Games Played through Agents *

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Abstract

We introduce a game of complete information with multiple principals and multiple agents. Each agent makes a decision that can affect the payoffs of all principals. Each principal offers monetary transfers to each agent conditional on the action taken by the agent. We characterize pure-strategy equilibria and we provide conditions – in terms of game balancedness – for the existence of an equilibrium with an efficient outcome. Games played through agents display a type of strategic inefficiency which is absent when either there is a unique principal or there is a unique agent.

1 Introduction

A *game played through agents* (GPTA) is a game where a set of players (*the agents*) take decisions that affect the payoffs of another set of players (*the principals*) and the principals can, by means of monetary inducements, try to influence the decisions of the agents. In other words, a game played though agents is a multi-principal multi-agent game.

The original principal-agent framework – which has one principal and one agent – has been extended in a general way in two directions: (1) Many principals and one agent (Bernheim and Whinston's [4] common agency); and (2) One principal and many agents (Segal's [22] contracting with externalities). The objective of this paper is to consider the general case with many principals and many agents. Multi-principal multi-agent problems arise in political economy, mechanism design, industrial organization, and labor markets:

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Lobbying A widespread way of modeling interest group politics is through common agency (e.g. Dixit, Grossman, and Helpman [9]). There are many lobbies (principals) and one politician (the agent). However, the assumption of a unique politician is unrealistic because modern democracies are characterized by a multiplicity of public decision-makers. This is true both in terms of organs (separation of powers) and in terms of organ members (many organs – such as parliaments – are collegial). It would be important to know how our understanding of the lobbying process is modified by the presence of multiple policy makers.[1]

Multiple Auctions Consider a set of auctioneers, each of whom has one object for sale and sells it through a first-price sealed bid auction. There is a set of bidders who put a value on each subset of objects, and these values are known to all bidders. To see that this game is a GPTA, interpret each bidder as a principal and each auctioneer as an agent whose action set consists in choosing which principal gets his object. The game could be extended to allow auctioneers to have more than one object and externalities among bidders (See Bernheim and Whinston [4] for a discussion on how multi-principal games can be seen as very general forms of first-price auctions). A natural question to ask is whether the allocation that arises in multiple auctions is efficient.

Vertical Restraints An industry with several firms (*sellers*) produces goods that are used by another set of players (*buyers*), who can be final consumers or intermediate producers. The sellers propose contracts to the buyers. A contract offered by one seller may be nonlinear and may cover not only the relation between that supplier and the buyer, but also the relation between the buyer and the other suppliers, such as an exclusive clause. While vertical restraints are sometimes viewed as anti-competitive, members of the Chicago School, in particular Bork [7, p. 280–309], have argued that the contractual arrangements that arise in equilibrium are efficient from a production point of view. Bernheim and Whinston [5] use common agency to show that the equilibrium contract maximizes the joint surplus of the sellers and the buyer. Is this efficiency result still true when there are multiple buyers?

Two-Sided Matching with Monetary Transfers Firms are looking to hire workers (or sport teams are looking to hire players). A firm can hire many workers. The output of each firm depends on what workers it employs, with the possibility of positive or negative externalities between workers. Workers may have preferences about which firm they work for, and of course they care about salary. Each firm makes a salary offer to each worker, and then each worker chooses a firm. Is the resulting match in any sense efficient? This model is taken from Pérez-Castrillo [18]. More about the connection with Pérez-Castrillo's work will be said in Section 6.2.[2]

[1] Groseclose and Snyder's [11] and Diermeier and Myerson [8] are exceptions in that they consider multiple policy-makers. In Section 4, we will consider their vote buying model in detail. See also Grossman and Helpman [12, 13] for models with multiple lobbies and multiple candidates.

[2] Another interesting example of games played through agents is provided by Besley and Seabright [6] in international taxation: national governments (principals) compete to attract international firms (agents) by offering subsidies and tax breaks to firms that relocate on their territory. In some practically relevant cases, this game has only inefficient equilibria

A GPTA is defined by a set of principals and a set of agents. Each agent must choose an action out of a feasible set of actions (policy choices in the case of lobbying, quantity orders in supply contracts, object allocations in auctions). Each principal offers to each agent a schedule of monetary transfers contingent on the agent choosing a certain action (campaign contributions, supply contracts, bids). Given the principals' transfer schedules, an agent chooses his action to maximize the sum of transfers he receives from the principals minus the cost of undertaking the action. A principal chooses her transfer schedules to maximize the utility from the agents' actions minus the sum of transfers he makes to agents.

The game is played in two stages. First all principals simultaneously choose their transfer schedules and then all agents observe the schedules and simultaneously choose their actions. In Section 6.5 we discuss possible sequential variants of the simultaneous game.

Our main focus is efficiency, which, in line with the other contributions in this area, is defined as surplus maximization. An outcome is efficient if it maximizes the sum of the payoffs of all agents and all principals. If there is a unique agent, Bernheim and Whinston have shown that there always exists an equilibrium (the *truthful equilibrium*) that produces an efficient outcome. If, instead, there is a unique principal, Segal shows that, if a certain type of externalities among agents' payoff functions is absent, then there always exists an efficient equilibrium. Hence, in both these polar cases, if there are no direct externalities among agents, efficient equilibria exist.

This efficiency result is important in the case of lobbying because it means that the outcome of the influence process will maximize the sum of the payoffs of all the players involved in the game, agent and principals. In many models this allows to find the outcome of lobbying even if we have difficulty finding the equilibrium campaign contributions. It is also important in supply contracts because it gives support to Judge Bork's Thesis.

In order to focus on a cause of inefficiency that is different from Segal's direct externalities, we assume that each agent cares only about the action he takes and the money he gets. However, simple examples show that, even in the absence of direct externalities, a multi-principal multi-agent game need not have an efficient equilibrium. The presence of multiple players on both sides creates a strategic externality that makes it impossible to achieve the efficient outcome. As it becomes apparent by looking at examples, efficiency is closely linked to the existence of pure-strategy equilibria. Indeed, the latter is a sufficient condition for a GPTA to have an efficient equilibrium. It is also possible to that, in a more restricted environment in which there are only two principals and agents have only two actions and only care about money, it is also a necessary condition.

The main result of this paper is to provide a general necessary and sufficient condition for the existence of a pure-strategy equilibrium. This condition relates to the cooperative concept of balancedness, which we extend – with some important differences – to our game.

In the present context, balancedness has a non-cooperative interpretation in terms of weighted deviations from the equilibrium outcome and sheds light on the nature of the strategic interaction between principals and agents. The balancedness of a game can be checked in a straightforward way by computing the value of a linear program. Balancedness can also be used to show that, if the principal's payoff functions are continuous and convex, then there always exists an efficient equilibrium. This last result gives rise to simple sufficient conditions under which Bork's Thesis is correct.

The paper also explores the connection with cooperative game theory. GPTA's provide a non-cooperative foundation for the core. Every cooperative game with transferable utility (TUG) can be put in correspondence with a special type of GPTA in which two identical principals compete to hire agents (the action of the agent consists in selecting one of the principals). The value of a coalition in the TUG becomes the payoff of principal who hires the agents in that coalition. We show that the core of a superadditive TUG is nonempty if and only if the corresponding GPTA has an efficient equilibrium.

The organization of the paper is as follows. We begin with the formal presentation of the game in Section 2. In Section 3 we give a characterization of pure-strategy equilibria that we will use in the rest of the paper. In Section 4, we focus on a simplified version of the game, in which each agent has only two possible actions and has no direct preference over actions. This simplification avoids issues of coordination among principals. The main result is a necessary and sufficient condition for the existence of a pure-strategy equilibrium. In Section 5 we examine the general case, in which agents can have more than two actions and care about actions. To deal with coordination problems, we introduce and study *weakly truthful equilibria*, which are an extension of Bernheim and Whinston's truthful equilibrium to games with many agents. We give necessary and sufficient conditions for their existence, again in terms of balancedness. Section 6 concludes by discussing several closely related issues: the existence and inefficiency of mixed-strategy equilibria; the connection between equilibria of GPTA and core allocations; the application of balancedness to convex environments and, hence, Bork's Thesis; outcome-contingent contracts (an agent is offered transfers conditional not only on what he does but also on what other agents do); sequential versions of the game; and direct externalities among agents.

2 Games Played through Agents

There is a set M of principals and a set N of agents. Let m denote the typical element of M and n the typical element of N. With an abuse of notation, the symbols M and N are sometimes used to denote also the respective cardinality of sets M and N. We emphasize that there is no natural relation between any of the principals and any of the agents. The game takes place in two stages: first the principals move simultaneously, then the agents move simultaneously.

Each agent has a finite pure set of *actions* S_n. Let $S \equiv \prod_{n \in N} S_n$ and H the disjoint union of the sets S_n over $n \in N$. The typical element of S_n is denoted by s_n, the typical element of S is a tuple $s = (s_1, ..., s_N)$ and is called an *outcome*. Each element of H is naturally associated to the pair (n, s_n) specifying the agent and the action of that agent. We write in the following an element of H as a pair (n, s_n). Each principal chooses a vector of nonnegative *transfers* $t^m \in \Re^{+H}$ which specifies the transfer from her to each agent for each action of that agent. Thus, $t^m_n(s_n)$ is the transfer of principal m to agent n conditional on agent n choosing action s_n. Agent n receives money only for the action that he actually chooses, but he may receive money from more than one principal.

We have assumed that the transfer from a principal to an agent can only be contingent on the action chosen by that agent. The transfer could depend also on the actions chosen by other agents, in which case we would write $t_n^m(s)$ instead of $t_n^m(s_n)$. Section 6.4 will examine this type of transfers, which we call *outcome-contingent*. While no general characterization is available, a simple example shows that allowing for outcome-contingent transfers does not solve the efficiency problem.[3]

Each agent cares about what action he chooses and how much money he gets. Agent n's payoff if principals offer t and he selects s_n is $F_n(s_n) + \sum_{m \in M} t_n^m(s_n)$. From Segal [22] we know that in the one-principal case, if agents care about the actions taken by other agents, then there need not be an efficient equilibrium. Thus, for our purpose it is interesting to restrict attention to the utility function of agent n to include his action but not actions taken by other agents. The term $F_n(s_n)$ can be interpreted as cost of effort in the principal-agent tradition.

Principals care both about money and the actions that agents choose. Let $G^m(s)$ be the gross payoff to Principal m if action s is chosen by the agents. The net payoff to Principal m if she offers transfers $\{t_n^m(s_n)\}_{(n,s_n)\in H}$ and agents choose $\hat{s}$ is $G^m(\hat{s}) - \sum_{n\in N} t_n^m(\hat{s}^n)$.[4]

Throughout the paper, we adopt the convention that superscripts denote principals, subscripts denote agents, while arguments inside brackets are reserved for actions or outcomes.

The extensive form game is as follows. First, each principal chooses her vector of transfers to the agents simultaneously and non-cooperatively. Second, the vectors of all principals are publicly announced to agents, who then choose their actions. Although it is not crucial, we assume that each agent also observes offers made to other agents.

We remark here – leaving the formal discussion to Section 6.1 – that a GPTA always has a subgame-perfect equilibrium, which may be in mixed strategy. In what follows we focus our interest on pure-strategy equilibria. The strategy set of Principal m is the subset $T^m \equiv \Re^{+H}$. A pure strategy for m is simply an element of T^m. Let $T \equiv \Pi_{m\in M} T^m$. The action set for agent n is S_n. A pure strategy for Agent n is $\sigma_n : T \to S_n$. A *pure-strategy equilibrium* is a subgame-perfect equilibrium of the two-stage game in which all agents and principals use pure strategies:

Definition 1 *A pure-strategy equilibrium of a GPTA is a pair* $(\hat{t}, \hat{\sigma})$ *(with* $\hat{t} = (\hat{t}_n^m(s_n))_{m\in M, n\in N, s_n\in S_n}$ *and* $\hat{\sigma} = (\hat{\sigma}_n)_{n\in N}$*) in which:*

(i) For every $n \in N$*, and every* $t \in T$*,*

$$\hat{\sigma}_n(t) \in \ argmax\ _{s_n\in S_n} F_n(s_n) + \sum_{m\in M} t_n^m(s_n)$$

(ii) For every $m \in M$*, given* $(\hat{t}^j)_{j\neq m}$*,* $\hat{t}^m$ *solves:*

$$\max_{t^m \geq 0} G^m(\hat{\sigma}(t^m, \hat{t}^{-m})) - \sum_{n\in N} t_n^m(\hat{\sigma}(t^m, \hat{t}^{-m}))$$

[3] One might also allow principals to use more complex mechanisms (Epstein and Peters [10] and Martimort and Stole [16]).

[4] The separability assumption for both principals and agents does not appear to be crucial to the results presented here, as it is not crucial to the results obtained in common agency (Dixit, Grossman, and Helpman [9]).

As in previous work on common agency, efficiency is defined as surplus maximization. An outcome s is efficient if it maximizes the sum of the net payoffs of all players (agents and principals). Transfers can be neglected, and the definition of efficiency is:

Definition 2 *The outcome s^* is efficient if*

$$\sum_{\mathsf{n}\in\mathsf{N}} F_{\mathsf{n}}(s^*_{\mathsf{n}}) + \sum_{\mathsf{m}\in\mathsf{M}} G^{\mathsf{m}}(s^*) \geq \sum_{\mathsf{n}\in\mathsf{N}} F_{\mathsf{n}}(s_{\mathsf{n}}) + \sum_{\mathsf{m}\in\mathsf{M}} G^{\mathsf{m}}(s) \tag{1}$$

for every $s \in S$.

We refer to the action profile that agents play in equilibrium as the *outcome* of that equilibrium. We sometimes say that an "equilibrium is efficient," meaning that the outcome of that equilibrium is efficient.

If we take our framework and let $M = \{1\}$, we have Segal [22] with no direct externalities, and we know from his Proposition 1 that the game always has an efficient equilibrium. If instead we take $N = \{1\}$, we get Bernheim and Whinston [4] and, again, we know that the game always has an efficient equilibrium. The question we ask here is whether efficiency also holds for a generic M and N.

3 A First Characterization of Pure-Strategy Equilibria

A pure-strategy equilibrium is characterized by three conditions, which are formally reported in Theorem 1 below. They are derived using the idea, common in principal-agent problems, that we may think of principals as choosing the action of the agents, provided they give the appropriate incentive to the agents. The conditions are:

1. Each agent chooses an action that maximizes his payoff, given the transfers of the principals. This is the condition (AM) (Agent Maximization) below.

2. Given the transfers of the other principals, Principal m can induce agents to choose any particular action provided that she puts high enough transfers on that action. The minimum cost for m to convince agent n to move from $\hat{s}_{\mathsf{n}}$ to s_{n} is $F_{\mathsf{n}}(\hat{s}_{\mathsf{n}}) + \sum_{\mathsf{j}\neq\mathsf{m}} \hat{t}^{\mathsf{j}}_{\mathsf{n}}(\hat{s}_{\mathsf{n}}) - F_{\mathsf{n}}(s_{\mathsf{n}}) - \sum_{\mathsf{j}\neq\mathsf{m}} \hat{t}^{\mathsf{j}}_{\mathsf{n}}(s_{\mathsf{n}})$. If $\hat{s}$ is an equilibrium, then the cost of a deviation must be greater than the benefit of a deviation for each m and each s, which is what Condition (IC) (Incentive Compatibility) requires.

3. Each principal sets her transfers so that the cost of implementing $\hat{s}$ is minimal. There cannot be a way in which principal m reduces the equilibrium transfers for $\hat{s}$ without deviating from $\hat{s}$. This is condition (CM) (Cost Minimization).[5]

Formally, this is the characterization:

[5] We call agent maximization what is usually called incentive-compatibility in principal-agent problems. However, it is useful here to reserve the term incentive compatibility for the principals' choices. While the agent maximization problem is defined by one set of conditions (AM), the principal maximization problem is defined by two sets of conditions (IC) and (CM). (IC) puts restrictions across actions, while (CM) puts restriction on the equilibrium transfer.

Theorem 1 *A pair $(\hat{t}, \hat{s})$ of transfers and outcome arises in a pure-strategy equilibrium if and only if the following conditions are satisfied:*

(AM) for every $n \in N$, $s_n \in S_n$,

$$F_n(\hat{s}_n) + \sum_{m \in M} \hat{t}^m_n(\hat{s}_n) \geq F_n(s_n) + \sum_{m \in M} \hat{t}^m_n(s_n);$$

(IC) for every $m \in M$, $s \in S$,

$$G^m(\hat{s}) + \sum_{n \in N} F_n(\hat{s}_n) + \sum_{n \in N} \sum_{j \neq m} \hat{t}^j_n(\hat{s}_n) \geq G^m(s) + \sum_{n \in N} F_n(s_n) + \sum_{n \in N} \sum_{j \neq m} \hat{t}^j_n(s_n);$$

(CM) for every $m \in M$, $n \in N$,

$$F_n(\hat{s}_n) + \sum_{j \in M} \hat{t}^j_n(\hat{s}_n) = \max_{s_n \in S_n} \left(F_n(s_n) + \sum_{j \neq m} \hat{t}^j_n(s_n) \right).$$

Proof: Condition (AM) is clearly necessary and sufficient for the action $\hat{s}_n$ to be a best response of the agent n to the transfers of the principals.

To prove the statement for the two remaining conditions, we characterize the best response of a principal m to the given choice $(\hat{t}^j)_{j \neq m}$ of transfers of the other principals. To lighten the notation, we write:

$$T^m_n(s_n) \equiv F_n(s_n) + \sum_{j \neq m} \hat{t}^j_n(s_n).$$

Principal m can induce agents to choose any outcome $s \in S$ provided he promises a transfer greater than

$$(\max_{a \in S_n} T^m_n(a)) - T^m_n(s_n)$$

to agent n for the action s_n. Principal m will not choose $\hat{s}$ (and $(\hat{s}, \hat{t})$ is not an equilibrium) unless $\hat{s}$ solves

$$\max_{s \in S} G^m(s) - \sum_{n \in N} [(\max_{a \in S_n} T^m_n(a)) - T^m_n(s_n)]. \tag{2}$$

But $\sum_{n \in N} \max_{a \in S_n} T^m_n(a)$ is a constant independent of s, so $\hat{s}$ solves problem (2) if and only if it satisfies (IC).

Finally we consider the condition (CM). The following lemma is very simple, but we state it for convenience. The proof is elementary.

Lemma 1 *For every outcome $\tilde{s}$ and every vector T^m, the cost minimization problem in $t^m = (t^m_n(s_n))_{n \in N, s \in S} \geq 0$*

$$\min_{t^m} \sum_{n \in N} t^m_n(\tilde{s}_n) \text{ subject to } t^m_n(\tilde{s}_n) + T^m_n(\tilde{s}_n) \geq t^m_n(s_n) + T^m_n(s_n), \text{ for every } n \in N, s \in S$$

has value $c(\tilde{s}, T^m) = \sum_{n \in N} [(\max_{a \in S_n} T^m_n(a)) - T^m_n(\tilde{s}_n)]$, and solution any $t^m_n(s_n) \geq 0$ such that:

$$t^m_n(\tilde{s}_n) = (\max_{a \in S_n} T^m_n(a)) - T^m_n(\tilde{s}_n),$$

$$t^m_n(s_n) \leq (\max_{a \in S_n} T^m_n(a)) - T^m_n(s_n), \text{ for every } s_n.$$

If $\tilde{s}$ is replaced with the candidate equilibrium outcome $\hat{s}$, we get that t^{m} is a solution of the cost minimization problem subject to $\hat{s}$ being implemented if and only if:

$$\hat{t}_{\mathsf{n}}^{\mathsf{m}}(\hat{s}_{\mathsf{n}}) = \max_{a \in \mathsf{S}_{\mathsf{n}}} \hat{T}_{\mathsf{n}}^{\mathsf{m}}(a) - \hat{T}_{\mathsf{n}}^{\mathsf{m}}(\hat{s}_{\mathsf{n}}) \tag{3}$$

and

$$\hat{t}_{\mathsf{n}}^{\mathsf{m}}(s_{\mathsf{n}}) \leq \max_{a \in \mathsf{S}_{\mathsf{n}}} \hat{T}_{\mathsf{n}}^{\mathsf{m}}(a) - \hat{T}_{\mathsf{n}}^{\mathsf{m}}(s_{\mathsf{n}}) \tag{4}$$

for every s. But (3) and (AM) imply (4). Since (3) is (CM), we have concluded our proof. ■

An immediate consequence of the theorem is:

Corollary 1 *For all n, either $\sum_{\mathsf{m} \in \mathsf{M}} \hat{t}_{\mathsf{n}}^{\mathsf{m}}(\hat{s}_{\mathsf{n}}) = 0$, or there is an $\tilde{s}_{\mathsf{n}} \neq \hat{s}_{\mathsf{n}}$ such that:*

$$F_{\mathsf{n}}(\hat{s}_{\mathsf{n}}) + \sum_{\mathsf{m} \in \mathsf{M}} \hat{t}_{\mathsf{n}}^{\mathsf{m}}(\hat{s}_{\mathsf{n}}) = F_{\mathsf{n}}(\tilde{s}_{\mathsf{n}}) + \sum_{\mathsf{m} \in \mathsf{M}} t_{\mathsf{n}}^{\mathsf{m}}(\tilde{s}_{\mathsf{n}}). \tag{5}$$

For every agent, either the agent gets no money for the equilibrium action, or he must be indifferent between choosing the equilibrium action and choosing some other action. Otherwise, some principal could reduce the transfers she offers for the equilibrium action.

Moreover, it is immediate from Corollary 1 and (CM), that, for every n and every m, there exists an action a (which could be $\hat{s}_{\mathsf{n}}$) such that

$$(i)\ F_{\mathsf{n}}(a) + \sum_{\mathsf{j} \in \mathsf{M}} t_{\mathsf{n}}^{\mathsf{j}}(a) = F_{\mathsf{n}}(\hat{s}_{\mathsf{n}}) + \sum_{\mathsf{j} \in \mathsf{M}} t_{\mathsf{n}}^{\mathsf{j}}(\hat{s}_{\mathsf{n}}), \text{ and } (ii)\ t_{\mathsf{n}}^{\mathsf{m}}(a) = 0. \tag{6}$$

Given a principal and an agent, there must be an action, that belongs to the set of actions the agent is indifferent between, for which the principal offers a zero transfer. This action could be the equilibrium action.

4 Agents with Two Actions and No Direct Preferences

In this section we introduce the main results of the paper in a simplified environment in which each agent has only two actions and he does not care directly about the action he chooses. We proceed as follows: Subsection 4.1 restates the characterization theorem in this simplified environment. Subsection 4.2 provides five examples of GPTA's, three of which do not have an efficient equilibrium. Subsection 4.3 states the main theorem: a necessary and sufficient condition for the existence of an efficient equilibrium.

4.1 Characterization

We denote by s'_{n} the action of n different from $\hat{s}_{\mathsf{n}}$. Assume that for each agent $F_{\mathsf{n}}(\hat{s}_{\mathsf{n}}) = F_{\mathsf{n}}(s'_{\mathsf{n}}) = 0$. The following is an immediate restatement of Theorem 1, combined with Corollary 1 in this simplified environment:

Proposition 1 *For every n, suppose that $\sharp S_{\mathsf{n}} = 2$ and that $F_{\mathsf{n}}(\hat{s}_{\mathsf{n}}) = F_{\mathsf{n}}(s'_{\mathsf{n}}) = 0$. The pair $(\hat{t}, \hat{s})$ arises in a pure-strategy equilibrium if and only if*

(AM) For every $n \in N$, $s \in S$

$$\sum_{\mathsf{m}\in\mathsf{M}} \hat{t}^{\mathsf{m}}_{\mathsf{n}}(\hat{s}_{\mathsf{n}}) = \sum_{\mathsf{m}\in\mathsf{M}} \hat{t}^{\mathsf{m}}_{\mathsf{n}}(s'_{\mathsf{n}});$$

(IC) For every $m \in M$, $s \in S$:

$$G^{\mathsf{m}}(\hat{s}) + \sum_{\mathsf{n}\in\mathsf{N}} \sum_{\mathsf{j}\neq\mathsf{m}} \hat{t}^{\mathsf{j}}_{\mathsf{n}}(\hat{s}_{\mathsf{n}}) \geq G^{\mathsf{m}}(s) + \sum_{\mathsf{n}\in\mathsf{N}} \sum_{\mathsf{j}\neq\mathsf{m}} \hat{t}^{\mathsf{j}}_{\mathsf{n}}(s_{\mathsf{n}});$$

(CM) For every $m \in M$, $n \in N$,

$$\textit{if } \hat{t}^{\mathsf{m}}_{\mathsf{n}}(\hat{s}_{\mathsf{n}}) > 0 \textit{ then } \hat{t}^{\mathsf{m}}_{\mathsf{n}}(s'_{\mathsf{n}}) = 0.$$

With two actions per agent, no principal can make at equilibrium a strictly positive transfer on more than one of the two actions for each agent, since this would immediately violate the condition (CM). Also by (AM) and the fact that agents do not care about actions directly, the sum of transfers for one action is exactly equal to the sum of transfers for the other action.

If there are only two actions per agent and agents do not have preferences over actions, pure-strategy equilibria have special significance because the corresponding outcome is efficient:

Proposition 2 *For every n, suppose that $\sharp S_{\mathsf{n}} = 2$ and that $F_{\mathsf{n}}(\hat{s}_{\mathsf{n}}) = F_{\mathsf{n}}(s'_{\mathsf{n}}) = 0$. Then, pure-strategy equilibria are efficient.*

Proof: If we add (IC) over $m \in M$ we get that for every s:

$$\sum_{\mathsf{m}\in\mathsf{M}} G^{\mathsf{m}}(\hat{s}) + (M-1) \sum_{\mathsf{n}\in\mathsf{N}} \sum_{\mathsf{m}\in\mathsf{M}} \hat{t}^{\mathsf{j}}_{\mathsf{n}}(\hat{s}_{\mathsf{n}}) \geq \sum_{\mathsf{m}\in\mathsf{M}} G^{\mathsf{m}}(s) + (M-1) \sum_{\mathsf{n}\in\mathsf{N}} \sum_{\mathsf{m}\in\mathsf{M}} \hat{t}^{\mathsf{j}}_{\mathsf{n}}(s_{\mathsf{n}}),$$

which, by (AM), implies $\sum_{\mathsf{m}\in\mathsf{M}} G^{\mathsf{m}}(\hat{s}) \geq \sum_{\mathsf{m}\in\mathsf{M}} G^{\mathsf{m}}(s)$. ■

With more than two actions or if agents have preferences, there can be inefficient pure-strategy equilibria due to coordination failures between principals (see Section 5).

4.2 Examples

We consider a few examples with $M = N = \{1, 2\}$. For convenience, the actions of the agents have the familiar labels $S_1 = \{T, B\}$ and $S_2 = \{L, R\}$. We adopt the convention of presenting the payoff matrix in the form:

$$\begin{array}{ccc} & L & R \\ T & G^1(TL), G^2(TL) & G^1(TL), G^2(TL) \\ B & G^1(BL), G^2(BL) & G^1(BR), G^2(BR) \end{array}$$

It is important to keep in mind that this is not the usual payoff matrix. The actions refer to agents while the payoffs refer to principals. Agents have no direct interest in the action they choose. Also, these are gross payoffs. The net payoffs will be given by the gross payoffs minus the transfers. The transfer vector t^{m} is written as $(t^{\mathsf{m}}_1(T), t^{\mathsf{m}}_1(B); t^{\mathsf{m}}_2(L), t^{\mathsf{m}}_2(R))$.

Prisoner's Dilemma The payoffs of the principals are:

	L	R
T	x, x	z, y
B	y, z	$0, 0$

with $y > x > 0 > z$ and $2x > y + z$. The efficient action is unique: (T, L). By Proposition 2, if a pure-strategy equilibrium exists, it must have (T, L) as outcome. By (CM) and (AM) it is easy to see that Principal 1 will not make a positive transfer on actions T or R, while Principal 2 will not make a positive transfer on B or L. By (AM), the payment on each action from the two principals must be the same; so the equilibrium transfers are pairs of the form:

$$t^1 = (0, a; b, 0) \qquad t^2 = (a, 0; 0, b).$$

The (IC) conditions for the two principal are, respectively:

$$\begin{aligned} G^1(TL) + a &\geq \max\{G^1(TR) + a + b, G^1(BL), G^1(BR) + b\}, \\ G^2(TL) + b &\geq \max\{G^2(BL) + a + b, G^2(TR), G^2(BR) + a\}, \end{aligned}$$

or

$$\begin{aligned} x + a &\geq \max\{z + a + b, y, b\}, \\ x + b &\geq \max\{z + a + b, y, a\}. \end{aligned}$$

The set of pure-strategy equilibria is given by any transfer with (a, b) such that $a, b \in [y - x, x - z]$ *and* $x \geq b - a \geq -x$. In particular, there exists a minimal transfer equilibrium in which $a = b = y - x$. The agents choose T, respectively L, whenever indifferent. The rent of each agent is at least the difference between the highest payoff and the cooperation payoff, $y - x$, and at most the difference between the cooperation payoff and the lowest payoff, $x - z$.

Coordination Game The payoffs of the principals are:

	L	R
T	x_1, y_1	$0, 0$
B	$0, 0$	x_2, y_2

with $x_1 + y_1 > x_2 + y_2$, $y_1 \leq y_2$, and $x_i, y_i \geq 0$ for $i = 1, 2$.

Again, there is a unique efficient outcome, (T, L), which must be the outcome of all pure-strategy equilibria. From Proposition 1 it is easy to see that there exists an equilibrium with outcome (T, L) if there are transfers $t^1 = (a, 0; b, 0)$, $t^2 = (0, a; 0, b)$ that satisfy

$$x_1 - x_2 \geq a + b \geq y_2 - y_1,$$

which is true for the parameters under consideration. The combined rent of the two agents is at least $y_2 - y_1$.

In the extreme "pure", coordination game with $x_1 = y_1 > x_2 = y_2$, there is an equilibrium in which only zero transfers are offered (but there are also equilibria with positive transfers).

The prisoners' dilemma and the coordination game have efficient pure-strategy equilibria. The next three examples instead show that there are games in which a pure-strategy equilibrium does not exist and all other equilibria are inefficient.

Opposite Interests Game The payoff matrix is

$$\begin{array}{ccc} & L & R \\ T & y,0 & 0,x \\ B & 0,x & 0,x \end{array}$$

with $\frac{1}{2}y < x < y$. By Proposition 2, the only possible pure-strategy equilibrium outcome is (T, L), and the second principal can only pay for the action B and R. So the possible transfers are $t^1 = (a, 0; b, 0)$ and $t^2 = (0, a; 0, b)$. The (IC) for the two principals are, respectively:

$$\begin{aligned} y &\geq \max\{a, b, a+b\} \\ a+b &\geq \max\{a+x, b+x, x\}. \end{aligned}$$

They imply $y \geq a+b$ and $2x \leq a+b$, which form a contradiction if $x > \frac{1}{2}y$. No pure-strategy equilibrium exists. The game has equilibria in mixed strategies which give rise to an outcome different from (T, L) with positive probability. Therefore, the game has only inefficient equilibria.[6]

The opposite interest game can be interpreted as an example of each of the four applications of GPTA's proposed in the Introduction. It can be a lobbying problem where Principal 1 is a lobby who wants to change the status quo and Principal 2 wants to keep things as they are. In order to change the status quo, Principal 1 needs unanimous approval from two governmental bodies, Agent 1 and Agent 2. The efficient outcome is to change the status quo. However, Principal 2 enjoys a strategic advantage because she only needs to convince one of the two agents to say no.

The game also corresponds to a first price auction over two objects (the agents). The value of two objects to the first principal is y, and the value of any other subset is 0; while the value of any non-empty set of objects to the second principal is x.

With some re-working, the Opposite Interest Game can also be interpreted as a very basic vertical contracting problem with two sellers (principals) and two buyers (agents). Each buyer needs exactly one unit of the input good produced by the sellers. The total

[6] One mixed-strategy equilibrium sees Principal 1 making transfers:

$$(t, 0, t, 0) \text{ according to the CDF } G(t) = \frac{x - \frac{3}{2}}{x - t}$$

with $t \in [0, 3/2]$. The second principal makes with probability $\frac{1}{2}$ a transfer

$$(0, s, 0, 0) \text{ according to the CDF } F(s) = \frac{s}{3-s}$$

and with probability $\frac{1}{2}$ a transfer

$$(0, 0, 0, s) \text{ according to the CDF } F(s) = \frac{s}{3-s},$$

in both cases with $s \in [0, 3/2]$. The mixed-strategy equilibrium of similar two-agent two-principal game is studied, independently, by Grossman and Helpman [13] in the context of lobbying. An interesting mixed strategy equilibrium, much harder to find, of a related game is in the paper by Szentes and Rosenthal, [25].

cost functions of the two sellers are as follows:

q	0	1	2
C^1	0	y	y
C^2	0	$y-x$	$2y-x$

Seller 1 has economies of scale while Seller 2 has diseconomies. The efficient allocation would be that 1 produces two units and 2 produces nothing. Let T represent Buyer 1 buying his unit from Seller 1 and let L represent Buyer 2 buying his unit from Seller 2. B and R are the opposite actions. Suppose that there is a 'fixed' price of y per unit but sellers can offer discounts (this is a quick, but heuristic, way to overcome the non-negativity constraint – see Section 6.3 for a more careful analysis of vertical contracting). For instance, $t_2^1(L)$ is the discount over the fixed price of y that Principal 1 offers to Agent 1 if he buys from her. Then, it is easy to check that this supply contract problem is exactly equivalent to the Opposite Interest Game examined above and, therefore, has no efficient equilibrium. In order to achieve efficiency, Principal 1 should sell to both buyers but Principal 2 can easily undercut her on one of the two buyers. The non-contractible externality here is that, if Principal 2 sells to Buyer 2, there is an increase in the cost of production for the good that Principal 1 is still selling to Agent 1.

One can also view the Opposite Interest Game as a two-sided matching problem with two firms and two workers. Firm 1 displays strong positive complementarities between workers, while firm 2 displays strong negative complementarities.

Matching Pennies Consider

	L	R
T	$x_1, 0$	$0, y_1$
B	$0, y_2$	$x_2, 0$

with x_1 the largest number. The only possible pure-strategy equilibrium outcome is (T, L), with transfers $t^1 = (a, 0; b, 0)$ and $t^2 = (0, a; 0, b)$. The (IC) for the first principal is $x_1 \geq \max\{a, b, x_2 + a + b\}$, for the second $a + b \geq \max\{y_1 + a, y_2 + b, 0\}$, which are equivalent to $x_1 - x_2 \geq a + b \geq y_1 + y_2$, so a pure-strategy equilibrium exists if and only if $x_1 - x_2 \geq y_1 + y_2$. The maximum total rent of the agents is the difference between the payoffs of the first principal, and the minimum total rent is the *sum* of the payoffs of the second principal.

In particular a pure-strategy equilibrium does not exist for the "true" matching pennies, with all the numbers equal to 1.

Voting Game Our last example has more than two agents and is related to Groseclose and Snyder [11]. There are two principals, $M = \{1, 2\}$, and an odd number $N = 2K + 1$ of agents. Each agent may vote for one of two alternatives, also labeled 1 and 2 and he may not abstain. The alternative with the larger number of votes is chosen. The payoff of Principal 1 is $x \geq 1$ if alternative 1 is chosen, and 0 if 2 is chosen. The payoff of Principal 2 is 1 if 2 is chosen and 0 otherwise. Thus, all outcomes such that $\sharp\{n \in N | s_n = 1\} \geq K + 1$ are efficient.

This game has no equilibrium in which alternative 1 is chosen for sure, and hence it has only inefficient equilibria. To see this, it is sufficient to look at the transfers that are paid in equilibrium. Suppose that an equilibrium exists, where alternative 1 is chosen for sure. In this equilibrium, Principal 2 must be paying no money to agents. If it were not so, Principal 2 would get a negative payoff while she can always ensure a zero payoff by offering zero to all agents. Also, Principal 1 will make a strictly positive transfer to exactly $K+1$ agents. If she paid less than $K+1$ agents, the other principal could buy off a majority for an infinitesimal price. If she paid more than $K+1$ agents, she would be wasting money. Thus, K agents receive zero offers on both alternatives. This leads to a contradiction because Principal 1 could stop paying some of the $K+1$ agents whom she is paying a positive amount and get the vote of some of the other K for an infinitesimal price.[7]

4.3 Necessary and sufficient condition

In this section we provide a necessary and sufficient condition for the existence of a pure-strategy equilibrium. We are interested in pure-strategy equilibria because of their connection with efficient equilibria. As we saw in Proposition 2, a pure-strategy equilibrium is efficient. We can also prove that, if there are only two principals, a GPTA has an efficient equilibrium if and only if it has a pure-strategy equilibrium (see Section 6.1). We introduce two notions:

Definition 3 *If agents have two actions and no preference over actions, the vector* $w = (w^{\mathsf{m}}(s))_{\mathsf{m}\in\mathsf{M},\mathsf{s}\in\mathsf{S}}$ *is said to be a collection of balanced weights if* $w^{\mathsf{m}}(s) \geq 0$ *for every* m *and* s, *and*

$$\text{for every } m \in M, n \in N \sum_{\{\mathsf{s}:\mathsf{s}_\mathsf{n} \neq \hat{\mathsf{s}}_\mathsf{n}\}} w^{\mathsf{m}}(s) = \sum_{\{\mathsf{s}:\mathsf{s}_\mathsf{n} \neq \hat{\mathsf{s}}_\mathsf{n}\}} w^{1}(s). \tag{7}$$

Definition 4 *A GPTA is balanced with respect to* $\hat{s}$ *if, for every collection of balanced weights* w,

$$\sum_{\mathsf{m}\in\mathsf{M}} \sum_{\mathsf{s}\in\mathsf{S}} w^{\mathsf{m}}(s)(G^{\mathsf{m}}(\hat{s}) - G^{\mathsf{m}}(s)) \geq 0. \tag{8}$$

The definition of balanced game starts with an arbitrary candidate outcome $\hat{s}$ and considers other outcomes s, which can be seen as deviations. The matrix w assigns a weight to every agent and every deviation (a positive weight on $s = \hat{s}$ is inconsequential). A deviation s involves some agents switching from $\hat{s}_\mathsf{n}$ to s'_n and may involve other agents staying at $\hat{s}_\mathsf{n}$. Condition (7) requires that for every agent n the sum of weights on deviations involving a switch on the part of n is constant across principals. A GPTA is balanced if, for every balanced w, the w-weighted sum of payoffs on $\hat{s}$ is at least as large as the w-weighted sum of payoffs on deviations.

The definition is reminiscent of that used in cooperative game theory (e.g. Scarf [21]) but of course it is different from it because of the distinction in our game between principals and agents.

It is easy to show that being balanced with respect to $\hat{s}$ is a necessary condition for a GPTA to have a pure-strategy equilibrium with outcome $\hat{s}$. To see this, suppose a

[7] Groseclose and Snyder [11] present the game in a sequential form. First Principal 1 makes offers. Then, Principal 2 observes the offers made by 1 and makes her offers. They show that a principal may want to buy a supermajority, that is, make a positive offer to strictly more than $K+1$ agents.

pure-strategy equilibrium with outcome $\hat{s}$ exists. For any collection of balanced weights w,

$$\begin{aligned}\sum_{m\in M}\sum_{s\in S} w^m(s)\left(G^m(\hat{s})-G^m(s)\right) &\geq \sum_{m\in M}\sum_{n\in N}\sum_{s\in S} w^m(s)\left(\sum_{j\neq m}\left(\hat{t}^j_n(s_n)-\hat{t}^j_n(\hat{s}_n)\right)\right)\\ &= \sum_{m\in M}\sum_{n\in N}\left(\sum_{\{s:s_n\neq\hat{s}_n\}} w^m(s)\right)\left(\sum_{j\neq m}\left(\hat{t}^j_n(s'_n)-\hat{t}^j_n(\hat{s}_n)\right)\right)\\ &= \sum_{n\in N}\left(\sum_{\{s:s_n\neq\hat{s}_n\}} w^1(s)\right)\sum_{m\in M}\left(\sum_{j\neq m}\left(\hat{t}^j_n(s'_n)-\hat{t}^j_n(\hat{s}_n)\right)\right)\\ &= 0,\end{aligned}$$

where the first inequality is obtained by summing over (IC), the first equality is a rearrangement, the second equality is because w is balanced, and the last equality is due to (AM). The resulting inequality is the requirement that the game is balanced given w.

The converse is true as well. A GPTA that is balanced has a pure-strategy equilibrium. This is proven in the following:

Theorem 2 *A GPTA where agents have two actions and no preference over actions has a pure-strategy equilibrium with outcome $\hat{s}$ if and only if it is balanced with respect to $\hat{s}$.*

Proof: From Proposition 1, a pure-strategy equilibrium exists if and only if the three conditions of that proposition hold. If we denote

$$d^j_n \equiv t^j_n(\hat{s}_n) - t^j_n(s'_n)$$

(AM) and (IC) may be rewritten as

$$\sum_{\{j:j\neq k\}}\sum_{\{n:s_n\neq\hat{s}_n\}} d^j_n \geq G^m(s)-G^m(\hat{s}) \qquad \forall s\in S, \forall k\in M, \tag{9}$$

$$\sum_{j\in M} d^j_n = 0 \qquad \forall n\in N. \tag{10}$$

The system (9) and (10) is a system of linear (in)equalities in the MN variables d^j_n. There are MS inequalities of the type in (9) and N equalities of the type (10).

We can find a d that solves (9) and (10) if and only if we can find a t that solves (AM), (IC), and (CM). The "if" part is by definition. The "only if" part can be seen as follows. Suppose we find a d that solves (9) and (10). Let $t^j_n(\hat{s}_n) = \max(0, -d^j_n)$ and $t^j_n(\hat{s}'_n) = \max(0, d^j_n)$. The resulting t satisfies (AM), (IC), and (CM).

The following result is well known, and reported here only for convenience:[8]

Theorem 3 (Farkas) *Exactly one of the following alternatives is true: (a) There exists a solution x to the linear system of (in)equalities given by $Ax \geq a$ and $Bx = b$; or (b) There exist vectors μ and ν such that: (i) $\mu A + \nu B = 0$; (ii) $\mu \geq 0$; and (iii) $\mu a + \nu b > 0$.*

[8]See for instance Mangasarian [14].

We now apply Farkas' Lemma to (9) and (10). Recall that d is a vector of MN elements. We construct a matrix A of dimensions $MS \times MN$, a matrix B of dimensions $N \times MN$, a vector a with MS elements, and a vector b with N elements. For $m, j \in M$, $i, n \in N$, $s \in S$, let

$$\begin{aligned} A_{(\mathsf{ms},\mathsf{jn})} &= \begin{cases} 1 \text{ if } j \neq m, s_{\mathsf{n}} \neq \hat{s}_{\mathsf{n}} \\ 0 \text{ otherwise;} \end{cases} \\ B_{(\mathsf{i},\mathsf{jn})} &= \begin{cases} 1 \text{ if } i = n \\ 0 \text{ otherwise.} \end{cases} \\ a_{\mathsf{ms}} &= G^{\mathsf{m}}(s) - G^{\mathsf{m}}(\hat{s}) \\ b_{\mathsf{i}} &= 0 \end{aligned}$$

Then, (9) and (10) rewrite as $Ad \geq a$ and $Bd = b$. By Farkas' Lemma a solution d of that system exists if and only if there is no solution in the variables $((w^{\mathsf{m}}(s)_{\mathsf{m}\in\mathsf{M},\mathsf{s}\in\mathsf{S}}, (\nu_{\mathsf{i}})_{\mathsf{i}\in\mathsf{N}})$ of the system:

$$\sum_{\mathsf{m}\in\mathsf{M}} \sum_{\mathsf{s}\in\mathsf{S}} w^{\mathsf{m}}(s) A_{(\mathsf{ms},\mathsf{jn})} + \sum_{\mathsf{i}\in\mathsf{N}} \nu_{\mathsf{i}} B_{(\mathsf{i},\mathsf{jn})} = 0 \qquad \forall j \in M, \forall n \in N; \tag{11}$$

$$w^{\mathsf{m}}(s) \geq 0 \qquad \forall m \in M, \forall s \in S;$$

and

$$\sum_{\mathsf{m}\in\mathsf{M}} \sum_{\mathsf{s}\in\mathsf{S}} w^{\mathsf{m}}(s)(G^{\mathsf{m}}(s) - G^{\mathsf{m}}(\hat{s})) > 0. \tag{12}$$

The system (11) may be rewritten as:

$$\text{for every } j \in M, n \in N, \sum_{\{\mathsf{s}:\mathsf{s}_\mathsf{n} \neq \hat{\mathsf{s}}_\mathsf{n}\}} w^{\mathsf{m}}(s) = -\nu_{\mathsf{n}}. \tag{13}$$

As this is the only restriction that the variables ν are imposing, (13) is true if and only if w is a vector of balanced weights.

Inequality (12) is the negation that the game is balanced for a particular vector of weights. The lack of nonnegative solution to the system (12) and (13) is equivalent to the requirement that for all balanced weights the inequality

$$\sum_{\mathsf{m}\in\mathsf{M}} \sum_{\mathsf{s}\in\mathsf{S}} w^{\mathsf{m}}(s)(G^{\mathsf{m}}(\hat{s}) - G^{\mathsf{m}}(s)) \geq 0$$

holds, and this is the statement we had to prove. ■

Theorem 2 is a duality result. Conditions (AM), (IC), and (CM) form a system of linear equalities and inequalities, which we view as the primal problem. Applying the theorem of the alternative, we know that exactly one of the following applies: either the primal problem has a solution or an appropriately defined dual problem has a solution. As the proof shows, the dual problem is the negation that the game is balanced. Thus, the primal has a solution if and only if the game is balanced.

The dual problem suggests a simple algorithm to ascertain whether a particular GPTA has an efficient equilibrium. Take $\hat{s}$ to be an efficient outcome (in the nongeneric case when there are many, one must examine each efficient outcome). Consider a minimization problem in which the objective function is $\sum_{\mathsf{m}\in\mathsf{M}}\sum_{\mathsf{s}\in\mathsf{S}} w^{\mathsf{m}}(s)(G^{\mathsf{m}}(\hat{s})-G^{\mathsf{m}}(s))$ and the control variables are the w. The minimization is subject to w being nonnegative and balanced (plus a constraint that guarantees that weights are bounded, like $\sum_{\mathsf{s}\in\mathsf{S}} w^1(s)=1$). The game has a pure-strategy equilibrium if and only if the value of the objective function is nonnegative.

The linear program tells us also something more. Each weight $w^{\mathsf{m}}(s)$ can be associated with the (IC) constraint that prevents Principal m from deviating to s. In the solution, zero weights correspond to constraints that are certainly not binding. We can thus restrict our attention to constraints associated to strictly positive weights. If the value of the program is negative, this identifies a minimal set of (IC) constraints that cannot be satisfied. If the value of the program is nonnegative, the weights help us compute the set of pure-strategy equilibria because we can disregard constraints corresponding to nonzero weights.

To illustrate this point, let us re-consider the Opposite Interest Game. The linear program associated to the dual problem is:

$$\min_{\mathsf{w}} y\left(w^1(TR)+w^1(BL)+w^1(BR)\right)-x\left(w^2(TR)+w^2(BL)+w^2(BR)\right),$$

subject to $w\geq 0$, and

$$w^1(TR)+w^1(BR)=w^2(TR)+w^2(BR);$$
$$w^1(BL)+w^1(BR)=w^2(BL)+w^2(BR);$$
$$w^1(TR)+w^1(BR)+w^1(TR)=1.$$

The solution is $w^1(BR)=w^2(BL)=w^2(TR)=1$ and the value is $y-2x$. The Opposite Interest Game has a pure-strategy equilibrium with outcome TL if and only if $x\leq\frac{1}{2}y$. The potentially binding (IC) constraints correspond to a "diagonal" deviation of Principal 1 on BR, a "horizontal" deviation of 2 on TR, and a "vertical" deviation of 2 on BL. If $x>\frac{1}{2}y$, these three constraints are incompatible, which formalizes our intuition that the game has no efficient equilibrium because the second principal can deviate in two directions.

If $x\leq\frac{1}{2}y$, a pure-strategy equilibrium exists, and we can find equilibrium transfers by looking at the three potentially binding (IC) constraints:

$$\begin{array}{rcl} y+t_1^2(T)+t_2^2(L) & \geq & t_1^2(B)+t_2^2(R), \\ t_1^1(T) & \geq & x+t_1^1(B), \\ t_2^1(L) & \geq & x+t_2^1(R). \end{array}$$

These three inequalities, combined with (AM) and (CM), fully determine the set of pure-strategy equilibrium transfers:

$$\begin{array}{rcl} t_1^1(T) & = & t_1^2(B)\in[x,y-v], \\ t_2^1(L) & = & t_2^2(R)\in[x,v], \\ t_1^1(B) & = & t_1^2(T)=t_2^1(R)=t_2^2(L)=0, \end{array}$$

with $v\in[x,y-x]$.

5 The General Case

We now leave the simplified environment where agents have only two actions and are not directly affected by the action they choose. We thus revert to the general model introduced in Sections 2 and 3.

If agents have more than two actions, a pure-strategy equilibrium need not be efficient. This is already true if $N = \{1\}$ (common agency). Consider the example (see [4]) where $M = \{1, 2\}$, $N = \{1\}$, $S_1 = \{a, b, c, d\}$, and $F(s_n) = 0$ (as there is only one agent, we omit the agent subscript), and the principals' payoffs are:

	a	b	c	d
G^1	8	6	0	1
G^2	0	6	7	1

Here $t^1 = (7, 0, 0, 0)$, $t^2 = (0, 0, 7, 0)$, and $\hat{s} = a$ is a pure-strategy equilibrium with an inefficient outcome. The main feature of this equilibrium is a failure of the two principals to coordinate on the efficient action. Principal 1 does not make an offer on action 2 because Principal 2 is not making an offer either, and vice versa. There exists another pure-strategy equilibrium in which $t^1 = (3, 1, 0, 0)$, $t^2 = (0, 2, 3, 0)$, and $\hat{s} = b$, which selects the efficient action and gives a higher payoff to both principals.

Inefficient pure-strategy equilibria arise also because of the agents' preferences. In this case, two actions are sufficient, as in the following example with one agent and two principals:

	a	b
G^1	2	0
G^2	2	0
F	-3	0

This game is similar to a problem of public good provision. The efficient action is a but there are equilibria in which b is chosen because principals do not contribute on the efficient action. For instance, $\hat{s} = b$, $\hat{t}^1 = \hat{t}^2 = (0, 0)$ is one. Of course, there are also efficient equilibria like $\hat{s} = a$, $\hat{t}^1 = \hat{t}^2 = (1.5, 0)$.

To overcome this multiplicity of equilibria, in common agency Bernheim and Whinston introduce the notion of truthful transfers. A transfer vector is truthful if, for all actions, it is equal to the principal's gross payoff minus a constant (save for the non-negativity constraint on transfers). Formally,

Definition 5 *If $N = \{1\}$, a transfer vector t^m is truthful relative to $\hat{s}$ if for every $s \in S$*

$$t^m(s) = \max(0, t^m(\hat{s}) + G^m(s) - G^m(\hat{s}))$$

A pure-strategy equilibrium giving $\hat{s}$ as equilibrium action is truthful if the transfer of every principal is truthful relative to $\hat{s}$.

In common agency, truthful equilibria play a fundamental role. They always exist, the equilibrium action is efficient ([4, Theorem 2]), and they are coalition proof ([4, Theorem 3]). The intuition is that truthful transfers restrict offers on out-of-equilibrium actions not too be too low with respect to the principals' payoffs and therefore exhaust all gains from coalitional deviations.

However, we cannot just extend the definition of truthful transfers to more than one agent because this would impose too many equality restrictions on the transfer matrix. We therefore choose to relax Definition 5 from equality to inequality. For $N = \{1\}$, a weaker condition is that for every $s \in S$, $t^{\mathsf{m}}(s) \geq t^{\mathsf{m}}(\hat{s}) + G^{\mathsf{m}}(s) - G^{\mathsf{m}}(\hat{s})$, or alternatively

$$G^{\mathsf{m}}(s) - t^{\mathsf{m}}(s) \geq G^{\mathsf{m}}(\hat{s}) - t^{\mathsf{m}}(\hat{s}).$$

This definition maintains the feature that offers on out-of-equilibrium actions cannot be too low and it can be extended to a GPTA with many agents:

Definition 6 *For principal m, t^{m} is weakly truthful relative to $\hat{s}$ if*

(WT) For every $s \in S$, $G^{\mathsf{m}}(\hat{s}) - \sum_{\mathsf{n} \in \mathsf{N}} t^{\mathsf{m}}_{\mathsf{n}}(\hat{s}^{\mathsf{n}}) \geq G^{\mathsf{m}}(s) - \sum_{\mathsf{n} \in \mathsf{N}} t^{\mathsf{m}}_{\mathsf{n}}(s_{\mathsf{n}})$.

A weakly truthful equilibrium with outcome $\hat{s}$ is a pure-strategy equilibrium with outcome $\hat{s}$ in which the transfer of every principal is weakly truthful relative to $\hat{s}$.

A consequence of this definition is that – like truthful equilibria of common agency games – weakly truthful equilibria of GPTA's are always efficient:

Proposition 3 *The outcome of a weakly truthful equilibrium is efficient.*

Proof: Sum (WT) over m. Sum the inequalities (AM) in Theorem 1 over n. Add the two resulting inequalities. The result is inequality (1), which defines efficiency. ■

The necessary and sufficient conditions for a weakly truthful equilibrium are the same as those in Theorem 1, except that (IC) is substituted with the stronger requirement that transfers be weakly truthful:

Proposition 4 *A pair $(\hat{t}, \hat{s})$ of transfers and outcome arises in a weakly truthful equilibrium if and only if it satisfies (WT), (AM) and (CM).*

Proof: Sum (AM) over n. To the resulting inequality, add (WT). The result is (IC). Hence, (WT) and (AM) imply (IC) and sufficiency is proven. Necessity is obvious because (AM) and (CM) are necessary by Theorem 1 and (WT) is necessary by the definition of weakly truthful equilibrium. ■

To check that the definition of weakly truthful equilibrium is consistent with the analysis of the previous section, consider what happens to weakly truthful equilibria if each agent has only two actions and cares solely about monetary payoff. In this case, it is immediate to check that (AM) and (IC) imply (WT) and, by Proposition 4:

Corollary 2 *If each agent has only two actions and cares solely about monetary payoff, then all pure-strategy equilibria are weakly truthful.*

Weak truthfulness eliminates inefficient equilibria that are due to coordination problems. If each agent has only two actions and no externalities, coordination problems do not arise.

We now move to the issue of whether a weakly truthful equilibrium exists. As in the previous section, we pose this question with respect to a particular outcome, that is, we ask whether, given $\hat{s} \in S$, there exists a weakly truthful equilibrium that produces outcome $\hat{s}$. We need to redefine balancedness:

Definition 7 *In a GPTA, the vectors w and z with respective dimensions MS and H are balanced weights with respect to $\hat{s}$ if all their elements are nonnegative, and*

$$\text{for every } m \in M, n \in N, a_n \in S_n/\hat{s}_n; \sum_{\{s: s_n = a_n\}} w^m(s) = z_n(a_n). \tag{14}$$

A GPTA is balanced with respect to $\hat{s}$ if and only if for every pair of vectors of balanced weights w and z we have:

$$\sum_{m\in M}\sum_{s\in S} w^m(s)(G^m(\hat{s}) - G^m(s)) + \sum_{n\in N}\sum_{s_n\in S_n} z_n(s_n)(F_n(\hat{s}_n) - F_n(s_n)) \geq 0 \tag{15}$$

If agents have only two actions, there is only one possible deviation for each agent. With more than two actions, the condition that principals put the same sum of weight must hold for every agent *and* every deviation that the agent has. Moreover, the definition of balancedness now includes weights on agents as well as principals, to take into account not only the benefit of principals but also that of agents. A game is balanced (with respect to a given outcome $\hat{s}$) if, for any vector of balanced weights, the sum of the direct change in payoffs for principals and agents of any possible deviation is negative. If agents do not care about actions and there are only two actions per agent, we recover the definition of balancedness used in the previous section.

As in the previous section, balancedness can be expressed as a linear program with variables w and v. The game is balanced if and only if the value of the linear program is nonnegative.

The main result of this section is:

Theorem 4 *A GPTA has a weakly truthful equilibrium with outcome $\hat{s}$ if and only if it is balanced with respect to $\hat{s}$.*

Proof: The proof, which is analogous to that of Theorem 2, is in the Appendix. ■

Remarks:

1. Games of common agency (Bernheim and Whinston [4]) are of course a special case of the games we are considering, where $N = \{1\}$. Condition (14) simply requires that for each action s_1 the weight $w^m(s_1)$ is the same for all principals. A weakly truthful equilibrium with outcome $\hat{s}_1$ exists if and only if, for all nonnegative $z_1(s_1)$,

$$\sum_{s_1\in S_1} z_1(s_1)\left(\sum_{m\in M}(G^m(\hat{s}_1) - G^m(s_1)) + F_1(\hat{s}_1) - F_1(s_1)\right) \geq 0.$$

   Hence, the set of outcomes that are supported by a weakly truthful equilibrium is the set of efficient outcomes.

2. The other extreme case is one principal and many agents, that is $M = \{1\}$ (Segal [22]). Condition (14) simplifies to

$$\text{for every } n \in N, a_n \in S_n/\hat{s}_n; \sum_{\{s: s_n = a_n\}} w^1(s) = z_n(a_n).$$

Then, for each n,

$$\begin{aligned}\sum_{s_n \in S_n} z_n(s_n)(F_n(\hat{s}_n) - F_n(s_n)) &= \sum_{s_n \in S_n} \sum_{\{\tilde{s}:\tilde{s}_n = s_n\}} w^1(\tilde{s})(F_n(\hat{s}_n) - F_n(s_n)) \\ &= \sum_{s \in S} w^1(s)(F_n(\hat{s}_n) - F_n(s_n)).\end{aligned}$$

Condition (14) becomes

$$\sum_{s \in S} w^1(s)\left(G^1(\hat{s}) - G^1(s)) + \sum_{n \in N}(F_n(\hat{s}_n) - F_n(s_n))\right) \geq 0.$$

The game is balanced if, for every vector w^1, which is restricted only to be nonnegative, the latter inequality holds. This in turn is true if and only if $\hat{s}$ is efficient.

3. Given a deviation $\tilde{s}$, a possible vector of balanced weights is, for every $m \in M$,

$$\begin{aligned} w^m(s) &= \begin{cases} 1 & \text{if } s = \tilde{s}; \\ 0 & \text{otherwise.} \end{cases} \\ z_n(s_n) &= \begin{cases} 1 & \text{if } s_n = \tilde{s}_n; \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

This collection of weights satisfies (14) because each principal is asking exactly the same deviation from all agents. Then, we get that a game is balanced only if, for every $\tilde{s} \in S$,

$$\sum_{m \in M}(G^m(\hat{s}) - G^m(\tilde{s})) + \sum_{n \in N}(F_n(\hat{s}_n) - F_n(\tilde{s}_n)) \geq 0.$$

that is, $\hat{s}$ is the efficient action. This is an indirect way of getting to Propositions 2 and 3. Of course, efficiency does not in general imply balancedness.

4. One question that is left open is whether there can be games that do not have a weakly truthful pure-strategy equilibrium but have a non-truthful pure-strategy equilibrium supporting the efficient outcome. The answer is positive, as illustrated by the following two-principal, two-agent, three-action-per-agent example:[9]

	L	C	R
T	$3, 0$	$0, 2$	$-10, 0$
M	$0, 2$	$0, 2$	$-10, 0$
B	$-10, 0$	$-10, 0$	$-10, 0$

By applying Theorem 4, we see that this game has no weakly truthful equilibrium with the efficient outcome TL. To see that balancedness is violated, set $w^1(MC) = w^2(ML) = w^2(TC) = 1$, $z_1(M) = z_2(C) = 1$, and all the other weights equal zero. Indeed, this game is just the Opposite Interest Game with the addition of a line and a column that are extremely bad for principal 1.

[9] We are grateful to Bruno Jullien for suggesting this example.

However, this game has a non-truthful pure-strategy equilibrium with outcome (T, L). Actually, there is a continuum of them. One is as follows. Principal 1 offers $t_1^1(T) = t_2^1(L) = 4$ and zero on all other actions. Principal 2 offers $t_1^1(B) = t_2^1(R) = 4$ and zero on all other actions. As usual, agents maximize revenues, and, in case of indifference, select (T, L). While somewhat unconvincing, this situation is an equilibrium because on one side any attempt by Principal 1 to save money would induce a payoff of -10, and on the other side Principal 2 finds it too expensive to deviate on M or C.

6 Discussion

We now deal with some questions that are closely related to the results just presented: the existence and properties of mixed-strategy equilibria, the connection between GPTA's and cooperative game theory, the special properties of convex GPTA's, and an extension of the model in which principals are allowed to offer outcome-contingent transfers.

6.1 Mixed-strategy equilibria

By now, we know that a GPTA may not have a pure-strategy equilibrium. However, we can show that a GPTA must have an equilibrium.

Given a transfer profile from the principals, the agents' best reply is almost everywhere unique except when transfers are such that at least one agent is indifferent between two or more actions. If we allow agents to use correlated strategies (the assumption that transfer offers are public becomes important), then we have that in the first stage of the game principals face a *payoff correspondence* that is upper-hemi continuous with compact and convex values. The game among principals satisfies the conditions of Simon and Zame [24]'s existence theorem for games with discontinuous payoffs and endogenous sharing rules. Therefore, we know that we can always find an appropriate sharing rule, in this case a tie-breaking rule for agents, such that the game among principals has an equilibrium. Then, we have:

Theorem 5 *Every GPTA has a subgame-perfect equilibrium (in which agents may use correlated strategies).*

Proof: The proof is a straightforward check of Simon and Zame's conditions and it is omitted. ■

We have seen that, when agents have only two actions and no preferences over actions, pure-strategy equilibria are efficient, and that, more generally, pure-strategy weakly truthful equilibria are efficient. Can a mixed-strategy equilibrium be efficient? Obviously, if the set of efficient outcomes is a singleton (the generic case) and the mixed strategy equilibrium involves at least one agent randomizing over actions, then it cannot be efficient. However, we may have mixed-strategy equilibria in which only principals randomize.

We prove here that when agents have two actions and do not have preferences over actions and in addition *there are only two principals*, if there is a mixed-strategy equilibrium in which only principals randomize, then there must be a payoff-equivalent pure-strategy equilibrium:

Theorem 6 *Assume that $\sharp S_n = 2$, $F_n = 0$ for every n, and $M = \{1, 2\}$. If a mixed strategy equilibrium has an outcome $\hat{s}$ which is constant almost surely then there is a pure-strategy equilibrium with the same outcome and the same equilibrium transfers.*

Proof: See Appendix. ■

The theorem implies that either a GPTA has a pure-strategy equilibrium or all mixed-strategy equilibria involve agents mixing over actions. Hence,

Corollary 3 *Consider a GPTA with $\sharp S_n = 2$, $F_n = 0$ for every n, and $M = \{1, 2\}$, and in which the set of efficient outcome is a singleton: $S^* = \{s^*\}$. The following three statements are equivalent: (i) The game has an efficient equilibrium; (ii) The game has a pure-strategy equilibrium with outcome s^*; (iii) The game is balanced with respect to s^*.*

In this restricted environment, balancedness is a necessary and sufficient condition for efficiency. Unfortunately, Corollary 3 does not extend beyond this restrictive environment (balancedness is only a sufficient condition: (iii) implies (ii) and (ii) implies (i)). This is due to two sets of reasons. First, as we saw, in Section 5, with more than two actions per agent or with agent preferences, balancedness is only a sufficient condition for the existence of an efficient pure-strategy equilibrium. Second, even if agents have only two action and they have no preferences, if there are more than two principals there are GPTA's that do not have efficient pure-strategy equilibria but have an efficient mixed-strategy equilibrium. In other words, Theorem 6 does not extend to more than two principals.

To see this fact, consider the following example with three principals and two agents with no preferences over actions. The payoff matrix is as in the earlier examples except that there are now three numbers in each box, denoting the payoffs of each of the three principals:

	L	R
T	$3, 0, 0$	$0, 2, 0$
B	$0, 0, -15$	$0, 0, 1.5$

To see that this game has no pure-strategy equilibrium, take the weights $w^1(BR) = w^2(BL) = w^2(TR) = w^3(BR) = 1$ with all the other weights equal to zero. The weights are balanced and the weighted sum of payoffs is $3 - 2 - 1.5 < 0$. By Theorem 2, the game has no pure-strategy equilibrium with outcome TL. However, the game has a mixed-strategy equilibrium with outcome TL in which Principal 1 offers 2 on L and zero on T, Principal 2 offers 2 with probability $p \in \left(\frac{2}{3}, \frac{9}{10}\right)$ on R and zero otherwise, and Principal 3 makes no offer. It is easy to see that these transfers are a best response for 1 and a (weak) best-response for 2. Principal 3 could convince agent 2 to deviate to B with an infinitesimal offer. However, this deviation is not profitable because with a probability of at least 1/10 it would give her a payoff of -15. Principal 3 could also get a joint deviation by both agents but the cost would be too high.

6.2 GPTA's and the core

The concept of balanced game that has been introduced in the previous section is reminiscent of the concept of balanced game as used in cooperative game theory. We now make this link explicit by showing that every transferable utility game (TUG) can be put in correspondence with a (very simple) GPTA and that the core of the TUG is nonempty if and only if the corresponding GPTA has a pure-strategy equilibrium.

We begin by recalling some basic notions of cooperative game theory (Osborne and Rubinstein [17, Ch. 13]). Let N be a finite set of players, and $v : 2^{N} \rightarrow R$ a value function. This function associates to each coalition I of players the value (or utility) that coalition can get. Assume without loss of generality that $v(\emptyset) = 0$. The core of the game (N, v) is the set of allocations $x \in R^{N}$ such that:

1. (Group rationality) $\sum_{n \in N} x_{n} = v(N)$;
2. (Coalition rationality) for all $I \subseteq N$, $\sum_{n \in I} x_{n} \geq v(I)$.

A TUG is called superadditive if, for any two disjoint sets of agents I and J,

$$v(I \cup J) \geq v(I) + v(J).$$

If a TUG is superadditive, total value is maximized when agents form the grand coalition $J = N$. This is required to make the core a meaningful concept.

Definition 8 *The game played through agents induced by the TUG (N, v) is defined by a set of principals $M = \{1, 2\}$, a set of agents N, an action set $S_{n} = \{1, 2\}$ for every agent, and, for $m = 1, 2$, payoff*

$$G^{m}(s) = v(I^{m}(s)), \tag{16}$$

where $I^{m}(s)$ is the set of agents "choosing" the principal m, namely, $I^{m}(s) \equiv \{n : s_{n} = m\}$.

To illustrate the definition, consider a well-known TUG: the *majority game*. There are $N = 2K + 1$ players, and the value is:

$$\begin{aligned} v(J) &= 0 \text{ if } \sharp J \leq K, \\ &= 1 \text{ if } \sharp J \geq K + 1. \end{aligned}$$

A possible interpretation is that players have to divide one dollar, and any coalition with the simple majority can vote to itself the dollar. It is easy to see that the *GPTA induced by the majority game* is the voting game presented earlier (the last example is Subsection 4.2), with the assumption that $x = 1$.

The connection between the TUG's and GPTA's is:

Theorem 7 *The TUG (N, v) has a core allocation $(\hat{x}^{n})_{n \in N}$ if and only if the GPTA induced by (N, v) has a pure-strategy equilibrium with outcome $\hat{s} = (1, ..., 1)$ and transfers $\hat{t}$ such that $\hat{t}^{1}_{n}(1) = \hat{t}^{2}_{n}(2) = \hat{x}^{n}$ and $\hat{t}^{1}_{n}(2) = \hat{t}^{2}_{n}(1) = 0$ for all agents.*

Proof: The induced GPTA has two actions per agent and no externalities. Proposition 1 applies. Let $\hat{s} = (1, ..., 1)$. Conditions (AM) and (CM) are satisfied if and only if $\hat{t}^1_n(1) = \hat{t}^2_n(2)$ and $\hat{t}^1_n(2) = \hat{t}^2_n(1) = 0$ for all agents. For every $J \subseteq N$, the (IC) for Principal 1 is

$$v(N) \geq v(N/J) + \sum_{n \in J} \hat{t}^2_n(2);$$

while the (IC) for 2 is:

$$\sum_{n \in N} \hat{t}^1_n(1) \geq v(J) + \sum_{n \notin J} \hat{t}^1_n(1).$$

The (IC) for $J = N$ implies $\sum_{n \in N} \hat{t}^2_n(2) = \sum_{n \in N} \hat{t}^1_n(1) = v(N)$. If $\sum_{n \in J} \hat{t}^2_n(2) = v(N)$ is added to both sides of the (IC) for 1, the (IC) for 1 becomes equal to the (IC) for 2. Thus, the (IC) rewrite as

$$\sum_{n \in J} \hat{t}^1_n(1) \quad \geq \quad v(J) \qquad \forall J \subseteq N. \tag{17}$$

$$\sum_{n \in N} \hat{t}^1_n(1) \quad = \quad v(N) \tag{18}$$

Hence, we have shown that a $\hat{t}$ is part of a pure-strategy equilibrium with outcome $(1, ..., 1)$ if and only if $\left\{\hat{t}^1_n(1)\right\}_{n \in N}$ satisfies (17) and (18). By letting $\hat{t}^1_n(1) = x_n$, (17) and (18) define a core allocation, and the theorem is proven. ∎

The core of a TUG is nonempty if and only the corresponding GPTA has a pure-strategy equilibrium in which all agents choose the same principal (as the game is symmetric between principals, the statement of Theorem 7 assumes it is Principal 1). This principal must pay at least $v(J)$ to every possible subset J of agents, otherwise the other principal could profitably "steal" J. To avoid negative profits, the sum of transfers paid by the winning principal must be $v(N)$. But these two conditions are the conditions for the core. Thus, each core allocation corresponds to a transfer profile of a pure-strategy equilibrium. In the majority game, an empty core corresponds to the lack of a pure-strategy equilibrium in the voting game.

Theorem 7 is related to Pérez-Castrillo [18]. He shows an equivalence between: (1) the set of subgame perfect equilibria of a game with multiple firms trying to hire multiple workers; and (2) the set of stable solutions of a cooperative game in which the set of players is the same as the set of workers in game (1). As in our setting, the value of a coalition of workers in the cooperative game is the profit of a firm that hires those workers in the noncooperative game. Pérez-Castrillo's noncooperative game is analogous to the GPTA's considered in this section, except that it has a somewhat different timing structure and specific rules to break ties.[10]

6.3 Convexity and Bork's Claim

Balancedness as defined in cooperative game theory has a useful connection to convexity. Scarf [21] shows that a market game where agents have convex preferences has a

[10] Another difference is that Pérez-Castrillo allows for a non-supermodular value function v. As the grand coalition may not be efficient, the appropriate solution is not the core but "core-like" stability. Theorem 7 could be extended in that direction.

nonempty core. This is achieved by proving that convexity is a sufficient condition for balancedness. A result in the same spirit can be derived for our definition of balancedness:

Theorem 8 *Assume that: For each agent n, the action space S_n is a convex set in $\Re^{k_n}$, where k_n is some natural number; For each agent n, the payoff function $F_n(s_n)$ is bounded, continuous, and concave in s_n; For each principal m, the payoff function $G^m(s)$ is bounded, continuous, and concave in s; There exists $\hat{s} \in \arg\max \sum_{n \in N} F_n(s_n) + \sum_{m \in M} G^m(s)$. Then, there exists a weakly truthful equilibrium with outcome $\hat{s}$.*

Proof: See Appendix. ∎

Theorem 4 can be used to evaluate the validity of Judge Bork's [7] claim that unregulated vertical contracting leads to productive efficiency. To do that, we introduce the *Vertical Contracting Game*. Let M be a set of sellers (upstream firms) and N a set of buyers (final consumers or downstream firms). Let $q^m \in Q^m = [0, \bar{q}^m]$ be the quantity produced by m. Let $q = (q^1, ..., q^M)$. The cost of production for m is $C^m(q)$, where C^m is assumed to be continuous, convex and bounded. The cost of production of m may depend on q^{-m} because sellers compete for the same inputs.

Let $q_n = (q_n^1, ..., q_n^M) \in Q^1 \times \cdots \times Q^M$ denote the vector of quantities that buyer n buys. The benefit (or revenue) that n derives from q_n is $R_n(q_n)$, and is assumed to be continuous, concave, and bounded in q_n.

Each seller offers a menu of contracts to each buyer. Let $p_n^m(q_n)$ be the price that m asks from n if n chooses vector q_n. This allows for nonlinear pricing and for exclusive clauses. For instance, seller m can impose an exclusive clause on competitor j by setting $p_n^m(q_n)$ at a prohibitively high level when both q_n^m and q_n^j are strictly positive (note that p_n^m need not be continuous). However, the buyer can always choose not to buy. Hence, we impose the restriction that $p_n^m(q_n) \leq 0$ whenever $q_n^m = 0$. We add the technical assumption that the total differentials of R and C are everywhere bounded above by a common constant.

The Vertical Contracting Game is not a GPTA as defined here because payments go from agents to principals. However, with a simple "trick" we can fit it in our framework. Instead of prices, we redefine contracts in terms of discounts from a prohibitively high linear pricing schedule. Thus:

Proposition 5 *The Vertical Contracting Game has an equilibrium that maximizes the joint surplus of sellers and buyers, that is, in which the outcome is*

$$\hat{q} = \arg\max_q \sum_{n \in N} R_n(q_n) - \sum_{m \in M} C^m(q).$$

Proof: Consider the following four games:

1. The Vertical Contracting game;
2. A GPTA with: $G^m(q) = -C^m(q)$; $F_n(q_n) = R_n(q_n)$; the action space of agents is the same of the Vertical Contracting Game; principals can offer transfers to agents of the usual form $t_n^m(q_n)$. But transfers are *not* restricted to be nonnegative everywhere: $t_n^m(q_n)$ must be nonnegative only if $q_n^m = 0$.

3. A GPTA with: $\tilde{G}^{\mathsf{m}}(q) = k\sum_{\mathsf{n}\in\mathsf{N}} q_{\mathsf{n}}^{\mathsf{m}} + G^{\mathsf{m}}(q)$ and $\tilde{F}_{\mathsf{n}}(q_{\mathsf{n}}) = F_{\mathsf{n}}(q_{\mathsf{n}}) - k\sum_{\mathsf{m}\in\mathsf{M}} q_{\mathsf{n}}^{\mathsf{m}}$, where k is a number; the action space of agents is the same of the Vertical Contracting Game; principals can offer transfers to agents of the usual form $t_{\mathsf{n}}^{\mathsf{m}}(q_{\mathsf{n}})$. Again, transfers are not restricted to be nonnegative everywhere: $t_{\mathsf{n}}^{\mathsf{m}}(q_{\mathsf{n}})$ must be nonnegative only if $q_{\mathsf{n}}^{\mathsf{m}} = 0$.

4. A GPTA with strategies and payoffs as in Game 3 (but with the nonnegativity constraint everywhere).

In all four games, the efficient outcome is $\hat{q}$. Game 4 satisfies the conditions of Theorem 8 and therefore it has a weakly truthful equilibrium supporting the efficient outcome. If k is high enough, $\tilde{F}_{\mathsf{n}}(q_{\mathsf{n}}) < 0$ everywhere except when $\sum_{\mathsf{m}\in\mathsf{M}} q_{\mathsf{n}}^{\mathsf{m}} = 0$. (the technical assumption guarantees that this is true even as $q_{\mathsf{n}}^{\mathsf{m}}$ becomes very small). Then, for k high enough, agent n would not choose $q_{\mathsf{n}}^{\mathsf{m}} \neq 0$ unless he is offered a strictly positive transfer. In equilibrium, the nonnegativity constraint on $t_{\mathsf{n}}^{\mathsf{m}}(q_{\mathsf{n}})$ can only be binding when $q_{\mathsf{n}}^{\mathsf{m}} = 0$. Thus, with k high enough, an equilibrium of Game 4 is also an equilibrium of Game 3.

However, Game 3 is strategically equivalent to Game 2. For every m, n, and s_{n}, let $\tilde{t}_{\mathsf{n}}^{\mathsf{m}}(q_{\mathsf{n}}) = t_{\mathsf{n}}^{\mathsf{m}}(q_{\mathsf{n}}) + kq_{\mathsf{n}}^{\mathsf{m}}$. The strategy profile (t, s) is feasible in 2 if and only if the strategy profile $(\tilde{t}, s)$ is feasible in 3. Moreover, the payoff generated by (t, s) in 2 is equal to the payoff generated by $(\tilde{t}, s)$ in 3 for every player. An equilibrium in 3 corresponds to an equilibrium in 2. This shows that Game 2 has an equilibrium with outcome $\hat{q}$.

Finally, it is immediate to see that Game 2 and Game 1 are strategically equivalent. Hence, we have shown that Game 1 has an equilibrium with outcome $\hat{q}$. ■

In this classical production environment, Bork's claim is correct in the sense that there is an equilibrium in which firms sign vertical contracts that ensure to productive efficiency (of course there could also be other equilibria).

To interpret this result in terms of welfare, we need to know who the buyers are. If they are final consumers, then Proposition 5 says that unrestricted vertical contracting will lead to an outcome that is optimal from a Utilitarian perspective. If buyers are downstream firms, it tells us that all the firms involved in the game – upstream and downstream – behave as if they were owned by the same person. We would need to know how the downstream firms relate to consumers in order to say whether this outcome is efficient. However, if the game between downstream firms and consumers satisfies, in turn, the assumptions of Proposition 5, then we know that the outcome is again optimal from a Utilitarian perspective.

A crucial assumption behind Proposition 5, and hence behind Bork's claim, is that there are no direct externalities among buyers. However, if buyers are downstream firms operating on related markets, one should expect demand interactions. If buyers are consumers, there could be network effects. There could also be externalities among buyers due to future competition, either because there are potential upstream entrants (Rasmusen, Ramseyer, and Wiley [20] and Segal and Whinston [23]) or because there are potential new buyers in noncoincident markets (Bernheim and Whinston [5]).

The other important assumption behind the proposition is, of course, that the cost functions of sellers are convex. An efficient equilibrium may not exist if some sellers have to sustain fixed costs in order to produce the first unit or, more generally, if they face economies of scale, as in the Opposite Interest Game example.

6.4 Outcome-contingent contracts

Throughout the paper we have assumed that a transfer between Principal m and Agent n can be conditional on the action taken by the agent s_n. However, the payment could also depend on the whole outcome s rather than only on the component under the control of Agent n. In this section, we discuss this possibility and we show that this addition does not solve the problem of non-existence of efficient equilibria.

A *GPTA with outcome-contingent contracts* is defined as in Section 2 except that the transfer offered by Principal m to Agent n is now dependent on s rather than s_n. We call this transfer $\tau_n^m(s)$.

In general, we may expect the set of equilibria to be modified because the conditions for agent maximization are changed. Now, the agent needs to know what other agents are doing before deciding what he should do. A characterization of pure-strategy equilibria along the line of Theorem 1 is not possible anymore. What cannot be written in a simple form are the (IC) conditions. With action-contingent contracts, principal m knows exactly how much money it takes to convince agent n to deviate from candidate equilibrium action $\hat{s}_n$ to alternative action s_n. With outcome-contingent contracts, this depends on what agent n expects other agents to do, which in turn depends on what the other agents expect n to do.

However, in the spirit of the paper, we would like to know whether the introduction of outcome-contingent contracts is sufficient to restore efficiency – that is, whether for any GPTA with outcome-contingent contracts there exists an equilibrium supporting an efficient outcome. The answer is negative. There are still games in which all equilibria are inefficient. To see this, we can utilize once more the Opposite-Interest Game:

Proposition 6 *The Opposite-Interest Game, played with outcome-contingent contracts, has no efficient equilibrium*

Proof: Suppose that there exists a pure-strategy equilibrium with outcome TL and let the equilibrium transfers of Principal 1 be given by $\hat{\tau}^1$. Principal 2 can guarantee herself a gross payoff of x by convincing at least one of the agents to deviate. If Principal 2 offers to Agent 1 $\tau_1^2(BL) > \hat{\tau}_1^1(TL) - \hat{\tau}_1^1(BL)$ and $\tau_1^2(BR) > \hat{\tau}_1^1(TL) - \hat{\tau}_1^1(BR)$, then it is a dominant strategy for Agent 1 to choose $s_1 = B$, independently of what Agent 2 does. Similarly, if Principal 2 offers to Agent 2 $\tau_2^2(TR) > \hat{\tau}_2^1(TL) - \hat{\tau}_2^1(TR)$ and $\tau_2^2(BR) > \hat{\tau}_2^1(TL) - \hat{\tau}_2^1(TL)$, then a deviation of 2 is guaranteed. Hence, in order for $\hat{\tau}^1$ to be an equilibrium transfer, it must be such that $\hat{\tau}_1^1(TL) \geq x$ and $\hat{\tau}_2^1(TL) \geq x$. Hence, $\hat{\tau}_1^1(TL) + \hat{\tau}_2^1(TL) \geq 2x$, which implies that the net payoff of Principal 1 is negative because, by Assumption, $y < 2x$. This shows that a pure-strategy equilibrium with outcome TL cannot exist.

So far, we have restricted attention to pure-strategy equilibria. However, it is easy to see that, if there exists a mixed-strategy equilibrium in which agents choose TL for sure, it must be the case that $\hat{\tau}_1^1(TL)$ and $\hat{\tau}_2^1(TL)$ are deterministic. Then, the above proof is still applicable. ■

Nevertheless, there are games in which the possibility of making outcome-contingent transfers creates efficient, if somewhat implausible, equilibria that do not exist if transfers are action-contingent. The point is illustrated by a variation of the Opposite Interest Game:

	L	R
T	$3, 0$	$0, 2$
B	$0, 2$	$0, -100$

in which the only difference is that Principal 2 receives a very negative payoff if both agents deviate from (T, L). The action-contingent version of this game does not have a pure-strategy equilibrium. This can be checked through the balancedness condition, using the weights $w^1(BR) = w^2(BL) = w^2(TR) = 1$. Instead, the outcome-contingent version does have a pure-strategy equilibrium with outcome (T, L). Principal 1 offers $\tau_1^1(BR) = \tau_2^1(BR) = 10$ and zero for all other outcomes. Principal 2 offers zero on all outcomes. Agents 1 and 2 face a coordination problem. We assume they coordinate on (TL) if and only if Principal 2 offers zero for all outcomes. Otherwise they coordinate on (BR). Of course, if Principal 2 were to offer more than 10 for either (BL) or (TR), she could get either of those outcomes, but such a deviation is not in her interest. Any lower deviation would be detrimental because it would induce the agent to coordinate on the very negative outcome (BR).

6.5 Other issues

Throughout the paper it has been assumed that principals make their offers simultaneously and that agents choose their actions simultaneously. One could instead consider the *principal-sequential* version of GPTA's, in which principals make their offers one after the other in a predetermined ordering and each principal observes the offers made before her. Alternatively, one could look at the *agent-sequential* version, in which all principals make their offers to Agent 1, Agent 1 chooses his actions which is observed by everybody, then all principals make their offers to Agent 2, and so on.[11]

Moving to sequential timing usually changes the set of subgame-perfect equilibria. However, it does not restore efficiency in general. In particular, it is easy to find examples in which there in no efficient subgame-perfect equilibrium, for no ordering of principals in the principal-sequential version and no ordering of agents in the agent-sequential version.

Another possible extension would be to assume that agents have externalities. The payoff function of Agent n is then written as $F_{\mathsf{n}}(s)$ rather than $F_{\mathsf{n}}(s_{\mathsf{n}})$. As in Segal [22], the difficulty is that now agent n's best response depends also on what the other agents are doing. It is therefore important to know whether offers are public or secret. The analysis developed here is still in part valid if one is willing to assume that offers are secret and that agents use passive beliefs, that is, if faced with an out-of-equilibrium offer from one principal, they believe that only that principal has changed her transfer profile and that she has changed it only with that particular agent (see Segal for a discussion of passive beliefs). In that case, one can prove a result analogous to Theorem 4: a GPTA has a pure-strategy weakly truthful passive-belief equilibrium if and only if a certain

[11]Prat and Rustichini [19] study the principal-sequential version of Bernheim and Whinston [4]. Bergemann and Välimäki [3] examine the multi-period version of Bernheim and Whinston.

balancedness condition is satisfied.[12]

However, as it was pointed out in the introduction, with direct agent externalities, the connection between pure-strategy equilibria and efficiency is severed. In general, it is not true anymore that a weakly truthful pure-strategy equilibrium supports an efficient outcome.

7 Appendix: Proofs

Proof of Theorem 4 By Proposition 4, we can focus on (WT), (AM), and (CM), which is a system of inequalities and equalities. However, we can further simplify the problem by showing that there exists a solution to (WT), (AM), and (CM) if and only if there exists a solution to another system, which contains only inequalities:

Lemma 2 *There exists a weakly truthful equilibrium with outcome $\hat{s}$ if and only if there exists $d \in R^{MH}$ that satisfies:*

(WTd) For all $s \in S$ and all $m \in M$, $\sum_{n:s_n \neq \hat{s}_n} d_n^m(s_n) \geq G^m(s) - G^m(\hat{s})$;

(AMd) For all $n \in N$ and all $s_n \in S_n$, $\sum_{m \in M} d_n^m(s_n) \leq F_n(\hat{s}_n) - F_n(s_n)$.

Proof Let $d_n^m(s_n) \equiv t_n^m(s_n) - t_n^m(\hat{s}_n)$. Then, (WTd) is (WT) and (AMd) is (AM). Hence, the "only if" part is immediate.

To prove sufficiency, suppose that a matrix d has been found that satisfies (WTd) and (AMd). Clearly, there exists a nonnegative matrix $\hat{t}$ that satisfies (WT) and (AM). Starting from $\hat{t}$, we now construct a nonnegative matrix $\tilde{t}$ that satisfies (AM), (WT), and (CM). For every n and m, define (the definition is recursive over m: fix n and then use the definition for $m = 1, 2, ..., M$):

$$\begin{aligned} b_n^m = \min[&\hat{t}_n^m(\hat{s}_n), F_n(\hat{s}_n) + \sum_{j=1}^{m-1} \tilde{t}_n^j(\hat{s}_n) + \sum_{j=m}^{M} \hat{t}_n^j(\hat{s}_n) \\ &- \max_{s_n \neq \hat{s}_n} \left(F_n(s_n) + \sum_{j=1}^{m-1} \tilde{t}_n^j(s_n) + \sum_{j=m+1}^{M} \hat{t}_n^j(s_n) \right)]; \\ \tilde{t}_n^m(s_n) = \max&\{0, \hat{t}_n^m(s_n) - b_n^m\} \qquad \forall s_n \in S_n. \end{aligned}$$

The new matrix $\tilde{t}$ is such that, for each m and n, the vector $\tilde{t}_n^m$ is a shifted down version of $\hat{t}_n^m$ (save for the nonnegativity constraint). The parameter shift b_n^m is, as we shall see, exactly enough to satisfy (CM) for that particular pair m and n.

This definition implies that $b_n^m \leq \hat{t}_n^m(\hat{s}_n)$ and hence

$$\tilde{t}_n^m(\hat{s}_n) = \hat{t}_n^m(\hat{s}_n) - b_n^m \tag{19}$$

We check that $\tilde{t}$ satisfies (AM), (WT), and (CM).

[12] This result was included in an earlier version of this paper, which is available from the authors.

We first show (AM) by proving that, for all $m \geq 2$ and n, if

$$F_n(\hat{s}_n) + \sum_{j=1}^{m-1} \tilde{t}_n^j(\hat{s}_n) + \sum_{j=m}^{M} \hat{t}_n^j(\hat{s}_n) \geq F_n(s_n) + \sum_{j=1}^{m-1} \tilde{t}_n^j(s_n) + \sum_{j=m}^{M} \hat{t}_n^j(s_n) \tag{20}$$

then

$$F_n(\hat{s}_n) + \sum_{j=1}^{m} \tilde{t}_n^j(\hat{s}_n) + \sum_{j=m+1}^{M} \hat{t}_n^j(\hat{s}_n) \geq F_n(s_n) + \sum_{j=1}^{m} \tilde{t}_n^j(s_n) + \sum_{j=m+1}^{M} \hat{t}_n^j(s_n) \tag{21}$$

Then, by letting $m = 1, 2, ..., M$, we can show that (AM) for $\hat{t}_n$ implies (AM) for $\tilde{t}_n$. To see that (20) implies (21), consider the two cases: $\tilde{t}_n^m(s_n) > 0$ and $\tilde{t}_n^m(s_n) = 0$. In the first case, $\hat{t}_n^m(\hat{s}_n) - \tilde{t}_n^m(\hat{s}_n) = \hat{t}_n^m(s_n) - \tilde{t}_n^m(s_n) = b_n^m$ and it is immediate to see that (20) implies (21). In the second case,

$$\begin{aligned}
F_n(s_n) + \sum_{j=1}^{m} \tilde{t}_n^j(s_n) + \sum_{j=m+1}^{M} \hat{t}_n^j(s_n) &= F_n(s_n) + \sum_{j=1}^{m-1} \tilde{t}_n^j(s_n) + \sum_{j=m+1}^{M} \hat{t}_n^j(s_n) \\
&\leq \max_{s_n \neq \hat{s}_n} (F_n(s_n) + \sum_{j=1}^{m-1} \tilde{t}_n^j(s_n) + \sum_{j=m+1}^{M} \hat{t}_n^j(s_n)) \\
&\leq F_n(\hat{s}_n) + \sum_{j=1}^{m-1} \tilde{t}_n^j(\hat{s}_n) + \sum_{j=m}^{M} \hat{t}_n^j(\hat{s}_n) - b_n^m \\
&= F_n(\hat{s}_n) + \sum_{j=1}^{m} \tilde{t}_n^j(\hat{s}_n) + \sum_{j=m+1}^{M} \hat{t}_n^j(\hat{s}_n).
\end{aligned}$$

where the second inequality is due to the definition of b_n^m and the last equality is due to (19). Again, (21) holds.

It is immediate to see that (WT) holds. The transfers on $\hat{s}_n$ are always reduced as much as the transfers on the other actions. For all m and n:

$$\hat{t}_n^m(\hat{s}_n) - \tilde{t}_n^m(\hat{s}_n) \geq \hat{t}_n^m(s_n) - \tilde{t}_n^m(s_n) \qquad \forall s_n \in S_n. \tag{22}$$

Finally, to prove (CM), note that, for every m and n, either $\tilde{t}_n^m(\hat{s}_n) = 0$, in which case (CM) for m and n is verified, or $b_n^m < \hat{t}_n^m(\hat{s}_n)$ and $\tilde{t}_n^m(\hat{s}_n) =$

$$\hat{t}_n^m(\hat{s}_n) - F_n(\hat{s}_n) - \sum_{j=1}^{m-1} \tilde{t}_n^j(\hat{s}_n) - \sum_{j=m}^{M} \hat{t}_n^j(\hat{s}_n) + \max_{s_n \neq \hat{s}_n} \left(F_n(s_n) + \sum_{j=1}^{m-1} \tilde{t}_n^j(s_n) + \sum_{j=m+1}^{M} \hat{t}_n^j(s_n) \right).$$

implying,

$$F_n(\hat{s}_n) + \sum_{j=1}^{m} \tilde{t}_n^j(\hat{s}_n) + \sum_{j=m+1}^{M} \hat{t}_n^j(\hat{s}_n) = F_n(\bar{s}_n) + \sum_{j=1}^{m-1} \tilde{t}_n^j(\bar{s}_n) + \sum_{j=m+1}^{M} \hat{t}_n^j(\bar{s}_n).$$

for some $\bar{s}_n \neq \hat{s}_n$. Combining the last inequality with (22), we have

$$F_n(\hat{s}_n) - \sum_{m \in M} \tilde{t}_n^m(\hat{s}_n) \leq F_n(\bar{s}_n) + \sum_{j \neq m} \tilde{t}_n^j(\bar{s}_n) \leq \max_{s_n \neq \hat{s}_n} \left(F_n(s_n) + \sum_{j \neq m} \tilde{t}_n^j(s_n) \right).$$

But (AM), which we proved above, implies

$$F_{\mathsf{n}}(\hat{s}_{\mathsf{n}}) - \sum_{\mathsf{m}\in\mathsf{M}} \tilde{t}^{\mathsf{m}}_{\mathsf{n}}(\hat{s}_{\mathsf{n}}) \geq \max_{\mathsf{s}_{\mathsf{n}}\neq\hat{\mathsf{s}}_{\mathsf{n}}} \left(F_{\mathsf{n}}(s_{\mathsf{n}}) + \sum_{\mathsf{m}\in\mathsf{M}} \tilde{t}^{\mathsf{m}}_{\mathsf{n}}(s_{\mathsf{n}}) \right),$$

which shows (CM). ■

With Lemma 2, we can focus on necessary and sufficient conditions for the existence of a vector d that solves (WTd) and (AMd). We use the following duality result:[13]

Theorem 9 *Given a matrix A and a vector a, either (i) there exists an x such that $Ax \leq a$; or (ii) there exists a y such that $yA = 0$, $ya < 0$, and $y \geq 0$.*

We rewrite (WTd) and (AMd) in a way that fits (i) of Theorem 9. Let

$$\begin{aligned}
B_{(\mathsf{ms},\mathsf{jna_n})} &= \begin{cases} -1 \text{ if } j = m, s_{\mathsf{n}} = a_{\mathsf{n}}, s_{\mathsf{n}} \neq \hat{s}_{\mathsf{n}} \\ 0 \text{ otherwise;} \end{cases} \\
C_{(\mathsf{ns_n},\mathsf{jia_i})} &= \begin{cases} 1 \text{ if } n = i, s_{\mathsf{n}} = a_{\mathsf{i}}, \\ 0 \text{ otherwise.} \end{cases} \\
b_{\mathsf{ms}} &= G^{\mathsf{m}}(\hat{s}) - G^{\mathsf{m}}(s) && (23) \\
c_{\mathsf{ns_n}} &= F_{\mathsf{n}}(\hat{s}_{\mathsf{n}}) - F_{\mathsf{n}}(s_{\mathsf{n}}) && (24)
\end{aligned}$$

Then B has dimensions (MS, MH), C (H, MH), b $(MS, 1)$, and c $(H, 1)$. If we let $x = d$,

$$A = \begin{bmatrix} B \\ C \end{bmatrix},$$

and

$$a = \begin{bmatrix} b \\ c \end{bmatrix},$$

we transform the problem of the existence of a d satisfying (AMd) and (WTd) into (i) of Theorem 9.

By Theorem 9, (i) is true if and only if there is no y such that (ii) is true. Let $y = [w, z]$, where w has dimensions $(1, M \times S)$ and z has dimensions $(1, H)$. Then (ii) says that $wB + zC = 0$, $wb + zc < 0$, $w, z \geq 0$.

Let $1_{(\cdot)}$ be the indicator function (which returns 1 if the argument is true and zero if it false). The system $wB + zC = 0$ can be rewritten as: for every $m \in M, n \in N$ $a_{\mathsf{n}} \in S_{\mathsf{n}}$:

$$-\sum_{\mathsf{j}\in\mathsf{M}} \sum_{\mathsf{s}\in\mathsf{S}} w^{\mathsf{m}}(s) 1_{(\mathsf{m}=\mathsf{j},\mathsf{s_n}\neq\hat{\mathsf{s}}_{\mathsf{n}},\mathsf{s_n}=\mathsf{a_n})} + \sum_{\mathsf{i}\in\mathsf{N}} \sum_{\mathsf{a_n}\in\mathsf{S_n}} z_{\mathsf{n}}(s_{\mathsf{n}}) 1_{(\mathsf{i}=\mathsf{n},\mathsf{s_n}=\mathsf{a_n})} = 0,$$

which we can write

$$\text{for every } m \in M, n \in N, a_{\mathsf{n}} \in S_{\mathsf{n}}/\hat{s}_{\mathsf{n}} : \sum_{\{\mathsf{s}:\mathsf{s_n}=\mathsf{a_n}\}} w^{\mathsf{m}}(s) = z_{\mathsf{n}}(a_{\mathsf{n}}).$$

which, together with the nonnegativity condition $w, z \geq 0$, corresponds to balancedness.

[13] See Mangasarian [14, p. 33].

The inequality $wb + zc < 0$ can be transformed into

$$\sum_{m \in M} \sum_{s \in S} w^m(s)(G^m(\hat{s}) - G^m(s)) + \sum_{n \in N} \sum_{s_n \in S_n} z^n(s_n)(F_n(\hat{s}_n) - F_n(s_n)) < 0,$$

This is the negation of the game being balanced.

We have shown that exactly one of the following two statements is true: the system (WTd) and (AMd) has a solution d, or there exist balanced vectors w and z that violate the condition for a balanced game. This proves the theorem.

Proof of Theorem 6 The strategies by the two principals are uncorrelated and can be represented as a vector of two independent random variables, taking values in the space of transfers. The probability space is the underlying probability space of these random variables; in particular "almost surely" refers to such space. The equilibrium strategy is denoted by the (now) random variable $(\hat{t}^m_n)_{m \in \{1,2\}, n \in N}$. For a random variable X, $\bar{X}$ denotes its essential supremum. Note that if X^m are finitely many independent random variables, then $\overline{\sum_j X^j} = \sum_j \overline{X^j}$.

The following properties are easy to prove directly, for mixed strategy equilibrium transfers, from the fact that the strategies are independent:

Lemma 3 *Assume that $\sharp S_n = 2$ and $F_n = 0$ for every n. If a mixed strategy equilibrium selects $\hat{s}$ almost surely, then*

1. $\hat{t}^m_n(\hat{s}_n)$ *is a constant almost surely;*
2. $\hat{t}^m_n(\hat{s}_n)\hat{t}^m_n(s'_n) = 0$ *almost surely, for all m and n;*
3. $\overline{\sum_{m \in \{1,2\}} \hat{t}^m_n(s'_n)} = \sum_{m \in \{1,2\}} \overline{\hat{t}^m_n(s'_n)} = \sum_{m \in \{1,2\}} \hat{t}^m_n(\hat{s}_n)$ *for every n.*

We claim that the strategy profile defined by:

$$\tilde{t}^m_n(s_n) \equiv \overline{\hat{t}^m_n(s_n)} \qquad \text{for every } m,\ n, \text{ and } s_n$$

is a pure-strategy equilibrium with outcome $\hat{s}$. The proof consists in verifying that the necessary and sufficient conditions (AM), (IC), and (CM) are satisfied. By Lemma 3 parts 2 and 3, (AM) and (CM) are easily verified. With two principals, (IC) is

$$G^m(\hat{s}) + \sum_n \tilde{t}^{-m}_n(\hat{s}_n) \geq G^m(s) + \sum_n \tilde{t}^{-m}_n(s_n) \qquad \text{for all } s.$$

Given that by (CM) $\tilde{t}^{-m}_n(s'_n) = \tilde{t}^m_n(\hat{s}_n)$, (IC) rewrites as

$$G^m(\hat{s}) - G^m(s) \geq \sum_{n: s_n \neq \hat{s}_n} (\tilde{t}^m_n(\hat{s}_n) - \tilde{t}^{-m}_n(\hat{s}_n)) \qquad \text{for all } s. \tag{25}$$

In the mixed-strategy equilibrium, the cost for principal m to induce agents to choose *for sure* outcome s is at most

$$\sum_{n: s_n = \hat{s}_n} \overline{\hat{t}^{-m}_n(s'_n)} + \sum_{n: s_n \neq \hat{s}_n} \overline{\hat{t}^{-m}_n(\hat{s}_n)} = \sum_{n: s_n = \hat{s}_n} \tilde{t}^m_n(\hat{s}_n) + \sum_{n: s_n \neq \hat{s}_n} \tilde{t}^{-m}_n(\hat{s}_n)$$

A necessary condition for the mixed-strategy equilibrium is then

$$G^{\mathsf{m}}(\hat{s}) - \sum_{\mathsf{n}} \tilde{t}_{\mathsf{n}}^{\mathsf{m}}(\hat{s}_{\mathsf{n}}) \geq G^{\mathsf{m}}(s) - \sum_{\mathsf{n}:\mathsf{s}_\mathsf{n}=\hat{\mathsf{s}}_\mathsf{n}} \tilde{t}_{\mathsf{n}}^{\mathsf{m}}(\hat{s}_{\mathsf{n}}) - \sum_{\mathsf{n}:\mathsf{s}_\mathsf{n}\neq\hat{\mathsf{s}}_\mathsf{n}} \tilde{t}_{\mathsf{n}}^{-\mathsf{m}}(\hat{s}_{\mathsf{n}}) \qquad \text{for all } s,$$

which rewrites as (25).[14]

Proof of Theorem 8 Theorem 4 is stated for a finite S. However, it is easy to see that the proof goes through for an infinite S provided the G's and the F's are bounded and continuous (for a version of the theorem of the alternative in a Banach space, see Aubin and Ekeland [2, Corollary 22, p 144]). Rather than introducing the notation for the infinite case, we note that, for any collection of balanced weights $\tilde{w}$ and $\tilde{z}$ that yields

$$\tilde{M} = \sum_{\mathsf{m}\in\mathsf{M}} \int_{\mathsf{s}\in\mathsf{S}} \tilde{w}^{\mathsf{m}}(s)(G^{\mathsf{m}}(\hat{s}) - G^{\mathsf{m}}(s))ds + \sum_{\mathsf{n}\in\mathsf{N}} \int_{\mathsf{s}_\mathsf{n}\in\mathsf{S}_\mathsf{n}} \tilde{z}_{\mathsf{n}}(s_{\mathsf{n}})(F_{\mathsf{n}}(\hat{s}_{\mathsf{n}}) - F_{\mathsf{n}}(s_{\mathsf{n}}))ds,$$

and for any positive number ϵ, there exists a collection of balanced weights w and z that yields at least $\tilde{M} - \epsilon$, but assigns strictly positive weights on only a finite number of elements of S . Hence, if there exists an "infinite" $\tilde{w}$ and $\tilde{z}$ that violates the condition for a balanced game, then there also exists a "finite" w and z that violates it.

To simplify notation, redefine without loss of generality S and G in a way that $\hat{s} = 0$ and, for all m and n, $G^{\mathsf{m}}(0) = F_{\mathsf{n}}(0) = 0$.

The proof proceeds by contradiction. We shall show that, if there exists a collection of weights that violates the condition for a balanced game, then there exists an outcome $\bar{s}$ that generates more surplus than the efficient outcome $\hat{s} = 0$. The outcome $\bar{s}$ is constructed as an "average" of outcomes weighted according to the collection of weights that violates the condition for a balanced game.

Suppose there exists no weakly truthful equilibrium with outcome $\hat{s}$. Then, there exists a collection of nonnegative weight w such that, for each agent n, there is a finite set $A_{\mathsf{n}} \subset S_{\mathsf{n}}/\{0\}$ such that

$$\forall m \in M, \forall n \in N, \forall a_{\mathsf{n}} \in A_{\mathsf{n}} : \sum_{\{\mathsf{s}:\mathsf{s}_\mathsf{n}=\mathsf{a}_\mathsf{n}\}} w^{\mathsf{m}}(s) = z_{\mathsf{n}}(a_{\mathsf{n}}), \tag{26}$$

and, letting $A = \prod_{\mathsf{n}\in\mathsf{N}} A_{\mathsf{n}}$,

$$\sum_{\mathsf{m}\in\mathsf{M}} \sum_{\mathsf{s}\in\mathsf{A}} w^{\mathsf{m}}(s)G^{\mathsf{m}}(s) + \sum_{\mathsf{n}\in\mathsf{N}} \sum_{\mathsf{s}_\mathsf{n}\in\mathsf{A}_\mathsf{n}} z_{\mathsf{n}}(s_{\mathsf{n}})F_{\mathsf{n}}(s_{\mathsf{n}}) > 0. \tag{27}$$

If necessary, re-scale the weights w and z in a homogeneous way (multiply all of them by the same scalar) so that $\sum_{\mathsf{n}\in\mathsf{N}} \sum_{\mathsf{s}_\mathsf{n}\in\mathsf{A}_\mathsf{n}} z_{\mathsf{n}}(s_{\mathsf{n}}) = 1$. This re-scaling does not unsettle the inequality (27) or the equalities (26), and it implies that

$$\sum_{\mathsf{s}\in\mathsf{A}} w^{\mathsf{m}}(s) \leq 1 \qquad \forall m \in M,$$

[14] The reason why this line of proof works only for $\mathsf{M} = \{1, 2\}$ is that with more than two principals it is not true that

$$\sum_{\mathsf{j}\neq\mathsf{m}} \sum_{\mathsf{n}:\mathsf{s}_\mathsf{n}=\hat{\mathsf{s}}_\mathsf{n}} \overline{\hat{\mathsf{t}}_{\mathsf{n}}^{\mathsf{j}}(\mathsf{s}'_{\mathsf{n}})} = \sum_{\mathsf{n}:\mathsf{s}_\mathsf{n}=\hat{\mathsf{s}}_\mathsf{n}} \tilde{\mathsf{t}}_{\mathsf{n}}^{\mathsf{m}}(\hat{\mathsf{s}}_{\mathsf{n}}).$$

$$\sum_{s_n \in A_n} z_n(s_n) \leq 1 \qquad\qquad \forall n \in N.$$

Let $\bar{s} = (\bar{s}_1, ..., \bar{s}_N)$ be defined by $\bar{s}_n = \sum_{a_n \in A_n} z_n(a_n)a_n$ for every n. By (26), for every m,

$$\bar{s}_n = \sum_{a_n \in A_n} \left(\sum_{\{s:s_n = a_n\}} w^m(s) \right) a_n = \sum_{a_n \in A_n} \left(\sum_{\{s:s_n = a_n\}} w^m(s)s_n \right) = \sum_{s \in A} w^m(s)s_n.$$

Note that, if $f : \Re^N \to R$ is a concave function and $\lambda_1, ..., \lambda_J$ is a collection of nonnegative numbers that add up to one, then

$$f\left(\sum_{j=1}^{J} \lambda_j x_{1,j}, ..., \sum_{j=1}^{J} \lambda_j x_{N,j} \right) \geq \sum_{j=1}^{J} \lambda_j f\left(x_{1,j}, ..., x_{N,j}\right).$$

Then, for every m, by concavity of G^m,

$$\begin{aligned} G^m(\bar{s}) &= G^m\left(\sum_{s \in A} w^m(s)s_1, ..., \sum_{s \in A} w^m(s)s_N \right) && (28) \\ &= G^m\left(\sum_{s \in A} w^m(s)s_1 + \left(1 - \sum_{s \in A} w^m(s)\right)0, ..., \sum_{s \in A} w^m(s)s_N + \left(1 - \sum_{s \in A} w^m(s)\right)0 \right) \\ &\geq \sum_{s \in A} w^m(s)G^m(s_1, ..., s_N) + \left(1 - \sum_{s \in A} w^m(s)\right) G^m(0) = \sum_{s \in A} w^m(s)G^m(s). \end{aligned}$$

Similarly, by concavity of F_n:

$$F_n(\bar{s}_n) \geq \sum_{s_n \in A_n} z_n(s_n)F_n(s_n) \qquad (29)$$

By summing (28) over m and summing (29) over n, and adding the two resulting inequalities, we get

$$\sum_{m \in M} G^m(\bar{s}) + \sum_{n \in N} F_n(\bar{s}_n) \geq \sum_{m \in M} \sum_{s \in A} w^m(s)G^m(s) + \sum_{n \in N} \sum_{s_n \in A_n} z_n(s_n)F_n(s_n).$$

By (27), this implies $\sum_{m \in M} G^m(\bar{s}) + \sum_{n \in N} F_n(\bar{s}_n) > 0$, which is a contradiction because $\sum_{m \in M} G^m(0) + \sum_{n \in N} F_n(0) = 0$ was assumed to be the maximum of $\sum_{m \in M} G^m(s) + \sum_{n \in N} F_n(s)$ over s.

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