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NATIONAL INCOME AND THE ENVIRONMENT

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[1]gmh1@columbia.edu and bengt.kristrom@home.se. The is first draft of this paper was prepared by Heal for the 1998 Ulvön Conference on Environmental Economics. Kriström then clarified the key relationship between the two measures, Hicksian National Income and National Wealth. We have benefitted from comments by Geir Asheim, Graciela Chichilnisky and Marty Weitzman. The usual disclaimer applies.

Abstract

There is widespread recognition that we need to revise our methods of measuring national income to incorporate better the impact of economic activity on environmental assets. Our aim here is to investigate alternative concepts of national income in a dynamic economy, one a generalization of Hicksian income and the other a generalization of the welfare economics concept of income as the value of output at equilibrium prices. The relationships between these concepts and the Hamiltonian of a dynamic optimization problem in a representative agent economy can be characterized fully and rather neatly:

1. the Hamiltonian is a stationary equivalent utility level, a weighted average of future utilities where the weights are discount factors.

2. a first order approximation to changes in the Hamiltonian is a Hicksian measure of the change in national income, equal in value to the real return available on the economy's stocks.

3. national wealth is the present value of consumption over time, valued at supporting prices.

4. if a consumption path changes slightly, then the change in Hicksian income is the change in wealth times the discount rate.

5. change in wealth equals the present value of change in income minus the present value of changes in stock accumulation.

The wealth concept seems more fundamental and in some ways more robust: it works for variable discount rates, zero discount rates, and the cake eating problem, whereas the income concept has problems in these cases. It also works for non-utilitarian objectives such as overtaking, the Chichilnisky criterion and the green golden rule, where as the income measure does not. A careful understanding of the two concepts is crucial to a proper treatment of changes in stocks in dynamic project evaluation: straightforward application of the income concept could lead to double counting.

Key Words: national income, national accounts, green accounting, environment, Hamiltonian.

1 National Income

There is widespread recognition that national income accounting practices are inadequate for reflecting environmental concerns. Current conventions were developed in the 1940s and 1950s to provide a statistical framework for the implementation of Keynesian macroeconomics. Although grown more sophisticated over the years, they are still dominated by the legacy of a social and intellectual milieu in which the natural environment was not an agenda item. They therefore have to be supplemented: economists have been working on that process since the work of Nordhaus and Tobin [22] and Weitzman [27]. This paper is an extension and refinement of that process.

Our aim here is to investigate alternative concepts of national income in a dynamic economy, one a generalization of Hicksian income and the other a generalization of the welfare economics concept of income as the value of output at equilibrium prices. The relationships between these concepts and the Hamiltonian of a dynamic optimization problem in a representative agent economy can be characterized fully and rather neatly:

- the Hamiltonian is a stationary equivalent utility level, a weighted average of future utilities where the weights are discount factors.
- a first order approximation to changes in the Hamiltonian is a Hicksian measure of national income, equal in value to the real return available on the change in the economy's stocks.[1]
- national wealth is the present value of consumption over time, valued at supporting prices.
- if a consumption path changes slightly, then the change in Hicksian income is the change in wealth times the discount rate.
- change in wealth equals the present value of changes in income minus the present value of changes in stock accumulation.

The wealth concept seems more fundamental and in some ways more robust: it works for variable discount rates, zero discount rates, and the cake eating problem, whereas the income concept has problems in these cases. It also works for non-utilitarian objectives such as overtaking, the Chichilnisky criterion and the green golden rule, where as the income measure does not. In these cases, paths are not ranked by present values. A careful understanding of the two concepts is crucial to a proper treatment of changes in stocks in dynamic project evaluation: straightforward application of the income concept could lead to double counting.

Hicks [19] states that[2] "Income No. 3 must be defined as the maximum amount of money which an individual can spend this week, and still expect to be able to spend the same amount in real terms in each ensuing week."[3] Hicks goes on to say that "We ask .... how much he would be receiving if he were getting a standard stream of the same present value as his actual expected receipts. This amount is his income."[4] So income is the expenditure which if kept constant would yield the same present value as

[1] This emphasizes the importance of measuring stocks of natural and human capital. On the impact of natural capital on human welfare see Daily [10]. On the economic value and role of such capital see Chichilnisky and Heal [8].

[2] We are very much indebted to Geir Asheim for a most instructive discussion of Hick's concept of income: his two papers [1] and [2] develop this concept in several interesting ways. This concept of income can also be found in earlier works of Fisher and Lindahl.

[3] Page 174 of Hicks [19].

[4] Page 184 of Hicks [19].

a person's actual future receipts. Proposed by Hicks as a definition of an individual's income, the concept has quite naturally been applied to the income of a society. It has the advantage of being explicitly dynamic. First formalized as a definition of national income by Weitzman [27], the concept has subsequently been developed further by Solow [26], Asheim [1] [2], Hartwick [17], Dasgupta [11] and Dasgupta, Kriström and Mäler [14]. Weitzman in a surprising result showed that the Hamiltonian of an optimal growth problem can be interpreted as a constant utility level equal in present value terms to the utility on an optimal path, i.e., as a weighted average of future utilities along the optimal path, the weights being the discount factors.

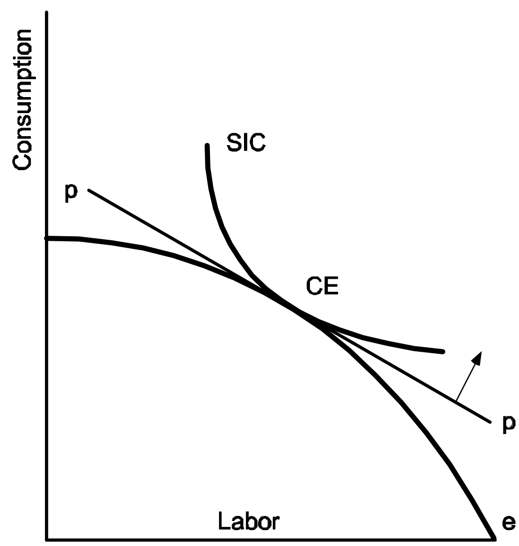

Figure 1: national income defined by prices that support the competitive allocation.

The alternative usage of national income is rooted in an attempt to construct an index number with welfare significance. The transformation frontier in figure 1 shows alternative combinations of labor and consumption available to an economy. CE is a competitive equilibrium, with SIC being the social indifference curve corresponding to the individual indifference curves attained at the equilibrium.[5] pp is a hyperplane separating the set bounded by the social indifference curve from the feasible set. The normal to this line represents the equilibrium price vector: national income is defined as the value of equilibrium consumption at these prices. This clearly has local welfare significance: any small move from CE which has a positive value at pp will move the economy above the social indifference curve through CE and is potentially Pareto improving. This is the concept of national income underlying cost-benefit analysis and was referred to in Heal [18] as national welfare or national wealth. In a dynamic context, it is the present value of incomes in all periods, the right hand side of the intertemporal budget constraint in a world of complete markets.

Both measures of national income depend upon the objectives of the economy, on its maximand. They thus depend on the discount rate and on the nature of the economy's

[5] This social indifference curve is the lower boundary of the set-theoretic sum of the individual preferred-or-indifferent sets corresponding to equilibrium consumption levels. For more discussion and a review of the literature on this topic, see Chichilnisky and Heal [7]

objective function. Until we have specified our valuation of future generations we cannot define dynamic national income. Intuitively this makes sense. Consider an economy endowed with a large number of very long-lived environmental assets each of which will provide a small flow of services indefinitely: it is rich if one takes a very long-term view but poor if one focuses mainly on the near future. This is a very real issue as many environmental assets (such as watersheds and biodiversity, see [8]) are capable of providing a flow of services indefinitely into the future, in contrast to physical and human capital. This highlights the importance in practical applications of assessing whether market prices reflect adequately social attitudes towards the future. By way of illustration, the services provided to New York City by the Catskills watershed could be replaced by a filtration plant at a capital cost of $8 billion [8]. This would have a life of a few decades, and would then need replacing, whereas the watershed could continue for ever, as it has for the last few millennia. So the total cost of the replacement, an upper bound on the value of the natural asset as a watershed,[6] is the present value of an indefinite sequence of $8 billion investments, clearly very sensitive to the discount rate.

In the next section we present a general model which is used to derive results that are subsequently specialized to more familiar cases. It is first applied to the concept of Hicksian income and then to that of national wealth. In the final section we draw some conclusions for the implementation of "green" accounting practices.

2 A General Model

Let the vector $c_t \in \Re^m$ be a vector of flows of goods consumed and giving utility at time t, and $s_t \in \Re^n$ be a vector of stocks at time t, also possibly but not necessarily sources of utility. Each stock $s_{i,t}, i = 1, .., n$, changes over time in a way which depends on the values of all stocks and of all flows:

$$\dot{s}_{i,t} = d_i (c_t, s_t), i = 1, ..., n \tag{1}$$

The economy's objective is to maximize the discounted integral of utilities (2):

$$\max \int_0^\infty u (c_t, s_t) e^{-\delta t} dt \tag{2}$$

subject to the rate-of-change equations (1) for the stocks. The utility function u is assumed to be strictly concave and the reproduction functions $d_i (c_t, s_t)$ are assumed to be concave. This is a very general and flexible formulation, and in the following we shall frequently specialize it to simple and familiar cases. For example, if u does not depend on s and if (1) takes the form $\dot{s}_{i,t} = -c_{i,t}$ then we have the Hotelling model. If $u = u(c)$ and $\dot{s} = f(s) - c$ then we have the Solow model.

To solve this problem we construct a Hamiltonian which takes the form

$$H_t = u (c_t, s_t) e^{-\delta t} + \sum_{i=1}^n \lambda_{i,t} e^{-\delta t} d_i (c_t, s_t) \tag{3}$$

where the $\lambda_{i,t}$ are the shadow prices of the stocks. The first order conditions for optimality can be summarized as

$$\frac{\partial u (c_t, s_t)}{\partial c_j} = -\sum_{i=1}^n \lambda_{i,t} \frac{\partial d_i (c_t, s_t)}{\partial c_j} \tag{4}$$

[6] It may play other roles, such as biodiversity support or provision of recreational facilities.

and

$$\dot{\lambda}_{i,t} - \delta\lambda_{i,t} = -\frac{\partial u(c_t, s_t)}{\partial s_i} - \sum_{k=1}^{n} \lambda_{k,t} \frac{\partial d_k(c_t, s_t)}{\partial s_i} \tag{5}$$

Within this framework, we shall now investigate the two alternative approaches to defining national income, the Hicksian first.

3 Hicksian Income and the Hamiltonian

Noted originally by Weitzman [27] in a simpler model, there is a rather surprising connection between the Hamiltonian in a dynamic optimization problem and the Hicksian concept of national income. Solow [26] and Asheim [1] extended the analysis. The value of the Hamiltonian at t (not discounted back to date zero) represents a utility level which if maintained for ever from t would give the same present value of utility as the present value of utility along an optimal path from t on. Let $\{c_t^*, s_t^*\}$ be the solution to the problem of maximizing (2) subject to (1). Also let CH_t be the Hamiltonian corresponding to this problem, not discounted to time zero: CH_t is the current value Hamiltonian. Then:

Proposition 1 *For any date t,*

$$\int_t^\infty CH_t(c_t^*, s_t^*)\, e^{-\delta(\tau - t)} d\tau = \int_t^\infty u(c_\tau^*, s_\tau^*)\, e^{-\delta(\tau - t)} d\tau$$

In words, a utility stream from t to infinity of a constant value equal to the Hamiltonian evaluated on the optimal path at t has the same present value as the utility stream from t to infinity associated with a solution to the problem of maximizing (2) subject to (1).

Proof. All proofs are in the Appendix.

The Hamiltonian is a measure of the "equivalent constant utility level" associated with an optimal path. It is a natural candidate for a measure of Hicksian national income. It is sometimes referred to as a "sustainable" utility level, which is in fact inaccurate. As we shall see in the discussion of the Hotelling case below, the equivalent constant utility level is not one that can actually be maintained for ever. In fact it is more convenient to work with a first order approximation to the Hamiltonian. What matters operationally is to know whether a given policy increases or decreases Hicksian income. A test of this for policies which involve small changes, is to see whether the changes have a positive inner product with the derivatives of the Hamiltonian, i.e., whether the policy increases the value of a linear approximation to the Hamiltonian.

3.1 The Linearized Hamiltonian

Take a linear approximation to changes in the current value Hamiltonian: we call this approximation ΔH.

$$\Delta H = \sum_{j=1}^{m} \Delta c_{j,t} \left\{ \frac{\partial u}{\partial c_{j,t}} + \sum_{i=1}^{n} \lambda_{i,t} \frac{\partial d_i}{\partial c_{j,t}} \right\} + \sum_{i=1}^{n} \Delta s_{i,t} \left\{ \frac{\partial u}{\partial s_{i,t}} + \sum_{k=1}^{n} \lambda_{k,t} \frac{\partial d_k}{\partial s_{i,t}} \right\} \tag{6}$$

We are now in a position to establish a simple but fundamental relationship between changes in the linearized Hamiltonian and changes in the values of the stocks in the economy and the returns available on them. The change in the linearized Hamiltonian is in fact just the real return on the change in the nation's stocks: it is the sum of the shadow values of stock changes each multiplied by the real rate of return on the stock.

Proposition 2 *Consider an economy whose operation is described by the maximization of* $\int_0^\infty u(c_t, s_t) e^{-\delta t} dt$ *subject to the constraints* $\overset{\bullet}{s}_{i,t} = d_i(c_t, s_t)$, $i = 1, ..., n$. *Assume that the Hamiltonian (3) has non-zero derivatives with respect to all stock variables* $s_i i = 1, ..$ *,* n *on an optimal path. Then the change in the linearized current value Hamiltonian is a return on the change in the economy's stocks: it is equal to the changes in the value of the stocks in the economy at time* t*, valued at the shadow prices at time* t*, multiplied by the discount rate minus the rate of appreciation of the shadow prices at time* t*. Formally,*

$$\Delta H_t = \sum_{i=1}^{n} \Delta s_{i,t} \lambda_{i,t} \left\{ \delta - \frac{\overset{\bullet}{\lambda}_{i,t}}{\lambda_{i,t}} \right\}$$

The linearized Hamiltonian is equivalent to the Hicksian concept of income as a return on stocks, with a generalization that deals with non-steady-state behavior. Consumption flows do not feature in this expression. This result is very general, as the formulation of maximizing (2) subject to (1) encompasses all of the conventional neoclassical formulations of optimal growth with or without natural resources.

The rate of return to be applied to the value of a stock equals the discount rate in a stationary state. Outside of a stationary state it is the real rate of return on the stock. Consider the expression for the change in the shadow price, which from equation (5) above is given by

$$\frac{\overset{\bullet}{\lambda}_{i,t}}{\lambda_{i,t}} + \frac{1}{\lambda_{i,t}} \left\{ \frac{\partial u(c_t, s_t)}{\partial s_i} + \sum_{k=1}^{n} \lambda_{k,t} \frac{\partial d_k(c_t, s_t)}{\partial s_i} \right\} = \delta \tag{7}$$

The left hand side here is the total return on a unit of the $i-$th. stock: the first term is the capital gain and the second represents the contribution made by an extra unit of the stock to utility and to the growth of all stocks, multiplied by their shadow prices. This second term is the real return: these terms represent the return to an increment of the stock used in the economy. In the long run, with full adjustment of stocks to their appropriate levels, one would expect the real return to equal the discount rate: indeed in a stationary state they are equal. The return applied to the value of stocks at each point in time, $\delta - \overset{\bullet}{\lambda}_{i,t} / \lambda_{i,t}$, equals the real return: at a stationary state, this equals the discount rate. The result in Proposition 2 is therefore in the spirit of the conventional definition of national income descending from Fisher, Lindahl and Hicks: recall that this definition of income states it to be the maximum amount which can be consumed without reducing capital, i.e., the return on capital.

In general, there is a difficulty in interpreting this concept of national income when the discount rate is zero. If the discount rate is always zero, the integrals used in defining Hicksian income, namely $\int_t^\infty H_t(c_t^*, s_t^*) e^{-\delta(\tau - t)} d\tau$ and $\int_t^\infty u(c_t^*, s_t^*) e^{-\delta(\tau - t)} d\tau$, are not well-defined, so that the equivalence of the Hamiltonian and Hicksian income cannot be established. This implies that we cannot use this approach if we adopt Ramsey's famous formulation of the optimal growth problem [23], or if we define optimality by overtaking or by the green golden rule [4]. It is impossible to extend this concept to non-constant discount rates, as the basic equivalence of proposition 1 does not hold in this case ([2], [18]).

In the next two sections, we study the application of the results on Hicksian national income to simple resource-based economies, firstly to economies with exhaustible resources, and then those with renewable resources. We see the exact implications of this approach to defining national income in these contexts.

3.1.1 Exhaustible Resources

We now set out in detail the implications of proposition 2 to a simple and familiar model of natural resource use that is a special cases of the general framework set out above. The model is:

$$\max \int_0^\infty u(c_t, s_t) e^{-\delta t} dt \text{ subject to } \int_0^\infty c_t dt \leq s_0$$

where s_0 is a given initial stock. The Hamiltonian will be $H_t = u(c_t, s_t) - \lambda_t c_t$ and using the first order condition for maximization of the Hamiltonian with respect to the level of consumption, a linear approximation to a change in this is

$$\Delta HNI_t = \Delta s_t u_s(c_t, s_t)$$

where HNI stands for Hicksian national income. So consumption flows net out and the value of ΔHNI at t is just the value of the flow of services from the resource stock, valued at the marginal utility of the stock. At a stationary solution s^* the marginal utility will satisfy $u_s = \delta u_c$.[7] With this relationship we can rewrite ΔHNI at a stationary state s^* as

$$\Delta HNI_t = \delta \Delta s^* \lambda_t \tag{8}$$

which is just the shadow value of the stock multiplied by the discount rate, as indicated by Proposition 2 above. This is, of course, the traditional definition of income: the flow of services from a capital stock. Away from a stationary state, the corresponding equation is

$$\Delta HNI_t = \{\delta - \frac{\dot{\lambda}}{\lambda}\} \lambda_t \Delta s_t \tag{9}$$

The change in the flow of utility from depleting the resource makes no contribution to HNI: the value of the flow $c_t u_c(c_t, s_t)$ is exactly offset by a term $\lambda_t c_t$ accounting for the depletion of the stock. Only the stock counts: if the stock were not valued in our economy, as in the Hotelling case, then ΔHNI would be zero. Any change in the stock, through discoveries or through sales, must be recorded in HNI and valued (in a stationary state) at the shadow price of the flow times the discount rate.

3.1.2 Renewable Resources

Similar results hold in the case of renewable resources, using the model above with the constraint $\dot{s}_t = r(s_t) - c_t$. The Hamiltonian is now $H_t = u(c_t, s_t) + \lambda_t \{r(s_t) - c_t\}$ so that

$$\Delta HNI_t = \{\Delta c_t u_c(c_t, s_t) + \Delta s_t u_s(c_t, s_t)\} + u_c \{\Delta s_t r'(s_t) - \Delta c_t\}$$

As before the terms in the flow c_t net out, so that once again, only terms relating to the stock appear in the expression for HNI:

$$\Delta HNI_t = \Delta s_t \{u_s(c_t, s_t) + u_c r'(s_t)\} = \Delta s_t \lambda_t \left(\delta - \frac{\dot{\lambda}_t}{\lambda_t}\right)$$

and in a stationary solution: $\Delta HNI_t = \delta \lambda_t \Delta s_t$.

[7] See Heal [18].

4 National Wealth

Turn now to the second approach to defining national income, that portrayed in figure 1 and associated with the use of the prices defining a separating hyperplane to judge whether a change is an increase in welfare. In the context of an intertemporal economy where consumption bundles are given by functions of time, prices must likewise be functions of time for the entire infinite horizon. We shall see that this approach, closer to the standard static interpretation of national income, avoids some of the counter-intuitive properties of the Hicksian approach. We set out the basic principles of measuring national welfare via the separating hyperplane approach in the context of the general model in equations (1) and (2). The first order conditions for optimality were given in (4) and (5). The use of arguments about separating hyperplanes in problems involving infinite time horizons is mathematically quite delicate, so we need to be precise about the framework to be used. We shall assume that the functions $d_i(c_t, s_t), i = 1, ..., n$ are such that the set of feasible paths for $c_{j,t}$ and $s_{i,t}$ are bounded: reasonable conditions sufficient for this are presented for the specific models used here in Heal [18]. Under this assumption, the paths of all variables, including utilities, are such that their integrals against a discount factor with a positive discount rate are finite.[8] A supporting hyperplane for a set S is then given by a function $h(t)$ such that everything in the set is above it in the sense of having at least as great a value at the prices defining the hyperplane:[9]

$$\langle s(t), h(t) \rangle = \int_0^\infty s(t)\, h(t)\, e^{-\delta t} dt \geq 0 \;\forall s(t) \in S.$$

A hyperplane which supports the optimal path is one that separates the set of paths preferred to an optimum from those which are feasible. This is a time path of prices for stocks and flows $p_{c,j}(t)$ and $p_{s,i}(t)$ which satisfies two conditions: any path at least as good as the optimum has a value at these prices at least as great as the optimal path, and any feasible path costs no more than the optimum.[10]

Definition 1 *A set of prices $p_{c,j}(t)$ and $p_{s,i}(t)$ supporting the optimal path*[11] *will be called optimal prices and will be used to define national wealth as follows: national wealth along the optimal path is*

$$\int_0^\infty \{\langle p_c(t), c_t^* \rangle + \langle p_s(t), s_t^* \rangle\} e^{-\delta t} dt.$$

This is just the inner product of consumption and stock prices with supporting prices, as in figure 1. By analogy with figure 1, we want to establish that any small change which increases this measure is a welfare improvement. In the next proposition we characterize a set of prices which are optimal prices in the sense of the above definition. These prices are quite intuitive: they are the marginal utilities of the stocks and flows along an optimal path. So price ratios are just marginal rates of substitution as usual. In effect these marginal utilities define the marginal rates of substitution between the different arguments of the maximand $\int_0^\infty u(c_t, s_t) e^{-\delta t} dt$, and are natural candidates for the role of defining a separating hyperplane.

[8] Formally, for any i and j, $\int_0^\infty c_{j,t} e^{-\delta t} dt < \infty$, $\int_0^\infty s_{i,t} e^{-\delta t} dt < \infty$ where $c_{j,t}$ and $s_{i,t}$ are real-valued functions of time. We can therefore regard the space of possible paths of consumptions levels and stocks as a weighted l_∞ space, with the norm $\|f(t)\| = \sup_t \left| f(t) e^{-\delta t} \right|$ and the inner product of two functions $f(t)$ and $g(t)$ being $\langle f, g \rangle = \int_0^\infty f(t)\, g(t)\, e^{-\delta t} dt$.

[9] If the function $s(t)$ is a $n-$vector-valued function defined on the real numbers, then likewise $h(t): \Re \to \Re^n$ and $s(t)\, h(t)$ is interpreted as the inner product of two vectors in $\Re^n$.

[10] For a formal definition see the appendix.

[11] Formally a separating hyperplane satisfying (16) and (17).

Proposition 3 *The sequence of prices defined by the derivatives of the utility function along an optimal path, i.e.,*

$$\{p_{c,j}(t), p_{s,i}(t)\} = \left\{ \frac{\partial u(c_t^*, s_t^*)}{\partial c_{j,t}}, \frac{\partial u(c_t^*, s_t^*)}{\partial s_{i,t}} \right\} \forall j, i, t$$

form a set of optimal prices.

We have now established that the derivatives of the utility function with respect to stocks and flows on an optimal path can be used to define a hyperplane which separates the set of paths preferred to an optimal path from the set of feasible paths. They can therefore be used to define a price system at which national income in the national welfare sense can be computed. It is of course immediate that any small change in a path which has a positive present value at these optimal prices will increase national welfare:

Corollary 1 *Let a small variation $\{\Delta c_t, \Delta s_t\}_0^\infty$ about an optimal path $\{c_t^*, s_t^*\}_0^\infty$ have positive present value at the optimal prices $\{p_{c,j}(t), p_{s,i}(t)\}_0^\infty$. Then the implementation of this variation leads to an increase in welfare. Conversely, if a small variation $\{\Delta c_t, \Delta s_t\}_0^\infty$ about an optimal path $\{c_t^*, s_t^*\}_0^\infty$ leads to an increase in welfare, then it has positive value at the optimal prices $\{p_{c,j}(t), p_{s,i}(t)\}_0^\infty$.*

4.1 National Wealth: Illustrations

It is natural to enquire in more detail about the relationship between the two definitions, and also about the connection, if any, which national welfare has with the Hamiltonian, which played so central a role in the definition of Hicksian income. This is best done by comparing the two definitions in the context of specific models.

4.1.1 National Wealth and Hicksian Income: the Hotelling Case

Consider briefly the present value of national wealth in the case of the classical formulation due to Hotelling. In this case, we seek to

$$\max \int_0^\infty u(c_t) e^{-\delta t} dt \text{ subject to } \int_0^\infty c_t dt = s_0$$

Here the remaining stock of the resource is not assumed to be a source of benefits to the economy. Denoting the optimal path of consumption be denoted by an asterisk, the present value national welfare would in this case be measured by

$$NW = \int_0^\infty c_t^* u'(c_t^*) e^{-\delta t} dt$$

Noting that $u'(c_t^*) e^{-\delta t}$ is a constant, equal to the initial value of the shadow price λ_0, this is simply the initial stock of the resource multiplied by the initial shadow price:

$$NW = \lambda_0 s_0$$

This is an extremely simple and natural measure of wealth: the welfare the economy can attain depends on its stock and the social value of this. Correspondingly, the change in NW resulting from a change in the stock is clearly $\Delta NW = \lambda_0 \Delta s_0$.

What is the present value of the change in the Hicksian measure of national income in this case? The Hamiltonian is $H = u(c_t) - \lambda_t c_t$ so that the $\Delta H = \Delta HNI = 0$ as $u' = \lambda$ by the first order conditions. Note that the Hotelling problem does not satisfy the conditions of proposition 2: it fails to meet the condition that the stock s is an argument of the Hamiltonian.

The Hicksian measure of the change in present value national income is zero, and national wealth is the shadow value of the initial stock of the resource. The change in Hicksian national income will remain zero even if the initial stock of the resource is multiplied by any finite number, whereas the national welfare measure will increase. The total insensitivity of the Hicksian measure to the initial stock gives one food for thought. The fact that changes in Hicksian income are zero is also paradoxical: this does not imply that the equivalent constant utility level is zero, as can easily be confirmed by taking the special case of $u(c) = \log c$. The optimal consumption path is $c_t = \delta s_0 e^{-\delta t}$ and the Hamiltonian is $H = \log c - \lambda c$ where $\lambda = u' = 1/c$. It is routine that $\int_0^\infty (\log c_0 - \lambda c_0) e^{-\delta t} dt = \int_0^\infty \log\left(\delta s_0 e^{-\delta t}\right) e^{-\delta t} dt$ so that the Hamilton at time zero is a constant utility level that has the same present value as the optimal path. Note that although $(\log c_0 - \lambda c_0)$ is a constant utility value with the same present value, it is not sustainable in the sense that it could be maintained for ever. Neither is zero, unlike the linearized Hamiltonian, which clearly fails to capture either aspect of income in this case. As noted, this is attributable to the stock not being an argument of the Hamiltonian.

4.1.2 Exhaustible Resources and National Wealth

Consider next an economy with only exhaustible resources as a source of consumption, which values these both as a stock and as a flow. Its optimal path in the utilitarian sense is defined by the problem

$$\max \int_0^\infty u(c_t, s_t) e^{-\delta t} dt \text{ subject to } \dot{s}_t = -c_t$$

Let $\{c_t^*, s_t^*\}$ be a solution to this problem: then the national wealth along this path is

$$NW = \int_0^\infty \{c_t^* u_c(c_t^*, s_t^*) + s_t^* u_s(c_t^*, s_t^*)\} e^{-\delta t} dt$$

This is automatically a present value, because the separating hyperplane defines prices covering the entire infinite horizon.

How does this expression relate to the Hicksian interpretation of national income? The Hicksian interpretation of national income is an instantaneous measure of income, holding at a particular time t : obviously if we consider a policy variation then the change in the instantaneous income level corresponding to NW is

$$\{\Delta c_t^* u_c(c_t^*, s_t^*) + \Delta s_t^* u_s(c_t^*, s_t^*)\}$$

On an optimal path the shadow price of the resource in the Hamiltonian equals the marginal utility of consumption: $\lambda_t = u_c(c_t^*, s_t^*)$. Furthermore, the marginal utility of the stock can be expressed as a function of λ_t and its rate of change: $u_s(c_t^*, s_t^*) = \delta\lambda_t - \dot{\lambda}_t$. Hence the change in the instantaneous value ΔNW is

$$\Delta NW_t = \Delta c_t^* \lambda_t + \Delta s_t^* \lambda_t \left(\delta - \frac{\dot{\lambda}_t}{\lambda_t}\right)$$

This is precisely the change in Hicksian national income ΔHNI *plus* the term $\Delta c_t^* \lambda_t$: it is the change in the return on the stock plus the change in the value of consumption. Equivalently it is the change in Hicksian national income without the inclusion of a stock depletion term.

4.1.3 Renewable Resources and National Wealth

Consider next the renewable problem

$$\max \int_0^\infty u(c_t, s_t) \text{ subject to } \dot{s}_t = r(s_t) - c_t$$

On the above definition, a change in national welfare is measured by

$$\Delta NW = \int_0^\infty \{u_c(c_t^*, s_t^*) \Delta c_t + u_s(c_t^*, s_t^*) \Delta s_t\} e^{-\delta t} dt$$

How does this compare with the Hicksian equivalent? For the same problem, the Hicksian measure is

$$\Delta HNI_t = \Delta s_t \lambda_t \left(\delta - \frac{\dot{\lambda}_t}{\lambda_t} \right)$$

Using the first order conditions for optimality, the instantaneous national welfare measure NW_t can be expressed as

$$\Delta NW_t = \Delta c_t \lambda_t + \Delta s_t \lambda_t \left(\delta - r' - \frac{\dot{\lambda}_t}{\lambda_t} \right)$$

The difference between ΔNW_t and ΔHNI_t is $\Delta c_t \lambda_t - \lambda_t \Delta s_t r'(s_t)$, which is the change in the flow of consumption minus the change in the return on the stock, evaluated at the marginal productivity of the stock in generating consumption flows. Clearly these two expressions are quite different, and are measuring different characteristics of the economy. As before, the difference arises from the inclusion in Hicksian income of the term $-\lambda \dot{s}$ reflecting stock changes, as $-\lambda \dot{s} = \lambda c - \lambda s r'$.

5 Hicksian Income and National Wealth

The time has now come to set out clearly the relationship between the two concepts, Hicksian national income and national wealth. In the examples we have seen that the difference between the two measures of national income originates in their treatment of the depreciation or augmentation of stocks. The Hamiltonian consists of the objective or utility function, which contains as arguments all goods contributing to utility, plus a term describing the changes in state variables, which in economic terms means the accumulation or decumulation of stocks. So in basing a measure on the Hamiltonian we are automatically adjusting consumption (and other variables affecting utility) for stock changes. This adjustment is not made in computing the national welfare measure. The next proposition formalizes this relationship between the two concepts:

Proposition 4 *The change in the instantaneous value of national wealth equals the change in the linearized current value Hamiltonian without terms reflecting changes in the state variables, i.e.,*

$$\Delta NW_t = \sum_{j=1}^{m} \frac{\partial u}{\partial c_{j,t}} \Delta c_{j,t} + \sum_{i=1}^{n} \frac{\partial u}{\partial s_{j,t}} \Delta s_{j,t}$$

which are the first two terms on the right hand side of expression (6) for the change in the linearized Hamiltonian.

Proof. The proof if this is immediate from the definition of NW and from proposition 3. ■

This difference arises naturally because income is an instantaneous concept, applying at a point in time only, whereas wealth is a measure applied to the entire time horizon. The former is a flow and the latter a stock. We expect that income is a return on wealth, and this is the next result.

Proposition 5 *Assume as in proposition 2 that the Hamiltonian (3) has non-zero derivatives with respect to all stock variables $s_i i = 1, .. , n$ on an optimal path. Consider a small variation about an optimal path of the problem of maximizing (2) subject to (1). Then the resulting changes in Hicksian national income and in national wealth are related as follows:*

$$\Delta HNI_0 = \delta \Delta NW$$

that is, the change in Hicksian national income is the interest on the change in national wealth.

Letting d_{ic} and d_{is} be the vectors of derivatives of the functions d_i with respect to consumption and stock levels and using Δc and Δs to denote changes in consumption and stock vectors and using the first order conditions (4) and (5) we can write express ΔNW as:

$$\Delta NW = \int_0^\infty \left[-\sum_i \lambda_{i,t} d_{ic,t} \Delta c_t - \sum_i \lambda_{i,t} d_{is,r} \Delta s_t + \sum_i \Delta s_{i,t} \left(\delta - \dot{\lambda}_{i,t} / \lambda_{i,t} \right) \right] e^{-\delta t} dt$$

so that

$$\Delta NW = \int_0^\infty \Delta HNI_t e^{-\delta t} dt - \int_0^\infty \sum_i \lambda_{i,t} \Delta d_{i,t} e^{-\delta t} dt \tag{10}$$

The change in national wealth is the integral of future changes in income minus the integral of the changes in the stocks valued at shadow prices, all discounted to the present. This is an important result in understanding intuitively the relationship between the income and wealth measures.[12] Because income at a point in time only incorporates consumption at that point in time, and at no other date, the impact of a policy change on consumption and welfare at later dates has to be captured in the changes in stocks that result from the change. Hence the presence of stocks in the definition of Hicksian national income. The definition of national wealth, in contrast, incorporates consumption levels at all dates, and therefore has no need to use stocks to proxy consumption changes at dates not considered. Incorporating changes in stocks in the wealth measure would therefore be double counting. This is why we have to subtract the integral of stock changes from the integral of income changes to arrive at the wealth change. There is an important policy implication here: when evaluating the welfare impact of a change which affects consumption over an interval of time, we can either value the changes in consumption at all dates, or we can value the change in consumption at the initial date and add to this the value of the changes in stocks at the initial date. To value all consumption changes and stock changes would be to double count.

6 Non-Utilitarian Optima

6.1 Chichilnisky's Criterion

The discussion of national income concepts so far has been within the framework of discounted utilitarianism. Now it is time to extend it to more general cases, which prob-

[12] This result holds for the Hotelling problem, as can be verified by substitution into (10).

ably capture better the concern for sustainability that motivates the current discussion of national income accounting. We focus first on the case of an economy which defines optimality according to Chichilnisky's criterion of sustainability [6], which involves the weighed average of an integral of utilities and a term depending on long-run or sustainable utility levels. We shall focus on the case of exhaustible resources. Consider the optimal use problem in the simplest version of this case:[13]

$$\max \alpha \int_0^\infty u(c_t, s_t) e^{-\delta t} dt + (1-\alpha) \lim_{t\to\infty} u(c_t, s_t) \text{ subject to } \dot{s}_t = -c_t \text{ and } s_t \geq 0 \forall t.$$

Let (c_t^*, s_t^*) be the optimal path for this problem. Then using an obvious generalization from the utilitarian case national welfare is now defined as

$$\left.\begin{array}{l} NW = \int_0^\infty \{c_t u_c(c_t^*, s_t^*) + s_t^* u_s(c_t^*, s_t^*)\} e^{-\delta t} dt \\ \quad + \lim\limits_{t\to\infty} \{c_t u_c(c_t^*, s_t^*) + s_t u_s(c_t^*, s_t^*)\} \end{array}\right\} \tag{11}$$

Marginal utilities are again used as prices. The definition of contains two elements: one the integral term, as in the case of the discounted utilitarian approach, and an extra element arising from the value placed by the objective on the limiting utility level. In this the limiting stock values and consumption levels are valued at limiting prices. The value assigned to a path depends both on the time path over finite horizons, via the present value term, and also on the limiting or sustainable values along the path. Formally, we are now defining the value of a sequence of consumption and stock levels (c_t, s_t) at prices $p_c(t), p_s(t)$, or equivalently defining the inner product of the consumption and stock sequences with the price sequences, as

$$\left.\begin{array}{c} \langle (c_t, s_t), (p_c(t), p_s(t)) \rangle = \\ \int_0^\infty \{c_t p_c(t) + s_t p_s(t)\} e^{-\delta t} dt + \lim_{t\to\infty} \{c_t p_c(t) + s_t p_s(t)\} \end{array}\right\} \tag{12}$$

We then define a supporting hyperplane for a set S of paths (c_t, s_t) of consumption and of the resource stock as functions $p_c(t), p_s(t)$ such that

$$\int_0^\infty \{c_t p_c(t) + s_t p_s(t)\} e^{-\delta t} dt + \lim_{t\to\infty} \{c_t p_c(t) + s_t p_s(t)\} \geq 0 \forall (c_t, s_t) \in S$$

With Chichilnisky's definition of optimality, the price system contains undiscounted terms because of the limiting term in the definition. So national welfare is measured in (11) as a present value plus a term reflecting long run or sustainable welfare. This term is not discounted: apart from this it has the same form as the other terms, namely stocks and flows evaluated at prices given by marginal valuations along an optimal path. The presence of this extra term is important, because it gives a reason for using in the measurement of national welfare prices which relate to the distant future yet are nevertheless not discounted.[14] This possibility has been discussed by several authors including Cline [9] and Bloom [**?**], but in the context of using only undiscounted valuations.

Definition 2 *A set of prices $p_c(t)$ and $p_s(t)$ at which paths preferred to the optimal path are at least as expensive and those that are feasible are no more expensive will be called optimal prices and will be used to define the welfare concept of national income as follows: national welfare along the optimal path is the inner product (12) of the optimal prices with the optimal paths of consumption and stocks:*

$$NW = \langle (c_t^*, s_t^*), (p_c(t), p_s(t)) \rangle$$

[13] For the solution of this problem see Heal [18].

[14] For a formal definition of a separating hyperplane in this case see the appendix.

$$= \int_0^\infty \{p_c(t)\, c_t^* + p_s(t)\, s_t^*\} e^{-\delta t} dt + \lim_{t\to\infty} \{p_c(t)\, c_t^* + p_s(t)\, s_t^*\}$$

By analogy with figure 1, we now want to establish that any small change which increases this welfare measure is a welfare improvement. In the next proposition we show that the marginal utilities of the stocks and flows along an optimal path are optimal prices in the sense of the above definition: the fact that a small change which leads to an increase in this national income measure is a welfare improvement, is then immediate.

Proposition 6 *The sequence of prices defined by the derivatives of the utility function along an optimal path, i.e.,*

$$\{p_{c,j}(t), p_{s,i}(t)\} = \left\{ \frac{\partial u(c_t^*, s_t^*)}{\partial c_{j,t}}, \frac{\partial u(c_t^*, s_t^*)}{\partial s_{i,t}} \right\} \forall j, i, t$$

form a set of optimal prices.

It is now immediate that any small change in a path which has a positive present value at these optimal prices, increases national welfare:

Corollary 2 *Let a small variation* $\{\Delta c_t, \Delta s_t\}$ *about an optimal path* $\{c_t^*, s_t^*\}$ *have positive inner product in the sense of (12) with, i.e., positive value at, the optimal prices* $\{p_{c,j}(t), p_{s,i}(t)\}$*. Then the implementation of this variation leads to an increase in welfare.*

In summary, the definition of national income implied by Chichilnisky's criterion of intertemporal optimality involves the use of prices to assign value to a path of the economy: the value will have two components, one a present value computed at a discount rate in the conventional fashion, and one an undiscounted value associated with the very long run properties of the path.

6.1.1 Application to Exhaustible Resources

With this criterion of optimality, national welfare is defined by (11). With the exhaustible resource model used in the previous illustrations, the instantaneous value of this expression at any point in time is exactly as for the previous welfare measure, namely $\{c_t^* u_c(c_t^*, s_t^*) + s_t^* u_s(c_t^*, s_t^*)\}$. However, there is a big difference from the previous national welfare case, which is that in this context the total national welfare along a path is not the present value of all instantaneous welfare levels, but exceeds this by the limiting terms $\lim_{t\to\infty} \{c_t^* u_c(c_t^*, s_t^*) + s_t^* u_s(c_t^*, s_t^*)\}$ reflecting sustainable welfare levels. There is no instantaneous welfare measure that reflects accurately the total contribution of the current configuration to the welfare value of a path. In evaluating any change, we have to consider both the effect on current welfare (and on future welfare levels at finite dates, which are captured in shadow prices) and the effect on limiting or sustainable welfare levels.

6.2 Sustainable Revenues and National Income

Suppose now that the economy is very future-oriented, in that the objective is to achieve the maximum sustainable utility. Consider in this case the renewable resource model used by Beltratti et al. [3] [4] to define the green golden rule:

$$\max \lim_{t\to\infty} u(c_t, s_t) \text{ subject to } \dot{s}_t + c_t = r(s_t)$$

The solution is described in figure 2: it involves moving to a point (c^*, s^*)in the $c-s$ plane at which the graph of $r(s)$ is tangent to an indifference curve of the utility function. We expect to be able to support such a point by facing agents with relative prices for the two commodities (the stock and the flow in this case) equal to the common slope of both sets at their point of tangency. Applying this in the present context, we could normalize the price of consumption to be unity, and set the price of the stock to be $p = r'(s^*)$. Maximizing

$$c + ps, \text{ where } p = r'(s^*) \tag{13}$$

over the set of $c - s$ points that can be maintained for ever (i.e., that form stationary solutions with $c = r(s)$) will lead to the green golden rule. In the renewable resource context, think of the following example. The owner of a resource stock can sell a flow generated from this, and is also paid a "rent" for maintaining the stock. The stock might be a forest: then the flow would be derived by cutting and selling a part of this, while the rental payment on the stock could be payments made by people using the forest for recreational purposes, a payment in recognition of the forest's carbon sequestration or biodiversity support roles. Then if the rent relative to the price of the flow is $r'(s^*)$, the combination of stocks and flows which maximizes total receipts is the green golden rule.

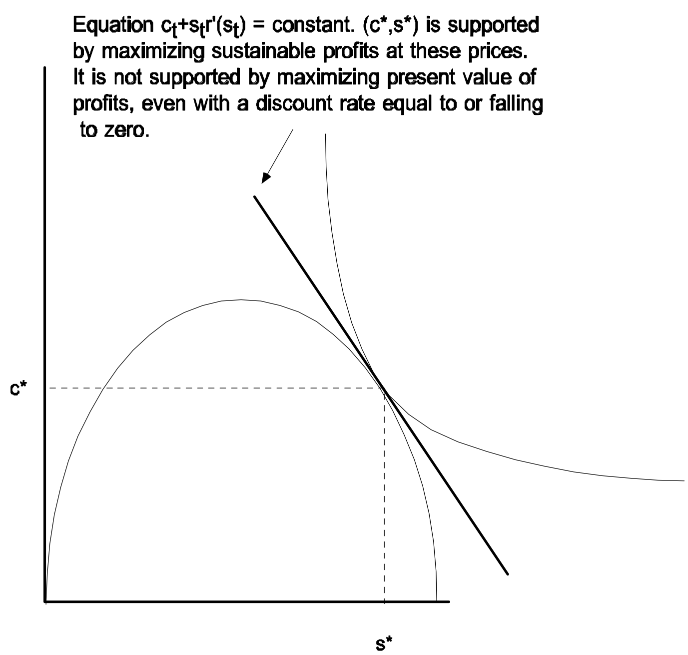

Figure 2: supporting the green golden rule.

There is a difficult point here, relating to the time dimension. The diagram and the analysis relate to a one period framework: we are interested in behavior which supports the green golden rule for ever. We would like to say that choosing (c^*, s^*) maximizes the sum of revenues from the resource over the long run, but we cannot say this, as this sum is clearly infinite, and there are many other feasible (c, s) combinations which will also give an infinite value. And we cannot say that (c^*, s^*) maximizes the present discounted value of revenues form the resource, because it does not: for any positive discount rate, the policy which maximizes the present value of profits will be non-stationary.

What then can we say? We can say that the green golden rule leads to the highest indefinitely maintainable level of revenues from the use of the resource: it maximizes "sustainable" or limiting or long-run revenues.

There is an important conclusion here: if society is so future-oriented as to wish to support the highest sustainable utility level, i.e., the green golden rule, then we need correspondingly future-oriented behavior on the parts of agents in the economy. We need firms to seek the highest sustainable profits (i.e., the maximum value of profits that can be maintained for ever) and resource owners to manage their resources so as to yield the highest sustainable revenues from the resources. Formally:

Proposition 7 *Consider the economy described by*

$$\left.\begin{array}{r} \max \lim_{t\to\infty} u\left(c_t, s_t\right) \\ s.t. \ \overset{\bullet}{s}_t = r\left(s\right) - c_t, \ s_t \geq 0 \forall t. \end{array}\right\}$$

Then there exists a price for the flow of the resource and a rental for the stock such that the green golden rule values of consumption and the resource stock lead to the maximum sustainable revenues from the use of the resource.

How do these observations relate to the previous discussions of national income in its several interpretations? As we have noted, with a zero discount rate, the Hicksian national income is not well-defined. What we are using here is again the separating hyperplane approach, as discussed in the previous section addressing national welfare and Chichilnisky's criterion, but now instead of the price system consisting of two parts, one defining a present value via an integral and the other reflecting the limiting behavior of the path, we now have only the latter term. The national welfare measure corresponding to this price system is now $\lim_{t\to\infty} \left\{c_t u_c\left(c_t^*, s_t^*\right) + s_t u_s\left(c_t^*, s_t^*\right)\right\}$ which can be rewritten as $\lim_{t\to\infty} \left\{c_t + p_t s_t\right\}$, where $p = r'(s^*) = u_s\left(c_t^*, s_t^*\right)/u_c\left(c_t^*, s_t^*\right)$. This is precisely the price system introduced above in (13), which we can now see as a particular form of our earlier concept of national welfare. Note that Hicksian national income is not appicable here, as there is no Hamiltonian to work with. Furthermore the Hicksian concept does not work for the overtaking concept. However we could apply the national wealth idea in an overtaking framework, by defining national wealth for each time horizon T and using the overtaking concept to rank paths by national wealth.

7 Summary

The following points have emerged in the discussion of national income and wealth:

1. There are two alternative approaches: the Hicksian measure of income, related to the Hamiltonian of a dynamic optimization problem, and National Wealth, a generalization of the normal welfare concept based on separating hyperplanes. There is a simple relationship between these: a perturbation in an optimal policy leads to changes in income and wealth, the former being interest on the latter. The change in wealth can also be expressed as the integral of changes in income minus the integral of the value of changes in stock accumulation. It is important to realize that the Hicksian measure is inappropriate if the discount rate is or falls to zero: it also produces a disturbing paradox in the Hotelling case.

2. In either case the appropriate measure depends on the objective of the economy and in particular on the discount rate. Different objectives and different discount rates give different numbers or even different formulae for national income. This is particularly important for environmental concerns, as many environmental assets are unusually long

lived and can in principle provide flows of valuable services to society indefinitely. Their valuation is thus very sensitive to the relative weights given to present and future.

3. Adopting Chichilnisky's criterion of intertemporal optimality gives a national wealth measure that involves the use of a present value and an undiscounted future value. This latter reflects the welfare level sustainable in the very long run and provides a justification for not discounting some of the services provided in the future by environmental assets.

4. When the economy's objective is the maximization of long-run welfare, i.e., obtaining the green golden rule, then the optimal configuration cannot be supported by the maximization of present value: we need to introduce the concept of maximum sustainable profits, and there exist prices at which the maximization of sustainable profits corresponds to and supports the maximization of sustainable utility.

Several points of practical significance are implied by these observations. Stocks are a sufficient statistic for the measurement of Hicksian national income. This emphasizes the importance of accurate measurement of all aspects of an economy's stocks, including its stocks of environmental assets, and of an understanding of how these affect welfare, reflected in the functional dependence of u on s in the models here. This is a task that we have barely begun: it is probably one of the biggest challenges facing environmental scientists.

There are two alternative ways of dealing with stocks: one is based on income at a point in time and involves including stocks, valuing them at shadow prices and attributing a return as in the expression $NI_t = \sum_{i=1}^{n} s_{i,t}\lambda_{i,t}\left\{\delta - \overset{\bullet}{\lambda}_{i,t} / \lambda_{i,t}\right\}$. Then a change in stocks automatically leads to a change in national income. The alternative is to measure national wealth as the value of the flow of goods and services contributing to utility, plus the value of services provided by stocks. The presence of the latter then makes further incorporation of stock changes inappropriate for a welfare measure. The same is true of project evaluation. Here we can value changes in all arguments of the utility function—both goods and services, including environmental services (ecosystems services)—for the entire duration of the project (this is the National Wealth approach), thus:

$$\Delta NW = \int_{t1}^{t2} \left\{\Delta c_t p_{c,t} + \Delta s_t p_{s,t}\right\} e^{-\delta(t-t1)}$$

Alternatively, we can value current changes in the arguments of the utility function and changes in current stocks, including stocks of environmental capital. Changes in the current stock proxy changes in future consumption. This is the linearized Hamiltonian approach:

$$\Delta HNI_t = \Delta c_t p_{c,t} + \Delta s_t p_{s,t} + \Delta\left[\frac{ds}{dt}\right] p_{s,t}$$

Within recent years there have been moves to recalculate national income on a "green" basis for several countries. For example, a study by Repetto et al. [24] for Costa Rica recently argued that when environmental costs and benefits are fully recognized, then the growth of Costa Rica's national income is reduced significantly. There has also been a similar study for Indonesia [25]. These studies suggest that the issues being analyzed here are important, and very probably underestimate the magnitude of the revisions because they take into account only a limited number of the relevant factors. Data limitations and the absence of a clear conceptual framework have held back the development of widely-accepted "green" measures of national income.

Finally, a caution on the use of national income or wealth for evaluating the consequences of far-ranging phenomena such as climate change: they are local measures of welfare changes, first order approximations. It seems quite within the bounds of scientific possibility that within the next century climate change could make non-local changes to our economic environment. National income as computed above is not an appropriate measure of the impact of such changes: the full impacts can only be captured by a general equilibrium model with a complete description of consumer demands.

8 Appendix

Proof of Proposition 1. Introduce a variable W_t which is the utility level which if maintained from t to infinity would have the same present value as $\{c_t^*, s_t^*\}_{t,\infty}$:

$$\int_t^\infty W_t e^{-\delta(\tau-t)} d\tau = \frac{W_t}{\delta} = \int_t^\infty u\left(c_\tau^*, s_\tau^*\right) e^{-\delta(\tau-t)} d\tau$$

Obviously we need to show that $W_t = H_t$: we shall do this by showing that both satisfy the same differential equation. Clearly

$$\frac{dW_t}{dt} = \delta\left\{-u\left(c_t^*, s_t^*\right) + W_t\right\} \tag{14}$$

so

$$W_t = u\left(c_t^*, s_t^*\right) + \frac{1}{\delta}\frac{dW_t}{dt}$$

Now turn to the Hamiltonian:

$$\frac{dCH_t}{dt} = \sum_i \frac{\partial u}{\partial c_i}\frac{dc_i}{dt} + \sum_i \frac{\partial u}{\partial s_i}\frac{ds_i}{dt} + \sum_i \frac{d\lambda_{i,t}}{dt}\frac{ds_i}{dt} + \sum_i \lambda_{i,t}\frac{d^2 s_i}{dt^2}$$

and by simplifying and using the first order conditions (4) and (5) this reduces to

$$\frac{dCH_t}{dt} = \delta \sum_i \frac{ds_i}{dt}\lambda_{i,t} \tag{15}$$

Recall that

$$CH_t = u\left(c_t^*, s_t^*\right) + \sum_{i=1}^n \lambda_{i,t}\frac{ds_i}{dt}$$

so that CH_t satisfies the differential equation

$$CH_t = u\left(c_t^*, s_t^*\right) + \frac{1}{\delta}\frac{dCH_t}{dt}$$

from which the desired result follows. ■

Proof of Proposition 2. Consider the change in the linearized Hamiltonian ΔH (6). Note that in the second expression for ΔH, the first term in parentheses is equal to zero by the first order conditions for optimality (4).[15] Consider now the second term:

$$\sum_{i=1}^n \Delta s_i \left\{\frac{\partial u}{\partial s_i} + \sum_{k=1}^n \lambda_{k,t}\frac{\partial d_k}{\partial s_i}\right\}$$

Then by the conditions (5) determining the rate of change of the shadow prices, this can be rewritten as

$$\sum_{i=1}^n \Delta s_i \left\{\delta\lambda_{i,t} - \dot{\lambda}_{i,t}\right\} = \sum_{i=1}^n \Delta s_i \lambda_{i,t}\left\{\delta - \frac{\dot{\lambda}_{i,t}}{\lambda_{i,t}}\right\}$$

which proves the proposition. ■

[15] Note that this is true even if we impose non-negativity constraints on the flows c_t and consider the corresponding complementary slackness conditions.

Definition 3 *Formally a separating hyperplane satisfies the following conditions (here an asterisk denotes the value of a variable along an optimal path):*

$$\begin{aligned}\int_0^\infty u(c_t,s_t)e^{-\delta t}dt &\geq \int_0^\infty u(c_t^*,s_t^*)e^{-\delta t}dt \Rightarrow \\ \int_0^\infty \{\langle p_c(t),c_t\rangle + \langle p_s(t),s_t\rangle\}e^{-\delta t}dt &\geq \int_0^\infty \{\langle p_c(t),c_t^*\rangle + \langle p_s(t),s_t^*\rangle\}e^{-\delta t}dt\end{aligned} \tag{16}$$

and

$$\begin{aligned}\{c_t,p_t\}\ \textit{feasible} &\Rightarrow \\ \int_0^\infty \{\langle p_c(t),c_t\rangle + \langle p_s(t),s_t\rangle\}e^{-\delta t}dt &\leq \int_0^\infty \{\langle p_c(t),c_t^*\rangle + \langle p_s(t),s_t^*\rangle\}e^{-\delta t}dt\end{aligned} \tag{17}$$

where $\langle p_c(t),c_t\rangle$ denotes the inner product of the price vector $p_c(t)$ with the consumption vector c_t.

Proof of Proposition 3. We need to show that these prices satisfy (16) and (17). Consider a path $\{c_t,s_t\}$ such that

$$\int_0^\infty \{u(c_t,s_t)-u(c_t^*,s_t^*)\}e^{-\delta t}dt \geq 0 \tag{18}$$

We need to show that in this case

$$\int_0^\infty \{\langle p_c,(t)\,c_t\rangle + \langle p_s,(t)\,s_t\rangle\}e^{-\delta t}dt \geq \int_0^\infty \{\langle p_c,(t)\,c_t^*\rangle + \langle p_s,(t)\,s_t^*\rangle\}e^{-\delta t}dt$$

Take a linear approximation to $u(c_t,s_t)$ about (c_t^*,s_t^*) along an optimal path, and use the concavity of the utility function:

$$\begin{aligned}u(c_t,s_t)-u(c_t^*,s_t^*) &\leq \sum_j \frac{\partial u(c_t^*,s_t^*)}{\partial c_{j,t}}(c_{j,t}-c_{j,t}^*) + \sum_i \frac{\partial u(c_t^*,s_t^*)}{\partial s_{i,t}}\left(s_{i,t}-s_{i,t}^*\right) \\ &= \sum_j p_{c,j}(t)(c_{j,t}-c_{j,t}^*) + \sum_i p_{s,i}(t)\left(s_{i,t}-s_{i,t}^*\right)\end{aligned}$$

Together with (18), this establishes the inequality needed, i.e., (16).

Now we need to establish the inequality (17). Consider the problem of choosing a program to maximize present value at the prices $\{p_{c,j}(t),p_{s,i}(t)\}$:

$$\max \int_0^\infty \{p_c(t)\,c_t + p_s(t)\,s_t\}e^{-\delta t}dt$$

$$\text{subject to } \dot{s}_{i,t} = d_i(c_t,s_t),\ i=1,...,n$$

The corresponding Hamiltonian is

$$H = \{\langle p_c,(t)\,c_t\rangle + \langle p_s,(t)\,s_t\rangle\}e^{-\delta t} + \sum_{i=1}^n \mu_{i,t}e^{-\delta t}d_i(c_t,s_t)$$

and a solution satisfies

$$p_{c,j}(t) = -\sum_{i=1}^n \mu_{i,t}\frac{\partial d_i(c_t,s_t)}{\partial c_j}$$

and

$$\overset{\bullet}{\mu}_{i,t} - \delta\mu_{i,t} = -p_{s,i}(t) - \sum_{k=1}^{n} \mu_{k,t} \frac{\partial d_k(c_t, s_t)}{\partial s_i}$$

Now note that given the definition of the optimal prices, these conditions are precisely the same as the conditions (4) and (5) which characterize a solution to the general optimization problem of maximizing (2) subject to (1). Hence a path which solves the overall optimization problem also solves the problem of maximizing the present value of the path at the optimal prices. This completes the proof. ■

Proof of Corollary 1. By assumption

$$\int_0^{\infty} \{\langle \Delta c_t, p_c(t)\rangle + \langle \Delta s_t, p_s(t)\rangle\} e^{-\delta t} dt > 0 \tag{19}$$

Let $\{\Delta c_t + c_t^*, \Delta s_t + s_t^*\}$ be the path resulting from implementing the variation in the optimal path. The welfare associated with this path is

$$\int_0^{\infty} \{u(c_t^*, s_t^*) + \sum_j \frac{\partial u(c_t^*, s_t^*)}{\partial c_{j,t}} \Delta c_{j,t} + \sum_i \frac{\partial u(c_t^*, s_t^*)}{\partial s_{i,t}} \Delta s_{i,t}\} e^{-\delta t} dt$$

which by (19) and the definition of the optimal prices is greater than that on the optimal path, as required. The proof of the converse is immediate. ■

Definition 4 *A hyperplane which separates the set of paths preferred to an optimum from those which are feasible is a time path of prices for stocks and flows $p_s(t)$ and $p_c(t)$ which must satisfy the following conditions (here an asterisk denotes the value of a variable along an optimal path):*

$$\left.\begin{array}{c} \int_0^{\infty} u(c_t, s_t) e^{-\delta t} dt + \lim\limits_{t\to\infty} u(c_t, s_t) \geq \\ \int_0^{\infty} u(c_t^*, s_t^*) e^{-\delta t} dt + \lim\limits_{t\to\infty} u(c_t^*, s_t^*) \Rightarrow \\ \langle (c_t, s_t), (p_c(t), p_s(t))\rangle \geq \langle (c_t^*, s_t^*), (p_c(t), p_s(t))\rangle \end{array}\right\} \tag{20}$$

and

$$\left.\begin{array}{c} \{c_t, p_t\} \text{ feasible implies} \\ \langle (c_t, s_t), (p_c(t), p_s(t))\rangle \leq \langle (c_t^*, s_t^*), (p_c(t), p_s(t))\rangle \end{array}\right\} \tag{21}$$

where $\langle p_c(t), c_t\rangle$ denotes the inner product (12) of the price vector $p_c(t)$ with the consumption vector c_t. ■

Proof of Proposition 5.

In the proof of proposition 1 we show that

$$\int_t^{\infty} W_t e^{-\delta(\tau - t)} d\tau = \frac{W_t}{\delta} = \int_t^{\infty} u(c_\tau^*, s_\tau^*) e^{-\delta(\tau - t)} d\tau$$

and then that $W = H$. Hence

$$H_0 = \delta \int_0^{\infty} u(c_\tau^*, s_\tau^*) e^{-\delta(\tau)} d\tau$$

This is the result that the Hamiltonian is the weighted average of future utility levels. Note that ΔHNI_0 is the change in the left hand side here resulting from a variation in the optimal path and ΔNW is the corresponding change in the right hand side. Hence

$$\Delta HNI_0 = \delta \Delta NW$$

■

Proof of Proposition 6. We need to show that these prices satisfy (20) and (21). Consider a path $\{c_t, s_t\}$ such that

$$\int_0^\infty \{u(c_t, s_t) - u(c_t^*, s_t^*)\} e^{-\delta t} dt + \lim_{t\to\infty} \{u(c_t, s_t) - u(c_t^*, s_t^*)\} \geq 0 \tag{22}$$

We need to show that in this case

$$\begin{aligned} &\int_0^\infty \{p_c(t) c_t + p_s(t) s_t\} e^{-\delta t} dt + \lim_{t\to\infty} \{p_c(t) c_t + p_s(t) s_t\} \geq \\ &\int_0^\infty \{p_c(t) c_t^* + p_s(t) s_t^*\} e^{-\delta t} dt + \lim_{t\to\infty} \{p_c(t) c_t^* + p_s(t) s_t^*\} \end{aligned}$$

Take a linear approximation to $u(c_t, s_t)$ about (c_t^*, s_t^*) along an optimal path, and use the concavity of the utility function:

$$\begin{aligned} u(c_t, s_t) - u(c_t^*, s_t^*) &\leq \sum_j \frac{\partial u(c_t^*, s_t^*)}{\partial c_{j,t}} (c_{j,t} - c_{j,t}^*) + \sum_i \frac{\partial u(c_t^*, s_t^*)}{\partial s_{i,t}} \left(s_{i,t} - s_{i,t}^*\right) \\ &= \sum_j p_{c,j}(t)(c_{j,t} - c_{j,t}^*) + \sum_i p_{s,i}(t)\left(s_{i,t} - s_{i,t}^*\right) \end{aligned}$$

Together with (22), this establishes the inequality needed, i.e., (20).

The inequality (21) can now be established using a very minor variation of the argument used for proposition 3 of section 4. ■

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