any fiscal policy with a level of debt $B\left(s^{t+1}|s^{t}\right)$ which exceeds (28) for a given long run distribution of consumption. Together with Theorem 2, Theorem 3 implies that although allocations respond persistently to shocks along the equilibrium path, they cease to do so in the long run.

The intuition for this result is that households which are borrowing constrained along the equilibrium path respond to these constraints by saving into the future, and eventually, they accumulate enough wealth that these constraints stop binding. Why does the government not tighten these constraints even further in the long run in order to borrow even more cheaply in response? Because borrowing constrained households would anticipate this tightening of credit and would save even more, and eventually borrowing constraints would stop binding. Moreover, such an action would reduce long run interest rates and reduce savings by the rich in the short run, undermining the government's ability to raise funds in the short run.

Mathematically, our result is a consequence of Proposition 4 which implies (27), so that the ratio of marginal utilities between households i and i^{*} is a monotonically decreasing sequence which must necessarily converge since it is bounded from below by zero.

As an aside, note that this long run characterization hinges on the assumption that state contingent bonds are available so that households are able to effectively save their way out of the borrowing constraints imposed by the government. A natural question for future research concerns the extent to which optimal policy in the presence of additional non-insurable shocks yields a similar result.

6 Extensions

In this section, we consider a number of implications of our theory. For simplicity, we describe these implications using small extensions of the simple example of Section 2.

6.1 Government Commitment

Our first implication concerns the issue of time-consistency of policy. Using a simple example, we can show that optimal policy is not time-consistent since the government would like to redistribute towards the poor ex-post by devaluing the current wealth of the rich. However, this example suggests that optimal policy can be made time-consistent with the use of long maturities.

Consider the economy of Section 2 with additional dates so that $G_0 = \overline{G}$ and $G_t = 0$ for $t = 1, ..., \infty$. From Proposition 4, we can write the government's problem as:

$$\max_{\left\{\left\{c_t^i\right\}_{i=L,H}\right\}_{t=0}^{\infty}} \frac{1}{2} \sum_{i=L,H} \sum_{t=0}^{\infty} \beta^t \log\left(c_t^i\right) \text{ s.t. } (20),$$

$$\sum_{t=0}^{\infty} \beta^t \frac{y^H - G_t - c_t^H}{c_t^H} = 0, \tag{34}$$

$$\text{and } \frac{c_t^H}{c_{t+1}^H} \geq \frac{c_t^L}{c_{t+1}^L} \; \forall t.$$

It can be easily verified that the solution to the above problem yields a solution for which $c_t^i = c_{t+1}^i$ for $t \geq 1$ for $i = L, H$, so that the government effectively only manipulates the interest rate at $t = 0$ in order to redistribute towards the poor.

Given this optimal allocation, imagine if the government were able to re-optimize at $t = 1$, taking into account that rich households are saving into date 1. In this situation, the analogous constraint to (34) given that $G_t = 0$ for $t \geq 1$ becomes:

$$\sum_{t=1}^{\infty} \beta^{t-1} \frac{y^H - c_t^H}{c_t^H} = -\frac{B_1}{c_1^H}. \tag{35}$$

Because c_1^H and c_t^H for $t > 1$ do not enter equally into constraint (35), a government re-optimizing at date 1 will not choose the same allocation as the period 0 government since it will choose $c_1^H > c_t^H$ for $t > 1$. In other words, the period 1 government would prefer to re-evaluate policy in the absence of commitment. Intuitively, the government wants to redistribute even more towards the poor ex-post by penalizing wealthy households by limiting the supply of debt and reducing interest rates.

An interesting implication of this result is that the optimal period 0 policy can be implemented in this example with the use of long maturities. Specifically, consider the implied level of c_t^H for $t \geq 1$ in the solution to the period zero problem, and denote this value by c^*. Imagine if the government at date 0 could issue debts ${}_0B_t$ which pay one unit of consumption at date t, where the price of such a claim at $t = 0$ must be ${}_0q_t = \beta^t c_0^H / c_t^H$. In this situation, the government could easily issue a set of long maturity debt at date 0 where ${}_0B_t = c^* - y^H$ for all t. As such, there would be no motive trading of government bonds from $t \geq 1$ onward, and rich households would receive a fixed debt repayment from the government forever. A government wishing to re-evaluate its policy at $t = 1$ would be bound by an analogous constraint to (34) which is:

$$\sum_{t=1}^{\infty}\beta^{t-1}\frac{y^H-c_t^H}{c_t^H}=-\sum_{t=1}^{\infty}\beta^{t-1}\frac{{}_0B_t}{c_t^H}=\sum_{t=1}^{\infty}\beta^{t-1}\frac{y^H-c^*}{c_t^H},$$

so that c_t^H enters equally into the constraint for all $t \geq 1$, implying that optimal policy involves $c_t^H = c_{t+1}^H$ for all $t \geq 1$. Therefore, optimal policy is time-consistent in the presence of long maturities.

Note that this result is in the spirit of Lucas and Stokey (1983) who show that optimal policies can be made time-consistent in the presence of long maturities in a distortionary tax setting. Our results can be generalized beyond this particular example using similar reasoning to that of Lucas and Stokey (1983). In particular, in any economy with two types of households in which one household type is credit constrained, optimal policy can be made time-consistent with the use of long maturities.[14]

6.2 Debt Denomination

Historically, governments have offered debt of various denominations and have provided preferential treatments to large denominated debt.[15] We ask in this section whether large denominated debt improves social welfare. To answer this question, consider our two period model, and imagine if the government only issues large denominated debt where the government picks both the quantity and the price of this debt.[16] In other words, every household can purchase an integer multiple of a government bond of size B_1 at a price q_1. Consider an equilibrium in which the government sells this bond and because of its size, only to the rich buy it and they each buy exactly one unit of it. Given (7) and (8), in order for the rich to weakly prefer buying this bond versus buying none of it, it must be that

$$\log\left(y^H-\overline{G}-q_1B_1\right)+\beta\log\left(y^H+B_1\right)\geq\log\left(y^H-\overline{G}+q_1B_1\right)+\beta\log\left(y^H-B_1\right), \quad (36)$$

and the poor must also prefer to buy none of it versus saving like the rich:

$$\log\left(y^L-\overline{G}+q_1B_1\right)+\beta\log\left(y^L-B_1\right)\geq\log\left(y^L-\overline{G}-q_1B_1\right)+\beta\log\left(y^L+B_1\right). \quad (37)$$

[14] Details available upon request.

[15] See Calomiris (1991) for a historical discussion of debt financing in the United States, for example.

[16] This is equivalent to the government choosing to default on all its small denominated debt so that it is never traded. Bryant and Wallace (1984) provide a related model in which the availability of large denominated bonds allows the government to price discriminate between agents.

Note that (36) and (37) look exactly like incentive compatibility constraints in dynamic private information economies (e.g., Atkeson and Lucas, 1992, Golosov, Kocherlakota, and Tsyvinski, 2003 and Albanesi and Sleet, 2006). In our particular setting, these restrictions do not guarantee that households prefer their equilibrium behavior to buying two or more units of debt, but as we will see, they are sufficient restrictions to impose on the utilitarian government. In particular, imagine if the planner chooses B_1 and q_1 which maximizes social welfare subject to (7), (8), (36), and (37). This problem effectively resembles the second best problem of private information economies, and it represents a relaxed form of the original problem described in Section 2.

It can be shown that optimal policy in the presence of large denominated debt yields higher equilibrium welfare relative to an economy without large denominated debt. This is because for any given amount of debt B_1, the government is able to charge the rich a higher price q_1 up until (36) binds. The government effectively issues large denominated debt at a high premium. At the going interest rates, the rich would like to save less, but they are unable to because they can only buy debt in bulk, and their only alternative is to not save altogether. This means that the government is able to finance the war by borrowing at an even cheaper rate than in the absence of large denominated debt. Thus, the government does not need to tax as much at the initial period since it is able to borrow easily, and it does not need to tax as much in the final date since interest payments are low.

One can see this mathematically by noting that (36) binds in the optimum so that the rich are indifferent between saving and not saving B_1. This is not true in the economy described in Section 2 since the rich are indifferent between saving a little bit more and a little bit less *on the margin*. More specifically, we can multiply both sides of (36) which binds by $1/(q_1 B_1)$, and given the concavity of the utility function, this implies

$$\frac{1}{y^H - \overline{G} - q_1 B_1} > \frac{\beta}{q_1} \frac{1}{y^H + B_1}, \tag{38}$$

which given (16) means the rich would like to save *less* on the margin if they could.

6.3 Imperfect Credit Markets

The analysis thus far as presumed the complete absence of credit for some households. In this section, we describe an extension which allows for imperfect credit markets, and we present an additional prediction with respect to the relationship between the credit spread and the stock of public debt. Imagine our two period economy where in addition to their holdings of public debt $b_1^i \geq 0$, households can also hold private debt $d_1^i \gtreqless 0$ which is traded at a price q_1/α where α represents the credit spread. While public debt is risk-free, private debt is subject to an aggregate shock, where with some probability $1-\theta$, all households in the economy default on their private debt. Because private debt is in zero net supply, it must be that $d_1^L + d_1^H = 0$.

Note that it can easily be shown that the characterization of competitive equilibria in Lemmas 1 and 2 and Corollary 1 hold in this alternative economy where c_1^i is a stochastic variable which depends on the realization of aggregate default. Moreover, Proposition 1 also holds for analogous reason as in the simpler economy.

The additional predictions involve the relationship between equilibrium levels of B_1 and the credit spread α and the volume of private trades d_1^H. Specifically, as the government increases B_1 from below B_1^*, α and d_1^H strictly decrease, and once B_1 passes B_1^*, then $\alpha = \theta$ and $d_1^H = 0$. In other words, as the government raises the supply of public debt, the credit spread and private lending drop. If public debt exceeds B_1^*, then there is no private lending and the credit spread reflects only the default risk.

Importantly, if public debt is below B_1^*, private debt is priced at a discount above and beyond that implied by default risk alone, and this emerges because any household interested in shorting government bonds is itself subject to credit risk and must pay private market borrowing rates. Any increase in the supply of government bonds at date 1 requires an increase in the return to these bonds in order to convince the rich savers to shift their savings away from private markets into public markets. As these savings shift away from private markets, the return to government bonds rises, thereby closing the return gap between government bonds and private bonds so that the credit spread declines. Moreover, as private savings decline, the size of defaults at date 1 also declines.

This negative relationship between public debt and the credit spread is consistent with the data. Krishnamurthy and Vissing-Jorgensen (2008) find a negative long run relationship between the levels of public debt and the credit spread in the United States. Our theory predicts that this relationship exists because of the government's unique ability to guarantee debts with its taxation ability. Moreover, consistent with the data, our theory implies that beyond a certain point, the level of public debt ceases to have an effect on the credit spread. One can therefore interpret our theory as considering how the government should exploit this negative relationship in its decision to use tax versus debt financing.

7 Conclusion

We have developed a theory of public debt management under aggregate shocks in which some households cannot borrow. We have shown that the government chooses the optimal intertemporal allocation of inequality, whereby some households are credit constrained along the equilibrium path. This generates allocations and policies which respond persistently to public spending shocks, despite the availability of contingent debt. Our theory also predicts that in the long run, allocations and policies cease to be history dependent and Ricardian Equivalence holds.

Our model makes a number of simplifying assumptions which leave room for further exploration. First, we have exogenously assumed that some households have access to credit whereas others do not. In practice, access to credit depends on a number of microeconomic considerations which we ignore and it is also not likely to be constant throughout the entire lifetime of a household. Second, we have ignored the presence of non-insurable idiosyncratic risk in order to focus on how our model departs from the benchmark environment with distortionary taxes of Lucas and Stokey (1983). Future work should consider the extent to which our results change once this is incorporated. Third, we have considered the social optimum under commitment which ignores realistic political distortions.[17]

[17] See for example Acemoglu, Golosov, and Tsyvinski (2008) and Yared (2009) for work which considers the role of political distortions.

8 Appendix

8.1 Proofs of Section 2

8.1.1 Proof of Lemma 1

Step 1. Substitution of (4) and (5) into (1) and (2) implies that

$$\left(c_0^L, c_1^L\right) = \left(y^L - \overline{G} - q_1\left(b_1^L - B_1\right), y^L + b_1^L - B_1\right), \text{ and} \tag{39}$$

$$\left(c_0^H, c_1^H\right) = \left(y^H - \overline{G} - q_1\left(b_1^H - B_1\right), y^H + b_1^H - B_1\right). \tag{40}$$

which after substitution into (3), yields:

$$b_1^L = \max\left\{0, B_1 + \frac{\beta\left(y^L - \overline{G}\right) - q_1 y^L}{q_1(1+\beta)}\right\}, \text{ and} \tag{41}$$

$$b_1^H = B_1 + \frac{\beta\left(y^H - \overline{G}\right) - q_1 y^H}{q_1(1+\beta)}. \tag{42}$$

Substitution into (6) yields the following equilibrium relationship between B_1 and q_1:

$$\frac{1}{2}\max\left\{0, B_1 + \frac{\beta\left(y^L - \overline{G}\right) - q_1 y^L}{q_1(1+\beta)}\right\} + \frac{1}{2}\left(B_1 + \frac{\beta\left(y^H - \overline{G}\right) - q_1 y^H}{q_1(1+\beta)}\right) = B_1. \tag{43}$$

Step 2. From (43), if $B_1 + \frac{\beta\left(y^L - \overline{G}\right) - q_1 y^L}{q_1(1+\beta)} > 0$, then $q_1 = \beta\left(Y - \overline{G}\right)/Y$, which implies that $B_1 > B_1^*$, and unique allocations are determined by substitution of (41) and (42) given q_1 into (39) and (40).

Step 3. From (43), if $B_1 + \frac{\beta\left(y^L - \overline{G}\right) - q_1 y^L}{q_1(1+\beta)} \leq 0$, then q_1 is determined according to (9) which implies that $B_1 \leq B_1^*$. Therefore, if $B_1 > B_1^*$, then $B_1 + \frac{\beta\left(y^L - \overline{G}\right) - q_1 y^L}{q_1(1+\beta)} > 0$.
Q.E.D.

8.1.2 Proof of Lemma 2 and Corollary 1

Step 1. From steps 2 and 3 of the proof of Lemma 1, if $B_1 < B_1^*$, then $B_1 + \frac{\beta\left(y^L - \overline{G}\right) - q_1 y^L}{q_1(1+\beta)} < 0$, and q_1 is determined according to (9), which after substitution into (39), (40), (41), and (42) yields a unique allocation of consumption characterized by (7) and (8), which completes the proof of the lemma.

Step 2. Part (i) of the corollary follows from (9). Parts (ii), (iii), and (iv) follow from step 1, from (41) and (42) which imply that $b_1^L = 0$ and $b_1^H = 2B_1$, and from (9) which implies that $q_1 B_1$ is increasing in B_1. **Q.E.D.**

8.1.3 Proof of Proposition 1

Step 1. If $B_1 \geq B_1^*$, then (11) must hold at B_1^* since from Lemmas 1 and 2, the allocation does not change with respect to B_1 for $B_1 \geq B_1^*$.

Step 2. From Lemmas 1 and 2, the left hand side of (11) is negative at $B_1 = B_1^*$, which means (11) cannot hold at $B_1 = B_1^*$. **Q.E.D.**

8.2 Proofs of Section 4

8.2.1 Proof of Proposition 2

Step 1. Consider a competitive equilibrium allocation. Construct a private lending equilibrium with the same sequence of allocations and the same sequence of prices in which every household i's holdings of debt $b^i\left(s^{t+1}|s^t\right)$ is equal to its holding of debt in the competitive equilibrium minus $B\left(s^{t+1}|s^t\right)$. From (21) all dynamic household budget constraints in the private lending equilibrium are satisfied. Consider a private lending equilibrium allocation under a debt limit sequence $\left\{\underline{b}\left(s^{t+1}|s^t\right)\right\}_{t=0}^{\infty}$ for $\underline{b}\left(s^{t+1}|s^t\right) = -B\left(s^{t+1}|s^t\right)$. Since (16) is satisfied in the competitive equilibrium, household allocations are optimal in the private lending equilibrium. Therefore, a competitive equilibrium allocation is a private lending equilibrium.

Step 2. Consider a private lending equilibrium allocation. Construct a competitive equilibrium with the same sequence of allocations and the same sequence of prices in which every household i's holdings of debt $b^i\left(s^{t+1}|s^t\right)$ is equal to its holding of debt in the private lending equilibrium plus $\underline{b}\left(s^{t+1}|s^t\right)$. Let the government choose public debt $B\left(s^{t+1}|s^t\right)$ equal to $-\underline{b}\left(s^{t+1}|s^t\right)$ and let $\tau\left(s^t\right)$ be chosen to satisfy (18) given this sequence of debt and given state prices. From (18) and (21), (13) is satisfied. Since (16) is satisfied in the private lending equilibrium, household allocations are optimal in the competitive equilibrium. Therefore, a private lending equilibrium allocation is a competitive equilibrium. **Q.E.D.**

8.2.2 Proof of Proposition 3

Step 1. Consider the unique allocation $\left\{\left\{c^i\left(s^t\right)\right\}_{i=1}^{N}\right\}_{t=0}^{\infty}$ which satisfies the below system of equations:

$$\frac{u'\left(c^i\left(s^t\right)\right)}{u'\left(c^i\left(s^0\right)\right)}=\frac{u'\left(Y-G\left(s_t\right)\right)}{u'\left(Y-G\left(s_0\right)\right)}\ \forall i,s^t, \tag{44}$$

$$\sum_{t=0}^{\infty}\sum_{s^t\in S^t}\beta^t\pi\left(s^t|s^0\right)\frac{u'\left(Y-G\left(s_t\right)\right)}{u'\left(Y-G\left(s_0\right)\right)}\left(y^i-G\left(s_t\right)-c^i\left(s^t\right)\right)=0\ \forall i. \tag{45}$$

This allocation corresponds to the unique allocation in an economy in which (16) is always binding $\forall i, s^t$. This is because if (16) is always binding, then from Assumption 2,

$$q\left(s^t|s^0\right)=\beta^t\pi\left(s^t|s^0\right)\frac{u'\left(Y-G\left(s_t\right)\right)}{u'\left(Y-G\left(s_0\right)\right)}, \tag{46}$$

which yields (44). Multiply both sides of (21) by $q\left(s^t|s^0\right)$ and the sum of all budget constraints (21) to generate (45). Since (44) and (45) have as many equations as unknowns, they are sufficient for determining the allocation. Given (44), $c^i\left(s^t\right)=c^i\left(s^l\right)$ if $s_t=s_l$. Note that under such an allocation,

$$B\left(s^{t+1}|s^t\right)-b^i\left(s^{t+1}|s^t\right)=\sum_{l=t+1}^{\infty}\sum_{s^l\in S^l}\beta^{l-t-1}\pi\left(s^l|s^{t+1}\right)\frac{u'\left(c^{i^*}\left(s^l\right)\right)}{u'\left(c^{i^*}\left(s^{t+1}\right)\right)}\left(y^i-G\left(s_l\right)-c^i\left(s^l\right)\right),$$

where $B\left(s^{t+1}|s^t\right)=B\left(s^{l+1}|s^l\right)$ if $s_{t+1}=s_{l+1}$.

Step 2. Given the allocation of step 1, define $B^*\left(s_{t+1}\right)$ as:

$$B^*\left(s_{t+1}\right)=\max_{i\text{ s.t. }a^i=0}\left\{\sum_{l=t+1}^{\infty}\sum_{s^l\in S^l}\beta^{l-t-1}\pi\left(s^l|s^{t+1}\right)\frac{u'\left(Y-G\left(s_l\right)\right)}{u'\left(Y-G\left(s_{t+1}\right)\right)}\left(y^i-G\left(s_l\right)-c^i\left(s^l\right)\right)\right\}.$$

Using Proposition 2, consider the individual consumer's problem in a private lending economy subject to $b^i\left(s^{t+1}|s^t\right)\geq -B\left(s^{t+1}|s^t\right)$ for $\left\{B\left(s^{t+1}|s^t\right)\right\}_{t=0}^{\infty}$ where $B\left(s^{t+1}|s^t\right)>B^*\left(s_{t+1}\right)\ \forall s^{t+1}$. Let us assume and later verify that this latter constraint does not bind. The solution to the relaxed consumer's problem yields the same allocation as described in step 1, and given the definition of $B^*\left(s_{t+1}\right)$, the constraint that $b^i\left(s^{t+1}|s^t\right)\geq -B\left(s^{t+1}|s^t\right)$ $\forall s^{t+1}$ is satisfied. This proves the first part of the proposition.

Step 3. Consider the individual household i's problem in a private lending economy where $a^i=0$ subject to $b^i\left(s^{t+1}|s^t\right)\geq -B\left(s^{t+1}|s^t\right)$ for $\left\{B\left(s^{t+1}|s^t\right)\right\}_{t=0}^{\infty}$ where $B\left(s^{t+1}|s^t\right)<B^*\left(s_{t+1}\right)$ for some s^{t+1}. The solution to the household's relaxed problem yields the same

allocation as in step 1 which does not satisfy the constraint that $b^i\left(s^{t+1}|s^t\right) \geq -B\left(s^{t+1}|s^t\right)$ $\forall s^{t+1}$. Therefore, $b^i\left(s^{t+1}|s^t\right) = -B\left(s^{t+1}|s^t\right)$ and (16) is slack for some s^{t+1}. Consider the household's problem subject to debt limits $\left\{B\left(s^{t+1}|s^t\right) + \epsilon\left(s^{t+1}\right)\right\}_{t=0}^{\infty}$ for $\epsilon\left(s^{t+1}\right)$ sufficiently small where $\epsilon\left(s^{t+1}\right) = 0$ for all s^{t+1} with the exception of the s^{t+1} for which (16) is slack in the original solution where $\epsilon\left(s^{t+1}\right) > 0$. Imagine if allocations were unchanged so that $q\left(s^{t+1}|s^t\right)$ is unchanged. Since (16) is slack in the original solution, but $b^i\left(s^{t+1}|s^t\right) > -B\left(s^{t+1}|s^t\right) - \epsilon\left(s^{t+1}|s^t\right)$, the original allocation is suboptimal in the perturbed problem, yielding a contradiction. This proves the second part of the proposition. **Q.E.D.**

8.2.3 Proof of Corollary 2

Step 1. Imagine if $B\left(s^{t+1}|s^t\right) < B^*\left(s_{t+1}\right)$ for some s^{t+1}. From step 3 of the proof of Proposition 3, (16) is slack for some i for some s^{t+1}. If (16) is an equality for all i for a given s^{t+1}, then (22) is an equality at s^{t+1} since the homotheticity of the utility function implies that the demand for $c^i\left(s^{t+1}\right)$ and $c^i\left(s^t\right)$ is linear in $c^i\left(s^t\right) + q\left(s^{t+1}|s^t\right) c^i\left(s^{t+1}\right)$.

Step 2. If (16) is an inequality for some i for a given s^{t+1}, then (22) is slack. Imagine if instead $q\left(s^{t+1}|s^t\right) \leq \beta\pi\left(s^{t+1}|s^t\right) u'\left(Y - G\left(s_{t+1}\right)\right) / u'\left(Y - G\left(s_t\right)\right)$. From (16), this implies that

$$\frac{u'\left(c^i\left(s^{t+1}\right)\right)}{u'\left(c^i\left(s^t\right)\right)} \leq \frac{u'\left(Y - G\left(s_{t+1}\right)\right)}{u'\left(Y - G\left(s_t\right)\right)} \tag{47}$$

for all i, where this inequality is slack for at least one i. However, given the homotheticity of the utility function, this implies that $\sum_{i=1}^{N} \lambda^i c^i\left(s^{t+1}\right) > Y - G\left(s_{t+1}\right)$ and $\sum_{i=1}^{N} \lambda^i c^i\left(s^t\right) < Y - G\left(s_t\right)$, which contradicts (20). **Q.E.D.**

8.2.4 Proof of Proposition 4

Step 1. Necessity of (20) follows by definition. (23) follows from the fact that $q\left(s^t|s^0\right) = q\left(s^t|s^{t-1}\right) \times ... \times q\left(s^1|s^0\right)$ and from (16) which is an equality for i^* for all s^t. (25) follows from analogous reasoning since (16) is an equality for i if $a^i = 1$. Multiply both sides of (21) by $q\left(s^t|s^0\right)$ and take the sum of all of these constraints which after the substitution of (23) yields (24). Analogous reasoning implies that

$$\sum_{l=t+1}^{\infty} \sum_{s^l \in S^l} \beta^{l-t-1} \pi\left(s^l|s^{t+1}\right) \frac{u'\left(c^{i^*}\left(s^l\right)\right)}{u'\left(c^{i^*}\left(s^{t+1}\right)\right)} \left(y^i - G\left(s_l\right) - c^i\left(s^l\right)\right) = B\left(s^{t+1}|s^t\right) - b^i\left(s^{t+1}|s^t\right), \tag{48}$$

so that given (23), (16) for $a^i = 0$ reduces to (26) for $\Gamma\left(s^{t+1}|s^t\right) = B\left(s^{t+1}|s^t\right)$.

Step 2. For sufficiency, consider an allocation which satisfies all of the properties. Let prices satisfy (23). Let the sequence of $B(s^{t+1}|s^t) - b^i(s^{t+1}|s^t)$ satisfy (48), and choose $B(s^{t+1}|s^t) = \Gamma(s^{t+1}|s^t)$. Choose $\tau(s^t)$ to satisfy (18). (13) is satisfied by Walras's law, and household optimality is guaranteed by (25) and (26). **Q.E.D.**

8.3 Proofs of Section 5

8.3.1 Proof of Theorem 1 and Corollary 3

Step 1. Consider an economy with $B(s^{t+1}|s^t) = B^*(s_{t+1})\ \forall s^{t+1}$ so that (16) is an equality for all i. There exists an s_0 such that $b^i(s^{t+1}|s^t)$ is increasing in y^i at s^{t+1}. This is because the homotheticity of the utility function implies given (48) that one can write $B(s^{t+1}|s^t) - b^i(s^{t+1}|s^t)$ as:

$$\sum_{l=t+1}^{\infty}\sum_{s^l\in S^l}\beta^{l-t-1}\pi\left(s^l|s^{t+1}\right)\frac{u'(Y-G(s_l))}{u'(Y-G(s_{t+1}))}\left(y^i-G(s_l)-\psi^i(Y-G(s_l))\right) \tag{49}$$

for

$$\psi^i = \frac{\sum_{l=0}^{\infty}\sum_{s^l\in S^l}\beta^l\pi\left(s^l|s^0\right)u'(Y-G(s_l))\left(y^i-G(s_l)\right)}{\sum_{l=0}^{\infty}\sum_{s^l\in S^l}\beta^l\pi\left(s^l|s^0\right)u'(Y-G(s_l))\left(Y-G(s_l)\right)},$$

which implies that $c^i(s^t)$ is increasing in y^i and which implies that $B(s^{t+1}|s^t)-b^i(s^{t+1}|s^t)$ is monotonic in y^i. By substitution, it is the case that $b^i(s^{t+1}|s^t)$ is monotonically increasing in y^i at s^{t+1} if

$$\frac{\sum_{l=0}^{\infty}\sum_{s^l\in S^l}\beta^l\pi\left(s^l|s^0\right)(Y-G(s_l))\,u'(Y-G(s_l))}{\sum_{l=0}^{\infty}\sum_{s^l\in S^l}\beta^l\pi\left(s^l|s^0\right)u'(Y-G(s_l))} < \frac{\sum_{l=t+1}^{\infty}\sum_{s^l\in S^l}\beta^{l-t-1}\pi\left(s^l|s^{t+1}\right)(Y-G(s_l))\,u'(Y-G(s_l))}{\sum_{l=t+1}^{\infty}\sum_{s^l\in S^l}\beta^{l-t-1}\pi\left(s^l|s^{t+1}\right)u'(Y-G(s^l))}$$

holds at s^{t+1}. This must be true starting from some s_0 since $M \geq 2$.

Step 2. Choose s_0 such that $b^i(s^{t+1}|s^t)$ is increasing in y^i at s^{t+1} where we denote s^{t+1} by $\widehat{s}^{t+1}$ and s^t by $\widehat{s}^t$. Construct an economy in which $B(s^{t+1}|s^t) >>> B^*(s_{t+1})$ $\forall s^{t+1} \neq \widehat{s}^{t+1}$ and $B(s^{t+1}|s^t) = B^*(s_{t+1}) - \epsilon$ for $s^{t+1} = \widehat{s}^{t+1}$ for $\epsilon > 0$ sufficiently small. Let $i^{\min}$ represent the identity of the credit constrained household with the lowest endowment so that it is constrained at $\widehat{s}^{t+1}$. This economy is characterized by (24) and

$$\frac{u'\left(c^{i^*}(s^{t+1})\right)}{u'\left(c^{i^*}(s^t)\right)} = \frac{u'\left(c^i(s^{t+1})\right)}{u'\left(c^i(s^t)\right)}\ \forall i, s^{t+1} \text{ except } i = i^{\min} \text{ at } \widehat{s}^{t+1}.$$

Given (48), the additional equation which completes the system is

$$\sum_{l=t+1}^{\infty}\sum_{s^l\in S^l}\beta^{l-t-1}\pi\left(s^l|s^{t+1}\right)\frac{u'\left(c^{i^*}\left(s^l\right)\right)}{u'\left(c^{i^*}\left(s^{t+1}\right)\right)}\left(y^{i^{\min}}-G\left(s_l\right)-c^{i^{\min}}\left(s^l\right)\right)=B^*\left(s_{t+1}\right)-\epsilon \text{ for } s^{t+1}=\widehat{s}^{t+1}.$$

From the Implicit Function Theorem, allocations which satisfy this system are differentiable functions of ϵ.

Step 3. Consider the derivative of (17) with respect to ϵ using Proposition 2 to abstract away from taxes:

$$\sum_{i=1}^{N}\sum_{t=0}^{\infty}\sum_{s^t\in S^t}\lambda^i\beta^t\pi\left(s^t|s^0\right)u'\left(c^i\left(s^t\right)\right)\times \tag{50}$$
$$\left(\begin{array}{c}\frac{db^i\left(s^t|s^{t-1}\right)}{d\epsilon}-\sum_{s^{t+1}\in S^{t+1}}q\left(s^{t+1}|s^t\right)\frac{db^i\left(s^{t+1}|s^t\right)}{d\epsilon}\\ -\sum_{s^{t+1}\in S^{t+1}}\frac{dq\left(s^{t+1}|s^t\right)}{d\epsilon}\left(b^i\left(s^{t+1}|s^t\right)-B\left(s^{t+1}|s^t\right)\right)\end{array}\right)$$

If $B\left(s^{t+1}|s^t\right)=B^*\left(s_{t+1}\right)\ \forall s^{t+1}$ is the optimal solution, then this term must be non-positive at $\epsilon=0$. From the proof of Corollary 2, the implied value of $dq\left(s^{t+1}|s^t\right)/d\epsilon=0$ if $s^{t+1}\neq\widehat{s}^{t+1}$, and $dq\left(s^{t+1}|s^t\right)/d\epsilon|_{\epsilon=0}>0$ if $s^{t+1}=\widehat{s}^{t+1}$. Using this fact together with the fact that (16) binds $\forall i, s^{t+1}$ if $\epsilon=0$, (50) reduces to a term which has the same sign as

$$-\frac{dq\left(\widehat{s}^{t+1}|\widehat{s}^t\right)}{d\epsilon}\sum_{i=1}^{N}\lambda^i u'\left(c^i\left(\widehat{s}^t\right)\right)\left(b^i\left(\widehat{s}^{t+1}|\widehat{s}^t\right)-B\left(\widehat{s}^{t+1}|\widehat{s}^t\right)\right). \tag{51}$$

$c^i\left(\widehat{s}^t\right)$ is strictly increasing in y^i and from step 1, $b^i\left(\widehat{s}^{t+1}|\widehat{s}^t\right)-B\left(\widehat{s}^{t+1}|\widehat{s}^t\right)$ is strictly increasing in y^i at $\widehat{s}^{t+1}$. Therefore, the sign of (51) is positive, which contradicts the fact that $B\left(s^{t+1}|s^t\right)=B^*\left(s_{t+1}\right)\ \forall s^{t+1}$ is the optimal solution. **Q.E.D.**

8.3.2 Proof of Theorem 2

Step 1. Imagine if this were not the case. Since $c^{i^*}\left(s^t\right)=c^{i^*}\left(s^l\right)$ if $s_t=s_l$ for $l\geq t$, then (23) implies that $q\left(s^t|s^0\right)=\beta^{t-l}\pi\left(s^t|s^0\right)q\left(s^l|s^0\right)/\pi\left(s^l|s^0\right)$.

Step 2. Proposition 2 implies the equilibrium is characterized by the individual household's maximization problem subject to a sequence of exogenous debt limits $\left\{-B\left(s^{t+1}|s^t\right)\right\}_{t=0}^{\infty}$ which hold if $a^i=0$. Household i's problem given that $a^i=0$ is to maximize (12) subject to (30) and (31). Let $\mu^i\left(s^0\right)$ and $\mu^i\left(s^t\right)$ represent the Lagrange multipliers for (30) and (31), respectively, and let $\Psi^i\left(s^t\right)=\Psi^i\left(s^{t-1}\right)+\mu^i\left(s^t\right)$ for $\Psi^i\left(s^t\right)=\mu^i\left(s^0\right)$.

Step 3. First order conditions with respect to $c^i\left(s^0\right)$ and $c^i\left(s^t\right)$ given step 1 yield (32), which given $c^i(s^t) = c^i(s^0)$, implies that $\mu^i(s^l) = 0$ for all s^l for $l \geq 1$ leading up to s^t. Given the assumption of full support, this implies that $\mu^i(s^l) = 0$ for all s^l. However, this contradicts Theorem 1 which states that (31) must bind somewhere for some i. **Q.E.D.**

8.3.3 Proof of Theorem 3

Proposition 4 implies that in any competitive equilibrium, (25) and (27) must hold. This means that $\left\{u'(c^i(s^t))/u'\left(c^{i^*}(s^t)\right)\right\}_{t=0}^{\infty}$ is a weakly decreasing stochastic sequence. Since $u'(c^i(s^t))/u'\left(c^{i^*}(s^t)\right)$ is bounded from below by 0, it must converge to a limit for every history s^∞. Therefore, it converges almost surely. **Q.E.D.**

8.4 Example with Endogenous Labor Supply

Consider the economy of Section 2 with the following modifications. Household i chooses labor n_t^i at $t = 0, 1$ and its utility function is $\sum_{t=0,1} \beta^t \left(\log(c_t^i) - \eta n_t^{i\gamma}/\gamma\right)$ for $\eta > 0$ and $\gamma > 1$. Each household produces $y_t^i = \omega^i n_t^i$ of goods and services at t which are taxed at a linear rate δ_t where $\omega^H > \omega^L$, capturing the fact that rich households are more productive. Markets are competitive so that a household i is paid a wage ω^i. Equations (1), (2), (4), and (5), respectively, are therefore replaced by

$$\begin{aligned} c_0^i + q_1 b_1^i &= y_0^i(1-\delta_0) - \tau_0, \\ c_1^i &= y_1^i(1-\delta_1) + b_1^i, \\ \overline{G} &= Y_0(1-\delta_0) + \tau_0 + \; q_1 B_1, \text{ and} \\ B_1 &= Y_1(1-\delta_1) + \tau_1, \\ \text{where } Y_t &= \frac{1}{2} y_1^L + \frac{1}{2} y_1^H. \end{aligned}$$

Note that the intratemporal condition for household i at t yields

$$\frac{\eta c_t^i n_t^{i\gamma-1}}{\omega^i} = 1 - \delta_t. \tag{52}$$

To simplify our argument, we assume that poor households have no access to financial markets, so that the constraint that $c_1^L/c_1^H \geq c_0^L/c_0^H$ can be ignored.

Analogous arguments to those of Proposition 4 taking (52) into account together with the arguments of Werning (2007) imply that an allocation $\left\{\{c_t^i, n_t^i\}_{i=L,H}\right\}_{t=0}^{1}$ is a competitive equilibrium if and only if it satisfies

$$C_t = \frac{1}{2}c_t^L + \frac{1}{2}c_t^H = Y_t - G_t \text{ for } t = 0, 1 \tag{53}$$

$$\sum_{t=0,1} \beta^t \left(1 - \eta n_t^{H\gamma}\right) = -\sum_{t=0,1} \beta^t \frac{\tau_t}{c_t^H} \tag{54}$$

$$\sum_{t=0,1} \beta^t \frac{c_t^L}{c_t^H} \left(1 - \eta n_t^{L\gamma}\right) = -\sum_{t=0,1} \beta^t \frac{\tau_t}{c_t^H} \tag{55}$$

$$\frac{c_t^L n_t^{L\gamma-1}}{\omega^L} = \frac{c_t^H n_t^{H\gamma-1}}{\omega^H} \tag{56}$$

for some appropriately chosen τ_0 and τ_1.

Equations (54) and (55) guarantee that the dynamic budget constraints of the rich and poor households are satisfied, respectively, and combined with the resource constraint (53), they guarantee that the government's dynamic budget constraint is satisfied. Equation (56) guarantees that both household types face the same linear labor tax rate.

Let ϕ_t represent the share of total consumption consumed by the poor so that $c_t^L = 2\phi_t C_t$ and $c_t^H = 2(1-\phi_t) C_t$. Substitution of (56) into (53) implies

$$\begin{aligned} n_t^H &= 2Y_t f(\phi_t) \text{ and} \\ n_t^L &= 2Y_t \left(1 - f(\phi_t)\omega^H\right)/\omega^L \text{ where} \\ f(\phi_t) &= \frac{1}{\omega^L \left(\frac{1-\phi_t}{\phi_t}\right)^{1/(\gamma-1)} \left(\frac{\omega^L}{\omega^H}\right)^{1/(\gamma-1)} + \omega^H}. \end{aligned}$$

One can therefore substitute these terms into (54) and (55) to write the social planner's problem as:

$$\max_{\{\phi_t, Y_t, \tau_t\}_{t=0,1}} \left\{ \sum_{t=0,1} \beta^t \left(\begin{array}{c} \log(\phi_t 2(Y_t - G_t)) + \log((1-\phi_t)2(Y_t - G_t)) \\ -\eta \frac{\left(2Y_t\left(1 - f(\phi_t)\omega^H\right)/\omega^L\right)^\gamma}{\gamma} - \eta \frac{(2Y_t f(\phi_t))^\gamma}{\gamma} \end{array} \right) \right\}$$

s.t.

$$\sum_{t=0,1} \beta^t \left(1 - \eta (2Y_t f(\phi_t))^\gamma\right) = \sum_{t=0,1} \beta^t \tau_t / ((1-\phi_t)2(Y_t - G_t)) \tag{57}$$

$$\sum_{t=0,1} \beta^t \frac{\phi_t}{1-\phi_t} \left(1 - \eta \left(2Y_t\left(1 - f(\phi_t)\omega^H\right)/\omega^L\right)^\gamma\right) = \sum_{t=0,1} \beta^t \tau_t / ((1-\phi_t)2(Y_t - G_t)) \tag{58}$$

Let φ^H and φ^L represent the Lagrange multipliers for constraints (57), and (58), respectively. Note that first order conditions for τ_t imply that $\varphi^H = -\varphi^L = \varphi$. Therefore, we can write the first order conditions for the other variables as:

$$
\begin{aligned}
Y_t &: \frac{2}{Y_t - G_t} + Y_t^{\gamma-1} v(\phi_t, \varphi) = 0 && (59) \\
\phi_t &: \frac{1}{\phi_t} - \frac{1}{1-\phi_t} - \varphi \frac{1}{(1-\phi_t)^2} + Y_t^{\gamma} v_\phi(\phi_t, \varphi) = 0, \text{ where} && (60) \\
v(\phi_t, \varphi) &= -\eta 2^\gamma \left[\left(\left(1 - f(\phi_t)\omega^H\right)/\omega^L \right)^\gamma \left(1 - \varphi\gamma \frac{\phi_t}{1-\phi_t}\right) + f(\phi_t)^\gamma (1 + \varphi\gamma) \right].
\end{aligned}
$$

Imagine if the solution admits $\phi_0 = \phi_1$. Substitution into (60) implies that $Y_0 = Y_1$. However, this violates (59) which implies that $Y_0 \neq Y_1$. Therefore, the solution does not admit $\phi_0 = \phi_1$. Note that one can easily construct numerical examples where the solution will admit $\phi_1 > \phi_0$ so that the constraint that the poor cannot save can be relaxed.

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