Asset Demand Based Tests of Expected Utility Maximization

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Abstract

In the neoclassical model of consumer behavior, considerable work has been done investigating when a consumer's demand behavior can be described as having been derived from utility maximization. However, most discussions are in a certainty world. We expand on prior analyses in an uncertainty setting by providing conditions under which contingent claim and asset demands will be consistent with state independent Expected Utility maximization. The question is addressed using two different traditional approaches. First given the analytical form of the demand functions, we derive necessary and sufficient conditions such that the consumer's behavior can be rationalized by an Expected Utility function. Second, we provide a necessary and sufficient condition for a finite set of observations on prices, probabilities and quantities to be consistent with Expected Utility maximization for the case of a single commodity in each state. This condition is shown to be analogous to the strong axiom of revealed preference in classical certainty demand theory. For both approaches, we consider the complete and incomplete asset market cases.

KEYWORDS. Expected utility, additive separability, state independence.
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1 Introduction

The neoclassical certainty model of consumer behavior postulates that a consumer's demand behavior can be described as having been derived from utility maximization subject to a budget constraint. One is then naturally led to ask what this model implies about observed behavior. This question has been addressed using two quite distinct approaches. The first, originating in the work of Slutsky (1915) and Antonelli (1886), derives necessary and sufficient conditions assuming knowledge of the consumer's demand functions. The second approach known as "revealed preference", following the classic work of Samuelson (1938), provides necessary and sufficient restrictions on a finite set of demand-price pairs such that the demand behavior of a consumer is consistent with utility maximization.

In this paper, we extend both approaches to a classic uncertainty setting, where consumers possess state independent Expected Utility preferences and there is a single commodity in each state. First, we derive a simple test based on contingent claim demand properties that is necessary and sufficient for preferences to be representable by a state independent Expected Utility function. Due to the equivalence between the contingent claim and asset portfolio settings in the complete market case, it is obvious that this test can be directly applied to asset portfolios. Because individuals can satisfy the classic Expected Utility axioms for some but not all asset portfolios – as we illustrate with a concrete example – we also consider local tests. Going back to at least Samuelson (1947), there is a substantial literature on this problem. Samuelson (1947) gives the necessary and sufficient condition for demand to be generated by additively separable utility. Our discussion focuses on the Expected Utility representation, which is not only additively separable but also is linear in the probabilities and is state independent. Dybvig (1983) gives a test for Expected Utility assuming the representation is additively separable. But while he assumes fixed probabilities, in our setting probabilities are allowed to change and his test does not apply.

For the incomplete market case, the contingent claim demand test cannot be applied to the asset portfolio setting because of the indeterminacy of the relationship between asset and contingent claim prices. To overcome this difficulty, we derive a new method for converting asset prices into contingent claim prices when asset markets are incomplete. Utilizing this result, our Expected Utility contingent claim test is extended to the transformed asset demand functions. The only other paper of which we are aware that addresses the incomplete markets case is Polemarchakis and Selden (1983). Although the authors derive necessary and

sufficient conditions for incomplete asset demands to be generated by Expected Utility preferences, the conditions are far more complex than those obtained in this paper.[1] Of course in all tests of investor behavior in Expected Utility settings, an obvious challenge is to observe the contingent claim-asset payoff matrix and state probabilities. We briefly consider this issue in the final Section where we discuss the application of recently developed methodologies for performing demand-based laboratory tests (see, for example Choi, et al. 2007 and Andreoni and Sprenger 2012a).

For the revealed preference approach, with the restriction of a single commodity in each state, we are able to derive a much simpler test for Expected Utility maximization than those that currently exists in the literature. For example, the formulation of Green and Srivastava (1986), as in the revealed preference models of Afriat (1967) and Varian (1983), postulates the existence of unknown utility levels and multipliers. Because we do not require these existential quantifiers, our condition is analogous to the strong axiom of revealed preference in classical demand theory.

The rest of the paper is organized as follows. Section 2 provides tests for contingent claim and asset demand functions to be rationalizable by Expected Utility functions. Both global and local tests are derived for contingent claim demands. These tests extend naturally to asset demand functions in the case of complete asset markets. For incomplete markets, an additional test is provided. In Section 3, we derive necessary and sufficient conditions such that observed demand-price pairs in complete and incomplete markets are consistent with Expected Utility maximization. The final Section discusses potential extensions including the possible design of laboratory demand tests of our Expected Utility maximization results.

2 Demand Function Tests

Our focus in this paper is on demand tests for financial assets. However since the next two Subsections assume complete asset markets where demands and prices are fully determined by contingent claim demands and prices, it is sufficient to formulate our tests in terms of the equivalent contingent claim demands. In Subsection 2.3, we consider incomplete markets and formally define the asset demand

[1] It should be noted that there is a substantial literature on the different although related question of when Expected Utility preferences can be recovered uniquely in complete and incomplete asset markets. See, for instance, Dybvig and Polemarchakis (1981) and the references cited therein.

optimization problem. Throughout this paper, we assume that there are S states and a single commodity in each state. Let x_s denote the quantity of contingent claims in state s and π_s the state probability. Assume $\mathbb{R}_+^S$ is the convex commodity space. Let p_s and I denote, respectively, the contingent claim price in state s and consumer income or wealth. It should be noted that throughout this Section, we assume that the Slutsky matrix for any given demands (asset or contingent claim) is symmetric and negative semidefinite, implying that the demands can always be rationalized by a utility function. The question is whether it takes the Expected Utility form.

2.1 Contingent Claims: Global Test

We start with the case where we are given a contingent claim demand function $\mathbf{x}(\mathbf{p}, \boldsymbol{\pi}, I)$ which solves

$$\max_{\mathbf{x}\in\mathbb{R}_+^s} U(\mathbf{x}; \boldsymbol{\pi}) \qquad S.T. \;\; \mathbf{p}\cdot\mathbf{x} \leq I, \tag{1}$$

where U is strictly quasi-concave, strictly increasing in $\mathbf{x}$ and twice continuously differentiable in $\mathbf{x}$ and $\boldsymbol{\pi}$. The utility $U(\mathbf{x}; \boldsymbol{\pi})$ is said to take the state independent Expected Utility form when

$$U(\mathbf{x}; \boldsymbol{\pi}) \;= \sum_{s=1}^{S} \pi_s u\left(x_s\right), \tag{2}$$

where u denotes the NM (von Neumann-Morgenstern) index. Unlike conventional contingent claim formulations, we will consider different state environments in which the number of states are fixed but their probabilities are allowed to change. We assume for all s, $p_s > 0$, $\pi_s \geq 0$ and $\sum_{s=1}^{S} \pi_s = 1$ and observe that the function $x_s\left(p_1, ..., p_S, \pi_1, ..., \pi_S, I\right)$ maps an open subset of positive prices, probabilities and income into contingent claim demands.[2] The following Theorem gives a necessary and sufficient condition for this utility to be ordinally equivalent to a state independent Expected Utility function.

Theorem 1 *Contingent claim demands $x_s\left(p_1, ..., p_S, \pi_1, ..., \pi_S, I\right) > 0$ $(s = 1, 2, ..., S)$ can be rationalized by a utility function that is ordinally equivalent to a state independent Expected Utility function as defined in eqn. (2) if and only if there is a function $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ such that the contingent claim demands $x_s\left(p_1, ..., p_S, \pi_1, ..., \pi_S, I\right)$ $(s = 1, 2, ..., S)$ satisfy $x_s = f\left(x_1, k_s\right)$, where k_s is defined by*

$$k_s =_{def} \frac{\pi_s p_1}{\pi_1 p_s}, \tag{3}$$

[2] It should be emphasized that the probabilities $(\pi_1, \pi_2, ..., \pi_S)$ given in the demand functions resulting from maximizing $U(\mathbf{x}; \boldsymbol{\pi})$ are variables and not constant coefficients.

$f(x_1, k_s)$ is strictly monotone in k_s and $f(x,1) = x$ for all x.

Proof. First we prove necessity. If U takes the form in eqn. (2), it follows from the first order conditions corresponding to (1) that

$$\frac{\pi_1 u'(x_1)}{\pi_s u'(x_s)} = \frac{p_1}{p_s} \qquad (s = 2, 3, ..., S), \tag{4}$$

implying that

$$u'(x_s) = \frac{u'(x_1)}{k_s}. \tag{5}$$

Since u' is monotone, we have

$$x_s = u'^{-1}\left(\frac{u'(x_1)}{k_s}\right) = f(x_1, k_s), \tag{6}$$

where $f(x_1, k_s)$ is also a strictly monotone function of k_s. Next prove sufficiency. The first order conditions for (1) is

$$\frac{\partial U/\partial x_1}{\partial U/\partial x_s} = \frac{p_1}{p_s} \qquad (s = 2, 3, ..., S). \tag{7}$$

For each $s \in \{1, 2, ..., S\}$, since $x_s = f(x_1, k_s)$ and $f(x_1, k_s)$ is a strictly monotone function of k_s, we have

$$\frac{\pi_s p_1}{\pi_1 p_s} = k_s = f_{x_1}^{-1}(x_s), \tag{8}$$

implying that the first order conditions can be rewritten as

$$\frac{\partial U/\partial x_1}{\partial U/\partial x_s} = \frac{\pi_1}{\pi_s} f_{x_1}^{-1}(x_s). \tag{9}$$

Since the above first order partial differential equation is homogeneous, the general solution is given by $T \circ U^*$, where U^* is a first integral and T is an arbitrary transformation (see, for example, Zachmanoglou and Thoe 1986 p. 29). Because a utility function is always defined up to a monotone transformation, we only need to solve the following two equations to derive the first integral (to simplify the notation, we use U instead of U^* to denote the first integral)

$$\frac{\partial U}{\partial x_1} = \pi_1 \varphi(x_1, x_s) \tag{10}$$

and

$$\frac{\partial U}{\partial x_s} = \pi_s \psi(x_1, x_s), \tag{11}$$

where φ and ψ are arbitrary functions satisfying

$$\frac{\varphi(x_1, x_s)}{\psi(x_1, x_s)} = f_{x_1}^{-1}(x_s). \tag{12}$$

Noticing that

$$\frac{\partial^2 U}{\partial x_1 \partial x_s} = \frac{\partial^2 U}{\partial x_s \partial x_1} \tag{13}$$

and utilizing eqns. (10) and (11), we have

$$\pi_1 \frac{\partial \varphi(x_1, x_s)}{\partial x_s} = \pi_s \frac{\partial \psi(x_1, x_s)}{\partial x_1}. \tag{14}$$

Since $\varphi(x_1, x_s)$ and $\psi(x_1, x_s)$ are not functions of probabilities, we must have

$$\frac{\partial \varphi(x_1, x_s)}{\partial x_s} = \frac{\partial \psi(x_1, x_s)}{\partial x_1} = 0, \tag{15}$$

and can define

$$u_1'(x_1) = \frac{1}{\pi_1} \frac{\partial U}{\partial x_1} \quad \text{and} \quad u'(x_s) = \frac{1}{\pi_s} \frac{\partial U}{\partial x_s}. \tag{16}$$

Therefore, the first order condition can be written as

$$\frac{\pi_1 u_1'(x_1)}{\pi_s u'(x_s)} = \frac{p_1}{p_s} \Leftrightarrow \frac{u_1'(x_1)}{u'(x_s)} = k_s. \tag{17}$$

Since when $k_s = 1$, $x_s = f(x_1, 1) = x_1$, we must have $u_1' = u'$, implying that the utility rationalizing the given demands is ordinally equivalent to that in eqn. (2).

■

Remark 1 *Recently, Andreoni and Sprenger (2012a) observe that in a two period discounted Expected Utility (DEU) model, intertemporal allocations will depend only on the ratio of the intertemporal probabilities π_1/π_2 and not on π_1 or π_2 separately, where π_1 and π_2 are the probabilities of the outcomes with nonzero utility value in periods one and two, respectively. They point out that this is a critical and testable implication of the DEU model. This conclusion uses a similar idea to eqn. (3) in Theorem 1, although their setting is very different from ours.*

Remark 2 *Dybvig (1983, Theorem 10) asserts that if U is additively separable, then it is a (state independent) Expected Utility function if and only if*

$$\left. \frac{p_1}{p_s} \right|_{x_1 = x_s} = \frac{\pi_1}{\pi_s}. \tag{18}$$

It should be noted that in a standard Arrow-Debreu setting such as assumed by Dybvig, the probabilities are fixed and treated as constants. However when probabilities are allowed to vary, the following utility

$$U(x_1, x_2; \pi_1, \pi_2) = -\pi_1 (\exp(-\pi_1 x_1) + \exp(-\pi_2 x_1)) - \pi_2 (\exp(-\pi_1 x_2) + \exp(-\pi_2 x_2)), \tag{19}$$

which is additively separable, state independent and satisfies (18) fails to be an Expected Utility function since the probabilities do not enter linearly as in eqn. (2) above.

We next consider an example which illustrates the application of Theorem 1.

Example 1 *Consider the following pair of demand functions*

$$x_1 = \left(\frac{1}{p_1 + p_2 \left(\frac{\pi_2 p_1}{\pi_1 p_2} \right)^2} + \frac{\pi_1}{p_1} \right) \frac{I}{2} \tag{20}$$

and

$$x_2 = \left(\frac{\left(\frac{\pi_2 p_1}{\pi_1 p_2} \right)^2}{p_1 + p_2 \left(\frac{\pi_2 p_1}{\pi_1 p_2} \right)^2} + \frac{\pi_2}{p_2} \right) \frac{I}{2}. \tag{21}$$

Defining the component of the Slutsky matrix as ϵ_{ij} $(i, j = 1, 2)$ *where*

$$\epsilon_{ij} = \frac{\partial x_i}{\partial p_j} + x_j \frac{\partial x_i}{\partial I}, \tag{22}$$

it can be verified that

$$\epsilon_{12} = \epsilon_{21}, \ \epsilon_{11} < 0, \ \epsilon_{22} < 0 \text{ and } \epsilon_{11}\epsilon_{22} = \epsilon_{12}\epsilon_{21}, \tag{23}$$

implying that the Slutsky matrix is symmetric and negative semidefinite. Therefore, the given demands can be rationalized by a utility function. Following the Hurwicz-Uzawa integration process (Hurwicz and Uzawa 1971), we obtain

$$U(x_1, x_2; \pi_1, \pi_2) = \frac{m}{\pi_1 \sqrt{\frac{p^{1+\pi_1}}{\pi_1^2 + p\pi_2^2}}}, \tag{24}$$

where the relative price $p = p_1/p_2$ *is given by*

$$p = \frac{\pi_1 \pi_2 (\pi_2 x_2 - \pi_1 x_1) + \sqrt{\pi_1^2 \pi_2^2 (\pi_1^2 x_1^2 + \pi_2^2 x_2^2 + 2x_1 x_2 (4 + \pi_1 \pi_2))}}{2x_1 \pi_2^2 (1 + \pi_2)} \tag{25}$$

and the normalized income $m = I/p_2$ *is given by*

$$m = \frac{\pi_1 \pi_2 (\pi_2 x_2 - \pi_1 x_1) + \sqrt{\pi_1^2 \pi_2^2 (\pi_1^2 x_1^2 + \pi_2^2 x_2^2 + 2x_1 x_2 (4 + \pi_1 \pi_2))}}{2\pi_2^2 (1 + \pi_2)} + x_2. \tag{26}$$

It would seem that the above form of utility (24) does not take the Expected Utility form. This is confirmed by the following application of the k*-test from Theorem 1. Noticing that*

$$\frac{x_2}{x_1} = \frac{\frac{1}{p_1 + p_2 \left(\frac{\pi_2 p_1}{\pi_1 p_2} \right)^2} + \frac{\pi_1}{p_1}}{\frac{\left(\frac{\pi_2 p_1}{\pi_1 p_2} \right)^2}{p_1 + p_2 \left(\frac{\pi_2 p_1}{\pi_1 p_2} \right)^2} + \frac{\pi_2}{p_2}} \tag{27}$$

is not a pure function of $k_2 = \frac{\pi_2 p_1}{\pi_1 p_2}$*, it follows from Theorem 1 that the demand functions (20)-(21) cannot be rationalized by an Expected Utility representation for all values of* x_1 *and* x_2*.*

Remark 3 *It should be noted that although the demands satisfying Theorem 1 are generated by the Expected Utility function (2), since we fix the number of states we can not necessarily test the strong independence axiom directly.*[3] *The strong independence axiom requires the set of distributions to be a mixture space (see Grandmont 1972) where one is able to form new lotteries or distributions based on convex combinations of any existing lotteries. However in our fixed state setting, these new distributions need not be in the set of distributions spanned by the existing set of states. To link the k-test in Theorem 1 to the lottery strong independence test one needs to find a consistent or compatible setting. One setting in which one can directly link the k-test and the conventional lottery based argument that preferences over distributions satisfy the strong independence axiom and are representable by an Expected Utility function is the Marschak-Machina probability triangle (see Marschak 1950 and Machina 1987). In the three state Marschak-Machina setting, the strong independence axiom holds if and only if the utility function takes the form*

$$U(x_1, x_2, x_3; \pi_1, \pi_2, \pi_3) = \sum_{s=1}^{3} \pi_s u(x_s) \tag{28}$$

(Mas-Colell, Whinston and Green 1995). Assume that an individual has the same preferences when facing lotteries and optimizing contingent claim demands. If the resulting demand functions pass the k-test in Theorem 1, then for any values of (x_1, x_2, x_3) and (π_1, π_2, π_3) her utility function must take the form in (28) and hence the strong independence axiom must hold.

2.2 Contingent Claims: Local Test

In the prior Subsection, we derived a global Expected Utility test which holds for all points in the contingent claim space. It is natural to wonder whether it is possible to conduct a similar test on a subregion of the contingent claim space. As a first step, one can apply the k-test from Theorem 1 to the full contingent claim space. Then if the test fails, the question becomes how to find possible subregions in which the consumer's utility takes the Expected Utility form.

To find such a subregion, we next consider the implications of Theorem 1 holding for the derivatives of contingent claim demand with respect to prices and probabilities.[4] It follows that when $x_s = f(x_1, \frac{\pi_s p_1}{\pi_1 p_s})$, x_s can be viewed as a

[3] A similar argument applies to Polemarchakis and Selden (1983) and Green and Srivastava (1986).

[4] One example of local demand conditions in the certainty case is the set of restrictions on

function of $(x_1, p_1, p_s, \pi_1, \pi_s)$. For $s = 1, 2, ..., S$, define

$$f_{1s} = \frac{\partial x_s}{\partial I} / \frac{\partial x_1}{\partial I} \tag{29}$$

and

$$f_{2s} = \frac{\pi_1 p_s^2}{\pi_s p_1} \left(f_{1s} \frac{\partial x_1}{\partial p_s} - \frac{\partial x_s}{\partial p_s} \right). \tag{30}$$

We use the normalization $\pi_S = 1 - \sum_{i=1}^{S-1} \pi_i$ and therefore need to treat the cases $s = 1, 2, ...S-1$ and $s = S$ separately. It is easy to verify that when $s \neq S$,[5]

$$\frac{\partial x_s}{\partial p_i} = f_{1s} \frac{\partial x_1}{\partial p_i} \quad i \neq 1 \text{ and } i \neq s, \tag{31}$$

$$\frac{\partial x_s}{\partial \pi_i} = f_{1s} \frac{\partial x_1}{\partial \pi_i} \quad i \neq 1 \text{ and } i \neq s, \tag{32}$$

$$\frac{\partial x_s}{\partial p_1} = f_{1s} \frac{\partial x_1}{\partial p_1} + \frac{\pi_s}{\pi_1 p_s} f_{2s}, \tag{33}$$

$$\frac{\partial x_s}{\partial \pi_1} = f_{1s} \frac{\partial x_1}{\partial \pi_1} - \frac{\pi_s p_1}{\pi_1^2 p_s} f_{2s} \tag{34}$$

and

$$\frac{\partial x_s}{\partial \pi_s} = f_{1s} \frac{\partial x_1}{\partial p_s} - \frac{\pi_s p_1}{\pi_1 p_s^2} f_{2s}. \tag{35}$$

When $s = S$, eqns. (31) and (33) continue to hold. But since π_S depends on $(\pi_1, \pi_2, ..., \pi_{S-1})$, eqns. (32), (34) and (35) must be revised as

$$\frac{\partial x_S}{\partial \pi_i} = f_{1S} \frac{\partial x_1}{\partial \pi_i} - \frac{p_1}{\pi_1 p_S} f_{2S} \quad i \neq 1 \text{ and } i \neq S \tag{36}$$

and

$$\frac{\partial x_S}{\partial \pi_1} = f_{1S} \frac{\partial x_1}{\partial \pi_1} - \frac{\left(1 - \sum_{i=2}^{S-1} \pi_i\right) p_1}{\pi_1^2 p_S} f_{2S}. \tag{37}$$

Further assume that

$$\frac{\partial x_s}{\partial I} > 0 \quad s = 1, 2, ..., S. \tag{38}$$

Directly differentiating $x_s = f_s(x_1, \frac{\pi_s p_1}{\pi_1 p_s})$ with respect to prices or probabilities shows that conditions (31)-(37) are necessary for $x_s = f_s(x_1, \frac{\pi_s p_1}{\pi_1 p_s})$ $(s = 1, 2, ..., S)$. Although eqn. (38) cannot be derived from $x_s = f_s(x_1, \frac{\pi_s p_1}{\pi_1 p_s})$, if the NM index u in eqn. (2) is strictly concave, then it must hold. In the following Theorem, we show that Conditions (31)-(38) are also sufficient for an Expected Utility representation.

components of the Slutsky matrix corresponding to separable utility provided by Goldman and Uzawa (1964).

[5] Eqns. (31) and (33) are consistent with $x_s = g_s(x_1, \frac{p_1}{p_s})$, where g_s is some arbitrary function. For the case where $S > 2$, this is analogous to the Samuelson (1947) additive separability test. The reason is that if the utility is not additively separable, then the left hand side of the first order condition (7) will depend on x_i ($i \neq 1$ and $i \neq s$). Hence it is not possible to obtain $x_s = g_s(x_1, \frac{p_1}{p_s})$ in general.

Theorem 2 *Conditions (31)-(38) together with the condition*

$$x_i = x_j \quad \text{if} \quad \frac{p_i/\pi_i}{p_j/\pi_j} = 1 \quad \forall i, j \in \{1, 2, ..., S\} \tag{39}$$

are necessary and sufficient conditions for contingent claim demands

$$x_s\left(p_1, ..., p_S, \pi_1, ..., \pi_S, I\right) > 0 \quad (s = 1, 2, ..., S) \tag{40}$$

to be rationalizable by a utility function that is ordinally equivalent to a state independent Expected Utility function as defined in eqn. (2).

Proof. Necessity is obvious. Next we prove sufficiency. Since the $S = 2$ case can be proved with the method discussed below, without loss of generality, assume that $S > 2$. Since

$$x_1 = h_1\left(p_1, p_2, ..., p_S, \pi_1, \pi_2, ..., \pi_{S-1}, I\right), \tag{41}$$

where h_1 is an arbitrary function, and $\frac{\partial x_1}{\partial I} > 0$, using the inverse function theorem, one can obtain

$$I = h_1^{-1}\left(x_1, p_1, p_2, ..., p_S, \pi_1, \pi_2, ..., \pi_{S-1}\right). \tag{42}$$

Moreover, for each $s = 2, ..., S-1$ there exists a function h_s such that

$$x_s = h_s\left(p_1, p_2, ..., p_S, \pi_1, \pi_2, ..., \pi_{S-1}, I\right), \tag{43}$$

implying that

$$x_s = g_s\left(x_1, p_1, p_2, ..., p_S, \pi_1, \pi_2, ..., \pi_{S-1}\right). \tag{44}$$

It follows from eqn. (31) that when $i \neq 1$ and $i \neq s$,

$$\frac{\partial x_s}{\partial p_i} = \frac{\partial g_s}{\partial x_1}\frac{\partial x_1}{\partial p_i} + \frac{\partial g_s}{\partial p_i} = f_{1s}\frac{\partial x_1}{\partial p_i}. \tag{45}$$

Noticing that

$$\frac{\partial x_s}{\partial I} = \frac{\partial g_s}{\partial x_1}\frac{\partial x_1}{\partial I} \Rightarrow \frac{\partial g_s}{\partial x_1} = \frac{\partial x_s}{\partial I}\Big/\frac{\partial x_1}{\partial I} = f_{1s}, \tag{46}$$

we have

$$f_{1s}\frac{\partial x_1}{\partial p_i} + \frac{\partial g_s}{\partial p_i} = f_{1s}\frac{\partial x_1}{\partial p_i}, \tag{47}$$

implying that $\frac{\partial g_s}{\partial p_i} = 0$. Similarly one can prove that $\frac{\partial g_s}{\partial \pi_i} = 0$. Therefore, we have

$$x_s = g_s\left(x_1, p_1, p_s, \pi_1, \pi_s\right) \quad (s = 2, 3, ..., S-1). \tag{48}$$

Next we need to show that $x_s = f_s(x_1, \frac{\pi_s p_1}{\pi_1 p_s})$. Notice that

$$\frac{\partial x_s}{\partial p_1} = f_{1s}\frac{\partial x_1}{\partial p_1} + \frac{\partial g_s}{\partial p_1} \Rightarrow \frac{\partial g_s}{\partial p_1} = \frac{\partial x_s}{\partial p_1} - f_{1s}\frac{\partial x_1}{\partial p_1} \tag{49}$$

and

$$\frac{\partial x_s}{\partial p_s} = f_{1s}\frac{\partial x_1}{\partial p_s} + \frac{\partial g_s}{\partial p_s} \Rightarrow \frac{\partial g_s}{\partial p_s} = \frac{\partial x_s}{\partial p_s} - f_{1s}\frac{\partial x_1}{\partial p_s}. \tag{50}$$

It follows from (33) that

$$\frac{\partial x_s}{\partial p_1} = f_{1s}\frac{\partial x_1}{\partial p_1} + \frac{p_s}{p_1}\left(f_{1s}\frac{\partial x_1}{\partial p_s} - \frac{\partial x_s}{\partial p_s}\right), \tag{51}$$

implying that

$$\frac{\partial g_s}{\partial p_1} = -\frac{p_s}{p_1}\frac{\partial g_s}{\partial p_s} \tag{52}$$

and hence

$$x_s = g_s\left(x_1, \frac{p_1}{p_s}, \pi_1, \pi_s\right) \quad (s = 2, 3, ..., S-1). \tag{53}$$

Similarly, using (34) and (35), one can show that

$$x_s = g_s\left(x_1, \frac{p_1}{p_s}, \frac{\pi_s}{\pi_1}\right) \quad (s = 2, 3, ..., S-1). \tag{54}$$

Therefore,

$$\begin{aligned} \frac{\partial x_s}{\partial p_1} &= f_{1s}\frac{\partial x_1}{\partial p_1} + \frac{1}{p_s}\frac{\partial g_s}{\partial (p_1/p_s)} = f_{1s}\frac{\partial x_1}{\partial p_1} + \frac{\pi_s}{\pi_1 p_s} f_{2s} \\ \Rightarrow \frac{\partial g_s}{\partial (p_1/p_s)} &= \frac{\pi_s}{\pi_1} f_{2s} \end{aligned} \tag{55}$$

and

$$\begin{aligned} \frac{\partial x_s}{\partial \pi_1} &= f_{1s}\frac{\partial x_1}{\partial \pi_1} - \frac{\pi_s}{\pi_1^2}\frac{\partial g_s}{\partial (\pi_1/\pi_s)} = f_{1s}\frac{\partial x_1}{\partial \pi_1} - \frac{\pi_s p_1}{\pi_1^2 p_s} f_{2s} \\ \Rightarrow \frac{\partial g_s}{\partial (\pi_s/\pi_1)} &= \frac{p_1}{p_s} f_{2s}, \end{aligned} \tag{56}$$

implying that

$$\frac{\partial g_s}{\partial (p_1/p_s)} = \frac{\pi_s p_s}{\pi_1 p_1}\frac{\partial g_s}{\partial (\pi_s/\pi_1)} \tag{57}$$

and hence

$$x_s = f_s(x_1, \frac{\pi_s p_1}{\pi_1 p_s}) \quad (s = 2, 3, ..., S-1). \tag{58}$$

When $s = S$, we have

$$x_S = g_S\left(x_1, \frac{p_1}{p_S}, \pi_1, \pi_2, ..., \pi_{S-1}\right). \tag{59}$$

It follows from (36) that when $i \neq 1$ and $i \neq S$,

$$\begin{aligned} \frac{\partial x_S}{\partial \pi_i} &= f_{1S}\frac{\partial x_1}{\partial \pi_i} + \frac{\partial g_S}{\partial \pi_i} = f_{1S}\frac{\partial x_1}{\partial \pi_i} - \frac{p_1}{\pi_1 p_S} f_{2S} \\ \Rightarrow \frac{\partial g_S}{\partial \pi_i} &= -\frac{p_1}{\pi_1 p_S} f_{2S} \end{aligned} \tag{60}$$

and it follows from (37) that

$$\begin{aligned}\frac{\partial x_S}{\partial \pi_1} &= f_{1S}\frac{\partial x_1}{\partial \pi_1}+\frac{\partial g_S}{\partial \pi_1}=f_{1S}\frac{\partial x_1}{\partial \pi_1}-\frac{\left(1-\sum_{i=2}^{S-1}\pi_i\right)p_1}{\pi_1^2 p_S}f_{2S}\\ \Rightarrow\ \frac{\partial g_S}{\partial \pi_1} &= -\frac{\left(1-\sum_{i=2}^{S-1}\pi_i\right)p_1}{\pi_1^2 p_S}f_{2S},\end{aligned} \tag{61}$$

implying that

$$\frac{\partial g_S}{\partial \pi_i}=-\frac{\pi_1}{\left(1-\sum_{i=2}^{S-1}\pi_i\right)}\frac{\partial g_S}{\partial \pi_1}\quad (i=2,3,...,S-1) \tag{62}$$

and hence

$$x_S=g_S\left(x_1,\frac{p_1}{p_S},\frac{1-\sum_{i=1}^{S-1}\pi_i}{\pi_1}\right). \tag{63}$$

Therefore,

$$\begin{aligned}\frac{\partial x_S}{\partial p_1} &= f_{1S}\frac{\partial x_1}{\partial p_1}+\frac{1}{p_S}\frac{\partial g_S}{\partial\left(p_1/p_S\right)}=f_{1S}\frac{\partial x_1}{\partial p_1}+\frac{1-\sum_{i=1}^{S-1}\pi_i}{\pi_1 p_S}f_{2S}\\ \Rightarrow\ \frac{\partial g_S}{\partial\left(p_1/p_S\right)} &= \frac{1-\sum_{i=1}^{S-1}\pi_i}{\pi_1}f_{2S}\end{aligned} \tag{64}$$

and

$$\begin{aligned}\frac{\partial x_S}{\partial \pi_1} &= f_{1S}\frac{\partial x_1}{\partial \pi_1}-\frac{\left(1-\sum_{i=2}^{S-1}\pi_i\right)}{\pi_1^2}\frac{\partial g_S}{\partial\left(\frac{1-\sum_{i=1}^{S-1}\pi_i}{\pi_1}\right)}\\ &= f_{1S}\frac{\partial x_1}{\partial \pi_1}-\frac{\left(1-\sum_{i=2}^{S-1}\pi_i\right)p_1}{\pi_1^2 p_S}f_{2S}\\ \Rightarrow\ \frac{\partial g_S}{\partial\left(\frac{1-\sum_{i=1}^{S-1}\pi_i}{\pi_1}\right)} &= \frac{p_1}{p_S}f_{2S},\end{aligned} \tag{65}$$

implying that

$$\frac{\partial g_S}{\partial\left(p_1/p_S\right)}=\frac{\left(1-\sum_{i=1}^{S-1}\pi_i\right)p_S}{\pi_1 p_1}\frac{\partial g_S}{\partial\left(\frac{1-\sum_{i=1}^{S-1}\pi_i}{\pi_1}\right)} \tag{66}$$

and hence

$$x_S=f_S(x_1,\frac{\left(1-\sum_{i=1}^{S-1}\pi_i\right)p_1}{\pi_1 p_S}). \tag{67}$$

Therefore $\forall s=2,3,...,S$, we always have $x_s=f_s(x_1,\frac{\pi_s p_1}{\pi_1 p_s})$. Further assuming that $x_i=x_j$ if

$$\frac{p_i/\pi_i}{p_j/\pi_j}=1\quad \forall i,j\in\{1,2,...,S\}, \tag{68}$$

then $x_s = f(x_1, \frac{\pi_s p_1}{\pi_1 p_s})$ with $f(x,1) = x$ for all x. It follows from Theorem 1 that contingent claim demands x_s $(s = 1, 2, ..., S)$ can be rationalized by a utility function that is ordinally equivalent to a state independent Expected Utility as defined in eqn. (2). ■

When applying Theorem 1, if the k-test fails globally one can test whether the conditions (31)-(38) hold for some subregion of the contingent claim space. The following example illustrates this process.

Example 2 *Assume the same demand functions (20)-(21) as in Example 1 which were shown to violate the k-test globally. To search for a possible subregion where preferences might locally take the Expected Utility form, first notice that $\frac{\partial x_s}{\partial I} > 0$ $(s = 1, 2)$. Then compute the following*

$$f_{12} = \frac{\frac{\partial x_2}{\partial I}}{\frac{\partial x_1}{\partial I}} = \frac{\frac{\left(\frac{\pi_2 p_1}{\pi_1 p_2}\right)^2}{p_1 + p_2\left(\frac{\pi_2 p_1}{\pi_1 p_2}\right)^2} + \frac{\pi_2}{p_2}}{\frac{1}{p_1 + p_2\left(\frac{\pi_2 p_1}{\pi_1 p_2}\right)^2} + \frac{\pi_1}{p_1}}, \tag{69}$$

$$\frac{\partial x_1}{\partial p_1} = -\left(\frac{1 + 2p_1 p_2\left(\frac{\pi_2}{\pi_1 p_2}\right)^2}{\left(p_1 + p_2\left(\frac{\pi_2 p_1}{\pi_1 p_2}\right)^2\right)^2} + \frac{\pi_1}{p_1^2}\right)\frac{I}{2}, \tag{70}$$

$$\frac{\partial x_1}{\partial p_2} = \frac{\left(\frac{\pi_2 p_1}{\pi_1 p_2}\right)^2 I}{2\left(p_1 + p_2\left(\frac{\pi_2 p_1}{\pi_1 p_2}\right)^2\right)^2}, \tag{71}$$

$$\frac{\partial x_2}{\partial p_1} = \frac{\left(\frac{\pi_2 p_1}{\pi_1 p_2}\right)^2 I}{2\left(p_1 + p_2\left(\frac{\pi_2 p_1}{\pi_1 p_2}\right)^2\right)^2} \tag{72}$$

and

$$\frac{\partial x_2}{\partial p_2} = -\left(\frac{\left(1 + 2p_1 p_2\left(\frac{\pi_1}{\pi_2 p_1}\right)^2\right)\left(\frac{\pi_2 p_1}{\pi_1 p_2}\right)^4}{\left(p_1 + p_2\left(\frac{\pi_2 p_1}{\pi_1 p_2}\right)^2\right)^2} + \frac{\pi_2}{p_2^2}\right)\frac{I}{2}. \tag{73}$$

It follows that

$$\begin{aligned} f_{22} &= \frac{\pi_1 p_2^2}{\pi_2 p_1}\left(f_{12}\frac{\partial x_1}{\partial p_2} - \frac{\partial x_2}{\partial p_2}\right) \\ &= \frac{\pi_1\left(\left(1+\pi_1\right)\pi_1^3 p_2^2 + 2\left(1+\pi_1\right)\pi_1\left(1+\pi_2\right)\pi_2 p_1 p_2 + \left(1+\pi_2\right)\pi_2^3 p_1^2\right) I}{2p_1\left(\pi_1^2 p_2 + \pi_2^2 p_1\right)\left(\left(1+\pi_1\right)\pi_1 p_2 + \pi_2^2 p_1\right)} \end{aligned} \tag{74}$$

and hence

$$\frac{\partial x_2}{\partial p_1} = f_{12}\frac{\partial x_1}{\partial p_1} + \frac{\pi_2}{\pi_1 p_2} f_{22}. \tag{75}$$

This is not surprising since

$$\frac{x_2}{x_1} = \frac{\frac{1}{\frac{p_1}{p_2}+\left(\frac{\pi_2 p_1}{\pi_1 p_2}\right)^2} + \frac{\pi_1}{p_1/p_2}}{\frac{\left(\frac{\pi_2 p_1}{\pi_1 p_2}\right)^2}{\frac{p_1}{p_2}+\left(\frac{\pi_2 p_1}{\pi_1 p_2}\right)^2} + \pi_2}, \tag{76}$$

implying that $x_2 = g\left(x_1, p_1/p_2\right)$*, where* g *is an arbitrary function and (33) holds.*[6] *Next computing the probability derivatives, since* $\pi_2 = 1 - \pi_1$*, we have*

$$\frac{\partial x_1}{\partial \pi_1} = \frac{\left(\pi_2^4 p_1^2 + \pi_1^4 p_2^2 + 2\pi_1 p_1 p_2 \left(1 - 2\pi_1^2 + \pi_1^3\right)\right) I}{2p_1 \left(\pi_2^2 p_1 + \pi_1^2 p_2\right)^2} \tag{77}$$

and

$$\frac{\partial x_2}{\partial \pi_1} = -\frac{\left(\pi_2^4 p_1^2 + \pi_1^4 p_2^2 + 2\pi_1 p_1 p_2 \left(1 - 2\pi_1^2 + \pi_1^3\right)\right) I}{2p_2 \left(\pi_2^2 p_1 + \pi_1^2 p_2\right)^2}. \tag{78}$$

It can be verified that

$$\begin{aligned} & f_{12}\frac{\partial x_1}{\partial \pi_1} - \frac{\left(1 - \sum_{i=2}^{1} \pi_i\right) p_1}{\pi_1^2 p_2} f_{22} - \frac{\partial x_2}{\partial \pi_1} \\ = & \frac{-\pi_2 \left(\pi_2 p_1 - \pi_1 p_2\right)^2 I}{2p_2 \left(\pi_2^2 p_1 + \pi_1^2 p_2\right)\left(\pi_2^2 p_1 + \pi_1^2 p_2 + \pi_1 p_2\right)}. \end{aligned} \tag{79}$$

It is clear that condition (37) is satisfied if and only if $\pi_2 p_1 - \pi_1 p_2 = 0$*. Notice that* $\pi_2 p_1 - \pi_1 p_2 = 0$ *implies* $k_2 = 1$*. Substituting into the demand function (20)-(21), it can be easily verified that* $x_2 = x_1$*. Since* $x_1 = f\left(x_1, 1\right)$*, along the* $45°$ *ray in contingent claim space, we have*

$$x_2 = x_1 = f\left(x_1, 1\right) = f\left(x_1, k_2\right), \tag{80}$$

implying that preferences are Expected Utility representable when $x_1 = x_2$*.*[7,8]

[6] Actually for the two commodity case, it follows from the first order condition (7) that x_2 can be always expressed as a function of x_1 and p_1/p_2. Therefore, eqn. (33) always holds for the two commodity case.

[7] Although it seems a little odd to say that preferences are Expected Utility representable for the certainty case, for a general non-Expected Utility function with probabilities in it, there is no guarantee that probabilities will cancel out when assuming all the states are the same. This property is satisfied by an Expected Utility representation and our example. In this sense, we say that preferences are Expected Utility representable when $x_1 = x_2$. In fact following Polemarchakis, et al. (1986), one can show that the demand functions of the CRRA (constant relative risk aversion) Expected Utility function

$$U(x_1, x_2) = 3\pi_1 x_1^{\frac{1}{3}} + 3\pi_2 x_2^{\frac{1}{3}}$$

match the demand functions of (20)-(21) in both zeroth and first order when $x_1 = x_2$.

[8] It is interesting to note that the demands (20)-(21) being consistent with Expected Utility

Example 2 illustrates the application of a local demand test for Expected Utility maximization. Although the way in which Expected Utility and non-Expected Utility behavior can be characterized in the choice space is quite interesting, the form of utility used is non standard. Schmidt (1998) created a specific preference model which would facilitate measuring the certainty effect associated with the widely documented laboratory violations of Expected Utility preferences. This $u - V$ model is based on an axiom system involving choices over lotteries where the utility u is used to evaluate choices involving degenerate lotteries (with certain outcomes) and V is the NM index of an Expected Utility representation used to evaluate non-degenerate lotteries. For this model it can be verified that the contingent claim expansion path is comprised of two branches – one corresponds to the set of certainty optima along the 45° ray and the other corresponds to a traditional Expected Utility expansion path. However for this case, unlike Example 2, the asset demand functions are not everywhere differentiable and the local test in this Section can not be applied.

2.3 Asset Demands: Incomplete Markets

In Subsections 2.1 and 2.2, we assumed the standard contingent claim setting. If financial asset markets are introduced and are complete, the contingent claim and financial asset settings are equivalent and the earlier results continue to apply. However, if markets are incomplete, the analysis becomes more involved.

Assume an incomplete asset market setting with J assets and S states, where $J < S$. Denote the payoff for asset j ($j \in \{1, 2, ..., J\}$) in state s ($s \in \{1, 2, ..., S\}$) by ξ_{js}. The quantities of assets and contingent claims are denoted by z_j and x_s and their respective prices are denoted by P_j and p_s. The asset optimization problem is given by

$$\max_{\mathbf{x}\in\mathbb{R}^S_+, \mathbf{z}\in\mathbb{R}^J} U(x_1, .., x_S; \pi_1, .., \pi_S) \qquad \text{S.T. } x_s = \sum_{j=1}^{J} \xi_{js} z_j \text{ and } \sum_{j=1}^{J} P_j z_j = I. \quad (81)$$

We assume that asset prices preclude arbitrage, i.e., there exist $p_s > 0$, $s = 1, ..., S$,

preferences only in the certainty region of the contingent claim space is effectively opposite to that of the "certainty effect", in which the Expected Utility model performs well for lotteries with distributions away from certainty, but fails for those that are degenerate or near certainty (Allais 1953 and Schmidt 1998). Indeed according to Allais (i) the strong independence axiom is "incompatible with a preference for security in the neighborhood of certainty" (Allais 2008) and (ii) individuals act as Expected Utility maximizers when valuing lotteries far from certainty (Andreoni and Sprenger 2012b).

satisfying the following

$$P_j = \sum_{s=1}^{S} \xi_{js} p_s \quad (j = 1, 2, , ..., J). \tag{82}$$

For the given optimal asset demands z_j as functions of asset prices and income, substituting eqn. (82) into $x_s = \sum_{j=1}^{J} \xi_{js} z_j$, we can express contingent claim demands as functions of contingent claim prices, probabilities and income. When markets are incomplete ($J < S$), $p_1, p_2, ..., p_S$ cannot be uniquely recovered from asset prices based on (82) since there are S variables but only J independent equations. Thus $p_1, p_2, ..., p_S$ are not independent and we cannot apply Theorem 1 directly.[9]

In order to derive the analogue of Theorem 1 for the incomplete markets case, it is useful to consider the inverse demand function which maps asset demands, probabilities and income into a support price vector. We denote this function by $P_j(z_1, ..., z_j, \pi_1, ..., \pi_S, I)$ $(j = 1, 2, ..., J)$. It is easy to see that this function is well defined. For our analysis below we need to use the partial derivatives of this function with respect to probabilities. It can be verified that $\frac{\partial P_j}{\partial \pi_s}$ is the (j, s) component of $J \times S$ matrix

$$-\left(\frac{\partial (z_1, z_2, ..., z_J)}{\partial (P_1, P_2, ..., P_J)}\right)^{-1} \frac{\partial (z_1, z_2, ..., z_J)}{\partial (\pi_1, \pi_2, ..., \pi_S)}, \tag{83}$$

where $\frac{\partial(z_1,z_2,...,z_J)}{\partial(P_1,P_2,...,P_J)}$ and $\frac{\partial(z_1,z_2,...,z_J)}{\partial(\pi_1,\pi_2,...,\pi_S)}$ are respectively the Jacobian matrix of asset demand partial derivatives with respect to probabilities and prices.

We next show that by combining the budget constraint and the price derivatives of inverse demand with respect to probabilities, it is possible to uniquely recover the contingent claim prices from the financial asset prices. If the contingent claim prices satisfy (82) and the induced contingent claim demands (from asset demands and the payoff matrix) pass the k-test in Theorem 1, then asset demands can be rationalized by an Expected Utility function.[10]

Theorem 3 *Given the financial asset demands* $z_j(\pi_1, ..., \pi_S, P_1, ..., P_J, I)$ *where* $\pi_S = 1 - \sum_{s=1}^{S-1} \pi_s$ *and the payoff matrix* $(\xi_{js})_{J \times S}$, *suppose that* $\xi_{js} - \frac{x_s P_j}{I} \neq 0$

[9] The point is that even if x_2 can be expressed as a function of $(x_1, p_1, ..., p_S, \pi_1, ..., \pi_S)$ based on (82), one cannot say the k-test in Theorem 1 fails since the first order condition and the budget constraint will impose additional restrictions on $p_1, ..., p_S$. We will see this in more detail in Example 3 below.

[10] In the following Theorem, we restrict attention to the case where $J > 1$ since if there is only a single asset it follows from the budget constraint that $z_1 = I/P_1$ no matter what form the utility function takes.

for all $j = 1, ..., J$ and all $s = 1, ..., S$. For any given $j \in \{1, 2, ..., J\}$, let $(p_1, p_2, ..., p_S)$ be the unique solution of the following set of equations[11]

$$\frac{\partial P_j}{\partial \pi_s} = \left(\xi_{js} - \frac{x_s P_j}{I}\right)\frac{p_s}{\pi_s} - \left(\xi_{jS} - \frac{x_S P_j}{I}\right)\frac{p_S}{\pi_S} \quad (s = 1, 2, ..., S-1) \tag{84}$$

and

$$I = \sum_{s=1}^{S} p_s x_s. \tag{85}$$

Then asset demand can be rationalized by a state independent Expected Utility function if and only if eqn. (82) holds and the induced contingent claim demands pass the k-test in Theorem 1.

Proof. We first show necessity. If preferences can be represented by a state independent Expected Utility function, then we have

$$U(x_1, .., x_S; \pi_1, .., \pi_S) = \sum_{s=1}^{S-1} \pi_s u(x_s) + \left(1 - \sum_{s=1}^{S-1} \pi_s\right) u(x_S), \tag{86}$$

where

$$x_s = \sum_{j=1}^{J} \xi_{js} z_j. \tag{87}$$

It follows from the first order conditions for (81) that

$$\sum_{s=1}^{S-1} \pi_s \xi_{js} u'(x_s) + \left(1 - \sum_{s=1}^{S-1} \pi_s\right) \xi_{jS} u'(x_S) = \mu P_j, \tag{88}$$

where μ is the Lagrange multiplier. Given the inverse demand function $P_j(\pi_1, ..., \pi_S, z_1, ..., z_J, I)$, differentiating eqn. (88) with respect to π_s ($s \in \{1, 2, ..., S-1\}$), one can obtain

$$\xi_{js} u'(x_s) - \xi_{jS} u'(x_S) = \frac{\partial \mu}{\partial \pi_s} P_j + \frac{\partial P_j}{\partial \pi_s} \mu. \tag{89}$$

Differentiating the budget constraint with respect to π_s ($s \in \{1, 2, ..., S-1\}$), it follows that

$$\sum_{j=1}^{J} \frac{\partial P_j}{\partial \pi_s} z_j = 0. \tag{90}$$

[11] Since eqns. (84) and (85) consist of S independent linear equations in $(p_1, p_2, ..., p_S)$, one can always find a unique solution no matter whether the demands are rationalizable by an Expected Utility function or not. However, if the corresponding demands cannot be rationalized by an Expected Utility function, the solution can fail to satisfy eqn. (82) or the associated contingent claim demands can fail to pass the k-test in Theorem 1.

Combining eqn. (89) with (90) yields

$$u'(x_s)\,x_s = u'(x_s)\sum_{j=1}^{J}\xi_{js}z_j = u'(x_S)\sum_{j=1}^{J}\xi_{jS}z_j + \frac{\partial\mu}{\partial\pi_s}\sum_{j=1}^{J}P_jz_j = u'(x_S)\,x_S + \frac{\partial\mu}{\partial\pi_s}I. \tag{91}$$

Substituting the above equation into (89) one obtains

$$\xi_{js}u_s'(x_s) - \xi_{jS}u_S'(x_S) = \frac{(u'(x_s)\,x_s - u_S'(x_S)\,x_S)\,P_j}{I} + \frac{\partial P_j}{\partial\pi_s}\mu, \tag{92}$$

or equivalently

$$\mu\frac{\partial P_j}{\partial\pi_s} = \left(\xi_{js} - \frac{x_sP_j}{I}\right)u'(x_s) - \left(\xi_{jS} - \frac{x_SP_j}{I}\right)u'(x_S). \tag{93}$$

The statement follows by defining $p_s = \frac{\pi_s u'(x_s)}{\mu}$.

To prove sufficiency, simply observe that any $p_1, ..., p_S$ that satisfy Equations (82) are potential contingent claim prices. If in addition the conditions of Theorem 1 hold, there must be a state independent Expected Utility function that generates the given asset demand function. ■

Next we use a concrete example to illustrate the above discussion.

Example 3 *Assume two assets and three states, with asset payoffs*

$$\xi_{11} = 1,\ \xi_{12} = 0,\ \xi_{13} = \frac{1}{2},\ \xi_{21} = 0,\ \xi_{22} = 1,\ \xi_{23} = \frac{1}{2}, \tag{94}$$

and financial asset demands are given by

$$z_1 = \frac{(1+\pi_1)\,P_1 - (1-\pi_2)\,P_2 - \sqrt{((1-\pi_1)\,P_1 - (1-\pi_2)\,P_2)^2 + 4\pi_1\pi_2P_1P_2}}{2P_1\,(P_1 - P_2)}I \tag{95}$$

and

$$z_2 = \frac{(1-\pi_1)\,P_1 - (1+\pi_2)\,P_2 + \sqrt{((1-\pi_1)\,P_1 - (1-\pi_2)\,P_2)^2 + 4\pi_1\pi_2P_1P_2}}{2P_2\,(P_1 - P_2)}I. \tag{96}$$

It is easy to derive the following relations between financial assets and contingent claims

$$x_1 = \xi_{11}z_1 + \xi_{21}z_2 = z_1, \tag{97}$$

$$x_2 = \xi_{12}z_1 + \xi_{22}z_2 = z_2, \tag{98}$$

$$x_3 = \xi_{13}z_1 + \xi_{23}z_2 = \frac{1}{2}(z_1 + z_2), \tag{99}$$

$$P_1 = \xi_{11}p_1 + \xi_{12}p_2 + \xi_{13}p_3 = p_1 + \frac{1}{2}p_3 \tag{100}$$

and

$$P_2 = \xi_{21}p_1 + \xi_{22}p_2 + \xi_{23}p_3 = p_2 + \frac{1}{2}p_3. \tag{101}$$

It follows that

$$x_1 = \frac{(1+\pi_1)\left(p_1 + \frac{1}{2}p_3\right) - (1-\pi_2)\left(p_2 + \frac{1}{2}p_3\right) - \sqrt{A}}{2\left(p_1 + \frac{1}{2}p_3\right)(p_1 - p_2)} I \tag{102}$$

and

$$x_2 = \frac{(1-\pi_1)\left(p_1 + \frac{1}{2}p_3\right) - (1+\pi_2)\left(p_2 + \frac{1}{2}p_3\right) + \sqrt{A}}{2\left(p_2 + \frac{1}{2}p_3\right)(p_1 - p_2)} I, \tag{103}$$

where

$$A = \left((1-\pi_1)\left(p_1 + \frac{1}{2}p_3\right) - (1-\pi_2)\left(p_2 + \frac{1}{2}p_3\right)\right)^2 + 4\pi_1\pi_2\left(p_1 + \frac{1}{2}p_3\right)\left(p_2 + \frac{1}{2}p_3\right). \tag{104}$$

Therefore, we have

$$x_2 = \frac{\left((1-\pi_1)\left(p_1 + \frac{1}{2}p_3\right) - (1+\pi_2)\left(p_2 + \frac{1}{2}p_3\right) + \sqrt{A}\right)\left(p_1 + \frac{1}{2}p_3\right)}{\left((1+\pi_1)\left(p_1 + \frac{1}{2}p_3\right) - (1-\pi_2)\left(p_2 + \frac{1}{2}p_3\right) - \sqrt{A}\right)\left(p_2 + \frac{1}{2}p_3\right)} x_1. \tag{105}$$

It seems that one cannot express x_2 as a function of x_1 and $\frac{\pi_2 p_1}{\pi_1 p_2}$. However since p_3 is not independent of p_1 and p_2, in order to apply Theorem 1 we must recover (p_1, p_2, p_3) from (P_1, P_2) uniquely as discussed in Theorem 3. To apply Theorem 3, first derive the following inverse asset demand functions from (95)-(96)

$$P_1 = \frac{(1-\pi_2)z_1 + \pi_1 z_2}{(z_1 + z_2)z_1} I \tag{106}$$

and

$$P_2 = \frac{(1-\pi_1)z_2 + \pi_2 z_1}{(z_1 + z_2)z_2} I. \tag{107}$$

It follows from Theorem 3 that

$$\frac{\partial P_1}{\partial \pi_1} = \frac{z_2 I}{(z_1 + z_2)z_1} = \left(\xi_{11} - \frac{x_1 P_1}{I}\right)\frac{p_1}{\pi_1} - \left(\xi_{13} - \frac{x_3 P_1}{I}\right)\frac{p_3}{1 - \pi_1 - \pi_2}, \tag{108}$$

and

$$\frac{\partial P_1}{\partial \pi_2} = -\frac{I}{z_1 + z_2} = \left(\xi_{12} - \frac{x_2 P_1}{I}\right)\frac{p_2}{\pi_2} - \left(\xi_{13} - \frac{x_3 P_1}{I}\right)\frac{p_3}{1 - \pi_1 - \pi_2}. \tag{109}$$

Combining the above equations with the budget constraint

$$\sum_{s=1}^{3} p_s x_s = I, \tag{110}$$

one can obtain contingent claim prices as

$$p_1 = \frac{P_1 - P_2}{1 - 1/B}, \tag{111}$$

$$p_2 = \frac{P_1 - P_2}{B - 1} \tag{112}$$

and

$$p_3 = \frac{2\left(P_2 - P_1/B\right)}{1 - 1/B}, \tag{113}$$

where

$$B = \frac{\pi_1 P_1 \left(\left(1 - \pi_1\right) P_1 - \left(1 + \pi_2\right) P_2 + \sqrt{\left(\left(1 - \pi_1\right) P_1 - \left(1 - \pi_2\right) P_2\right)^2 + 4\pi_1\pi_2 P_1 P_2} \right)}{\pi_2 P_2 \left(\left(1 + \pi_1\right) P_1 - \left(1 - \pi_2\right) P_2 - \sqrt{\left(\left(1 - \pi_1\right) P_1 - \left(1 - \pi_2\right) P_2\right)^2 + 4\pi_1\pi_2 P_1 P_2} \right)}. \tag{114}$$

Therefore, we have

$$\frac{p_1}{p_2} = B \qquad \textit{and} \qquad \frac{p_1}{p_3} = \frac{P_1 - P_2}{2\left(P_2 - P_1/B\right)}, \tag{115}$$

implying that[12]

$$x_2 = \frac{\pi_2 p_1}{\pi_1 p_2} x_1 \qquad \textit{and} \qquad x_3 = \frac{\left(1 - \pi_1 - \pi_2\right) p_1}{\pi_1 p_3} = \frac{\pi_3 p_1}{\pi_1 p_3} x_1. \tag{116}$$

Since $x_s = k_s x_1$ and $x_s = x_1$ when setting $k_s = 1$ $(s = 2, 3)$, it follows from Theorem 1 that the demands in (95) and (96) are consistent with Expected Utility maximization where the contingent claims prices are given by (111)-(113).[13]

3 A Revealed Preference Test

In this Section we consider the revealed preference approach, where markets are first assumed to be complete and then incomplete. In both cases, we provide

[12] To obtain the relation between x_1 and x_2, first solve for I as a function of x_1 (this is possible since contingent claims must be normal goods in an Expected Utility setting). Next substitute for I in the x_2 demand function, yielding x_2 as a function of x_1, P_1, P_2 and probabilities. Based on eqns. (111) and (112) solving for P_1 and P_2 as functions of p_1 and p_2, x_2 can be expressed as a function of x_1, p_1, p_2 and probabilities. Then one can apply the k-test in Theorem 1. The relations in eqn. (116) can be easily derived following this process.

[13] As the reader will no doubt surmise from eqn. (116), the demands (95) and (96) can be rationalized by

$$U\left(z_1, z_2, \pi_1, \pi_2, \pi_3\right) = \sum_{s=1}^{3} \pi_s \ln\left(\xi_{1s} z_1 + \xi_{2s} z_2\right).$$

necessary and sufficient restrictions on a finite set of demand-price-probability triples such that the demand behavior of a consumer is consistent with state independent Expected Utility maximization.

3.1 Contingent Claims

Green and Srivastava (1986) derive a complete market revealed preference test of Expected Utility maximization in a setting where there are multiple goods in each state. As in the prior Section, we consider the simpler case where there is a single good in each state. This will enable us to eliminate the assumption of existential qualifiers required in Green and Srivastava (1986) and obtain a result that is essentially analogous to the strong axiom of revealed preference.

Using the same notation as in the prior Section when markets are complete, we have $J = S$. Suppose we are given an invertible $S \times S$ payoff matrix A and we have N observations on asset demands, prices, as well as probabilities $(\mathbf{z}^i, \mathbf{P}^i, \boldsymbol{\pi}^i)_{i=1}^N$. Since markets are complete, we can transform our observations into $(\mathbf{x}^i, \mathbf{p}^i, \boldsymbol{\pi}^i)_{i=1}^N$, where $\mathbf{x} = A\mathbf{z}$ and $\mathbf{P}^T = \mathbf{p}^T A$. Without loss of generality, we assume throughout this Section that $x_s^i \neq x_{s'}^j$ for all i, j and all s, s' (the case where $x_s^i = x_{s'}^j$ for some i, j and some s, s' can be easily considered separately). In order to eliminate the assumption of existential qualifiers required in Green and Srivastava (1986), it will prove convenient to define $\rho_s^i = p_s^i / \pi_s^i$ for all $i = 1, ..., N$, $s = 1, ..., S$, and the operator

$$L(i,j) = \max_{s,s':\, x_s^i > x_{s'}^j} \frac{\rho_s^i}{\rho_{s'}^j}, \tag{117}$$

where we set $L(i,j) = 0$ if $x_s^i < x_{s'}^j$ for all s, s'.

Definition 1 *We say that $(\mathbf{x}^i, \boldsymbol{\rho}^i)_{i=1}^N$ satisfy the strong axiom of revealed expected utility (SAREU) if for each i and all s, s'*

$$x_s^i > x_{s'}^i \Rightarrow \rho_s^i < \rho_{s'}^i \tag{118}$$

and for any $m > 1$ all $i_1, ..., i_m \in \{1, ..., N\}$ we have

$$L(i_1, i_2) \cdot L(i_2, i_3) \cdot ... \cdot L(i_{m-1}, i_m) \cdot L(i_m, i_1) < 1. \tag{119}$$

Comparing our analysis with the certainty formulation of Afriat (1967), it can be observed that SAREU plays the role of Afriat's cyclic consistency which as Varian (1982) argues is a version of the general axiom of revealed preference (GARP). As in Afriat our condition (119) must hold for all cycles. However our condition involves the product of the risk neutral prices (i.e., $\frac{p_s}{\pi_s}$) rather than a sequence of price-quantity inequalities.

We have the following theorem.[14]

Theorem 4 *The following three statements are equivalent.*

(i) *There exists a continuous, increasing and strictly concave NM index* $u : \mathbb{R}_+ \to \mathbb{R}$ *such that*

$$\mathbf{x}^i \in \arg\max \sum_{s=1}^{S} \pi_s^i u(x_s) \quad S.T. \ \mathbf{p}^i \cdot \mathbf{x} \leq \mathbf{p}^i \cdot \mathbf{x}^i. \tag{120}$$

(ii) *There exist real numbers* $\lambda^i > 0$ *for* $i = 1, ..., N$ *such that for all* $i, j = 1, ..., N$ *and any* $s, s' = 1, ..., S$,

$$x_s^i > x_{s'}^j \Rightarrow \lambda^i \rho_s^i < \lambda^j \rho_{s'}^j. \tag{121}$$

(iii) *The observations* $(\mathbf{x}^i, \boldsymbol{\rho}^i)_{i=1}^N$ *satisfy SAREU.*

Proof. The equivalence between (i) and (ii) is standard. To see that (i) implies (ii), consider for example the necessary and sufficient conditions in Green and Srivastava (1986). Their result, together with the construction in Chiappori and Rochet (1987) show that (i) is equivalent to the existence of $(u_s^i)_{i=1...N, s=1...S}$ and $(\lambda^i)_{i=1}^N$ such that

$$u_{s'}^j - u_s^i \leq \lambda^i \rho_s^i (x_{s'}^j - x_s^i) \ \text{ for all } i, j, s, s', \tag{122}$$

with the strict inequality holding in the one-dimensional case, whenever $x_{s'}^j \neq x_s^i$. Adding up the inequalities for j, s' and i, s and the one for i, s and j, s', we obtain

$$0 \leq \lambda^i \rho_s^i (x_{s'}^j - x_s^i) + \lambda^j \rho_s^j (x_s^i - x_{s'}^j) \tag{123}$$

which is obviously equivalent to (ii) if we assume that $x_s^i \neq x_{s'}^j$. Since the x_s^i are scalars, the converse is also true. This is easily seen by observing that given x_s^i, ρ_s^i and λ^i for all i, s, we can construct a piecewise linear and concave utility function where $\lambda^i \rho_s^i$ is taken to be the slope at x_s^i.

Next show that (ii) implies (iii). From (ii), for $i = j$, clearly we have

$$x_s^i > x_{s'}^i \Rightarrow \lambda^i \rho_s^i < \lambda^i \rho_{s'}^j \Rightarrow \rho_s^i < \rho_{s'}^j, \tag{124}$$

[14] It follows from Varian (1988) that a necessary and sufficient condition for an Expected Utility function to exist that rationalizes some choices $(\mathbf{p}, \mathbf{x})$ is that x_s is a decreasing function of p_s/π_s. However, this test can be only applied to one set of data (since it is equivalent to Theorem 4(ii) when $i = j$) and our Theorem 4 is more general.

implying that eqn. (118) of SAREU holds. For all $i \neq j$ if $x_s^i > x_{s'}^i$, we must have that $\frac{\lambda^j}{\lambda^i} > L(i,j)$. Since $\frac{\lambda_{i_2}}{\lambda_{i_1}} \cdot \frac{\lambda_{i_3}}{\lambda_{i_2}} ... \cdot \frac{\lambda_{i_1}}{\lambda_{i_m}} = 1$, we must have that eqn. (119) of SAREU holds.

Finally we show that (iii) implies (ii). Given that SAREU holds, we can construct $(\lambda^i)_{i=1}^N$ as follows. Define $\lambda^1 = 1$ and for $j = 2, 3, ..., N$,

$$\lambda^j = \max_k \max_{(1,i_2,...,i_{k-1},j) \in \mathcal{I}_k} L(1, i_2) \cdot ... \cdot L(i_{k-1}, j), \tag{125}$$

where $\mathcal{I}_k = \{ (i_1, i_2, ..., i_{k-1}, i_k) \mid \text{ for } \forall m \in \{1, 2, ..., k\}, \, i_m \in \{1, 2, ..., N\}\}$. Note that SAREU guarantees that each λ^j is well defined. In effect the maximum is only taken over vectors of at most length N, since SAREU implies that cycles where $i_l = i_k$ for some l, k can never be optimal. To prove that (iii) implies (ii), it suffices to show that $\lambda^i > \lambda^j \frac{\rho_s^j}{\rho_{s'}^i}$ whenever $x_s^j > x_{s'}^i$. Note that there must be some k and some $i_2, ..., i_{k-1}$ so that $\lambda^j = L(1, i_2) \cdot \ldots \cdot L(i_{k-1}, j)$. Furthermore by definition, we have

$$\lambda^i \geq L(1, i_2) ... \cdot L(i_{k-1}, j) \cdot L(j, i) \geq \lambda^j \frac{\rho_s^j}{\rho_{s'}^i}. \tag{126}$$

Since one of the inequalities has to hold strictly, this proves the Theorem. ■

Remark 4 *Since probabilities are not required to be constant across the set of demand-price observations, it is natural to conclude that the data will correspond to different utility functions $U(\mathbf{x}; \boldsymbol{\pi})$ and one cannot apply the approach of Afriat (1967) to test whether the observations are consistent with utility maximization. However our proof utilizes the analysis of Green and Srivastava (1986), where the test is conducted on just the NM index u and not the full utility U. Since the variation in probabilities does not affect the form of u, the conclusion of Theorem 4 holds even if probabilities change. Of course, Theorem 4 holds for fixed probabilities and is then comparable to Theorem 1 in Green and Srivastava (1986).*

As argued above in the proof except for the fact that we allow probabilities to change, conditions (i) and (ii) of Theorem 4 are essentially the same as conditions (i) and (ii) of Theorem 1 in Green and Srivastava (1986). But since we impose the more restrictive assumption that there is only one good in each state, we are able to derive the new result Theorem 4(iii), which does not require the existential quantifiers $\boldsymbol{\lambda}$. In this sense, Theorem 4(iii) is analogous to Afriat's cyclic consistency and is analogous to the strong axiom of revealed preference.

3.2 Asset Demands: Incomplete Markets

In order to extend the Theorem 4 revealed preference test as well as that of Green and Srivastava (1986) to incomplete asset markets, one encounters the same question of how to identify the unique contingent claim prices discussed in Subsection 2.3. However since only discrete market data are observed, Theorem 3 which requires the analytical form of asset demand functions cannot be applied. Instead, to recover the contingent claim prices, it is necessary to obtain additional data. Before characterizing the process for deriving the contingent claim prices, it will prove convenient to introduce the following definition.

Definition 2 *Assume that the asset payoff matrix and income are fixed. For a given set of data* $(\mathbf{z}, \mathbf{P}, \boldsymbol{\pi})$*, a* χ*-observation is a set of asset price-probability data* $(\mathbf{P}', \boldsymbol{\pi}')$ *such that the optimal demand* $\mathbf{z}$ *is unchanged.*

In the following Theorem, we provide a sufficient condition for obtaining the contingent claim prices corresponding to $\mathbf{P}$ required to apply the test in Theorem 4 to the given observations $(\mathbf{z}, \mathbf{P}, \boldsymbol{\pi})$.

Theorem 5 *Suppose there are* J *(*$J \geq 2$*) assets and* S *(*$S > J$*) states. Assume that the asset payoff matrix* $\left(\xi_{js}\right)_{J\times S}$ *is known. For a given set of data* $(\mathbf{z}, \mathbf{P}, \boldsymbol{\pi})$*, suppose* $S-1$ *independent* χ*-observations can be obtained. If the contingent claim prices derived from these observations satisfy eqn. (82) and the induced contingent claim demands pass the test in Theorem 4, then the data* $(\mathbf{z}, \mathbf{P}, \boldsymbol{\pi})$ *are consistent with Expected Utility maximization.*

Proof. If the given asset demand-price data can be rationalized by an Expected Utility $\sum_{s=1}^{S} \pi_s u(x_s)$, then it must satisfy the following first order condition

$$\mu P_j = \sum_{s=1}^{S} \pi_s u'(x_s) \xi_{js}, \tag{127}$$

where μ is the Lagrange multiplier. For the ith $(i \in \{1, 2, ..., S-1\})$ χ-observation, we have

$$\mu^{(i)} P_j^{(i)} = \sum_{s=1}^{S} \pi_s^{(i)} u'(x_s) \xi_{js}, \tag{128}$$

where $\mu^{(i)}$ is the Lagrange multiplier. Subtracting eqn. (128) from (127) yields

$$P_j - \frac{\mu^{(i)}}{\mu} P_j^{(i)} = \sum_{s=1}^{S} \left(\pi_s - \pi_s^{(i)}\right) \xi_{js} \frac{u'(x_s)}{\mu}. \tag{129}$$

Noticing that

$$I = P_j z_j = P_j^{(i)} z_j \quad \text{and} \quad x_s = \sum_{j=1}^{J} \xi_{js} z_j, \tag{130}$$

multiplying z_j on both sides of eqn. (129) and summing from 1 to J, yields

$$\frac{\mu^{(i)}}{\mu} = 1 - \sum_{s=1}^{S} \left(\pi_s - \pi_s^{(i)}\right) x_s \frac{u'(x_s)}{\mu I}. \tag{131}$$

Substituting eqn. (131) into (129) yields

$$P_j - \left(1 - \sum_{s=1}^{S} \left(\pi_s - \pi_s^{(i)}\right) x_s \frac{u'(x_s)}{\mu I}\right) P_j^{(i)} = \sum_{s=1}^{S} \left(\pi_s - \pi_s^{(i)}\right) \xi_{js} \frac{u'(x_s)}{\mu}, \tag{132}$$

which holds for any $i \in \{1, 2, ..., S-1\}$. Since

$$p_s = \frac{\pi_s u'(x_s)}{\mu}, \tag{133}$$

the budget constraint for the contingent claim setting can be written as

$$\sum_{s=1}^{S} \frac{\pi_s u'(x_s)}{\mu} x_s = I. \tag{134}$$

Combining eqn. (134) with (132), we have S independent equations for S unknowns $\frac{u'(x_s)}{\mu}$ $(s = 1, 2, ..., S)$. Therefore, we can obtain a unique solution, which can be transformed into contingent claim prices based on eqn. (133). ■

Remark 5 *It is clear that the conditions in Theorem 5 are very demanding. If one wants to apply Theorem 4 to N sets of asset price-demand data in an incomplete market, then $N \times (S-1)$ additional χ-observations are required. However obtaining real world price-probability observations would not seem reasonable. Hence rather than conducting a conventional revealed preference test, it will be necessary to design a specifically designed laboratory test as briefly discussed at the end of the next Section.*[15] *Absent such a lab test, Theorem 5 should be viewed as a theoretical result.*

4 Potential Lab Tests

Given that the demand tests developed in the prior Sections along with those of Varian (1983) and (1988), Polemarchakis and Selden (1983) and Green and

[15] For instance, this lab test could involve asking participants to provide for each demand-price observation alternative asset prices corresponding to different probabilities such that their allocation of assets doesn't change.

Srivastava (1986) require knowledge of the state probabilities and asset payoff matrix which are not obtainable from real world market data, it is natural to ask how these tests might be applied in practice. To address this question, we offer some observations on how the new laboratory methodologies of Choi, et al. (2007) and Andreoni and Sprenger (2012a) might be applied in our setting. The idea would be for participants in a lab setting to be confronted with specific demand problems designed to test whether their responses are consistent with particular preference properties. In our setting, such an approach would allow the researcher to know the state probabilities, asset payoff matrix and asset prices.[16] It would also allow the researcher, as suggested in footnote 16, to potentially obtain a substantially larger sample of observations.

We next sketch out briefly a possible lab test design for Expected Utility maximization based on the Theorem 1 k-test and then comment on testing the Theorem 4 revealed preference conditions. Assume for simplicity that there is one risk free asset and one risky asset, where respectively the quantities demanded are denoted by z_1 and z_2. Each participant or subject in the test is given initial wealth or income of $I = 1$ and is informed that the risk free asset pays ξ_f in each state of nature. For the Theorem 1 test, assume complete markets where the risky asset pays ξ_{2s} $(s = 1, 2)$ with probability π_s in state s. Participants are aware of the simple bivariate return distribution for the risky asset. P_1 and P_2 denote, respectively, the price of the risk free and risky assets. Define the risky asset expected (gross) rate of return and the risk free asset (gross) rate of return as follows

$$E\widetilde{R} = \frac{E\widetilde{\xi}}{P_2} = \frac{\pi_1\xi_{21} + \pi_2\xi_{22}}{P_2} \qquad \text{and} \qquad R_f = \frac{\xi_f}{P_1}. \tag{135}$$

Each subject is asked to construct her optimal portfolio (z_1, z_2) subject to the budget constraint

$$P_1 z_1 + P_2 z_2 = 1. \tag{136}$$

[16] In contrast to laboratory demand tests for Expected Utility maximization, the typical approach for testing whether an individual's preferences are representable by an Expected Utility function is to offer subjects binary choices over sets of lotteries. Whether it be Expected Utility or other competing models such as Prospect Theory, as one paper observes "these theories tend to be systematically disconfirmed by the data" (Choi, et al. 2007, p. 1922). One important advantage of the new demand lab tests over lottery tests is that choices from convex budget sets in the former provide more information about preferences than do discrete binary choices in the latter. As a result, direct application of the demand lab tests where the subjects make portfolio choices subject to the traditional budget constraints would seem to be quite promising. Finally, one interesting open question is whether the behavior of individuals when facing market simulated asset choices is consistent with their choices when confronting abstract lotteries. Will an individual whose asset demands pass the k-test in Theorem 1 also make lottery choices that are consistent with Expected Utility?

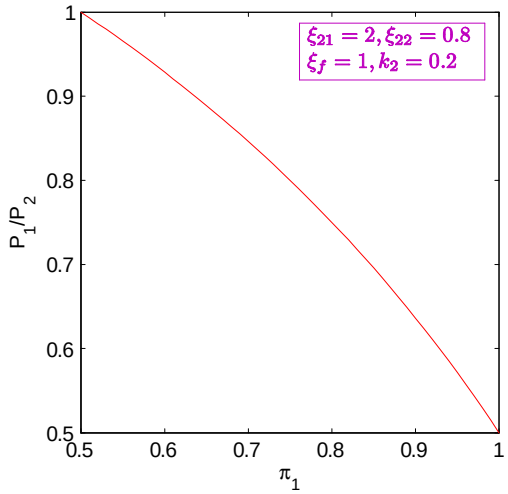

Figure 1:

Assuming the probabilities of the risky asset payoffs are changed from π_s to π'_s, each subject is asked to provide a new ratio risk premium $E\widetilde{R}/R_f$ such that she has no desire to change the portfolio from (z_1, z_2) that was chosen based on π_s.[17] If a participant's preferences are consistent with the Expected Utility model, then it follows from Theorem 1 that $x_2 = f(x_1, k_2)$, where

$$x_1 = \xi_f z_1 + \xi_{21} z_2 \quad \text{and} \quad x_2 = \xi_f z_1 + \xi_{22} z_2 \tag{137}$$

and

$$k_2 = \frac{\pi_2 p_1}{\pi_1 p_2} = \frac{\pi_2 \left(\xi_f P_2 - \xi_{22} P_1\right)}{\pi_1 \left(\xi_{21} P_1 - \xi_f P_2\right)} = \frac{\pi_2 \left(\xi_f - \xi_{22} P_1/P_2\right)}{\pi_1 \left(\xi_{21} P_1/P_2 - \xi_f\right)}. \tag{138}$$

Given that (z_1, z_2) is required to remain the same when probabilities change, (x_1, x_2) will not change and k_2 must be constant, implying that

$$\frac{P_1}{P_2} = \frac{(\pi_1 k_2 + \pi_2)\, \xi_f}{\pi_1 k_2 \xi_{21} + \pi_2 \xi_{22}}. \tag{139}$$

It is clear that the participant's demand behavior will be consistent with Expected Utility maximization if and only if her selected price ratio P_1/P_2 satisfies this relation.

For example fixing

$$\xi_{21} = 2, \ \xi_{22} = 0.8 \text{ and } \xi_f = 1 \tag{140}$$

[17] To test Theorem 1, the price ratio P_1/P_2 is required. However we have substituted the risk premium $E\widetilde{R}/R_f$ based on the belief that it would be easier for the participants to provide. One can easily recover the price ratio P_1/P_2 from the return ratio $E\widetilde{R}/R_f$ since the probabilities and payoffs are known.

and providing the parameters

$$P_1 = P_2 = 1 \quad \text{and} \quad \pi_1 = \pi_2 = 0.5, \tag{141}$$

the participants can be asked to construct their optimal portfolio. Then when varying π_1 in the range $[0.5, 1]$, the P_1/P_2 ratio must be on the curve shown in Figure 1 if the participant's demand behavior is consistent with Expected Utility maximization.

The discussion above is based on the demand function approach. For the revealed preference approach, we can test the SAREU following a procedure paralleling that in Choi, et al. (2007). In the case of complete markets, the application of Theorem 4 is straightforward since we can simply vary the prices of the financial assets or probabilities and observe the corresponding asset demands of the participants. Then one can readily verify whether or not the SAREU holds. In the incomplete market case, we should follow Theorem 5 in conducting additional lab tests to recover the contingent claim prices from the asset prices and then test SAREU.

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