Equilibrium with Derivatives: Price-Contingent Securities and Resource Allocation

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Abstract

We establish an extension of the classical general equilibrium treatment of uncertainty about exogenous states of nature to the phenomenon of price uncertainty. Traders do not know the prices at which trade will occur, but have expectations over possible prices. They trade derivatives, price-contingent securities, to insure against the risks arising from this uncertainty. We establish three results: one is a set of necessary and sufficient conditions for the existence of equilibrium, called an equilibrium with price insurance, in such a framework; another is the fact that equilibria with price insurance are Pareto efficient and agents insure themselves optimally against the price uncertainty represented by their price expectations; and finally we show that in this framework agents' price expectations matter, in the sense that they affect the equilibrium allocation of resources. Completeness of the securities markets requires an uncountable number of securities, one contingent on each possible goods price vector.

Key Words: derivatives, price uncertainty, endogenous uncertainty, general equilibrium, Hilbert space, price expectations.

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1 Price Uncertainty

Uncertainty about future prices has a real welfare cost, just as does uncertainty about exogenous events such as earthquakes and hurricanes. Future price movements may make us rich or poor, and there is naturally a demand for insurance against this risk. In advanced economies uncertainty about prices is probably more extensive and more costly than uncertainty about states of nature (as argued by Shiller [18]).

Many markets exist to meet this demand for insurance against price risks: options and futures and swap markets, indeed derivative markets in general, are used very extensively to insure against price uncertainty. In this respect reality has been ahead of economic theory: until recently there have been few attempts to incorporate price uncertainty into the classical general equilibrium model, although in the finance literature there has been extensive and illuminating discussion of the use and valuation of derivatives to hedge price risks. Finance models, however, are not designed to shed light on issues such as the impact of derivative markets on the overall allocation of risks in the economy, or the interactions between derivative markets, the markets for underlying assets and goods markets. These questions are the focus of this paper: the aim is to establish a framework within which one can think systematically about them. The issue of how derivatives affect the allocation of risks within the economy is particularly pertinent in view of the interest of regulators in the possible systemic risks associated with the extensive use of derivative markets. And the fact that for many securities trade in derivatives now regularly exceeds that in the underlying asset highlights the need to model the links between derivative markets and underlying securities markets in general equilibrium. The extent of derivative trade suggests that the existence of derivative markets may be influencing prices in the underlying markets, calling into question the premises underlying some option valuation models. The results here suggest that the introduction of derivative markets in a general equilibrium model with price uncertainty leads to an improvement in resource allocation, with agents benefitting from the ability to insure otherwise uninsurable risks. There is no evidence of the types of concerns about multiplication of risks and increase in systemic risks that have worried regulators, although some of these might arise from capital market imperfections that are omitted from this model.

Recent work has suggested that uncertainty about prices and other endogenous variables is conceptually more complex than uncertainty about exogenous states of nature. The implication is that it is difficult if not impossible to use state-contingent contracts to manage these risks.[1] We show here that there is in fact a very natural and conceptually simple extension of the classical general equilibrium framework to price uncertainty, involving the use of derivative contracts conditioned on the price index. This extension provides a satisfactory account of the welfare role of derivative markets in hedging price uncertainty in a complete general equilibrium framework.

We consider an exchange economy augmented by the possibility of trading price

[1] For more details, see[5], [10]. Dreze [7] is a good survey of many of the issues raised.

contingent securities which pay a specified sum if and only if the equilibrium price vector assumes a specified value (called price contingent securities or PCSs).[2] Prices play the role played by "states of nature" in the Savage framework. This means that the determination of the state is endogenous to the economic system, whereas in the Arrow-Debreu model, following Savage, states are determined exogenously. It is this endogeneity of states that has led commentators to suggest that there are fundamental differences between the two cases.

In the present model, agents have expectations about possible prices. Before an equilibrium price emerges agents trade price contingent securities, shifting income across price states from states in which they are well off to states in which they are badly off. This changes their endowments in the exchange economy: their endowments are now price contingent, as they depend on the prices that are realized via the PCSs that they hold.

We can think of agents making choices in two stages. One stage involves choosing portfolios of PCSs in the light of their endowments and price expectations. They choose a portfolio of securities which, given their expectations of equilibrium prices and their endowments of goods, will maximize expected utility over equilibrium prices. After they have chosen portfolios of PCSs, equilibrium prices are realized.[3] Given realized equilibrium prices, agents choose consumption bundles to maximize utility subject to their budgets, which depend on their holdings of PCSs. At this point they face no uncertainty, and are behaving as in a standard exchange economy except for the dependence of endowments on prices.

In modelling this we derive an indirect utility function relating agents' expected utilities from equilibrium consumption to their holdings of securities: agents select their portfolios of securities so as to maximize this function. Together with securities prices, this generates demands for and supplies of securities: the net endowment is zero. Market clearing prices in the securities markets equate supply and demand.

An equilibrium of the entire system, called an *equilibrium with price insurance,* is an equilibrium in both the goods and the PCS markets. We establish the following results: (1) conditions on preferences and price expectations necessary and sufficient for the existence of such an equilibrium, (2) equilibria are Pareto efficient and provide agents with insurance against the uncertainty represented by their price expectations and (3) that agents' price expectations influence the equilibrium allocation of resources. This is of course in the context of a complete set of markets, and

[2] Such securities were discussed in the context of a temporary equilibrium model by Svensson [20], and earlier in a growth model by Stigum [19]. More recently they have been discussed inWalrasian general equilibrium models by the authors cited above, and have been discussed as well as by Kurz [15], and Henrotte [13]. None of these latter authors consider these instruments in a standard Walrasian model. An approach within the mainstream general equilibrium framework is in Detemple and Selden [8], which considers the interactions between markets for securities and for options on those securities.

[3] These depend on the portfolios people have chosen, inter alia, but consistent with the competitive assumption they do not take this into account.

so is different from the phenomena which arise in models with sunspots or temporary equilibria.

If goods and contracts contingent on their prices are traded in the same markets, then arbitrage will prevent the securities markets from playing a positive role. Once the prices of goods are announced by an auctioneer, then it is immediately clear that only certain price-contingent securities can have non-zero prices. To avoid this, one needs to separate the markets and prevent arbitrage.[4] In real markets, options and other PCSs are traded before goods are traded: this is a natural arbitrage-free structure, which we use here.

In the next section I provide a simple geometric illustration of the main concepts and results of the paper. Then I set out the general model and results. Technical definitions and proofs are in the appendix.

2 Example

In this section I give two examples that illustrate how the introduction of price contingent securities can affect the equilibria of a Walrasian economy: they also indicate how the general model works. These illustrations use a two-person two-good economy which can be understood via the geometry of an Edgeworth box.[5]

The economy we consider has two risk-averse agents with Leontief indifference curves. There are infinitely many equilibrium allocations when this economy is considered as an exchange economy of the Arrow-Debreu type. The introduction of price uncertainty and of PCSs, traded before goods prices are determined, leads to a unique equilibrium allocation in the trading of goods and the removal of all price risk.

There are two agents with preferences $u_i = \log\min\{c_{1i}, c_{2i}\}, i = 1,2$ where c_{ji} is the consumption of good j by agent i. These preferences combine right-angled indifference curves with risk aversion. The total endowment of each good is one unit, so that $c_{11} + c_{12} = 1$ and $c_{21} + c_{22} = 1$. Agents' endowments are $\Omega_1 = (1,0)$ and $\Omega_2 = (0,1)$, placing the initial endowments at the lower right corner of the Edgeworth box. Agent i holds PCSs $s_i(p_1, p_2)$, where $s_i \lesseqgtr 0$ is the amount paid if and only if the equilibrium price vector is p_1, p_2. At given goods prices p_1, p_2 agents' choice problems are

$$\max u_i = \log\min\{c_{1i}, c_{2i}\} \text{ s.t. } p_1 c_{1i} + p_2 c_{2i} = p_i + s_i\left(p_1, p_2\right)$$

Demands are given by

$$c_{21} = c_{11} = \frac{p_1 + s_1}{p_1 + p_2}, c_{22} = c_{12} = \frac{p_2 + s_2}{p_1 + p_2}$$

[4] As in Arrow [1] or Radner [16].

[5] Note that this example does not satisfy the assumption that preferences are smooth, used in the theoretical development. This assumption is not central to the analysis and can be dropped at the cost of some technical complexity. Here the use of Leontief preferences facilitates the computation of solutions.

Given that $p_1 + p_2 = 1$ and that $s_1(p_1, p_2) + s_2(p_1, p_2) = 0 \; \forall p_1, p_2$, it is clear that these demands clear markets at any prices. As individuals always consume goods 1 and 2 in equal proportions, we can write their utilities as functions just of their consumption of the first good, which is $\frac{p_1+s_1}{p_1+p_2} = p_1 + s_1$ for 1. Consider first the case without price uncertainty or PCSs, so that $s_1 = s_2 = 0$. In this case, *any* point on the diagonal of the Edgeworth box is a competitive equilibrium (see figure 1).

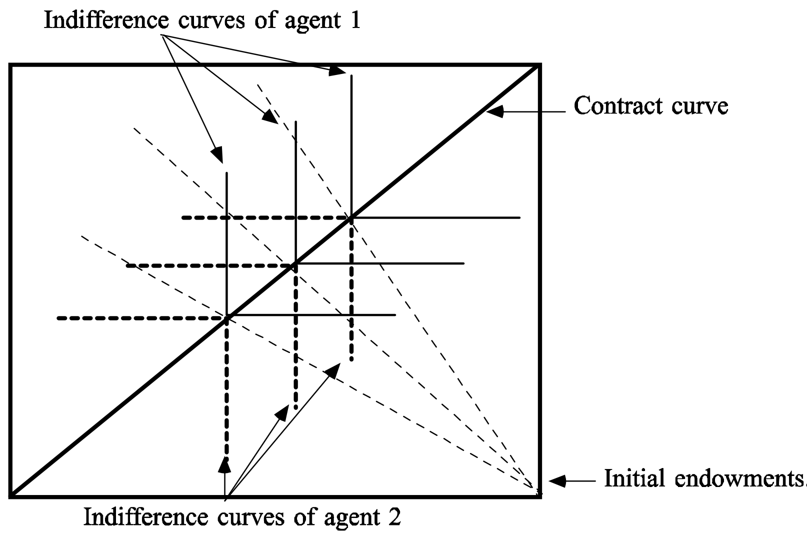

Figure 1: all points on the contract curve of the Edgeworth box are Walrasian equilibria of the exchange economy.

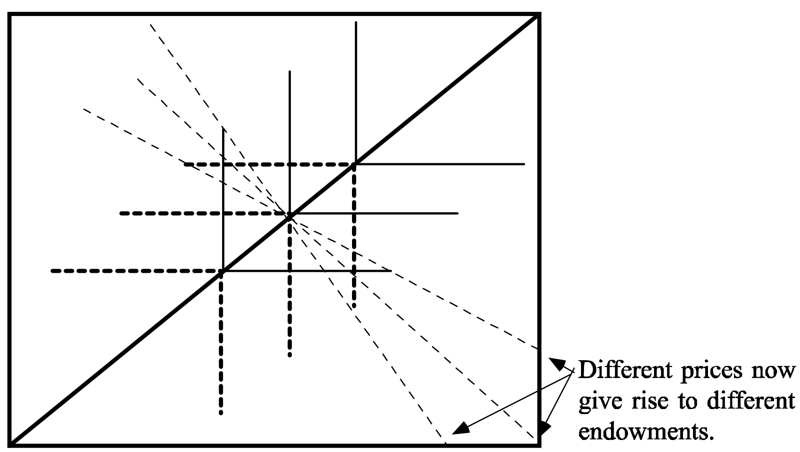

Figure 2: after trading PCSs, there is a unique consumption equilibrium at the mid point of the contract curve. Consumption is independent of prices, and payoffs from price-contingent securities vary so as the move the budget lines and maintain consumption constant.

In the case of markets with price uncertainty and PCSs, the securities holdings are chosen to maximize expected utility given agents' price expectations. We consider two cases. In the simpler, which we analyze first, agents' expectations over equilibrium prices are point distributions. Both expect the prices to be either $(1/4, 3/4)$or $(3/4, 1/4)$: agent 1 believes that these prices are equally likely, whereas two assigns them probabilities of 0.75 and 0.25 respectively. Agent 1 in seeking to maximize expected utility solves the problem

$$\max 0.5 \log \left(0.25 + s_1 \left(0.25\right)\right) + 0.5 \log \left(0.75 + s_1 \left(0.75\right)\right)$$

subject to the securities budget constraint

$$\pi \left(0.25\right) s_1 \left(0.25\right) + \pi \left(0.75\right) s_1 \left(0.75\right) = 0$$

Here $\pi \left(p_1\right)$ is the price of a PCS that pays a unit of account if and only if the equilibrium price of good 1 is p_1 (and so that of good 2 is $p_2 = 1 - p_1$). It is easy to verify that the securities holdings are

$$s_1 \left(0.25\right) = -0.25 + \left(2\lambda_1 \pi \left(0.25\right)\right)^{-1}, s_1 \left(0.75\right) = -0.75 + \left(2\lambda_1 \pi \left(0.75\right)\right)$$

where λ_1 is the shadow price on 1's budget constraint. Similarly, 2's holdings of securities are

$$s_2 \left(0.25\right) = -0.75 + \left(1.33\lambda_2 \pi \left(0.25\right)\right)^{-1}, s_2 \left(0.75\right) = -0.25 + \left(4\lambda_2 \pi \left(0.75\right)\right)$$

Given that $s_1 \left(0.25\right) + s_2 \left(0.25\right) = 0$ and $s_1 \left(0.75\right) + s_2 \left(0.75\right) = 0$, it follows that $\pi \left(0.25\right) = \pi \left(0.75\right)$: securities prices are independent of the underlying goods prices. From this it follows that agents' welfare levels, given by $p_i + s_i$, are independent of the realized goods prices after trading PCSs. So agents trade PCSs in such a way that payoffs from these exactly offset variations in the underlying goods prices, and in so doing they insure themselves fully against price uncertainty.

Next we consider a similar but more general case. Assume agents' price expectations to be uniform, so that the ratios $\frac{p_i}{p_1+p_2}, i = 1, 2$, are uniformly distributed between zero and one. Agent one now seeks to maximize

$$\int_0^1 \log \left(p_1 + s_1 \left(p_1\right)\right) dp_1 \text{ subject to } \int_0^1 s_1 \left(p_1\right) \pi \left(p_1\right) dp_1 = 0$$
$$\text{and } p_1 + s_1 \geq 0$$

recalling that $p_1 + p_2 = 1$. Provided that the non-negativity constraint is never binding (which we assume), the solution to this is

$$s_1 \left(p_1\right) = K_1 - p_1 - \left(\lambda_1 \pi \left(p_1\right)\right)^{-1}$$

where K_1 is a positive constant and λ_1 the shadow price on the integral constraint. This expresses agent 1's demand for PCSs as a function of their price π and the goods price p_1. By analogous arguments $s_2(p_1) = K_2 - 1 + p_1 - (\lambda_2 \pi(p_1))^{-1}$ and the condition that $s_1 + s_2 = 0$ implies that $\pi(p_1)$ is a constant. In this case holdings of PCSs are linear in the price p_1 for both agents. Clearly $s_1(p_1) + p_1$ is a constant, implying that consumption is independent of prices: agents are fully insured at an equilibrium. There is now a unique equilibrium consumption allocation with price insurance, independent of goods prices, involving each agent consuming a half unit of each good. This unique consumption allocation is realized whatever the equilibrium prices, and endowments vary with prices, via the trading of PCSs, so as to make this possible, as illustrated in figure 2. So in both of these cases the introduction of PCSs has radically changed the set of equilibrium allocations in the goods markets, narrowing it from a continuum to a singleton and so completely removing all price risk.

3 The model

The economy $\mathcal{E}$ which we consider can be defined formally as follows.

1. There are n markets for goods and services, and in addition a complete set of markets for price contingent securities, i.e., markets for securities which will pay a unit of account contingent on the realization of any goods price vector in the simplex $\Delta \subset \Re_+^n$.

2. A portfolio of securities is an element of the space of functions from the price simplex to the real line: it assigns to each price vector a value. The value assigned to a price vector is the holding of securities contingent on the specified price vector. A security pays one unit if and only if the specified price vector is realized as an equilibrium A portfolio specifies the quantity of securities held contingent on each possible equilibrium price vector. We take the space of portfolios to be the Hilbert space L_2. Securities prices are elements of the dual to this space. As L_2 is self-dual, both portfolios and securities prices are in L_2.

3. There are I agents, $i \in I$. Agents are characterized by endowments $\Omega_i \in \Re_+^n$, preferences $u_i : \Re_+^n \to \Re$ and price expectations $e_i : 2^\Delta \subset \Re_+^n \to \Re$, where 2^Δ is the σ-algebra of subsets of Δ and $e_i(p)$ is a probability measure representing agent i's expectations over possible prices in the simplex. $e_i(p)$ is measurable with respect to the σ-algebra. Agents choose portfolios of PCSs in the securities markets, and these are a part of their endowments in the goods markets.

4. Consumption sets are the non-negative orthant $\Re_+^n$ for goods and L_2 for securities. So unbounded short sales are allowed for securities but not for commodities. This seems a natural combination, although all other possible combinations could equally well be covered.

Agents are unsure of the prices at which trade in goods in $\Re^n$ will occur. They face no other uncertainty. Prior to trading goods they trade price-contingent securities (PCSs) which allow them to shift income between alternative price states, where price states are defined as possible trading prices in the goods markets. Securities pay in units of numeraire, and the securities market is complete in that it is possible to trade contingent on any price vector in the simplex. The payoff from PCSs purchased by an agent i if trade in the goods market occurs at price vector p is $s_i(p) \lesseqgtr 0$, where we use s_i to denote the holdings by agent i of price contingent securities. The securities traded have payoffs contingent on price vectors, and thus (by Lemma 1 section VII of [5]) can be interpreted as contingent on a price index, and indeed are a form of "exotic option" (Rubinstein [17]).

The utilities are assumed to be continuous, strictly concave and to satisfy the following condition of uniformly bounded rates of increase:

Assumption 1 *Each trader i has a preference represented by a strictly concave continuously differentiable function $u_i : N(\Re^n_+) \to \Re$ (where $N(\Re^n_+)$ is a neighborhood of $\Re^n_+$) such that $u_i(0) = 0$, $\sup_{x \in \Re^n_+} u_i(x) = \infty$, and $\exists \epsilon, \delta > 0$ such that $\forall x \in N(\Re^n_+)$:*

$$0 < \epsilon \leq \|Du_i(x)\| \leq \delta$$

where $Du_i(x)$ is the first derivative of the function u_i at x.

Assumption 2 *If an indifference surface through a positive consumption bundle c intersects a boundary ray r of $\Re^n_+$, then every indifference surface containing points preferred to c also intersects the ray r.*

Assumption 2 is satisfied by many standard preferences on $\Re^n_+$, such as Cobb-Douglas, CES, linear preferences, most Leontief preferences, and smooth utilities defined on a neighborhood of the positive orthant which are transversal to its boundary.[6]

4 Behavior

4.1 The goods market

Consider first behavior in the goods market once a price vector $p \in \Delta$ is known. At this point agent i solves the following problem

$$\max u_i(c_i) \text{ subject to } p.c_i \leq p.\Omega_i + s_i(p) \quad (1)$$

Here with a given price vector p utility is maximized subject to a budget constraint of the standard type augmented by the payoff from PCSs. Denote the maximum of

[6] These assumptions are introduced and used in Chichilnisky [3].

utility subject to this constraint by $w_i(p, s_i(p), \Omega_i)$. As Ω_i will be considered fixed in the following, we suppress dependence on this and write

$$w_i(p, s_i(p)) = \max u_i(c_i) : p.c_i \leq p.\Omega_i + s_i(p) \tag{2}$$

$w_i(p, s_i(p))$ denotes the maximum utility attainable if the goods market price is p, as a function of the payoff in this state from PCSs purchased by agent i. It is a "securities utility function" relating welfare to prices and holdings of PCSs. It can be written $w_i^p(s_i(p))$ to show that it can also be interpreted as a state-dependent utility function when the states are defined by price vectors. Clearly this is a variant on the standard concept of an indirect utility function, being used to reflect the dependence of utilities on the portfolios of securities chosen as well as on market prices. We can also derive a demand function $D_i(p, s_i(p))$ denoting $i's$ demand vector if equilibrium prices are p. Clearly

$$w_i(p, s_i(p)) = \max u_i(c_i) : p.c_i \leq p.\Omega_i + s_i(p) = u_i(D_i(p, s_i(p))) \tag{3}$$

that is, indirect utility given p and s_i is the utility of the demand vector chosen at p and s_i.

4.2 The securities market

As the price p is perceived as a random variable, the agent seeks to pick securities so as to maximize the expected utility derived from trading goods, given the wealth resulting from securities holdings. The agent thus seeks to maximize with respect to securities portfolios the expectation of $w_i(p, s_i(p))$ over all possible prices p. The expectation is with respect to the probability distribution representing the agents' expectations of prices. Agents therefore maximize

$$W_i(s_i(p)) \equiv \int_{p \in \Delta} w_i^p(s_i(p))\, de_i = \int_{p \in \Delta} u_i(D_i(p, s_i(p)))\, de_i$$

This is the expected utility from consumption in the goods market, given the agent's price expectations and holdings of securities. Agents maximize this with respect to the function $s_i(p) : \Delta \to \Re$ defining the portfolio of price-contingent securities held. There is a budget constraint on the choice of price-contingent securities. Let $\pi(p)$ be the price of a security which pays a unit of account if and only if the goods price vector is p: as agents' initial endowments of PCSs are zero, the budget constraint in the securities market is now $\int_\Delta \pi(p)\, s_i(p)\, dp \equiv \langle \pi(p), s_i(p) \rangle = 0$ where $\langle , \rangle$ denotes the inner product of a portfolio vector with a price vector in the space of portfolios and securities prices, see below. In addition, an agent cannot sell more income in a state than he or she has in that state: $p.\Omega_i + s_i(p) \geq 0 \forall p$. This is a "no short sales" condition. Hence the agent's optimal choice of PCSs solves the problem

$$\left.\begin{array}{l}\max\limits_{s_i(p)} \int_{p\in\Delta} w_i\left(p, s_i\left(p\right)\right) de_i (= \int_{p\in\Delta} u_i\left(D_i\left(p, s_i\left(p\right)\right)\right) de_i) \\ \text{subject to} \int_\Delta \pi\left(p\right) s_i\left(p\right) dp = 0 \text{ and } p.\Omega_i + s_i\left(p\right) \geq 0.\end{array}\right\} \tag{4}$$

This maximization problem is a generalization to a high dimensional domain of a classic problem in the calculus of variations, an isoperimetric problem.[7] A solution to such a problem is a set of holdings of PCSs, i.e. a portfolio of securities, which transfers income between price states so as to maximize the expected utility from consuming goods and services. Let $s_i^*\left(p\right)$ be the solution: then in the goods market the agent's optimal behavior solves

$$\max u_i\left(c_i\right) \text{ subject to } p.c_i \leq p.\Omega_i + s_i^*\left(p\right) \tag{5}$$

In the formulation (4) we have formulated the agent's choice problem in a way that collapses both choices, the choice of goods given a price and the choice of securities given price expectations, into one problem of maximizing utility subject to a budget constraint and constraints on short sales. This suggests that, as we shall see later, the problem has the same underlying structure as the basic Arrow-Debreu model, with the expectation being taken over states which in the present case are also prices.

5 General equilibrium

Agents' portfolios of price contingent securities are functions $s_i\left(p\right)$ from the price simplex to the real line, indicating the amount of PCS purchased for each possible price vector in the simplex: $s_i\left(p\right) : \Delta \rightarrow \Re$. They are solutions to the variational problem (4). As such, they satisfy differential equations, the Euler conditions [9], and so are C^1 functions from $\Delta \subset \Re^n_+$ to $\Re$. The space of portfolios of PCSs therefore has to contain C^1 functions from $\Delta \subset \Re^n_+$ to $\Re$. We assume this space to be L_2, the Hilbert space of measurable square integrable functions from $\Delta \in \Re^n_+$ to the real line. A Hilbert space has the norm

$$\|s\| = \left[\int_\Delta s\left(p\right)^2 dp\right]^{1/2}$$

and inner product $\langle x, y\rangle = \left[\int_\Delta x\left(p\right) y\left(p\right) dp\right]$. Prices of price contingent securities are then the dual to the portfolio space: if π is a set of PCS prices, then π is a continuous linear function on the space of PCSs, so that $\pi : L_2 \rightarrow \Re$ and π is in the dual of L_2, which is self dual, so that securities prices lie in the same space as securities. We can therefore use the inner product of a portfolio vector and a price vector to give the value of a portfolio.

The expected utility from securities, W_i, is assumed to satisfy conditions equivalent to Assumption 1.

[7] For a characterization of the solutions of such problems, see Gelfand and Fomin [9], chapter 7.

Assumption 3 *The indirect utility function $w_i^p(s_i(p))$ is normalized so that the expected utility $W_i(s_i(p))$ satisfies $W_i(0) = 0$ and $\sup_{s \in L_2} W_i = \infty$.*[8]

Assumption 4 *The expected indirect utility function W_i satisfies one of the following two conditions: either each indifference surface is bounded below or the set of supports to each indifference surface is a closed set.*

Note the following:

(1) by standard arguments for each agent i, the real-valued function of a real variable $w_i^p(s_i(p))$ is concave in s_i.

(2) from Lemma 2 of Chichilnisky and Heal [6] $W_i(s)$ has a uniformly bounded rate of increase, i.e., $\forall \epsilon > 0 \exists N, \delta \geq 0$ such that $\forall s\ \exists y(s) : W_i(s+y) \geq W_i(s) + \epsilon\delta$, $\|y(s)\| \leq \epsilon$, and $W_i(s+y) \geq W_i(s) + \epsilon\delta \Rightarrow \|y\| \geq \epsilon/N$. This condition of a uniformly bounded rate of increase is equivalent to that in Assumption 1, but applied to a function which is not necessarily differentiable.

We will use extensively the condition of limited arbitrage introduced in Chichilnisky [2] [3] [6], and applied to economies with infinite dimensional commodity spaces by Chichilnisky and Heal in [6].[9] This condition will be applied in the goods markets, where it will be called limited goods arbitrage, and in the securities markets, where it will be called limited securities arbitrage. The definitions are set out in detail in the appendix. Limited arbitrage is a condition that restricts the permissible differences between individuals preferences and expectations. This is similar in concept to the condition of overlapping or similar expectations used as a sufficient condition for the existence of equilibrium in temporary equilibrium models and securities markets models (see Hart [12] and Hammond [11]). The basic intuition is that to ensure existence of an equilibrium we need a condition on preferences that will prevent any two agents from wishing to take unbounded and opposite positions (see Werner [21]). This is attained by defining a cone of rays along which an agent's utility or expected utility increases, and then requiring that all such cones intersect: this in essence is limited arbitrage. The precise definition of this cone is extremely delicate, especially in the case of bounded consumption sets.

5.1 Equilibrium

An equilibrium with price insurance in the economy $\mathcal{E}$ is

1. a set of PCS prices at which PCS markets clear, and such that the associated PCS holdings for each agent maximize the agent's expected utility according to her price expectations in goods markets, as indicated in (4), and

[8] The exact value of the common supremum of all securities utility functions EW_i does not matter: it is taken to the infinity for definitiveness only. What matters is merely that there is such a common supremum.

[9] This condition is shown to be necessary and sufficient for existence of equilibrium, non-emptiness of the core, and the existence of social choice rules, see e.g. [3].

2. a set of goods prices and goods consumptions such that goods markets clear and agents maximize utility as in (1). Formally,

Definition 1 *An equilibrium with price insurance in the economy* $\mathcal{E}$ *is a set of PCS prices* $\pi^* \epsilon L_2$*, of goods prices* $p^* \in \Delta$*, of PCS holdings* $s_i^*(p) \epsilon L_2, i = 1, ..., I$*, and consumptions* $c_i^* \in \Re_+^n, i = 1, .., I$ *such that (1)* c_i^* *maximizes* $u_i(c_i)$ *subject to* $p^* c_i^* \leq p^* \Omega_i + s_i^*(p^*)$*, (2)* $\sum_i c_i^* = \sum \Omega_i$*, (3)* $s_i^*(p)$ *maximizes* $\int_\Delta w_i(p, s_i^*(p)) de_i(p)$ *subject to* $\int_\Delta s_i^*(p) \pi^*(p) dp = 0$ *and* $p.\Omega_h + s_i^*(p) \geq 0$ *and (4)* $\sum_i s_i^*(p) = 0 \forall p \in \Delta$*.*

6 Existence and Efficiency of Equilibrium

6.1 Existence

The main theorem establishes that generalizations of the conditions that are necessary and sufficient for the existence of a competitive equilibrium are also necessary and sufficient for the existence of an equilibrium with price insurance. In this sense the equilibrium with price insurance is a natural extension of the competitive equilibrium concept to the case of price uncertainty.

Theorem 1 *Under Assumptions 1, 2 and 3, the economy* $\mathcal{E}$ *has an equilibrium with price insurance if and only if it satisfies limited arbitrage in goods and securities markets.*

Proof. In the appendix.

Note the following important points:

Remark 1: The equilibrium with price insurance is in general *not* a Walrasian equilibrium of the exchange economy defined by preferences $u_i : \Re_+^n \to \Re$, endowments $\Omega_i \in \Re_+^n$, the commodity space $\Re^n$ and consumption sets $\Re_+^n$.

Remark 2: The equilibrium with price insurance depends upon agents' price expectations $e_i(p)$, as these influence their positions in the PCS markets and thus their endowments in price-contingent states. So price expectations matter and affect the ultimate allocation of resources. This occurs in a context of complete markets and perfect competition in the usual Walrasian framework, as formalized by Arrow and Debreu. We have simply added the possibility of agents insuring against uncertainty about equilibrium prices, an uncertainty which in any realistic analysis of a competitive system they must face.

6.2 Efficiency

What are the welfare properties of an equilibrium? Firstly, note that it is Pareto efficient.

Theorem 2 *Any equilibrium with price insurance in the economy $\mathcal{E}$ is Pareto efficient.*

Proof. Assume that an equilibrium price has been realized in the exchange economy: clearly the equilibrium that then occurs in the exchange economy market will be an efficient allocation in that economy: this is just the first theorem of welfare economics. So equilibria are clearly ex post efficient. That they are ex ante efficient, i.e. efficient from the perspective of agents' preferences before any positions are taken in securities markets, follows from the structure of the problem (4), which is the same as the standard competitive equilibrium in an exchange economy with preferences $\int_{p\in\Delta} u_i\left(D_i\left(p, s_i\left(p\right)\right)\right) de_i$. ■

In general, the equilibria with price insurance are neither unique not fully insured: there may be multiple equilibria, so that trading PCSs as described here does not remove price uncertainty and produce a unique outcome, and in this respect is weaker than the framework for trading such securities in Chichilnisky Dutta and Heal [5]. The key point is that the assumptions there are stronger: they involve (1) common price expectations, (2) price expectations restricted to the set of Walrasian equilibrium prices, and (3) the trading of many levels of derivatives. Finally, note that whatever price occurs, agents have insured optimally against it according to their price expectations. However, if agents have common price expectations, i.e., if $e_i\left(p\right) = e_j\left(p\right) \forall i, j$, then all equilibria are fully insured:

Theorem 3 *Equilibria with price insurance in the economy $\mathcal{E}$ are fully insured if agents have common price expectations, i.e., if $e_i\left(p\right) = e_j\left(p\right) \forall i, j$.*

Proof. This follows immediately from Lemma 2 of Chichilnisky Dutta and Heal [5]. ■

7 Rational expectations

So far, I have taken expectations to be exogenous, part of the data describing individuals. A natural progression is to endogenize agents' beliefs. That remains a task for further work: here I make some simple observations on this general matter.

It is natural to enquire about rational expectations in the present framework: what is a rational expectation here? One interpretation would be a set of point price expectations (probability measures putting all weight at one price vector in the simplex, Dirac measures) which are fulfilled at the equilibrium. Intuitively, we expect that a Walrasian equilibrium of the underlying exchange economy (the exchange economy defined by preferences $u_i : \Re^n_+ \rightarrow \Re$ and endowments $\Omega_i \in \Re^n_+$) will be realized if expected with probability one by all agents, so that this equilibrium and the expectations giving probability one to it form a rational expectations equilibrium. This can be shown to be correct, although we should note that if all agents expect the same price vector p' to be realized with probability one, there will be no trade in the PCS market.

A more general and interesting case arises if there are several Walrasian equilibria of the underlying economy, and agents have expectations over these, and only over these: the set of Walrasian equilibria is the support of the distribution of price expectations, as in [5]. In this case, *none* of the Walrasian equilibria will be realized, even if all agents have identical expectations. This is proven in Chichilnisky Dutta and Heal [5]. It is easy to see why this is: agents will trade income between possible equilibrium prices, and in so doing will change their endowments and so the equilibria of the underlying economy. The key point is that in this case there will be activity in the PCS market, and this will change endowments. This shows that with multiple Walrasian equilibria, there is no rational expectations equilibrium, not a new conclusion (see also Hahn [10]).

Finally let us ask what happens in the model of this paper if agents hold general non-point expectations, the operation of the economy is repeated many times, and agents revise their expectations between repetitions in the light of the equilibria which are realized. A general treatment must await another paper, but the following observation may be of interest. Take an extreme case of expectation revision in the light of outcomes, and assume that once an equilibrium price p_1 has been realized in the first operation of the economy, agents revise their expectations to give probability one to p_1. This is a case of myopic expectations. Will the expected price vector p_1 now be realized in further operations of the economy? As all agents hold the same point expectation, there will be no trade in the PCS market. Hence agents' endowments will be unaltered from the underlying exchange economy, and so the only possible competitive equilibria are those of that economy. Hence the expected p_1 is realized if and only if it is a Walrasian equilibrium of the underlying economy. Assume that p_1 is not such a Walrasian equilibrium: then the outcome, call this p_2, must be a Walrasian equilibrium, and if all agents now once more form another myopic point expectation that p_2 will be the next equilibrium price, then this expectation *will* be fulfilled. So in the extreme case of myopic expectation formation after the first operation of the economy, it will converge rapidly to a Walrasian equilibrium of the underlying economy. It is of course an open question whether this will happen for more general expectation processes.

8 Conclusions

A relatively simple extension of the conventional general equilibrium framework allows it to incorporate uncertainty about the values of equilibrium prices, a form of uncertainty which is faced by most participants in modern economies yet which is not captured in the current formulations of general equilibrium models. The extension is achieved by the introduction of price-contingent securities, which are traded within a framework that prevents arbitrage between the market for these and the goods markets. Conditions for the existence of such an equilibrium are obvious extensions of the conditions know to be necessary and sufficient for existence of a competitive equilibrium.

Several extensions of this work seem natural. The present model is one of complete markets, in the sense that it is possible to trade securities contingent on *any* possible price vector. Incompleteness in this context will take the form of PCSs restricted to a subset of possible prices, or perhaps of PCSs whose payoffs have a different functional relationship with prices than those used here. The PCSs considered in this paper have payoffs represented by Dirac delta-functions: they pay only at one point in the price simplex. Conventional options, which are also PCSs, pay varying but non-zero amounts on a set of prices of positive measure. It would be interesting to know whether efficiency could be attained by the use of PCSs of this form.

Another interesting question concerns the determinants of the prices of PCSs: how are the prices of these derivative securities related to the prices of underlying goods and the expectations that agents hold over these? An answer to this question could provide a general equilibrium model of the pricing of derivative securities. We know that the prices of securities must be in the intersection of the cones $\widetilde{D}_i$ defined in the appendix, so this already gives some bounds on possible equilibrium derivative prices, bounds related quite naturally to agents' preferences and expectations. Tightening these bounds would clearly be an interesting approach to derivative pricing.

9 Appendix

This appendix contains the definitions of limited goods and securities arbitrage, and the proof of the main theorem. These parallel those used by Chichilnisky and Chichilnisky and Heal in [3] and [6].

Definition 2 *The goods global cone A_i of agent i consists of all directions in the space of goods along which utility increases without bound:*

$$A_i\left(\Omega_i\right) = \left\{z \epsilon \Re^n : \forall y \in \Re^n, \exists \lambda > 0 : u_i\left(\Omega_i + \lambda z\right) > u_i\left(y\right)\right\}$$

Definition 3 *The cone G_i is the set of directions in the space of goods along which utility never ceases to increase:*

$$G_i\left(\Omega_i\right) = \left\{z \in \Re^n : \forall \mu \exists \lambda : u_i\left(\Omega_i + \lambda z\right) > u_i\left(\Omega_i + \mu z\right) \;\; if\; \lambda > \mu \geq 0\right\}$$

Under assumption 1 the cone $G_i\left(\Omega_i\right)$ contains the cone $A_i\left(\Omega_i\right)$ and in addition contains some of the boundary rays of $A_i\left(\Omega_i\right)$.

Definition 4 *The goods market cone ∂D_i is defined as follows. Let D_i be the dual to the cone G_i:*

$$D_i = \left\{p \in \Re^n_+ : \forall z \in G_i, p.z > 0\right\}$$

Then

$$\partial D_i = D_i \cap S(A) \text{ if } S(A) \subset N \text{ and } \partial D_i = D_i \text{ otherwise}$$

where $S(A)$ is the set of prices which support efficient individually rational and affordable allocations and N the set of prices which assign some trader zero wealth.

Formally,

$$A = \left\{ \begin{array}{c} c_i \in \Re^n_+ : \sum_i c_i \leq \sum_i \Omega_i, \forall i, u_i(c_i) \geq u_i(\Omega_i), \nexists y_i \in \Re^n_+, \\ \sum y_i \leq \sum_i \Omega_i \ \& \ \forall i, u_i(y_i) \geq u_i(c_i) \& \exists j : u_j(y_j) > u_j(c_j) \end{array} \right\}$$

$$S(A) = \{p \in \Delta : p \text{ supports allocations in } A \text{ s.t. } p.c_i = p.\Omega_i + s_i(p)\}$$

$$N = \left\{v \in \Delta \subset \Re^n_+ : \exists j \ s.t. \ v.\Omega_j + s_j(v) = 0\right\}$$

In words, if at all supports to feasible efficient affordable individually rational allocations, some agent j has zero income (including income from securities transactions), then D_j consists of all those supporting prices at which only limited increases in utility can be afforded from initial endowments. Otherwise it is the dual of the set D_i of directions in which utility never ceases to increase.

Definition 5 *The economy satisfies limited arbitrage in the goods market if there is a price vector at which no agent can always derive increased utility from arbitrarily large zero-cost trades in goods: formally, there exists a price $p \in \Re^n_+$ assigning strictly positive value to all vectors in G_i, for $i = 1, ..., I$, i.e.,*

$$\bigcap_{1 \leq i \leq I} \partial D_i \neq \emptyset$$

The condition of limited arbitrage states, roughly, that if all supporting prices to efficient individually rational affordable allocations imply zero income for some trader, then there is one supporting price at which only limited increases in utility are affordable from initial endowments. Otherwise, there exists a price at which only limited increases in utility are affordable.

The definitions of global cone, market cone and limited arbitrage given for the exchange economy have to be modified slightly to apply to the trading of securities in the economy with price insurance. The commodity spaces are different, in fact infinite dimensional, and the consumption sets are unbounded below. For the securities market these definitions are:

Definition 6 *The securities global cone $\widetilde{A}_i$ of agent i consists of all directions in the space of securities along which expected utility W_i increases without bound:*

$$\widetilde{A}_i(s_i) = \{z \epsilon L_2 : \forall y \in L_2, \exists \lambda > 0 : W_i(s_i + \lambda z) > W_i(y)\}$$

Definition 7 *The cone* $\widetilde{G}_i$ *is the set of directions in securities space along which expected utility* W_i *never ceases to increase:*

$$\widetilde{G}_i\left(s_i\right) = \left\{z \in L_2 : \forall \mu \exists \lambda : W_i\left(s_i + \lambda z\right) > W_i\left(s_i + \mu z\right) \text{ if } \lambda > \mu \geq 0\right\}$$

Definition 8 *The securities market cone* $\widetilde{D}_i$ *is the set of securities prices assigning positive value to all trades in directions along which expected utility* W_i *never ceases to increase:*

$$\widetilde{D}_i = \left\{\pi \in L_2 : \forall z \in G_i, \langle \pi, z \rangle > 0\right\}$$

Definition 9 *The economy satisfies limited arbitrage in securities markets if there is a securities price vector at which no agent can always derive increased expected utility from arbitrarily large zero-cost securities trades: formally, there exists a price* $\pi \in L_2$ *assigning strictly positive value to all vectors in* $\widetilde{G}_i$*, for* $i = 1, ..., I$*, i.e.,*

$$\bigcap_{1 \leq i \leq I} \widetilde{D}_i \neq \emptyset$$

Note that while limited goods arbitrage places restrictions on differences between agents' preferences for goods, limited securities arbitrage places restrictions on differences between preferences for goods *and* on differences between price expectations. Even if limited goods arbitrage is satisfied, limited securities arbitrage may fail if agents' price expectations are very different.

A difference between our use of limited goods arbitrage and that in Chichilnisky [3][2] is in the set N, the set of prices at which some agent has zero income. Here this set is endogenous, in the sense that an agent's income at a given price vector depends not only on endowments but also on his or her position in the PCS market, and so on expectations and the agent's optimal income transfers across price states. To appreciate the difference, note that if agent i assigns probability zero to price vector p', then he will set $s_i\left(p'\right) = -p'.\Omega_i$ and sell all of his income in price state p'. If the agent were wrong in these expectations and the equilibrium price were p' then his income would be zero: this could prevent p' from being an equilibrium price, although Chichilnisky [2][3] shows that under certain conditions there may be a competitive equilibrium at which some agents have zero income.

It is proven in Chichilnisky [3] and Chichilnisky and Heal [6] that if the economy satisfies the condition (Assumption 1) of uniformly bounded preferences and also Assumption 4 and an equivalent assumption on the indifference surfaces of preferences over goods, then the cones $A_i, \widetilde{A}_i, D_i, \widetilde{D}_i, G_i$ and $\widetilde{G}_i$ are the same for any endowment vectors: they are translation invariant.

Lemma 4 *Let* $s_i^*\left(p\right)$ *be the solution to the problem (4)*

$$\max_{s_i(p)} \int_{p \in \Delta} w_i\left(p, s_i\left(p\right)\right) de_i\left(p\right)$$

$$where \int_{p\in\Delta} \pi(p)\, s_i(p) = 0 \; and \; p.\Omega_h + s_i(p) \geq 0$$

Then $s_i^*(p)$ is a continuous function of the goods price vector p.

Proof. This follows immediately from the fact that $s_i^*(p)$, being a solution to a variational problem, is the solution of a differential equation and so continuous. ■

There are two stages to the proof of the main result. First, we prove that for a given set of PCS holdings $s_i^*(p), i = 1, ..., I$ the goods economy has a competitive equilibrium if and only if it satisfies limited goods arbitrage. This is a relatively simple extension of the proof in Chichilnisky [2]: the economy satisfies limited arbitrage in goods, which is necessary and sufficient for the existence of an equilibrium in an exchange economy. The only difference from a standard competitive exchange economy is that agents have holdings of price-contingent securities which augment their endowments.

The second stage of the proof is to show that the entire economy has an equilibrium with price insurance if in addition the condition of limited securities arbitrage is satisfied.

Lemma 5 *For a given set of PCS holdings, the economy has a competitive equilibrium if and only if it satisfies limited goods arbitrage.*

Proof. We can use the proof in Chichilnisky [2] with one modification. On page 97, in the proof of Lemma 3, the correspondence φ from the simplex Δ^I in the space of utility levels to the set $T = \left\{y \in \Re^I : \sum_{i=1}^I y_i = 0\right\}$ must be modified so that for each $r \in \Delta^I$,

$$\varphi(r) = \{p.[\Omega_1 + s_1(p) - x_1(r)], .., p.[\Omega_I + s_I(p) - x_I(r)] : p \in P(r)\}$$

Given that $s_i(p)$ is continuous (Lemma 1 above), upper semi continuity of φ can now be established exactly as in Chichilnisky [2] page 97, and the proof of existence of an equilibrium in that paper can be used to establish the Lemma. ■

Proof of the main theorem.

We can now complete the proof of the main result. This is a direct application of Theorem 5 page 12 of Chichilnisky and Heal [6]. Consider the exchange economy $\mathcal{H}$ with commodity space and consumption sets L_2, preferences $W_i(s_i(p)) : L_2 \to \Re$, and initial endowments $0 \in L_2$. This is an exchange economy which satisfies all the conditions of Theorem 5 of [6] and for which existence of a competitive equilibrium is therefore (by Theorem 5 of [6]) equivalent to limited arbitrage, which in the present context is precisely what has been called limited securities arbitrage. Hence under the conditions of the main theorem the economy $\mathcal{H}$ has a competitive equilibrium. This means that there exists a set of prices for price-contingent securities $\pi^*(p)$ and a set of securities holdings for each agent $s_i^*(p)$ such that markets for price-contingent securities clear, $\sum_i s_i^*(p) = 0 \forall p \in \Delta$, and all agents are optimizing subject to their budget constraints, i.e., for all i, $s_i^*(p)$ maximizes $\int_\Delta w_i^p(s_i^*(p))\, e_i(p)\, dp$ subject to $\int_\Delta s_i^*(p)\, \pi^*(p)\, dp = 0$ and $p.\Omega_h + s_i(p) \geq 0$.

Now, by Lemma 5, given any set of PCS holdings, and so in particular the equilibrium holdings $s_i^*(p)$ in the economy $\mathcal{H}$, the goods economy has a competitive equilibrium. Let the prices at this equilibrium be p^* and the consumptions $c_i^*, i = 1, ..., I$. So the securities holdings $s_i^*(p), i = 1, ...I$, securities prices $\pi^*(p)$, goods prices p^* and goods consumption vectors c_i^* satisfy all the conditions of the definition of an equilibrium with price-contingent securities. This completes the proof of theorem 1. ■

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