Portfolio Choice with Illiquid Assets[*]

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Abstract

We investigate how the inability to continuously trade an asset affects portfolio choice. We extend the standard model of portfolio choice to include an illiquid asset which can only be traded at infrequent, stochastic intervals. Because investors cannot insure against the inability to trade, illiquidity induces endogenous additional time-varying risk aversion above the curvature in their utility functions. This additional risk aversion creates under-investment in both the liquid and illiquid risky assets relative to the standard Merton (1969) case; the optimal investment and consumption policies are time-varying and depend on the liquidity mix of assets in the investor's portfolio. The presence of liquidity risk distorts the allocation of the liquid and illiquid assets even when liquid and illiquid asset returns are uncorrelated and the investor has log utility.

JEL Classification: G11, G12

Keywords: Asset Allocation, Liquidity, Alternative Assets, Endowment Model

1   Introduction

Investors seeking to buy or sell assets that are not traded on centralized exchanges can face substantial difficulty in finding a counterparty or an opportunity to trade. From the investor's perspective, this inability to continuously trade represents an additional source of risk that cannot be hedged. This inability to find a willing counter-party every instant can arise for several reasons. First, trading the asset may require specialized knowledge that is in limited supply, as is the case for certain securitized fixed income and structured credit products. Second, the asset may have unique characteristics, so it may be time-consuming to find an investor willing to trade in this particular asset, for example as in real estate or certain private equity investments. Third, private equity and venture capital limited partner investments have uncertain exit and re-investment timing because the timing of the exit from the underlying investments is uncertain. Finally, markets sometimes shut down, as was the case for many fixed income markets that froze during the financial crisis in 2008/09.[1] Thus, even if an investor so desires, certain assets cannot be traded or liquidated for significant periods of time. We view the inability to trade frequently as one of the defining characteristics of liquidity, and we investigate its effects on asset allocation.

We examine the optimal portfolio policies of a long-lived constant relative risk aversion (CRRA) investor able to trade in two risky assets – a liquid and an illiquid security – as well as a liquid risk-free asset. The illiquid asset can only be traded at infrequent but randomly occurring times. We interpret these stochastically occurring trading times as the outcome of an un-modeled search process: the investor must find an appropriate counter-party, and such counter-parties are either not freely identifiable or are difficult to locate. We model the arrival of trading opportunities as an i.i.d. Poisson process, and so the waiting time before a counter-party is found is random and represents a source of risk that the investor cannot hedge.

We find that illiquidity causes the investor to behave in a more risk-averse manner with respect to *both* liquid and illiquid holdings. This increase in risk aversion arises through two channels. First, the waiting time until the next trading opportunity is random; this additional source of risk reduces the allocation to the illiquid asset. Second, the investor's immediate obligations (consumption or payout) can only be financed through liquid wealth. If the investor's liquid wealth drops to zero, these obligations cannot be met until after the next rebalancing opportunity. The investor is willing to reduce her allocation to both the liquid and illiquid risky assets in order to minimize the probability that a state with liquid

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[1]This is not simply a question of a seller reducing prices to a level where a buyer is willing to step in. As Tirole (2011) and Krishnamurthy, Nagel, and Orlov (2011) comment, there were no bids, at any price, representing "buyers' strikes" in certain markets where whole classes of investors simply exited markets.

wealth (as opposed to zero total wealth) is reached.

The distinction between liquid and illiquid wealth has important implications for asset allocation. When the illiquid asset cannot be traded, the ratio of liquid to illiquid securities in the investor's portfolio is not under the investor's control, and the investor's portfolio moves away from her optimal position. Therefore, this ratio is a state variable in the investor's asset allocation problem. The investor's ability to fund intermediate consumption depends on her liquid wealth, thus her effective level of risk aversion endogenously increases in the fraction of wealth held in illiquid securities. In fact, the investor's concern over the mix of liquid and illiquid securities affects portfolio policies even when the liquid and illiquid asset returns are uncorrelated and under log utility. This concern corresponds to real-world situations where investors or investment funds are insolvent, not because their assets under management have hit zero, but because they cannot fund their immediate obligations.

In addition, the investor need not fully take advantage of opportunities that might look like an arbitrage, for instance a situation where the returns to the liquid and illiquid asset are perfectly correlated, yet the two assets have different expected returns. The reason is that taking advantage of this arbitrage involves a strategy that causes the investor's liquid wealth to drop to zero with positive probability.

The effect of illiquidity on portfolio choice can be economically large. We compare the investor's optimal allocation in the presence of illiquidity and with a two-risky-asset Merton (1969, 1971) economy, where all assets can be traded continuously. We find that for realistic parameter values, an investor would be willing to accept a 1% lower risk premium on the illiquid asset in order to make the illiquid asset fully liquid. Intuitively, the effect of illiquidity is significant because the shadow cost of illiquidity is unbounded; liquidity cannot be generated, e.g. a counter-party found, simply by paying a cost. Hence, our definition of liquidity is intrinsically different than a definition based on transactions cost models, where trade can always take place at a cost. In contrast to our paper, in these models the shadow cost of illiquidity is bounded by the level of transaction costs.

Our analysis falls into a large literature dealing with asset choice and various aspects of the investor's unwillingness or inability to continuously rebalance part of her total endowment.[2] The most closely related papers to our analysis are Dai, Li and Liu (2008), Longstaff (2009), and de Roon, Guo and ter Horst (2009). In these papers, the period in which the investor

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[2]This literature considers transaction costs (Amihud and Mendelson, 1986; Constantinides, 1986; Vayanos, 1998; Huang, 2003; Lo, Mamaysky and Wang, 2004), the inability to trade arbitrarily large amounts (Longstaff, 2001), market shutdowns (Rogers and Zane, 2002; Kahl, Liu, and Longstaff, 2003; Dai, Li and Liu, 2008; Longstaff, 2009; de Roon, Guo and ter Horst, 2009), search frictions associated with finding counterparties to trade (Duffie, Gârleanu and Pedersen, 2005, 2007; Vayanos and Weill, 2008), and unhedgeable labor income or business risk (Heaton and Lucas, 1996; Koo, 1998).

cannot trade is deterministic, whereas in our model the illiquid period is recurring and of stochastic duration, which introduces an additional, unhedgeable, source of risk from the investor's perspective. Longstaff (2001) allows investors to trade continuously, but with only bounded variation, which makes illiquid assets partially marketable at all times and the model closer to the literature on transaction costs. Finally, de Roon, Guo and ter Horst (2009) do not consider recurring periods of illiquidity and set the horizon of their portfolio choice problem to the expiration of the lock-up period of illiquid assets.

Our work is related to the literature on transaction costs, since illiquidity is often viewed as an implicit transaction cost which investors pay when rebalancing. Our work is similar in the sense that in the presence of fixed transaction costs the investor is *unwilling* to rebalance continuously. However, in our setting the investor is *unable* to trade continuously, even at a cost. Thus, our setup corresponds to a situation without a centralized market, where investors need to search for suitable counter-parties. In addition, the fact that the illiquid asset can be traded at infrequent intervals implies that total wealth can drop sharply between rebalancing times. This is economically similar to situations in which there is a jump component in prices, as in Liu, Longstaff and Pan (2003). A key difference between our setting and the jump-diffusion setting of Liu, Longstaff and Pan (2003) is that in our model of illiquidity, risk aversion is time-varying and portfolios drift away from optimal diversification leading to time variation in investment and consumption policies even when returns are i.i.d. Our work is also related to the literature on unhedgeable human capital risk in that part of the investor's total wealth cannot be traded, which introduces a motive to hedge using the set of tradeable securities. We differ in that our illiquid asset can be traded, though not frequently.

It is still an open question whether illiquidity has large or small effects in equilibrium. On the one hand, Lo, Mamaysky and Wang (2004), Longstaff (2009), and Dai, Li and Liu (2008), show that the presence of illiquidity can have large effects on prices in equilibrium models. On the other hand, transactions costs and other measures of illiquidity have small or negligible effects in the models of Constantinides (1986), Vayanos (1998), and Gârleanu (2009). Intuitively, even though individual agents' asset holdings and trading patterns may be significantly affected by the presence of illiquidity, other agents may be unconstrained, or the effects of illiquidity may wash out in aggregate, leaving average prices little changed by certain agents not having the ability to trade. Endogenizing the risk premiums to be a function of the degree of illiquidity is outside the scope of our paper, but we provide some calculations illustrating that the cost of illiquidity may be large in terms of certainty equivalent wealth.

Finally, our paper is related to the "endowment model" of asset allocation for insti-

tutional long-term investors made popular by David Swensen's work, *Pioneering Portfolio Management*, in 2000. Swensen's thesis is that highly illiquid markets, such as private equity and venture capital, have large potential payoffs to research and management skill, which are not competed away because most managers have short horizons. Leaving aside whether there are superior risk-adjusted returns in alternative investments, the endowment model does not consider the illiquidity of these investments. Recently, Siegel (2008) and Leibowitz and Bova (2009) recognize that the inability to trade illiquid assets should be taken into account in determining optimal asset allocation weights, but only investigate scenario or simulation-based procedures and do not solve for optimal asset holdings. In addition to economically characterizing the impact of illiquidity risk on portfolio choice, our certainty equivalent calculations are quantitatively useful for investors to take into account the effect of illiquidity on risk-return trade-offs.

The rest of this paper is organized as follows. Section 2 sets out the model and discusses the calibrated parameter values. We solve the model in Section 3 and show that illiquidity induces endogenous risk aversion. We discuss time-varying optimal portfolio and consumption policies in Section 4. Section 5 considers how the characteristics of the illiquid asset affect optimal asset allocation and computes illiquidity risk premiums using certainty equivalents. Section 6 concludes. All proofs are in the appendix.

2   Model

2.1   Information

The information structure obeys standard technical assumptions. Specifically, there exists a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ supporting the vector of two independent Brownian motions $Z_t = (Z_t^1, Z_t^2)$ and an independent Poisson process (N_t). $\mathcal{P}$ is the corresponding measure and $\mathcal{F}$ is a right-continuous increasing filtration generated by $Z \times N$.

2.2   Assets

There are three assets in the economy: a risk-free bond B, a liquid risky asset S, and an illiquid risky asset P. The riskless bond B appreciates at a constant rate r:

$$dB_t = r\, B_t\, dt \tag{1}$$

The second asset S is a liquid risky asset whose price follows a geometric Brownian motion with drift μ and volatility σ:

$$\frac{dS_t}{S_t} = \mu \, dt + \sigma \, dZ_t^1. \tag{2}$$

The first two assets are liquid and holdings can be rebalanced continuously.

The third asset P is an illiquid risky asset, for which the price process evolves according to a geometric Brownian motion with drift ν and volatility ψ:

$$\frac{dP_t}{P_t} = \nu \, dt + \psi \rho \, dZ_t^1 + \psi \sqrt{1 - \rho^2} \, dZ_t^2, \tag{3}$$

where ρ captures the correlation between the returns on the two risky assets.

The illiquid asset P differs from the first two assets B and S in two important ways. The first distinction is that asset P can only be rebalanced at infrequent, stochastic intervals. In particular, the illiquid asset P can only be traded at stochastic times τ, which coincide with the arrival of a Poisson process with intensity λ. Thus, the expected time between rebalancing is $1/\lambda$. When a trading opportunity arrives, the investor is able to rebalance her holdings of the illiquid asset up to any amount. Note that P_t reflects the fundamental value of the illiquid asset, which varies randomly irrespective of whether trading in the asset is possible.

Our specification of illiquidity is motivated by the literature on search and asset prices, e.g. Duffie, Gârleanu and Pedersen, (2005, 2007). We interpret the illiquid asset P as an asset which is not traded in a centralized exchange. In this case, investors who are willing to trade in this asset need to search for a counterparty. This search process might be time-consuming, since in many cases the number of market participants with the required expertise, capital, and interest in these illiquid assets is small. Hence, the average waiting time $1/\lambda$ captures the expected period needed to find a suitable counterparty to trade the illiquid asset. Examples of such illiquid assets are hedge funds, venture capital, private equity, structured credit, and real estate. Some of these assets are traded in over-the-counter markets, but in others investors need to search directly for a counterparty in order to rebalance a position. For instance, Ang and Bollen (2010) illustrate that investors redeeming directly from hedge funds after lockup provisions have expired may face gates, which restrict their withdrawal of capital.

The second way in which the illiquid asset P differs from the liquid assets B and S is that it cannot be pledged as collateral for borrowing. If investors could borrow using the illiquid asset as collateral, they could convert the illiquid asset into liquid wealth and thus implicitly

circumvent the illiquidity friction. This non-pledgeability assumption is significant, so one way to interpret asset P is as the fraction of illiquid wealth that cannot be collateralized.[3] For instance, in the case of real estate, we could interpret the illiquid asset P as the fraction of the value of the property that cannot be used as collateral against a mortgage or a home equity line. Furthermore, in many cases, finding a counterparty who is willing to lend cash using illiquid assets as collateral may be difficult. For instance, Krishnamurthy, Nagel, and Orlov (2011) find evidence suggesting that money market mutual funds, which are the main providers of repo financing, were unwilling to accept private asset-backed securities as collateral between the third quarter of 2008 and the third quarter of 2009.

In summary, we parsimoniously introduce illiquidity risk into a standard Merton (1969, 1971) setting by the addition of one parameter, λ, which controls how often, on average, the illiquid asset can be rebalanced. The Merton model is the limiting case where λ approaches infinity and the illiquid asset can be rebalanced at every instant. The illiquidity friction introduces a difference between the investor's liquid and illiquid wealth, since only the former can be used to finance intermediate obligations such as consumption or payout to investors.

Finally, we will assume the standard discount rate restriction from the Merton one-asset model

$$\beta > (1-\gamma)r + \frac{1-\gamma}{2\gamma}\left(\frac{\mu-r}{\sigma}\right)^2. \tag{4}$$

and that the illiquid asset has at least as high a Sharpe ratio as the liquid asset in order to focus discussion on the more interesting case:

$$\frac{\nu-r}{\phi} \geq \frac{\mu-r}{\sigma}; \quad \rho > -1 \tag{5}$$

2.3 Investor

The investor has CRRA utility over sequences of consumption, C_t, given by:

$$\max_{\{C_t\}} \mathrm{E}\left[\int_0^\infty e^{-\beta t} U(C_t) dt\right], \tag{6}$$

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[3]This interpretation assumes that the amount that the asset can be collateralized does not vary over time and that the constraint is always binding. We could extend the model to allow the investor to endogenously choose the amount of collateralized borrowing every period, up to a limit. Hence, this model is equivalent to a hybrid model of infrequent trading and transaction costs. In this case the quantitative effects of illiquidity are mitigated but the qualitative effects remain.

where β is the subjective discount factor and $U(C)$ is given by

$$U(C) = \begin{cases} \dfrac{C^{1-\gamma}}{1-\gamma} & \text{if} \quad \gamma > 1 \\ \log(C) & \text{if} \quad \gamma = 1. \end{cases} \tag{7}$$

We focus on the case $\gamma > 1$ and present the results for log utility, $\gamma = 1$, in the appendix. Despite our investor having preferences that exhibit constant relative risk aversion with respect to consumption, we show that her relative risk aversion with respect to wealth is time varying. Previous authors employing discrete trading, such as Lo, Mamaysky and Wang (2004) and Gârleanu (2009), have used exponential rather than CRRA utility to rule out wealth effects. As we show below, wealth effects play an important role. Gârleanu (2009) further restricts agents to hold only illiquid assets rather than optimizing over the liquid-illiquid asset mix.

Investors face a single intertemporal budget constraint. However, in our case, the agent's illiquid wealth (the amount invested in the illiquid asset) cannot be immediately converted into liquid wealth (the amount invested in the liquid risky asset and the risk-free asset). Thus, we model the agent's liquid wealth and illiquid wealth separately. The joint evolution of the investor's liquid, W_t, and illiquid wealth, X_t, is given by:

$$\frac{dW_t}{W_t} = \left(r + (\mu - r)\,\theta_t - c_t\right)dt + \theta_t \sigma dZ_t^1 - \frac{dI_t}{W_t} \tag{8}$$

$$\frac{dX_t}{X_t} = \nu dt + \psi \rho dZ_t^1 + \psi \sqrt{1-\rho^2}dZ_t^2 + \frac{dI_t}{X_t} \tag{9}$$

The agent invests a fraction θ of her liquid wealth into the liquid risky asset, while the remainder $(1-\theta)$ is invested in the bond. The agent consumes (C_t) out of liquid wealth, so $c_t = C_t/W_t$ is the ratio of consumption to liquid wealth. When a trading opportunity arrives, the agent can transfer an amount dI from her liquid wealth to the illiquid asset.

Following Dybvig and Huang (1988) and Cox and Huang (1989), we restrict the set of admissible trading strategies, $\{\theta_t\}$, to those that satisfy the standard integrability conditions.

Our first result is that trading risk eliminates any willingness by the investor either to short the illiquid asset or to net borrow in liquid wealth to fund long purchases of the illiquid asset. Thus, without loss of generality, we restrict our attention to solutions with $W_t > 0$ and $X_t \geq 0$:

Proposition 1 *Any optimal policies will have $W > 0$ and $X \geq 0$ a.s.*

Proof. Consumption is out of liquid wealth only and the illiquid asset cannot be pledged, so $W_t \leq 0$ implies zero consumption before the next trading day, leaving the objective

function (6) at $-\infty$. For $|\rho| < 1$, $X_t < 0$ implies that under any admissible investment and consumption policy, there is a positive probability that at the next trading time $W_{t'} + X_{t'} \leq 0$, violating limited liability, implying zero consumption, and leaving the objective function (6) at $-\infty$. For $\rho = 1$, $X_t < 0$ is ruled out by assuming that the illiquid asset has a higher Sharpe ratio than the liquid asset (5). ∎

We will discuss the behavior of the investor with different asset correlations (including the apparent but not realizable arbitrage of $\rho = 1$) in Section 5.

2.4 Calibrated Parameters

We select our parameters so that the liquid asset can be interpreted as an investment in the aggregate stock market and the illiquid asset can be interpreted roughly as an alternative asset class such as private equity, buyout funds, or venture capital funds. Table 1 reports some statistics on the S&P500 and illiquid asset returns reported by Venture Economics and Cambridge Associates, which they loosely group into private equity, buyout, and venture capital funds. We report data from September 1981 to June 2010. Because of the unusually large, negative returns of many assets over the financial crisis over 2007-2008, we also report summary statistics ending in December 2006. We construct an artificial "illiquid investment", which is an equally-weighted average of private equity, buyout, and venture capital funds.

We set the parameters of the liquid equity return to be $\mu = 0.12$, $\sigma = 0.15$, and set the risk-free rate to be $r = 0.04$. Table 1 shows that this set of parameters closely matches the performance of the S&P500 before the financial crisis. The mean of the S&P500 including 2008-2010 falls to 0.10 and slightly more volatile, at 0.18, but our calibrated values are still close to these estimated values. We work mostly with the risk aversion case $\gamma = 6$, which for an investor allocating money between only the S&P500 and the risk-free asset paying $r = 0.04$ produces an equity holding of $(\mu - r)/(\gamma\sigma^2) = 0.59$. This is very close to a classic 60% equity, 40% risk-free bond portfolio used by many institutional investors.

Table 1 shows that the returns on illiquid investments have similar characteristics to equity. For example, over the full sample (1981-2010), the mean log return on the illiquid investment is 0.11 with a volatility of 0.17. This is close to the S&P500 mean and volatility of 0.10 and 0.18, respectively, over that period. Table 1 shows that the returns on liquid and illiquid investments are even closer in terms of means and volatilities before the financial crisis. This suggests setting the parameters of the illiquid asset, ν and ψ, to be the same as the parameters on the S&P500.

Interpreting the returns of the illiquid assets, however, must be done with extreme cau-

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tion. The data from Venture Economics and Cambridge Associates are not indexes, but a collation and grouping of data from private capital firms willing to supply the data providers with NAV and IRR data. Phalippou (2010) discusses many pitfalls in the "point-to-point" method used in computing these returns: they are very dissimilar to actual, investable returns. Indeed, the returns on individual fund investments reported in the literature, especially those studies which work with realized cashflows rather than NAVs, find very different characteristics of private equity, buyout, and venture capital returns.

For most of our analysis, we take a conservative approach and set $\nu = 0.12$ and $\psi = 0.15$ to be the same mean and volatility, respectively, as the liquid asset. This has the advantage of isolating the effects of illiquidity rather than obtaining results due to the higher Sharpe ratios of the illiquid assets. Second, even for individual funds this assumption is not unrealistic, at least for some illiquid asset classes. Driessen, Lin and Phalippou (2008) and Gottschalg and Phalippou (2009) estimate private equity fund alphas, with respect to equity market indexes, close to zero. Both Kaplan and Schoar (2005) and Gottschalg and Phalippou (2009) report that private equity fund performance is very close to the S&P. On the other hand, Cochrane (2005), Korteweg and Sorensen (2010), and Phalippou (2010), among others, report that the alphas and betas of venture capital funds are potentially very different from zero and one, respectively.[4]

We take a baseline case of $\lambda = 1$, or that the average waiting time to rebalance the illiquid asset is one year. Individual private equity, buyout, and venture capital funds can have average fund lives of approximately 10 years, which would correspond to $\lambda = 1/10$, but these are often held in portfolios of many alternative asset funds, which would mitigate the illiquidity risk of a single fund. For portfolios of illiquid assets, λ could be calibrated to the average turnover. In this case, an appropriate interpretation is that the Poisson liquidity event is when assets are recovered from a previous fund and the investor can re-invest those proceeds in a new fund or convert them to liquid assets. Since λ is a crucial parameter, we take special care to show the portfolio and consumption implications for a broad range of λ. Fortunately, the economics behind the solution are immune to the particular parameter values chosen, as we now detail.

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[4]Phalippou (2010) computes a beta around three for both individual venture capital and buyout funds. Cochrane (2005) and Korteweg and Sorensen (2010) also report betas around two or three for venture capital funds. Venture capital fund returns have extremely high volatility, often exceeding 100%, causing arithmetic return alphas to be very large but when log returns are used, alphas are closer to zero. The log return is appropriate for our portfolio choice setup. In our parameter choice, we choose a value of $\psi \ll 100\%$ in order to satisfy the participation conditions below. Thus, our illiquid asset should be interpreted as a diversified portfolio of alternative assets rather than individual venture capital deals with high volatilities. To our knowledge there are no publicly available, long time series of returns on institutional portfolios holding diversified, but illiquid, portfolios of many venture capital and buyout funds.

3 Solution

Because markets are not dynamically complete, we use dynamic programming techniques to solve the investor's problem. First, we establish some basic properties of the solution. Then, we compute the investor's value function and optimal portfolio and consumption policies.

3.1 Value function

We define the value function as

$$e^{-\beta t} F(W_t, X_t) = \max_{\{\theta, I, c\}} E_t \left[\int_t^\infty e^{-\beta s} U(C_s) ds \right], \tag{10}$$

where $U(C)$ is defined in equation (7).

The first step is to establish bounds on the value function. The trader's value function must be bounded below by the problem in which the illiquid asset does not exist, and the value function must be bounded above by the problem in which the entire portfolio can be continuously rebalanced. We refer to these as the Merton (1969, 1971) one-stock and two-stock problems, respectively. Hence, there exist constants K_1 and K_2 such that

$$K_1 W^{1-\gamma} \le F(W, X) \le K_2 (W + X)^{1-\gamma} \le 0. \tag{11}$$

Since the Merton one-asset value function exists (4), our value function is bounded between the one-asset solution and zero.

Since the utility function is homothetic and the return processes have constant moments, it must be the case that F is homogeneous of degree $1 - \gamma$. Thus, there exists a function g with $g(x) = F(1, x)$ so that

$$F(W, X) = W^{1-\gamma} g \left(\frac{X}{W} \right). \tag{12}$$

From equation (11), we obtain that g is bounded from above and below.

The next step involves characterizing the value function at times when the agent can rebalance between her liquid and illiquid wealth. When the Poisson process hits and the agent rebalances her portfolio, the value function may discretely jump. Denote the new, higher, value function just before rebalancing occurs as F^*, so that the total amount of the jump is $F^* - F$. At the Poisson arrival, the agent is free to make changes to her entire

portfolio. Thus, we have that

$$F^*(W_t, X_t) = \max_{I \in [-X_t, W_t)} F(W_t - I, X_t + I). \tag{13}$$

Since F^* must also be homogeneous of degree $1 - \gamma$, there exists a function g^* such that $F^* = W^{1-\gamma} g^* \left(\frac{X}{W} \right)$. In addition, since rebalancing the illiquid asset is costless when possible, we also have $(W - \delta)^{1-\gamma} g^* \left(\frac{X+\delta}{W-\delta} \right) = W^{1-\gamma} g^* \left(\frac{X}{W} \right)$ for any $-X \le \delta < W$. Differentiating both sides with respect to δ and setting $\delta = 0$ yields $g^{*\prime}(x)(1 + x) = (\gamma - 1)g^*(x)$. Integrating yields

$$F^*(W_t, X_t) = GW_t^{1-\gamma} \left(1 + \frac{X_t}{W_t} \right)^{1-\gamma}, \tag{14}$$

where G is a constant.

Equation (14) is the value function when a rebalancing opportunity arrives. We now characterize the behavior of the value function $F(W, X)$ as X and W change between rebalancing times:

Proposition 2 *The solution is characterized by the function $g(x)$ and constants G and x^* with*

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\begin{aligned}
0 = \max_{c, \theta} & \left[\frac{1}{1 - \gamma} c^{1-\gamma} + \lambda G(1 + x)^{1-\gamma} \right. \tag{15} \\
& + g(x) \left(-\beta - \lambda + (1 - \gamma)(r + (\mu - r)\theta - c) - \frac{1}{2}\gamma(1 - \gamma)\sigma^2\theta^2 \right) \\
& + g'(x)x \left(\nu - (r + (\mu - r)\theta - c) - \gamma\psi\theta\rho\sigma + \gamma\theta^2\sigma^2 \right) \\
& \left. + g''(x)x^2 \left(\frac{1}{2}\theta^2\sigma^2 + \frac{1}{2}\psi^2 - \psi\theta\rho\sigma \right) \right].
\end{aligned}
$$

and

$$g(x^*) = G(1 + x^*)^{1-\gamma} \tag{16}$$
$$g'(x^*) \le G(1 - \gamma)(1 + x^*)^{-\gamma}. \tag{17}$$

with equality for $x^* > 0$.

When a trading opportunity occurs at time τ, the trader selects I_τ so that $\frac{X_\tau}{W_\tau} = x^*$.

Define the constants K_0 and K_∞, $K_0 < K_\infty < 0$, so that K_0 is the solution to

$$0 = -\beta + \gamma((1 - \gamma)K_0)^{-\frac{1}{\gamma}} + (1 - \gamma)r + \frac{1}{2}(1 - \gamma)\frac{(\mu - r)^2}{\gamma\sigma^2} + \lambda \left(\frac{(1 - \gamma)G}{(1 - \gamma)K_0} - 1 \right) \tag{18}$$

and K_∞ is given by

$$K_\infty = \frac{1}{1-\gamma} \left[\frac{1}{\gamma} \left(\beta + \lambda + (\gamma - 1)r + \frac{1}{2}(\gamma - 1)\gamma \left(\frac{\mu - r}{\gamma\sigma} \right)^2 \right) \right]^{-\gamma}. \tag{19}$$

Then the solution satisfies

$$
\begin{aligned}
\lim_{x \to 0} g(x) &= K_0 \\
\lim_{x \to \infty} g(x) &= K_\infty \\
\lim_{x \to 0} g^{(n)}(x)x^n &= 0 \\
\lim_{x \to \infty} g^{(n)}(x)x^n &= 0
\end{aligned}
\tag{20}
$$

for $n = \{1, 2\}$.

The homogeneity of the value function implies that when a trading opportunity arrives, the investor rebalances her portfolio so that the fraction of illiquid to liquid wealth equals x^*. At x^*, $F(W, x^*W) = F^*(W, x^*W)$ and $F_W(W, x^*W) = F_X(W, x^*W)$ by equation (13). These two conditions lead to the value matching and smooth pasting optimality conditions in equations (16) and (17), respectively, which jointly determine x^* and G. We solve the investor's value function numerically, which we detail in the Appendix.

An important comment is that the $g(x)$ function is bounded from above by $K_\infty < 0$, rather than by zero as in the Merton case. This implies that even as $X \to \infty$ the investor's utility is strictly below the Merton benchmark – the investor cannot achieve bliss with even an unboundedly large endowment of illiquid wealth. This is intuitive because illiquid wealth cannot be used immediately: an investor can access only liquid wealth for consumption during non-trading intervals. In contrast to illiquid wealth, liquid wealth can be used to achieve bliss, defined as $\lim_{W \to \infty} F = 0$.

The inability to shift wealth from liquid to illiquid assets also plays an important role in the relative value an investor places on liquid and illiquid wealth. In our setup, liquid and illiquid wealth are not perfect substitutes, particularly when illiquid holdings are large:

Proposition 3

$$\lim_{X \to \infty} \frac{F_X X}{F_W W} = 0 \tag{21}$$

and

$$\lim_{X \to \infty} \frac{F_{WX} X}{F_{WW} W} = 0. \tag{22}$$

In contrast, in the standard Merton problem in which the investor can freely rebalance, $F = K_2(W + X)^{1-\gamma}$, and so both limits are infinite. Intuitively, the first equation (21) means that the relative shadow value of illiquid to liquid wealth $(\frac{F_X}{F_W})$ is near zero. The second equation (22) characterizes non-substitutability of risk preferences: the investor cannot use risks taken with illiquid wealth to offset risks taken with liquid wealth, even if those risks are correlated. We show below this has important effects on optimal policies; in particular, the liquid asset portfolio policy, θ, reverts to the myopic value for large endowments of the illiquid asset, even when the liquid and illiquid assets are correlated.

3.2 Discussion and Intuition

Propositions 2 and 3 prove that illiquidity can have a substantial effect on the investor's optimal investment and consumption decisions. The fact that the investor is only allowed to trade infrequently leads to a separation of her decision problem into two parts: what to do before she can trade and what to do after she can trade the illiquid asset.

To gain some intuition on Propositions 2 and 3, consider an approximation to the investor's objective function. Using the continuation value at the first rebalancing time $F^*(W_\tau, X_\tau)$, we decompose the investors value function into the utility she derives from consuming until the first rebalancing date and her continuation value thereafter:

$$F(X_t, W_t) = \mathrm{E}_t \left[\int_t^\tau e^{-\beta(s-t)} U(C_s) ds + e^{-\beta(\tau-t)} F^*(W_\tau, X_\tau) \right]. \tag{23}$$

We can approximate the value function as

$$F(X_t, W_t) \approx \tilde{F}(X_t, W_t) \equiv K_\infty W_t^{1-\gamma} + (K_0 - K_\infty)(W_t + X_t)^{1-\gamma}. \tag{24}$$

This approximation is exact for $X = 0$ and $X = \infty$ and reasonably accurate for intermediate values using our parameters.[5]

The first component in the approximation (24) corresponds to the part of the value function capturing the utility of consumption until the next trading day, $E_t \left[\int_t^\tau e^{-r(s-t)} U(C_s) ds \right]$. This depends only on liquid wealth, W, because the investor can only instantaneously con-

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[5]This approximation generates an approximation error, defined as

$$\int_0^\infty \left[F(X_t, W_t) - \tilde{F}(X_t, W_t) \right]^2 / F(X_t, W_t)^2 \mu(x) dx,$$

where $\mu(x)$ is the invariant distribution of $x = X/W$, of less that 1%. While this is a good approximation for the level of the value function, it cannot necessarily be used to generate good approximations of the optimal policies.

sume out of her liquid holdings. The second term in equation (24) corresponds to the investor's continuation value immediately after the next trading time. At that instant, the investor can freely convert her illiquid holdings into liquid assets and vice versa. At this point, the continuation value is $F^*(X, W) = G(X + W)^{1-\gamma}$. This second component is very close to the current expectation of the continuation value.

Thus, to an approximation, illiquid wealth affects the level of the value function only through the continuation value F^* at the trading time $t = \tau$. This explains the non-substitution results in Proposition 3: illiquid wealth can only be used to fund consumption after τ, but liquid wealth is used for consumption both before and after τ. When the illiquid endowment is large, this non-substitutability is particularly acute because variation in liquid wealth becomes unimportant for long-run consumption. When $X \gg W$, the continuation value after rebalancing comes almost entirely from the value of illiquid wealth and so

$$F(X, W) \approx K_\infty W^{1-\gamma} + (K_0 - K_\infty)X^{1-\gamma}.$$

Then, the value function completely separates, with liquid wealth being used to fund immediate consumption and illiquid wealth being used to fund future consumption. Since consumption preferences are time separable, so is the value function. As a consequence, when X is large, the hedging demand disappears and the correlation between the liquid and illiquid asset returns does not matter for portfolios.

The approximation (24) also makes clear why the agent cannot achieve bliss through an increasing allocation of the illiquid asset:

$$\lim_{X \to \infty} F(X, W) < 0 = \lim_{W \to \infty} F(X, W).$$

The first term in equation (24) bounds the value function away from zero for large values of X: the illiquid asset cannot be used to fund immediate consumption and illiquid wealth is inaccessible until after the first trading time. In contrast, the value function is not bounded away from zero for large values of W because liquid wealth can be used for consumption today.

Finally, the approximation demonstrates how the illiquid asset creates additional high-marginal-utility states. In contrast to the standard Merton model, the investor's marginal value of wealth is high in two types of states: states where total wealth is low and states where liquid wealth is low. If the investor has high total wealth but low liquid wealth, she cannot fund immediate consumption, leading to high marginal utility. As we now show, this induces additional curvature in the value function – effective risk aversion – and frequent under-investment relative to the Merton benchmark.

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3.3 Effective Risk Aversion

Even though the utility coefficient of risk aversion is constant, the presence of the illiquid asset endogenously induces additional curvature in the value function with respect to W and X. We are interested in three different curvatures. First, the agent's curvature with respect to liquid wealth, $\frac{F_{WW}W}{F_W}$, which describes her willingness to accept gambles over W. Second, the agent's curvature with respect to illiquid wealth, $\frac{F_{XX}X}{F_X}$, which describes her willingness to accept gambles over X. Third, the agent's joint curvature, $\frac{F_{WW}W}{F_W} + \frac{F_{XX}X}{F_X}$, which describes her willingness to accept a gamble that affects both liquid and illiquid wealth.

Figure 1 graphs these three measures of risk aversion for the $\gamma = 6$ case. The utility coefficient of risk aversion, γ, is represented by the horizontal gray line. We plot the curvature with respect to illiquid wealth, X, as a dotted line, which increases from zero when illiquid wealth is zero to six when illiquid wealth comprises all wealth. The curvature with respect to liquid wealth, W, is shown in the dashed line. This starts at six when $X = 0$, decreases as illiquid wealth constitutes a greater fraction of total wealth, and then converges to six again when illiquid wealth dominates in the portfolio. The black solid line plots the total curvature of the value function with respect to X and W. This is total effective risk aversion, which increases from six when $X = 0$ and ends at 12 when illiquid wealth constitutes all wealth.

For the Merton two-asset problem in which rebalancing is continuous, the value function is proportional to $(W + X)^{1-\gamma}$ and so the three curvatures respectively equal $\gamma \frac{X}{W+X}$, $\gamma \frac{W}{W+X}$, and γ. For small amounts of illiquid wealth, the curvatures are the same as the Merton curvature. Figure 1 shows that illiquidity induces effective risk aversion to be different from the utility coefficient of risk aversion. Effective risk aversion also changes with the amount of illiquid assets held in the portfolio.

The endogenous risk aversion is driven by the presence of an additional "default" state where all future consumption is zero: if either type of wealth becomes negative, the investor faces a positive probability of zero consumption. If liquid wealth is negative, the investor cannot fund immediate consumption, while if illiquid wealth is negative, the inability to rebalance implies a positive probability that total wealth will become negative. As the investor must now be concerned with multiple types of default, she is now more averse to gambles.[6]

To gain further intuition for the endogenous risk aversion induced by illiquidity, consider

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[6]This is similar to the endogenous risk aversion arising in Panageas and Westerfield (2009) where a risk-neutral investor chooses a risk-averse portfolio to avoid default and to maximize the possibility of future consumption. The difference in our model is that the consumption and default boundaries are exogenously specified and held fixed. In our model, however, the rebalancing policy is endogenous and its timing exogenous, whereas in Panageas and Westerfield the optimal portfolio policy is simply a constant.

again the approximate value function (24). The agent's risk aversion for gambles over illiquid wealth is the curvature of the second part of the approximate value function, $\gamma \frac{X}{W+X}$. This comes from the continuation utility at rebalancing, ranges from 0 and γ, and increases in the fraction of the agent's total wealth that is invested in illiquid assets. When the holdings of illiquid assets are small, the illiquid assets do not contribute much to continuation utility and so a small gamble over illiquid assets has a very small impact on the value function. As the agent's wealth becomes increasingly concentrated in illiquid assets, the bet over illiquid wealth becomes closer to a bet over total wealth. Since illiquid wealth funds the agent's consumption after the trading time, the investor is as risk averse over illiquid wealth as over future consumption.

In contrast, the agent's risk aversion over liquid wealth arises from two sources: liquid wealth funds immediate consumption and also affects the continuation value after the first trading time. In fact, the curvature of the agent's approximate value function is a weighted sum of these two effects:

$$\frac{\tilde{F}_{WW} W}{\tilde{F}_W} \approx \gamma \left(\frac{K_\infty W^{-\gamma}}{K_\infty W^{-\gamma} + (K_0 - K_\infty)(W_t + X_t)^{-\gamma}} \right)$$
$$+ \gamma \frac{W}{W+X} \left(\frac{(K_0 - K_\infty)(W_t + X_t)^{-\gamma}}{K_\infty W^{-\gamma} + (K_0 - K_\infty)(W_t + X_t)^{-\gamma}} \right)$$

The first term comes from the agent's risk aversion with respect to immediate consumption, which can only be funded out of liquid wealth. The curvature is equal to γ and constant. The weight put on this term represents the relative importance of marginal immediate consumption compared to marginal long-term consumption. When liquid wealth is close to zero, the marginal value of immediate consumption is very high and the weight is near one; when liquid wealth is high, the weight declines as immediate consumption is more easily funded. The second term comes from the agent's willingness to accept gambles over wealth at the next trading time, $\gamma \frac{W}{W+X}$. The intuition here is that when liquid wealth is large, gambles over liquid wealth are large as well and resemble gambles over all wealth. Thus, the agent's risk aversion increases in the value of liquid wealth. In addition, when liquid wealth is high and illiquid wealth is low, the marginal value of future consumption is high relative to the marginal value of current consumption. This causes the weight on the second term, future consumption, to increase to one.

The curvature of the agent's value function with respect to gambles that affect both X and W is simply the sum of the two individual curvatures. Total risk aversion increases with illiquid wealth, X, for two reasons. First, immediate consumption is harder to fund – the marginal value of additional immediate consumption is high – and so the agent is sensitive to gambles over W. Second, illiquid wealth represents the majority of total wealth and so

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illiquid wealth funds most of long-term consumption.

Consumption for individuals or immediate funding for institutions is thus intimately linked with illiquidity: funding immediate obligations becomes increasingly difficult when a large fraction of wealth is tied up in illiquid securities. In the standard Merton problem, the investor cares about her total wealth. With illiquid assets, the investor's utility drops to $-\infty$ if *either* total or liquid wealth falls to zero.

4   Optimal Policies

In this section we characterize the investor's optimal asset allocation and consumption policies. Even though the investment opportunity set is constant, the optimal policies vary over time and depend on the amount of illiquid assets held in the investor's portfolio.

4.1   Participation

Before characterizing the optimal allocation, we first find sufficient conditions for the investor to have a non-zero holding of the illiquid asset. An investor prefers holding a small amount of the illiquid asset to holding a zero position if $F_X(W, X = 0) \geq F_W(W, X = 0)$. Thus, a sufficient condition for participation in the illiquid asset market is

Proposition 4 $F_X(W, X = 0) \geq F_W(W, X = 0)$ *if and only if*

$$\frac{\nu - r}{\psi} \geq \rho \frac{\mu - r}{\sigma}. \tag{25}$$

These conditions for participation are identical to the Merton two-asset case and depend only on the mean-variance properties of the two securities. The degree of illiquidity, λ, does not affect the decision to invest a small amount in the illiquid asset because of the infinite horizon of the agent: a trading opportunity will eventually arrive where the illiquid asset can be converted to liquid wealth and eventual consumption. Although the conditions for participation are the same as the standard Merton case, the optimal holdings of the illiquid and liquid assets are very different, which we now discuss.

4.2   Illiquid Asset Holdings

In the presence of infrequent trading, the fraction of wealth invested in the illiquid asset can vary substantially. Figure 2 plots the stationary distribution of an investor's holding of the illiquid asset, X/W. The optimal holding of illiquid assets when rebalancing is possible is

0.37 and is shown by the vertical gray line. Because rebalancing is infrequent, the range of illiquid asset allocations is large: the 20%-80% range is 0.36 to 0.45 while the 1% to 99% range is 0.30 to 0.65. As Figure 2 shows, the holdings of illiquid assets can vary significantly in an investor's optimal portfolio and the agent can be away from optimal diversification for a long time.

Figure 2 plots the optimal holdings of the illiquid asset when the rebalancing time arrives, at 0.37. This is lower than the optimal two-asset Merton holding, which is 0.60. Not surprisingly, this is due to illiquidity. The distribution in Figure 2 is also positively skewed, with a normalized skewness of 1.9. This is because illiquid wealth grows faster on average than liquid wealth, despite the fact that both risky assets have the same mean return, $\mu = \nu = 0.12$: liquid wealth is only partially allocated to the risky asset (the rest goes to the bond) and consumption is taken out only from liquid wealth. Knowing this, the investor optimally chooses an allocation to the illiquid asset that is less than what she would end up holding on average. Thus, the optimal holding at rebalancing $\frac{x^*}{1+x^*} < E(X/(X+W))$. In this example, the mean holding is 0.41, compared to a rebalancing value of 0.37.

4.3   Liquid Asset Holdings

Figure 3 plots the agent's allocation to the liquid risky asset as a function of the illiquid asset's share of the agent's wealth for $\gamma = 6$. The solid black lines represent the optimal allocation to the liquid asset as a fraction of total wealth, $X + W$, and the dashed lines represent the optimal allocation to the liquid asset as a fraction of liquid wealth, W. The horizontal gray line corresponds to the allocation to the risky asset in the one- or two-asset Merton setup, $\frac{\mu - r}{\gamma \sigma^2}$. The risk that the investor will be unable to trade for a long period of time – illiquidity waiting risk – affects the optimal allocation to liquid assets. Optimal allocation to the liquid asset between rebalancing times depends on the investor's current illiquid holdings, X_t. If either illiquid holdings are zero ($X = 0$) or illiquid assets constitute all wealth ($X/(X+W) = 1$), then the division of liquid assets between the stock and the bond are the same as the Merton benchmark. In the intermediate cases, liquid wealth is more heavily allocated to liquid risky asset holdings than in the case of continuous rebalancing. However, as a fraction of total wealth, the investor usually under-allocates to the liquid risky asset relative to the Merton benchmark.

Figure 3 illustrates the central result of asset allocation to the liquid risky asset: relative to the Merton benchmark, the allocation is higher as a fraction of liquid wealth, but not as high as the Merton benchmark expressed as a fraction of total wealth. This means that liquid wealth is more exposed to shocks to liquid assets than in the Merton benchmark,

but total wealth is less exposed. In other words, the agent partially compensates for the presence of liquidity risk by taking less risky asset value risk, even though the illiquid and liquid risks are structurally independent. We also have a hump-shaped allocation function: if liquid wealth is held constant and illiquid wealth is increased, allocation to the liquid asset increases and then declines.[7]

Figure 3 is produced with $\rho = 0$ and shows that the liquid portfolio weight tends to the Merton benchmark as $X \to 0$ or $X \to \infty$. Surprisingly, this behavior occurs *irrespective* of the value of ρ:

Proposition 5 *The agent's optimal investment policy is such that for any ρ*

$$\lim_{X \to 0} \theta(W, X) = \lim_{X \to \infty} \theta(W, X) = \frac{\mu - r}{\gamma \sigma^2}. \tag{26}$$

As a corollary,

$$\lim_{X \to \infty} \theta(W, X) \frac{W}{W + X} = 0. \tag{27}$$

In addition, for $\rho = 0$,

$$\theta(W, X = W x^*) \leq \frac{\mu - r}{\gamma \sigma^2} (1 + x^*). \tag{28}$$

To give some intuition on these results, we write the investor's optimal allocation to the liquid risky asset as a fraction of liquid wealth, θ_t, as

$$\theta_t = \frac{\mu - r}{\sigma^2} \left(-\frac{F_W}{F_{WW} W} \right) + \rho \frac{\psi}{\sigma} \left(-\frac{F_{WX} X}{F_{WW} W} \right), \tag{29}$$

which is the first order condition to equation (15) with respect to θ. As a fraction of her total wealth, the investor allocates $\theta_t \frac{W}{W+X}$ to the liquid asset.

First consider the case where the liquid and illiquid asset returns are uncorrelated, $\rho = 0$. Allocation to the risky assets is governed both by effective risk aversion and the fraction of total wealth in the liquid asset, $-\frac{F_W}{F_{WW} W} \frac{W}{W+X}$. If W and X were interchangeable as in the two-asset Merton problem, this term would simply equal γ, and the dashed line in Figure 3 would increase monotonically. However, in our case, this term goes to zero as X increases – the effective risk aversion goes to γ while the liquid fraction goes to zero. Note that this is due entirely to illiquidity risk because there are no conventional hedging motives as the

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[7]The appendix shows that allocation differs from the Merton benchmark even in the case where $\gamma = 1$ and assets are uncorrelated.

assets are uncorrelated.

In the case where the liquid and illiquid asset are correlated, $\rho \neq 0$, there is an additional element that influences the demand for the liquid asset, namely the desire to hedge changes in the value of the illiquid asset. The strength of this motive depends on the strength of the correlation, ρ, and how much the investor perceives liquid and illiquid wealth to be substitutes, $(F_{WX}X/F_{WW}W)$.[8] When X is large, Proposition 3 shows that the liquid and illiquid assets are not substitutes and so this hedging demand is near zero, even when the liquid and illiquid assets are correlated. When illiquid holdings are zero, there is nothing to hedge, and so the agent exhibits standard behavior. For intermediate values of illiquid holdings, the investor understands that the two risky assets are correlated, but only partially substitutable, and so she uses the liquid asset to smooth some of the risk in her illiquid position.

4.4   Consumption

Figure 4 plots the agent's optimal consumption choice as a function of the illiquid asset's share of the agent's wealth. For comparison, we also plot the one- and two- asset Merton consumption levels, which are shown by the horizontal gray lines. Consumption is fairly flat over a wide range of illiquid asset shares, but declines to zero when the illiquid asset share becomes close to one.

The consumption policy in Figure 4 is formalized by the following proposition:

Proposition 6 *The optimal consumption policy is such that*

$$
\begin{aligned}
\lim_{X \to 0} c(W, X) &= ((1 - \gamma)K_0)^{-\frac{1}{\gamma}} \\
\lim_{X \to \infty} c(W, X) &= ((1 - \gamma)K_\infty)^{-\frac{1}{\gamma}}
\end{aligned}
\tag{30}
$$

and

$$
c(W, X = Wx^*) \geq (1 + x^*)((1 - \gamma)K_0)^{-\frac{1}{\gamma}}.
\tag{31}
$$

As a corollary, $\lim_{X \to \infty} c(W, X)\frac{W}{W+X} = 0.$

Consumption is lower than the two-asset Merton case because the second asset is illiquid. In the Merton problem, consumption is a constant fraction of wealth; with illiquid assets, consumption depends on the fraction of illiquid assets. Although illiquid wealth cannot be immediately consumed, an investor with more illiquid wealth is still richer and can consume

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[8]If both assets were perfectly liquid then X and W are perfect substitutes and $F_{WX}/F_{WW} = -1$.

more after rebalancing. To smooth overall consumption, the agent increases consumption as a fraction of liquid wealth today. Equivalently, her marginal valuation of liquid wealth falls as illiquid wealth X increases, $F_{WX} < 0$. At the same time, illiquid wealth is not a perfect substitute for liquid wealth because of infrequent rebalancing, so the marginal valuation of liquid wealth does not fall with X as fast as perfect substitution would imply, $\frac{F_{WX}}{F_{WW}} > -1$. Hence, while consumption as a fraction of liquid wealth rises with X, consumption as a fraction of total wealth tends to fall when X rises.

5 Illiquidity and Correlation

In this section, we further investigate the effects of two key characteristics of the illiquid asset: the frequency of trading, λ, and how it covaries with the liquid asset, ρ.

5.1 Illiquidity

Figure 5 plots optimal allocations at the rebalancing time as a function of $1/\lambda$. The dashed line represents the allocation to the liquid risky asset, $\frac{\theta(x^*)}{1+x^*}$, and the solid line represents the allocation to illiquid risky asset, $\frac{x^*}{1+x^*}$. For comparison, the Merton benchmark is 0.6 for both assets. Not surprisingly, as the expected waiting time between trades increases, the holdings of the illiquid risky asset fall. The effect is large: the holdings of the illiquid asset fall from 0.37 for an average rebalancing interval of 1 year to 0.05 for an average rebalancing interval of 10 years. In addition, changes in illiquidity have the largest effect on allocations when existing liquidity is high (when the expected waiting time to rebalance is small).

Figure 5 confirms the theoretical results in Section 2 and the optimal policies in Section 4 that the optimal liquid holding is, on average, below the Merton benchmark. But, it appears that the optimal holding of the liquid asset is relatively unaffected by the frequency of rebalancing. However, Figure 5 plots optimal allocations only at the rebalancing point. In Figure 6, we characterize liquid asset positions during non-trading periods.

Figure 6 plots the optimal allocation to the liquid asset as a function of the liquid to illiquid composition of total wealth in the black lines. The plot displays three levels of λ: 1/4, 1, and 4, shown in the solid line, the dashed line, and the dotted line, respectively moving from relatively illiquid to more liquid. The vertical gray lines with the same line type correspond to the optimal allocation to the liquid asset at the time of rebalancing. Note that these are close to the Merton level of 0.6 even though the average holding of liquid assets is shifting downwards as illiquidity increases.

There are two competing effects in Figure 6. First, as λ declines, the entire curve of

holdings in the liquid asset shifts down (moving from the black dotted line to the dashed line to the solid line). This is because illiquidity reduces the willingness of the investor to hold any risky asset, even liquid ones. Thus, the investor shifts out of the liquid risky asset and into the riskless asset for any given wealth composition. Second, as illiquidity increases the investor optimally rebalances so that less of her wealth is illiquid. This compositional shift towards liquid assets balances the effect of illiquidity on the liquid risky assets. The net result is that illiquidity causes the agent to shift from the illiquid risky asset to the liquid riskless (as opposed to the risky) asset. This results in the fairly flat allocation to liquid asset at the trading time shown in Figure 5.

As illiquidity increases, the investor's wealth can deviate more from its optimal level for increasingly long periods. We illustrate this effect in Table 2, which reports the proportion of illiquid assets at the rebalancing point and moments of the stationary distribution of the illiquid composition of total wealth, $\frac{X_t}{X_t + W_t}$. Not surprisingly, the optimal holdings of illiquid assets at the trading time decrease as illiquidity increases: being able to rebalance once every ten years, on average, has an optimal holding of illiquid assets at the rebalancing point of 0.05 compared to 0.50 for an asset with an average turnover of four times a year. The dispersion of the composition of illiquid wealth is also high and the dispersion increases as illiquidity increases; much of the mass of the stationary distribution is far from optimal diversification.

Table 2 shows that the stationary distribution of illiquid wealth is highly right-skewed, with normalized skewness ranging from 1.23 to 2.66. This skewness is mainly due to the amount allocated to the liquid risky asset being, over most of its range, a declining function of the fraction of illiquid wealth. Thus the mean growth rate and volatility of liquid wealth both decline as a function of the fraction of wealth that is illiquid. Note that when she rebalances, the agent chooses an allocation to the illiquid asset that is lower, often much lower, than the allocation she expects on average due to this skewness. For example, for $\lambda = 0.1$, the optimal rebalance value is 0.05 compared to a mean holding of illiquid assets of 0.17.

How much would an investor pay to make the illiquid asset fully tradeable? We answer this question in Table 3. We report the fraction of wealth the investor is willing to give up in order to be able to continuously rebalance the illiquid asset ("Certainty Equivalent Wealth"). For the case of $\rho = 0$ and the investor is able to rebalance once a year, she is willing to give up 5.7% of her wealth. For $\lambda = 0.1$ and rebalancing once every ten years, on average, the investor is willing to give up a staggering 28.7% of her wealth in order to continuously rebalance the illiquid asset.

We also report the premium the illiquid asset must command in order for the investor to

have the same utility as holding two fully liquid assets ("Liquidity Premium"). For example, for $\rho = 0$ and $\lambda = 1$, an investor holding two liquid assets with drift 0.120 has the same utility as an investor holding a liquid asset with drift $\mu = 0.120$ and an illiquid asset with drift $\nu = 0.111$. We define the difference $0.120 - 0.111 = 0.009$ as the liquidity premium: it is the premium the investor requires to hold the illiquid asset if a fully liquid asset with the same volatility and correlation characteristics is available. Given that the equity premium in these calibrations is 0.08, these premiums are economically significant. In particular, the liquidity premium for rebalancing once per ten years, on average, is an extremely large 0.06.

In the bottom panel of Table 3, we repeat the calculations for $\rho = 0.6$, which is approximately the correlation of a portfolio of alternative assets with the S&P500 (see Table 1). For this level of correlation, the certainty equivalent wealth and the liquidity premiums are both lower than the $\rho = 0$ case. This is because increasing the correlation between risky assets, while holding expected returns constant, lowers the maximum Sharpe ratio. As a result, an investor allocates a lower fraction of wealth to both assets, even in the fully liquid economy. Thus, the illiquid asset is less valuable to hold because it offers fewer diversification benefits and there is less value in making the illiquid asset liquid. For the $\rho = 0.6$ and $\lambda = 1$ case, an investor is willing to give up 1.1% of wealth to make the illiquid asset liquid, or equivalently demand a liquidity premium of 0.2%. For $\rho = 0.6$ and being able to rebalance once every ten years, on average, the certainty equivalent wealth is a large 12.4% and the liquidity premium is 2.2%.

5.2 Correlation

In order to illustrate the effects of correlation, we break the symmetry between the two assets, setting $\nu = 0.2 > \mu = 0.12$. This implies the Sharpe ratio for the illiquid asset is twice the Sharpe ratio for the liquid asset so the investor has a large incentive to create a leveraged portfolio which goes long the illiquid asset and (partially) hedges the risk with a reduced position in the liquid asset.

Figure 7 graphs the liquid and illiquid holdings as a function of ρ at the Poisson trading time. Figure 7 shows that as the correlation increases, the investor chooses a larger position in the illiquid asset, which has the higher expected return. However, increasing ρ from zero to one results in an increase in the optimal illiquid holding from 0.36 to 0.42. The corresponding liquid asset weight drops from 0.57 to 0.18. Both of these changes are small: in the Merton two-stock problem, the allocation to the illiquid asset would move from 1.19 to ∞ while the weight in the liquid asset would go from 1.19 to $-\infty$. In addition, the fact that the liquid risky asset weight drops more quickly than the illiquid asset weight implies

that the agent is shifting wealth towards the riskless asset.

The small effect of correlation on optimal holdings and the reluctance of the investor to exploit the very large differences in Sharpe ratios by employing leverage are direct consequences of illiquidity. Consider the extreme case of perfectly correlated liquid and illiquid asset returns, $\rho = 1$. In the standard Merton setup, assets with perfect correlation represent an arbitrage opportunity. There is no arbitrage with illiquidity because the two assets are not close substitutes.

Suppose we construct a portfolio that requires zero initial outlay and has no risk in terms of total wealth. This portfolio requires taking an extreme position in the liquid risky asset. There is a positive probability that the level of liquid wealth drops to zero for a finite time, resulting in the utility of consumption being equal to negative infinity. This is not offset by higher consumption in the future since utility is bounded above.

The portfolio that requires zero initial outlay and has no risk in terms of total wealth starts by borrowing an amount $\hat{B}_t$ in the riskless asset (bond) and then allocating it across the two risky assets and the bond. We invest an amount X_t in the illiquid asset and a fraction $\theta_t = -\frac{\psi}{\sigma}\frac{X_t}{\hat{B}_t - X_t}$ of liquid wealth $(\hat{B}_t - X_t)$ in the liquid risky asset. The total value of this portfolio evolves as

$$\frac{d\hat{B}_t}{\hat{B}_t} - rdt = \frac{dX_t}{\hat{B}_t} + \frac{d(\hat{B}_t - X_t)}{\hat{B}_t} - rdt = \left(\nu - r - \psi\frac{\mu - r}{\sigma}\right)\frac{X_t}{\hat{B}_t}dt \qquad (32)$$

As long as $\frac{\nu - r}{\psi} > \frac{\mu - r}{\sigma}$, this portfolio costs zero and would return a strictly positive amount at the next opportunity the investor has to rebalance. Thus, it would seem that this represents an arbitrage opportunity.

This argument is, however, incomplete. To see why, consider the evolution of the *liquid* wealth portion of the candidate arbitrage:

$$\frac{d(\hat{B}_t - X_t)}{(\hat{B}_t - X_t)} - rdt = -\frac{\psi}{\sigma}(\mu - r)\frac{X_t}{(\hat{B}_t - X_t)}dt - \psi\frac{X_t}{(\hat{B}_t - X_t)}dZ_t^1. \qquad (33)$$

Over any finite interval, liquid wealth can become unboundedly negative. Since the investor consumes out of her liquid wealth and cannot rebalance immediately, utility has a positive probability of achieving negative infinity.[9] Thus, the presence of illiquidity greatly constrains leveraged positions which exploit highly correlated asset positions compared to a fully tradable setup.

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[9]Alternatively, the agent's portfolio policy θ can become arbitrarily large between rebalancing times, violating standard integrability requirements.

6   Conclusion

We solve the optimal asset allocation and consumption problem faced by a CRRA investor who has access to two risky securities, liquid and illiquid, and a risk-free asset. We find that illiquidity, modeled as the ability to trade only at randomly occurring discrete points in time, has large effects on policies. Illiquidity induces a wedge between the marginal values of liquid and illiquid wealth. The investor is worried that her liquid wealth, and not just her total wealth, may drop to zero. She therefore chooses her optimal allocation to liquid assets to minimize this possibility and implies that an investor's total effective risk aversion varies over time and is greater than the constant risk aversion coefficient of utility. Thus, illiquidity affects both liquid and illiquid optimal holdings, and the optimal asset allocation and consumption policies also depend on the proportion of illiquid and liquid wealth in an investor's portfolio.

Our model could be extended along a number of dimensions. First, we could allow for endogenous trading times by allowing the investor to pay a cost in order to increase the frequency of trading in the illiquid security. If this cost is paid out of illiquid wealth, this is very similar to price impact or an implicit transaction cost. If the cost is paid in terms of liquid wealth, similar to a direct search cost, the effect on overall utility is ambiguous: in the current model the trader is able to trade, albeit infrequently, for free. This will lead to a cap on the shadow cost of the inability to trade as some liquidity could be generated when it is most valuable, which will moderate the effect of illiquidity. Another extension is to let the frequency of trading be correlated with asset returns. If the interval of non-trading increases when asset prices fall, there will be a reduction in the optimal holdings of illiquid securities relative to our model.

Appendix

A Proofs for $\gamma \neq 1$

A.1 Proof of Proposition 2

We begin with the agent's problem for $X = 0$. If $X = 0$, the agent's problem is to maximize

$$\max_{\{\theta,\, c\}} \mathrm{E} \left[\int_0^\tau e^{-\beta t} \frac{1}{1-\gamma} C_t^{1-\gamma} dt + e^{-\beta \tau} F^*(W_\tau, X_\tau) \right] \tag{A-1}$$

when the next trading day arrives at random time τ and subject to (8). The Hamilton-Jacobi-Bellman (HJB) equation is

$$0 = \max_{c,\theta} \left[-\beta F + \frac{1}{1-\gamma}(cW)^{1-\gamma} + F_W W \left(r + (\mu - r)\theta - c \right) + \lambda \left(F^* - F \right) + \frac{1}{2} F_{WW} W^2 \theta^2 \sigma^2 \right] \tag{A-2}$$

Since $F^* = G W^{1-\gamma}$, a standard verification argument shows that the solution is $F = K_0 W^{1-\gamma}$ where K_0 is the solution to

$$0 = -\beta + \gamma((1-\gamma)K_0)^{-\frac{1}{\gamma}} + (1-\gamma)r + \frac{1}{2}(1-\gamma)\frac{(\mu-r)^2}{\gamma \sigma^2} + \lambda \left(\frac{(1-\gamma)G}{(1-\gamma)K_0} - 1 \right) \tag{A-3}$$

The general HJB equation for F between rebalancing times is

$$
\begin{aligned}
0 = \ & \max_{c,\theta} \left[-\beta F + \frac{1}{1-\gamma}(cW)^{1-\gamma} + F_W W \left(r + (\mu - r)\theta - c \right) + F_X X \nu \right. \\
& \left. + \lambda \left(F^* - F \right) + \frac{1}{2} F_{WW} W^2 \theta^2 \sigma^2 + \frac{1}{2} F_{XX} X^2 \psi^2 + F_{WX} W X \psi \sigma \rho \theta \right]
\end{aligned} \tag{A-4}
$$

Substituting (12) and setting $x = \frac{X}{W}$, we obtain

$$
\begin{aligned}
0 = \ & \max_{c,\theta} \left[\frac{1}{1-\gamma} c^{1-\gamma} + \lambda G(1+x)^{1-\gamma} \right. \\
& + g(x) \left(-\beta - \lambda + (1-\gamma)(r + (\mu - r)\theta - c) - \frac{1}{2}\gamma(1-\gamma)\sigma^2\theta^2 \right) \\
& + g'(x)x \left(\nu - (r + (\mu - r)\theta - c) - \gamma \psi \theta \rho \sigma + \gamma \theta^2 \sigma^2 \right) \\
& \left. + g''(x)x^2 \left(\frac{1}{2}\theta^2\sigma^2 + \frac{1}{2}\psi^2 - \psi \theta \rho \sigma \right) \right]
\end{aligned} \tag{A-5}
$$

Next, we characterize what happens when the agent can rebalance. From (13) and (12), the agent chooses dI so that $\frac{X_\tau}{W_\tau} = x^*$, where x^* is a constant. The smooth pasting condition (first order condition with respect to dI) and the value matching condition ($F^*(x^*) = F(x^*)$)

are

$$g(x^*) = G(1 + x^*)^{1-\gamma}$$
$$g'(x^*) = G(1 - \gamma)(1 + x^*)^{-\gamma}$$

Next, we observe that F is continuous at $X = 0$. Then,

$$\lim_{X \to 0} \frac{\partial^n F(W, X)}{\partial W^n} = K_0 \frac{\partial^n [W^{1-\gamma}]}{\partial W^n} \tag{A-6}$$

The left hand side is the limit of the derivatives with respect to W as X approaches zero, while the right hand side is the value the limit takes. Substituting in (12) yields

$$\lim_{X \to 0} \frac{\partial^n \left[W^{1-\gamma} g\left(\frac{X}{W}\right)\right]}{\partial W^n} = K_0 \frac{\partial^n [W^{1-\gamma}]}{\partial W^n} \tag{A-7}$$

The derivatives can then be checked iteratively to show

$$\lim_{x \to 0} g^{(n)}(x) x^n = 0 \tag{A-8}$$

Next, we take limits as $x \to \infty$, remembering the bounds given in (11). First set $z = \ln(x)$ and define h so that $h(z) = g(x)$. Then, since g is bounded from above (by zero), so is h. Taking derivatives with respect to x, we have $h'(z)\frac{1}{x} = g'(x)$. Since $g'(x) > 0$ and $x > 0$, we have $h'(z) > 0$. Since h is increasing, bounded from above, and continuously differentiable, we have $\lim_{z \to \infty} h'(z) = \lim_{x \to \infty} g'(x)x = 0$. Second, take the second derivative with respect to x to obtain $h''(z) + h'(z)(-1) = g''(x)x^2$. Then, since $\lim_{z \to \infty} h'(z) = 0$ and $h'(z)$ is bounded from below, decreasing, and continuously differentiable, we also have $\lim_{z \to \infty} h''(z) = 0$. Then, $\lim_{x \to \infty} g''(x)x^2 = 0$. For the value of K_∞, plug the first order conditions for c and θ back into (A-5) and apply the limit conditions.

Given the bounds on the value function (11), a standard verification argument can be used to show that these equations characterize the solution.

A.2 Proof of Proposition 3

Using (12), we have that

$$F_{XW}(W, X) = -W^{-\gamma-1}\left(\gamma g'\left(\frac{X}{W}\right) + g''\left(\frac{X}{W}\right)\frac{X}{W}\right)$$

$$F_{WW}(W, X) = W^{-\gamma-1}\left(\gamma(\gamma-1)g\left(\frac{X}{W}\right) + 2\gamma g'\left(\frac{X}{W}\right)\frac{X}{W} + g''\left(\frac{X}{W}\right)\frac{X^2}{W^2}\right)$$

$$F_X(W, X) = W^{-\gamma} g'\left(\frac{X}{W}\right)$$

$$F_W(W, X) = -W^{-\gamma}\left(g'\left(\frac{X}{W}\right)\frac{X}{W} + (\gamma-1)g\left(\frac{X}{W}\right)\right)$$

From (20), we have $\lim_{X \to \infty} g^{(n)}(\frac{X}{W})(\frac{X}{W})^n = 0$ for $n \in \{1, 2\}$. Together, these imply the limits given in the statement of the proposition.

A.3   Proof of Proposition 4

We begin by showing that $\frac{\nu - r}{\psi} - \rho\frac{\mu - r}{\sigma} \leq 0$ implies $F_X(W, X = 0) \leq F_W(W, X = 0)$.

Assume that we have $\{W_0, X_0 = \epsilon\}$, which gives rise to an optimal portfolio policy in number of shares equal to π_t along paths for $t \in [0, \tau]$, where τ is the next trading time. $\{W_0, X_0 = \epsilon\}$ also gives rise to a consumption policy C_t along those same paths. Then, total discounted wealth at the next trading time equals

$$
\begin{aligned}
e^{-r\tau}(W_\tau + X_\tau) &= W_0 + \epsilon + \int_0^\tau e^{-rt}\left[\pi_t(\mu - r)S_t + (\nu - r)X_t - C_t\right] dt \\
&\quad + \int_0^\tau e^{-rt}\left[\pi_t\sigma S_t + \psi\rho X_t\right] dZ_t^1 + \int_0^\tau e^{-rt}\left[\psi\sqrt{1 - \rho^2}X_t\right] dZ_t^2
\end{aligned}
$$

Now consider the starting point $\{\hat{W}_0 = W_0 + \epsilon, \hat{X}_0 = 0\}$ and use the previous consumption policy state-by-state (feasible because consumption is out of liquid wealth). The portfolio policy is now $\hat{\pi}_t = \pi_t + \frac{\psi\rho X_t}{\sigma S_t}$. Then,

$$
\begin{aligned}
e^{-r\tau}\left(\hat{W}_\tau + \hat{X}_\tau\right) &= W_0 + \epsilon + \int_0^\tau e^{-rt}\left[\pi_t(\mu - r)S_t + \frac{\psi\rho X_t}{\sigma}(\mu - r) - C_t\right] dt \\
&\quad + \int_0^\tau e^{-rt}\left[\pi_t\sigma S_t + \psi\rho X_t\right] dZ_t^1.
\end{aligned}
$$

The drift in the second ($\{\hat{W}_0 = W_0 + \epsilon, \hat{X}_0 = 0\}$) minus the drift in the first ($\{W_0, X_0 = \epsilon\}$) equals

$$
\int_0^\tau e^{-rt}\left[\psi\rho X_t\frac{\mu - r}{\sigma} - (\nu - r)X_t\right] dt,
$$

which is positive if $\psi\rho\frac{\mu - r}{\sigma} - (\nu - r) \geq 0$. Thus, the second initial condition produces higher expected wealth and lower volatility, path by path, with a possibly sub-optimal portfolio and consumption strategy. Since the value function at rebalancing (F^*) is concave, this proves that $\rho\frac{\mu - r}{\sigma} - \frac{\nu - r}{\psi} \geq 0$ implies $F(W_0 + \epsilon, 0) \geq F(W_0, \epsilon)$.

Next we will show that $\frac{\nu - r}{\psi} - \rho\frac{\mu - r}{\sigma} \geq 0$ implies $F_X(W, X = 0) \geq F_W(W, X = 0)$. Consider a deviation in which a trader starting with $\{W_0, 0\}$ is able to move an amount ϵ into X, and then withdraws it at the next trading day. This results in higher utility if

$$
0 \leq -F_W(W_0, 0)\epsilon + \mathrm{E}\left[e^{-\beta\tau}F_W(W_\tau, 0)\epsilon\frac{X_\tau}{X_0}\right],
$$

with W_t following the optimal portfolio and consumption policies (as a function of W_t) for

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$X_t = 0$. Plugging in the value function at $X = 0$, we obtain

$$1 \leq \mathrm{E} \left[e^{-\beta\tau} \left(\frac{W_\tau}{W_0} \right)^{-\gamma} \frac{X_\tau}{X_0} \right].$$

Direct calculation yields the result.

A.4 Proof of Propositions 5 and 6

First, $X = 0$. Continuing from the proof of Proposition 2, we substitute $F = K_0 W^{1-\gamma}$ into the HJB (A-2) and obtain

$$0 = \max_{c,\theta} \left[-\beta K_0 + \frac{1}{1-\gamma} c^{1-\gamma} + (1-\gamma)K_0 \left(r + (\mu - r)\theta - c \right) \right.$$
$$\left. + \lambda \left(G - K \right) - \frac{1}{2}(1-\gamma)\gamma K_0 \theta^2 \sigma^2 \right],$$

and so $\theta = \frac{\mu-r}{\gamma\sigma^2}$ and $c^{-\gamma} = (1-\gamma)K_0$. Since the jump is positive, we have $G \geq K_0$.

Next, taking the first order conditions in (A-5) (using $\rho = 0$ for θ), we obtain

$$c(x) = \left(-g'(x)x + (1-\gamma)g(x) \right)^{-\frac{1}{\gamma}}$$
$$\theta(x) = \frac{\mu - r}{\sigma^2} \frac{-g'(x)x + (1-\gamma)g(x)}{\gamma(1-\gamma)g(x) - 2g'(x)x\gamma - g''(x)x^2}.$$

At x^*, optimality in (16) implies $g(x^*) = G(1+x^*)^{1-\gamma}$ and $g'(x^*) = G(1-\gamma)(1+x^*)^{-\gamma}$. Optimality (concavity at x^*) also implies that $g''(x^*) \leq G(1-\gamma)(-\gamma)(1+x^*)^{-\gamma-1}$. Define $\xi \geq 1$ so that $g''(x^*) = \xi G(1-\gamma)(-\gamma)(1+x^*)^{-\gamma-1}$. Then,

$$c(x^*) = (1+x^*) \left(G(1-\gamma) \right)^{-\frac{1}{\gamma}}$$
$$\theta(x^*) = \frac{\mu - r}{\sigma^2} \frac{1+x^*}{\gamma(1 + x^{*2}(\xi - 1))}.$$

Since $0 > G \geq K_0$ implies $0 < (1-\gamma)G \leq (1-\gamma)K_0$, comparison to the results for $X = 0$ gives the result for consumption. $\xi \geq 1$ and comparison to the results for $X = 0$ gives the result for the portfolio policy.

Next, we find the values as $x \to \infty$. The result for θ can be obtained directly by taking the first order condition in (A-5), allowing $\rho \neq 0$ and then taking the limit using (20). For consumption, the first order condition implied by (A-5) yields $c(x) = \left(-g'(x)x + (1-\gamma)g(x) \right)^{-\frac{1}{\gamma}}$. Using the limit conditions (20) yields the result.

B The Log Case

The results for $\gamma = 1$ are analogous to those for $\gamma \neq 1$. Figure A-1 demonstrates that the log utility investor has a pattern of effective risk aversion that is qualitatively similar to that of

the $\gamma = 6$ investor. Figure A-2 shows that the log utility investor's allocation to the liquid risky security is also qualitatively similar to that of the $\gamma = 6$ investor.

B.1   Characterization of the Solution

We follow Section 3 and Proposition 2.

The value function is bounded above and below by the as the Merton one-stock and two-stock problems, respectively. Thus, there exist constants L_1 and L_2 such that

$$\frac{1}{\beta}\log(W) + \frac{1}{\beta}L_1 \leq F(W, X) \leq \frac{1}{\beta}\log(W + X) + \frac{1}{\beta}L_2. \tag{B-1}$$

Given that the value function is bounded by known functions, it is sufficient to find a solution to the HJB equation. Because we are using log utility and the returns processes have constant moments, the agent's value function takes the form $F(\alpha W, \alpha X) = F(W, X) + \frac{1}{\beta}\log(\alpha)$. Thus, there exists a function g with $g(x) = \beta F(1, x)$ so that

$$F(W, X) = \frac{1}{\beta}\log(W) + \frac{1}{\beta}g\left(\frac{X}{W}\right). \tag{B-2}$$

From (B-1), we obtain that g is bounded: $L_1 \leq g(x) \leq \log(1 + x) + L_2$.

When the Poisson process hits and the agent can rebalance her portfolio, her value function may make a discrete jump. Denote the new, higher, value function (before rebalancing occurs) as F^*, so that the total amount of the jump is $F^* - F$. Then, we have

$$F^*(W_t, X_t) = \max_{I \in [-X_t, W_t]} F(W_t - I, X_t + I). \tag{B-3}$$

Since F^* must also be such that $F^*(\alpha W, \alpha X) = F^*(W, X) + \frac{1}{\beta}\log(\alpha)$, there exists a function g^* such that $F^* = \frac{1}{\beta}\log(W) + \frac{1}{\beta}g^*\left(\frac{X}{W}\right)$. Since rebalancing is free when available, we must also have $\frac{1}{\beta}\log(W + \delta) + \frac{1}{\beta}g^*\left(\frac{X+\delta}{W-\delta}\right) = \frac{1}{\beta}\log(W) + \frac{1}{\beta}g^*\left(\frac{X}{W}\right)$ for any $-X < \delta < W$. Differentiating both sides with respect to δ and setting $\delta = 0$ yields $g^{*\prime}(x)(1 + x) = 1$. Integrating yields

$$F^*(W_t, X_t) = \frac{1}{\beta}\log(W + X) + \frac{1}{\beta}G, \tag{B-4}$$

where G is a constant.

Define the constant L_0 so that

$$L_0 = \frac{1}{\beta + \lambda}\left(\beta\log(\beta) + r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 - \beta + \lambda G\right).$$

Then, the agent's value function is characterized by a function $g(x)$ and constants x^* and G

such that

$$
\begin{aligned}
0 \;=\; \max_{c,\,\theta} \Bigg[&\beta \log(c) + \left(r + (\mu - r)\theta - c - \frac{1}{2}\theta^2 \sigma^2 \right) + \lambda \left(\log(1+x) + G \right) \\
&+ g(x)\left(-\beta - \lambda\right) + g'(x)x\left(\nu - (r + (\mu - r)\theta - c) - \psi\theta\rho\sigma + \theta^2\sigma^2\right) \\
&+ g''(x)x^2 \left(\frac{1}{2}\theta^2\sigma^2 + \frac{1}{2}\psi^2 - \psi\theta\rho\sigma \right) \Bigg].
\end{aligned}
\tag{B-5}
$$

The boundary conditions are

$$
\begin{aligned}
\lim_{x \to 0} g(x) &= L_0 \\
\lim_{x \to 0} g^{(n)}(x)x^n &= 0.
\end{aligned}
\tag{B-6}
$$

When a trading opportunity occurs at time τ, the trader changes I_τ so that $\frac{X_\tau}{W_\tau} = x^*$. The optimality conditions are

$$
\begin{aligned}
g(x^*) &= \log(1 + x^*) + G \\
g'(x^*) &= \frac{1}{1 + x^*}.
\end{aligned}
\tag{B-7}
$$

Proposition 4 follows as stated for the $\gamma > 1$ case.

The agent's optimal investment policy is such that for any ρ,

$$
\lim_{X \to 0} \theta(W, X) = \frac{\mu - r}{\sigma^2}.
$$

In addition, for $\rho = 0$,

$$
\theta(W, X = Wx^*) \leq \frac{\mu - r}{\sigma^2}(1 + x^*).
$$

The optimal consumption policy is such that

$$
\lim_{X \to 0} c(W, X) = \beta
$$

and

$$
c(W, X = Wx^*) = \beta(1 + x^*).
$$

B.2   Proof of the Characterization

First, we find the solution for $X = 0$. If $X = 0$, the agent's problem is to maximize

$$
\max_{\{\theta,\,c\}} \mathrm{E} \left[\int_0^\tau e^{-\beta t} \log(C_t)dt + e^{-\beta\tau} F^*(W_\tau, X_\tau) \right]
$$

when the next trading day arrives at random time τ and subject to (8).

The HJB equation is

$$0 = \max_{c,\theta} \left[-\beta F + \log(cW) + F_W W \left(r + (\mu - r)\theta - c \right) + \lambda \left(F^* - F \right) + \frac{1}{2} F_{WW} W^2 \theta^2 \sigma^2 \right].$$

(B-8)

Since $F^* = \frac{1}{\beta} \log(W) + G$, a standard verification argument shows that the solution is $F = \frac{1}{\beta} \log(W) + L_0$ where

$$L_0 = \frac{1}{\beta + \lambda} \left(\beta \log(\beta) + r + \frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2 - \beta + \lambda G \right).$$

Since the overall value function is bounded for any (W_0, X_0) pair, we can write down the general HJB equation for F between rebalancing times:

$$\begin{aligned}
0 = {} & \max_{c,\theta} \left[-\beta F + \log(cW) + F_W W \left(r + (\mu - r)\theta - c \right) + F_X X \nu + \lambda \left(F^* - F \right) \right. \\
& \left. + \frac{1}{2} F_{WW} W^2 \theta^2 \sigma^2 + \frac{1}{2} F_{XX} X^2 \psi^2 + F_{WX} WX \psi \sigma \rho \theta \right].
\end{aligned}$$

After substituting (B-2), and setting $x = \frac{X}{W}$, we obtain

$$\begin{aligned}
0 = {} & \max_{c,\theta} \left[\beta \log(c) + \left(r + (\mu - r)\theta - c - \frac{1}{2} \theta^2 \sigma^2 \right) + \lambda \left(\log(1 + x) + G \right) \right. \\
& + g(x) \left(-\beta - \lambda \right) \\
& + g'(x)x \left(\nu - (r + (\mu - r)\theta - c) - \psi \theta \rho \sigma + \theta^2 \sigma^2 \right) \\
& \left. + g''(x)x^2 \left(\frac{1}{2} \theta^2 \sigma^2 + \frac{1}{2} \psi^2 - \psi \theta \rho \sigma \right) \right].
\end{aligned}$$

(B-9)

Next, we characterize what happens when the agent can rebalance. From (B-2) and (B-3), the agent chooses dI so that $\frac{X}{W} = x^*$, where x^* is a constant. The smooth pasting condition (first order condition with respect to dI) and the value matching condition ($F^*(x^*) = F(x^*)$) are

$$\begin{aligned}
g(x^*) &= \log(1 + x^*) + G \\
g'(x^*) &= \frac{1}{1 + x^*}.
\end{aligned}$$

Next, observe that F is continuous at $X = 0$. Then,

$$\lim_{X \to 0} \frac{\partial^n F(W, X)}{\partial W^n} = \frac{1}{\beta} \frac{\partial^n \left[\log(W) \right]}{\partial W^n}.$$

The left hand side is the limit of the derivatives with respect to W as X approaches zero,

while the right hand side is the value the limit takes. Substituting in (B-2) yields

$$\lim_{X \to 0} \frac{\partial^n \left[\frac{1}{\beta} \log(W) + \frac{1}{\beta} g\left(\frac{X}{W}\right) \right]}{\partial W^n} = \frac{1}{\beta} \frac{\partial^n \left[\log(W) \right]}{\partial W^n}$$

and so

$$\lim_{x \to 0} g^{(n)}(x) x^n = 0.$$

A standard verification argument can be used to show that these equations characterize the solution.

Next, we show the policy results. First, $X = 0$. Continuing from the results of the previous section, we substitute in $F = \frac{1}{\beta} \log(W) + \frac{1}{\beta} L_0$ to the HJB (B-8) and obtain

$$0 = \max_{c, \theta} \left[-\beta L_0 + \log(c) + \frac{1}{\beta} \left(r + (\mu - r)\theta - c \right) + \lambda \left(G - L_0 \right) - \frac{1}{2} \theta^2 \sigma^2 \right],$$

so $\theta = \frac{\mu - r}{\sigma^2}$ and $c = \beta$.

For the values at x^*, we take the first order conditions (using $\rho = 0$ for θ) in (B-9) to obtain

$$c(x) = \beta \frac{1}{1 - g'(x)x}$$

$$\theta(x) = \frac{\mu - r}{\sigma^2} \frac{g'(x)x - 1}{g''(x)x^2 + 2g'(x)x - 1}.$$

At x^*, optimality given in equation (B-7) implies $g(x^*) = \log(1 + x^*) + G$ and $g'(x^*) = \frac{1}{1 + x^*}$. Optimality (concavity at x^*) also implies that $g''(x^*) \leq -\frac{1}{(1+x^*)^2}$. Define $\xi \geq 1$ so that $g''(x^*) = -\xi \frac{1}{(1+x^*)^2}$. Then,

$$c(x^*) = \beta(1 + x^*)$$

$$\theta(x^*) = \frac{\mu - r}{\sigma^2} \frac{1 + x^*}{(1 + x^{*2}(\xi - 1))}.$$

Comparison to $X = 0$ gives the result.

C   Numerical Methods

We solve the HJB Equation characterizing the solution using value function iteration. Our solution method is based on Kushner and Dupuis (1992). We illustrate the solution for $\gamma \neq 1$.

First, we perform a change of variables, denoting $z = \ln x$. The value function then

becomes

$$
\begin{aligned}
0 \;=\; \max_{c,\theta} \Big[& \frac{1}{1-\gamma}c^{1-\gamma} + \lambda G(1+e^z)^{1-\gamma} \\
& -g(z)\left(\beta + \lambda + (\gamma-1)(r+(\mu-r)\theta - c) - \frac{1}{2}\gamma(\gamma-1)\sigma^2\theta^2 \right) \\
& +g'(z)\left(\nu - (r+(\mu-r)\theta - c) - (\gamma-1)\psi\theta\rho\sigma + \frac{1}{2}(2\gamma-1)\theta^2\sigma^2 - \frac{1}{2}\psi^2 \right) \\
& +g''(z)\left(\frac{1}{2}\theta^2\sigma^2 + \frac{1}{2}\psi^2 - \psi\theta\rho\sigma \right) \Big].
\end{aligned}
$$

We separate the positive and negative portions of the first derivative (to ensure later that our implied probabilities are non-negative) to obtain

$$
\begin{aligned}
0 \;=\; \max_{c,\theta} \Big[& \frac{1}{1-\gamma}c^{1-\gamma} + \lambda G(1+e^z)^{1-\gamma} \\
& -g(z)\left(\beta + \lambda + (\gamma-1)(r+(\mu-r)\theta - c) - \frac{1}{2}\gamma(\gamma-1)\sigma^2\theta^2 \right) \\
& +g'_+(z)\left(\nu + c + \frac{1}{2}(2\gamma-1)\theta^2\sigma^2 \right) + g'_-(z)\left(-(r+(\mu-r)\theta) - (\gamma-1)\psi\theta\rho\sigma - \frac{1}{2}\psi^2 \right) \\
& +g''(z)\left(\frac{1}{2}\theta^2\sigma^2 + \frac{1}{2}\psi^2 - \psi\theta\rho\sigma \right) \Big].
\end{aligned}
$$

We discretize the state space, creating a grid for z and g with $h = \Delta z$. For our calibrations, we use $h = 1/100$ or finer. Then, we use the following approximations:

$$
\begin{aligned}
g'_+(z_n) &= \frac{g_{n+1} - g_n}{h} \\
g'_-(z_n) &= \frac{g_n - g_{n-1}}{h} \\
g''(z_n) &= \frac{g_{n+1} + g_{n-1} - 2\,g_n}{h^2}
\end{aligned}
$$

Plugging these approximations into the HJB equation and solving for g_n yields

$$
\begin{aligned}
g_n = \max_{c_n,\theta_n} \Big\{ & p_n^d(c_n,\theta_n)g_{n-1} + p_n^u(c_n,\theta_n)g_{n+1} \\
& + \left(\frac{1}{1-\gamma}c_n^{1-\gamma} + \lambda G(1+e^{z_n})^{1-\gamma} \right) \Delta t_n(c_n,\theta_n) \Big\}
\end{aligned}
$$

where

$$\Delta t_n(c,\theta) = \frac{h^2}{C_1\,h + C_2\,h^2 + \sigma^2\theta^2 - 2\psi\rho\theta\sigma + \psi^2}$$

$$p_n^d(c,\theta) = \frac{\left((\psi\rho\sigma(\gamma-1)+\mu-r)\theta+r+\frac{1}{2}\psi^2\right)h + \frac{1}{2}\sigma^2\theta^2 - \psi\rho\theta\sigma + \frac{1}{2}\psi^2}{h^2}\Delta t_n(c,\theta)$$

$$p_n^u(c,\theta) = \frac{(\sigma^2(\gamma-\frac{1}{2})\theta^2+\nu+c)h + \frac{1}{2}\sigma^2\theta^2 - \psi\rho\theta\sigma + \frac{1}{2}\psi^2}{h^2}\Delta t_n(c,\theta),$$

and

$$C_1 \equiv \sigma^2\left(\frac{1}{2}-\gamma\right)\theta^2 + (\psi\rho\sigma(\gamma-1)+\mu-r)\,\theta + c + \frac{1}{2}\psi^2 + r + \nu$$

$$C_2 \equiv (\gamma-1)\left(r+(\mu-r)\theta - c - \frac{1}{2}\gamma\sigma^2\,\theta^2\right) + \beta + \lambda.$$

Our numerical algorithm starts from an initial guess on the value function g^0 and a candidate $x^* = x^0$:

1. Given x^i compute the value of G^{i+1} based on

$$g^i(z_n = x^i) = G^{i+1}(1+e^{x^i})^{1-\gamma}.$$

2. Given value function iteration g^i, compute the optimal policies for step $i+1$ at each grid point n based on

$$c_n^{i+1} = \arg\max_c \left\{ p_n^d(c,\theta)\,g_{n-1}^i + p_n^u(c,\theta)\,g_{n+1}^i \right.$$

$$\left. + \left(\frac{1}{1-\gamma}c^{1-\gamma} + \lambda G^{i+1}(1+e^{z_n})^{1-\gamma}\right)\Delta t_n(c,\theta)\right\}$$

$$\theta_n^{i+1} = \arg\max_\theta \left\{ p_n^d(c,\theta)\,g_{n-1} + p_n^u(c,\theta)\,g_{n+1} \right.$$

$$\left. + \left(\frac{1}{1-\gamma}c^{1-\gamma} + \lambda G^{i+1}(1+e^{z_n})^{1-\gamma}\right)\Delta t_n(c,\theta)\right\}.$$

3. Given policy functions c^{i+1} and θ^{i+1} compute the next value function iteration g^{i+1}

$$g_n^{i+1} = \left\{ p_n^d(c_n^{i+1},\theta_n^{i+1})\,g_{n-1}^i + p_n^u(c_n^{i+1},\theta_n^{i+1})\,g_{n+1}^i \right.$$

$$\left. + \left(\frac{1}{1-\gamma}c_n^{i+1}{}^{1-\gamma} + \lambda G^{i+1}(1+e^{z_n})^{1-\gamma}\right)\Delta t_n(c_n^{i+1},\theta_n^{i+1})\right\}.$$

4. Repeat steps 1-3 until convergence of g. Denote $G^*(x^i)$ as the value of G given x^i.

5. Repeat steps 1-4 for $x^i - h$ and $x^i + h$. If $G^*(x^i+h) < G^*(x^i) > G^*(x^i-h)$ stop.

6. Iterate $x^{i+1} = x^i + a\,c\,h$, where $a = \mathrm{sgn}[G^*(x^i+h) - G^*(x^i-h)]$, and c is a constant. Go to step 1.

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Table 1: Liquid and Illiquid Asset Returns

	1981Q3 – 2010Q2			1981Q3 – 2006Q4		
	Mean	Stdev	Corr	Mean	St Dev	Corr
Equity	0.103	0.182	1.000	0.125	0.157	1.000
Illiquid Assets						
Private Equity	0.103	0.229	0.629	0.110	0.231	0.605
Buyout	0.092	0.134	0.267	0.097	0.110	0.010
Venture Capital	0.133	0.278	0.557	0.143	0.286	0.548
Illiquid Investment	0.109	0.165	0.674	0.117	0.159	0.623

The table reports summary statistics on excess returns on liquid and illiquid assets. Liquid equity returns are total returns on the S&P500. Data on private equity, buyout, and venture capital funds are obtained from Venture Economics and Cambridge Associates. We construct annual horizon log returns at the quarterly frequency. We compute log excess returns using the difference between log returns on the asset and year-on-year rollover returns on one-month T-bills expressed as a continuously compounded rate. The column "Corr" reports the correlation of excess returns with equity. The illiquid investment is a portfolio invested with equal weights in private equity, buyout, and venture capital and is rebalanced quarterly.

Table 2: Asset Holdings and Wealth Composition

Average Turnover	λ	Optimal Rebalance Value	Stationary Distribution		
			Mean	St Dev	Skew
10 years	0.1	0.0483	0.1659	0.1855	2.3967
5 years	0.2	0.1053	0.1875	0.1273	2.6560
2 years	0.5	0.2423	0.2962	0.0854	2.2373
1 year	1.0	0.3729	0.4076	0.0633	1.8724
1/2 year	2.0	0.4403	0.4584	0.0422	1.5690
1/4 year	4.0	0.4963	0.5051	0.0283	1.2308

The table summarizes the effect of illiquidity on the moments of asset holdings. The optimal rebalance value is $\left(\frac{X}{X+W}\right)^*$, while the mean, standard deviation (st dev), and normalized skewness (skew) are all taken with respect to the stationary distribution of the ratio of illiquid wealth to total wealth, $\frac{X_t}{X_t+W_t}$. The table is computed using the following other parameter values: $\gamma = 6$, $\mu = \nu = .12$, $r = .04$, $\sigma = \psi = .15$, and $\rho = 0$.

Table 3: Illiquidity Premiums

Average Turnover	λ	Certainty Equivalent Wealth	Liquidity Premium
$\rho = 0$			
10 years	0.1	0.2866	0.0600
5 years	0.2	0.2148	0.0433
2 years	0.5	0.1140	0.0201
1 year	1.0	0.0572	0.0093
1/2 year	2.0	0.0415	0.0066
1/4 year	4.0	0.0397	0.0063
$\rho = 0.6$			
10 years	0.1	0.1235	0.0224
5 years	0.2	0.0692	0.0141
2 years	0.5	0.0197	0.0041
1 year	1.0	0.0106	0.0022
1/2 year	2.0	0.0098	0.0020
1/4 year	4.0	0.0096	0.0020

The table summarizes the effect of the trading frequency, λ, on certainty equivalents of holding the illiquid asset. The column labeled "Certainty Equivalent Wealth" reports the fraction of wealth the agent is willing to give up in order to make the illiquid asset liquid (taken as an expectation over the stationary distribution of wealth). The column labeled "Liquidity Premium" is a certainty equivalent comparison, so a liquidity premium of 0.02 means that the utility level in the economy with liquid assets with an expected returns of 12% and 10% is equal to the utility level in an economy with one liquid and one illiquid asset, both with expected returns of 12%. The numbers are computed taking expectations over the stationary distribution with the following other parameter values: $\gamma = 6$, $\mu = \nu = .12$, $r = .04$, and $\sigma = \psi = .15$.

Figure 1: Effective Risk Aversion
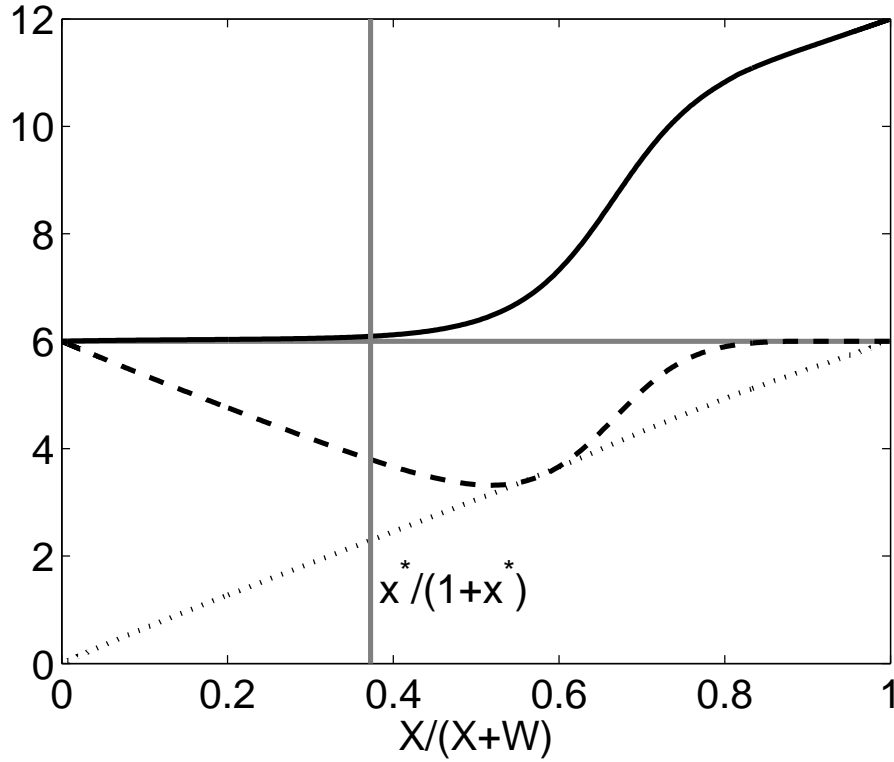
The figure plots the relative curvature of the value function for $\gamma = 6$. The solid line represents the total relative curvature, $\frac{F_{WW}W}{F_W} + \frac{F_{XX}X}{F_X}$, the dashed line represents the curvature with respect to W, $\frac{F_{WW}W}{F_W}$, and the dotted line represents the curvature with respect to X, $\frac{F_{XX}X}{F_X}$. The horizontal gray line is the point $x^*/(1+x^*)$, which is the optimal holding of illiquid assets relative to total wealth at the arrival of the trading time. The curves are plotted with the following parameter values: $\gamma = 6$, $\mu = \nu = .12$, $r = .04$, $\lambda = 1$, $\sigma = \psi = .15$, and $\rho = 0$.

Figure 2: Distribution of Illiquid Holdings
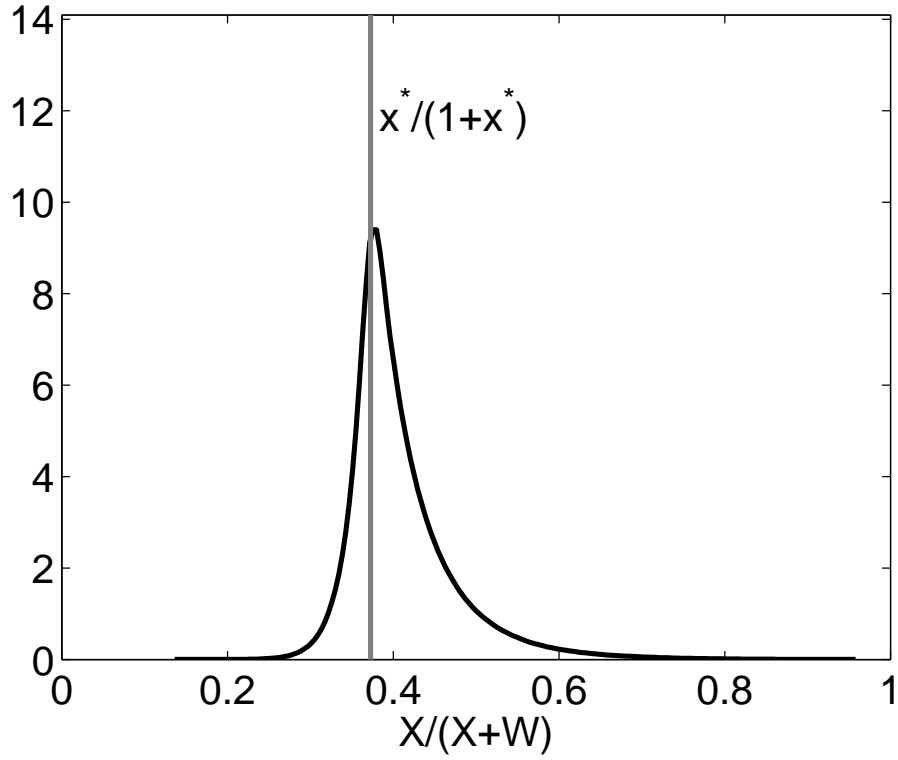
The figure plots the stationary distribution of allocation to the illiquid asset as a fraction of total wealth, $x = \frac{X}{X+W}$. The vertical solid gray line corresponds to the value of the optimal rebalancing point $x^*/(1+x^*)$, which is the desired allocation to the illiquid asset as a fraction of total wealth at the time of rebalancing. The figure uses $\gamma = 6$, $\mu = \nu = .12$, $r = .04$, $\lambda = 1$, $\sigma = \psi = .15$, and $\rho = 0$.

Figure 3: Optimal Allocation to the Liquid Risky Asset
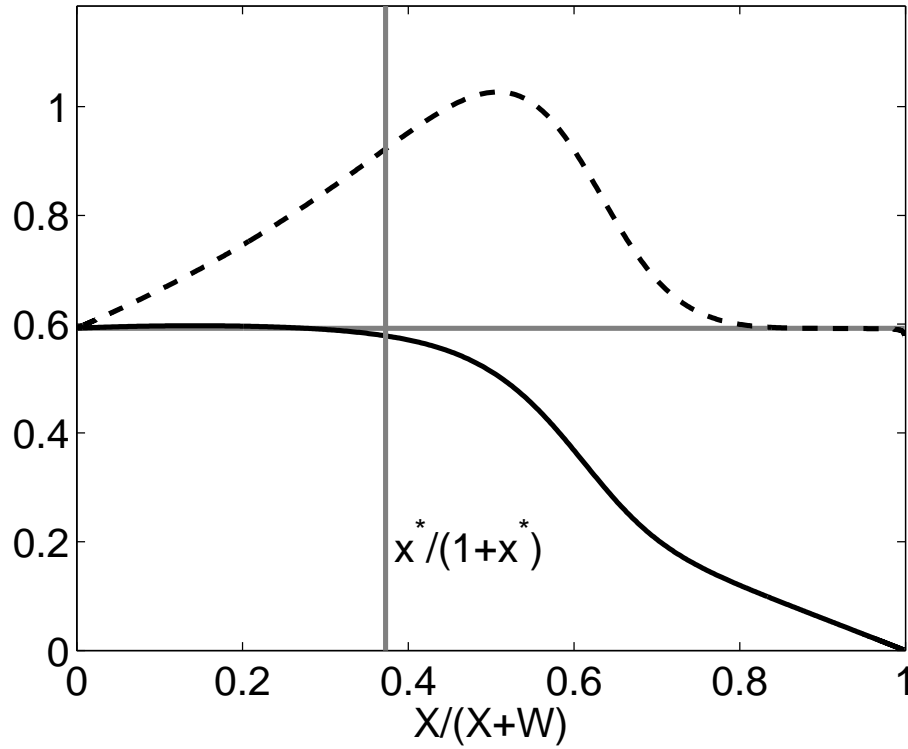
The figure displays the optimal allocation to the liquid assets. The solid black lines represent allocation to the liquid risky asset taken as a fraction of total wealth, $\theta/(1+x)$, whereas the dashed lines represent the allocation to the liquid risky asset as a fraction of liquid wealth only, θ. The gray horizontal line corresponds to the allocation to the risky asset in the one- and/or two-asset Merton economy. The vertical gray line is the point $x^*/(1+x^*)$, which is the optimal holding of illiquid assets relative to total wealth at the arrival of the trading time. The dashed line can be above one because the investor can use the liquid (but not illiquid) risky asset as collateral, as in the standard Merton problem. The solid line must remain between zero and one because the illiquid asset cannot be so used. The curves are plotted with the following parameter values: $\gamma = 6$, $\mu = \nu = .12$, $r = .04$, $\lambda = 1$, $\sigma = \psi = .15$, and $\rho = 0$.

Figure 4: Optimal Consumption
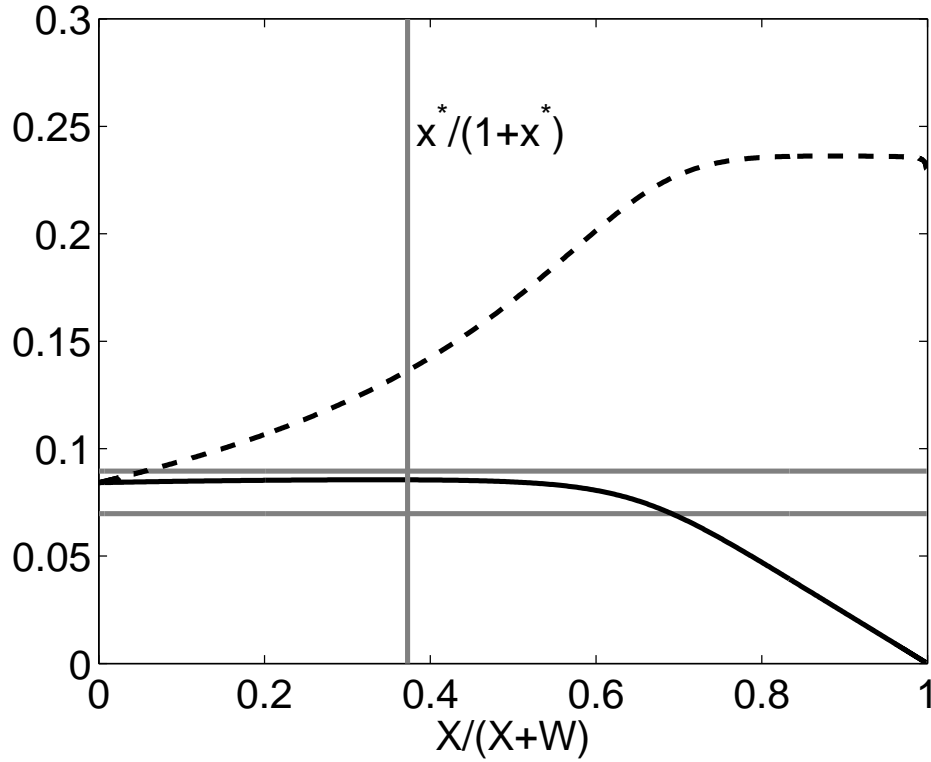
We plot the optimal consumption policy. The solid black line is the consumption policy as a fraction of total wealth, $c/(1 + x)$, and the dashed line depicts consumption policy as a fraction of liquid wealth only, c. The horizontal gray lines correspond to consumption in the one- and two-asset Merton benchmarks (consumption is higher in the two-asset case). The vertical solid gray line corresponds to the value of the optimal rebalancing point, $x^*/(1 + x^*)$, which is the desired allocation to the illiquid asset as a fraction of total wealth at the time of rebalancing. The curves are plotted with the following parameter values: $\gamma = 6$, $\mu = \nu = .12$, $r = .04$, $\lambda = 1$, $\sigma = \psi = .15$, and $\rho = 0$.
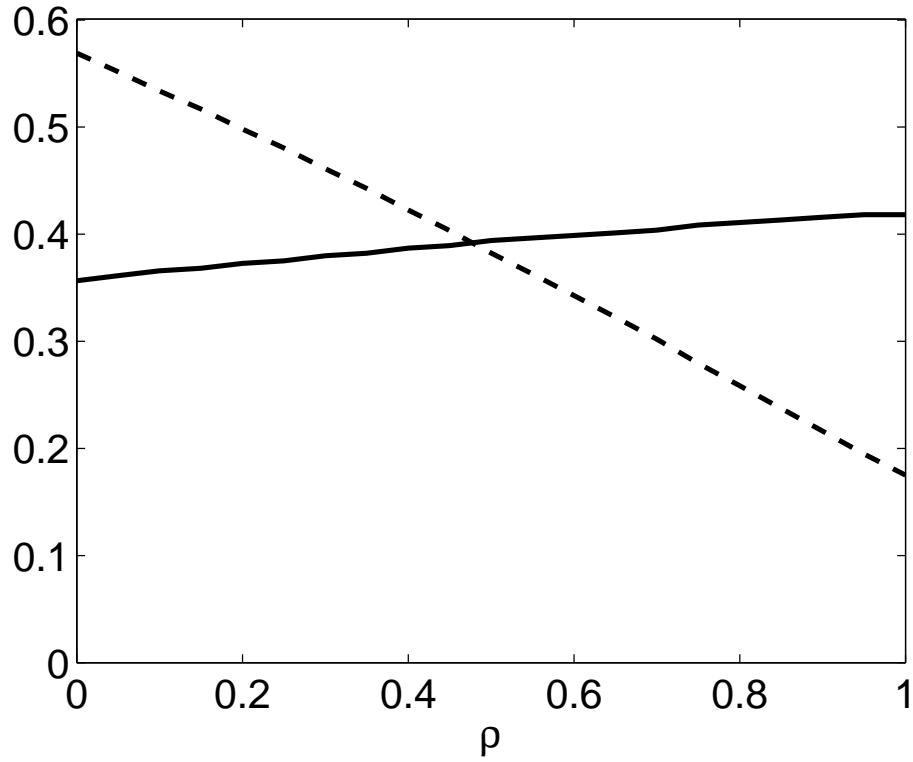
Figure 5: Effect of Illiquidity on Asset Holdings at Trading Times
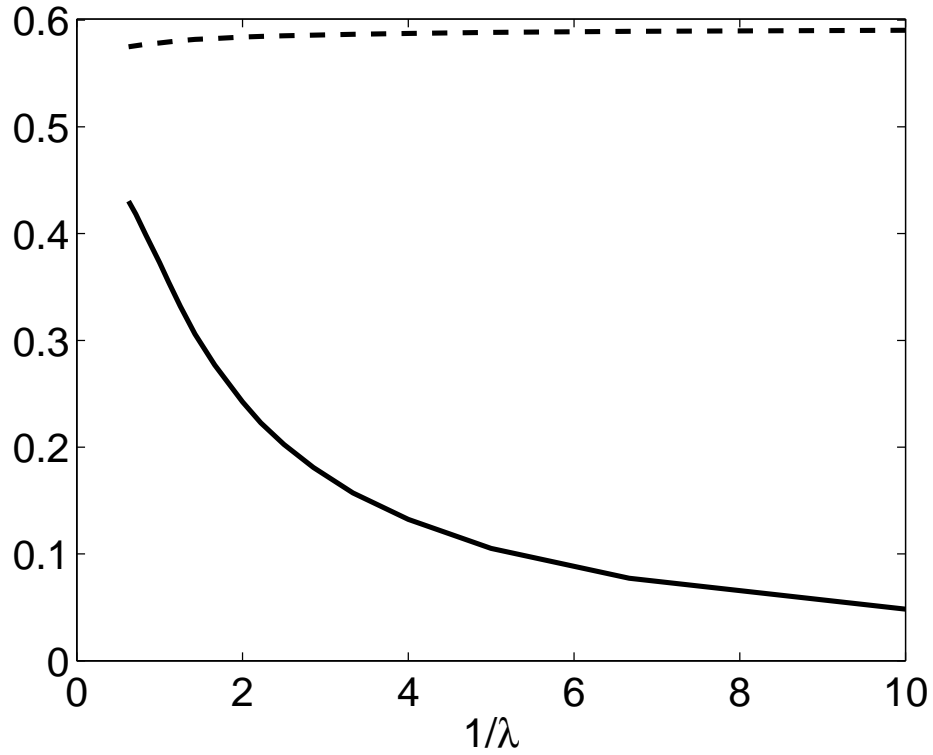
We plot the optimal allocations to the liquid risky asset as a fraction of total wealth (dashed line) and the illiquid risky asset as a fraction of total wealth (solid line) at the rebalancing time, both as a function of $1/\lambda$. The remainder is allocated to the riskless asset. The curves are plotted with the following other parameter values: $\gamma = 6$, $\mu = \nu = .12$, $r = .04$, $\rho = 0$, and $\sigma = \psi = .15$.

Figure 6: Effect of Illiquidity on Asset Holdings Between Trading Times
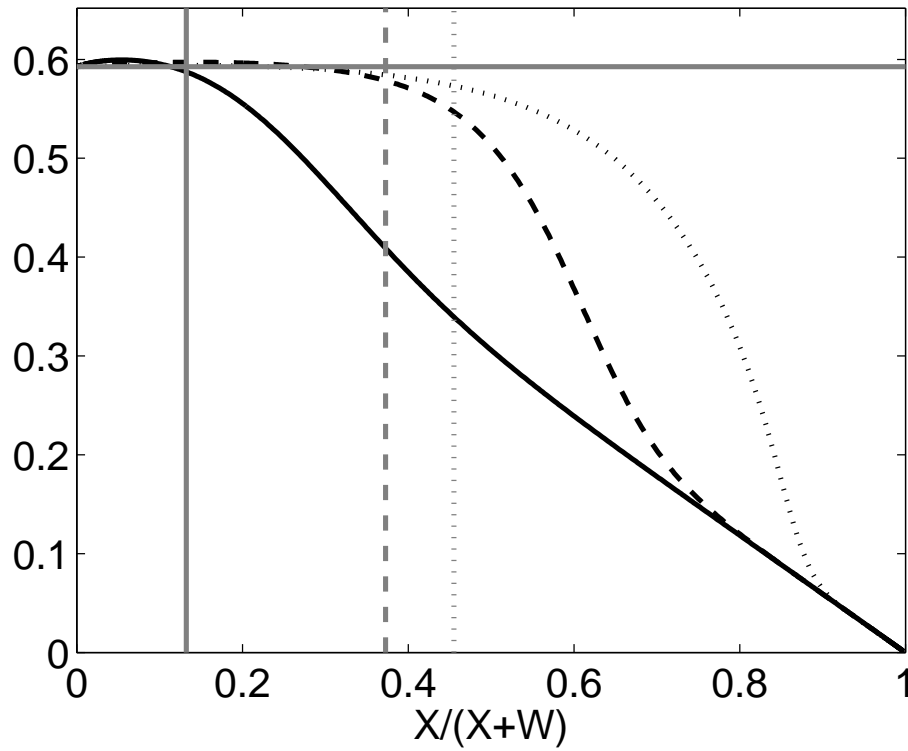
We plot the optimal allocations to the liquid risky asset as a fraction of total wealth as a function of the liquid/illiquid composition of total wealth. There are three black curves which display holdings: $\lambda = 1/4$, $\lambda = 1$, and $\lambda = 4$ correspond to the solid line, the dashed line, and the dotted line, respectively. The vertical gray lines with the same line style are the corresponding optimal rebalance levels. The curves are plotted with the following other parameter values: $\gamma = 6$, $\mu = \nu = .12$, $r = .04$, $\rho = 0$, and $\sigma = \psi = .15$.

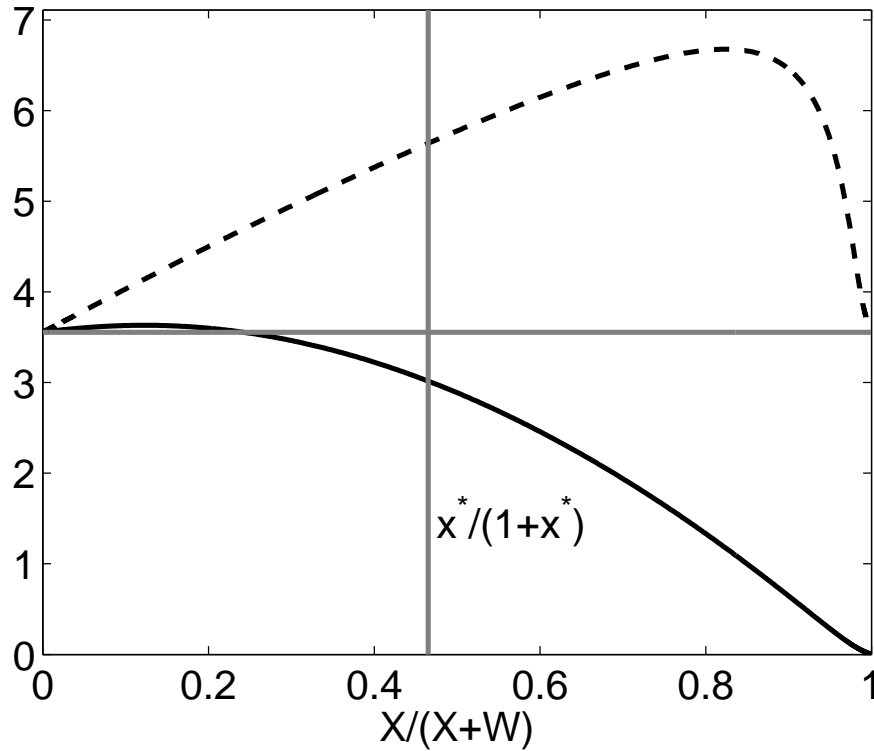Figure 7: Effect of Correlation on Asset Holdings

We plot the optimal allocations to the liquid risky asset as a fraction of total wealth (dashed line) and the illiquid risky asset as a fraction of total wealth (solid line) at the rebalancing time, both as a function of ρ. The remainder is allocated to the riskless asset. The curves are plotted with the following other parameter values: $\gamma = 6$, $\mu = .12$, $\nu = .20$, $r = .04$, $\lambda = 1$, and $\sigma = \psi = .15$.

Figure A-1: Effective Risk Aversion: $\gamma = 1$
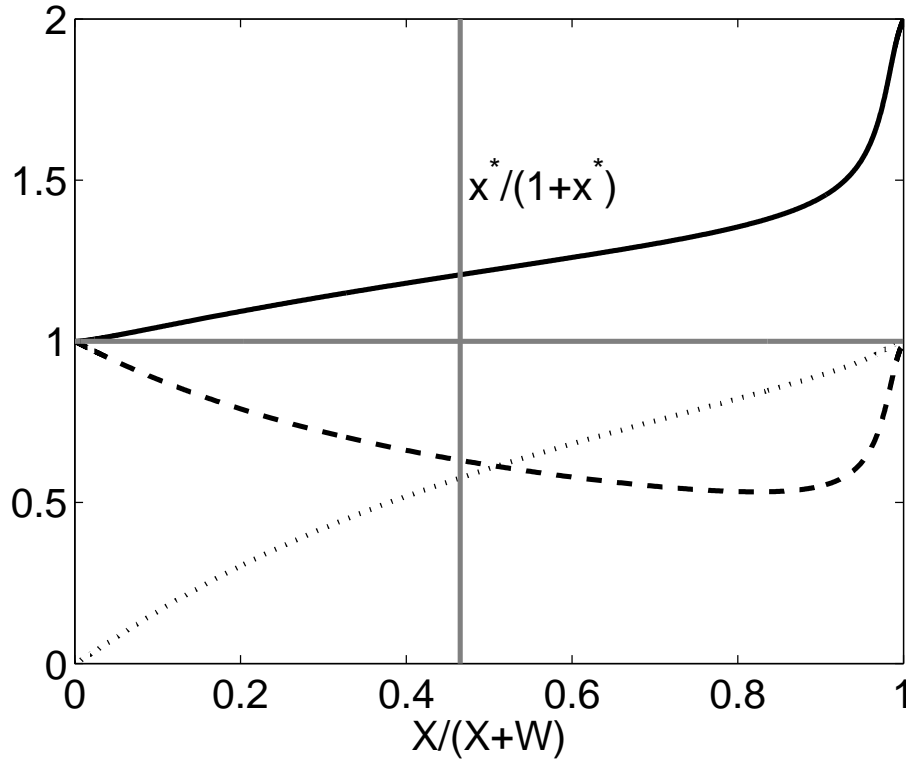
The figure plots the relative curvature of the value function for $\gamma = 1$. The solid line represents the total relative curvature, $\frac{F_{WW}W}{F_W} + \frac{F_{XX}X}{F_X}$, the dashed line represents the curvature with respect to W, $\frac{F_{WW}W}{F_W}$, and the dotted line represents the curvature with respect to X, $\frac{F_{XX}X}{F_X}$. The horizontal gray line is the point $x*/(1+x^*)$, which is the optimal holding of illiquid assets relative to total wealth at the arrival of the trading time. The curves are plotted with the following parameter values: $\gamma = 1$, $\mu = \nu = .12$, $r = .04$, $\lambda = 1$, $\sigma = \psi = .15$, and $\rho = 0$.

Figure A-2: Optimal Allocation to the Liquid Risky Asset: $\gamma = 1$

The figure displays the optimal allocation to the liquid assets. The solid black lines represent allocation to the liquid risky asset taken as a fraction of total wealth, $\theta/(1+x)$, whereas the dashed lines represent the allocation to the liquid risky asset as a fraction of liquid wealth only, θ. The gray lines correspond to the allocation to the risky asset in the one- and/or two-asset Merton economy. The horizontal gray line is the point $x*/(1+x^*)$, which is the optimal holding of illiquid assets relative to total wealth at the arrival of the trading time. The curves are plotted with the following parameter values: $\gamma = 1$, $\mu = \nu = .12$, $r = .04$, $\lambda = 1$, $\sigma = \psi = .15$, and $\rho = 0$.