Risk, Returns, and Optimal Holdings of Private Equity

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I. Introduction

Private equity (PE) investments are investments in privately-held companies, trading directly between investors, not on organized exchanges. The investments can be direct equity investments into individual companies. More commonly, though, they are made through a PE fund organized as a limited partnership with the investor as a limited partner (LP) and the PE firm as the general partner (GP), overseeing and managing the investments in the individual companies. Depending on the type of companies they invest in, PE funds are typically classified as buyout (BO) funds, venture capital (VC) funds, mezzanine debt funds, or some other type specializing in other illiquid non-listed investments. Additionally, it is not uncommon for LPs to make co-investments directly into the portfolio companies in parallel to the investments made through the PE fund.

PE is often considered a distinct asset class, and it differs from typical liquid, public equity investments in several fundamental ways: First, there is no active market for PE positions, making these investments illiquid and difficult to value. Second, the investments are long-term investments. PE funds typically have horizons of ten to thirteen years during which the invested capital cannot be redeemed. Third, partnership agreements specifying the funds’ governance are complex documents,
typically specifying the GP’s compensation as a combination of regular fees (Management Fee), a profit share (Carried Interest), and other fees.

This article surveys the academic research about the risks and returns of PE investing and the optimal holdings of PE in an investor’s portfolio. It should be noted that researchers have had very limited access to information about the nature and performance of PE investments, and that the research in this area is preliminary and often inconclusive. Research into many important aspects of these investments, such as the performance of PE in the recession, the secondary market for LP positions, and LPs’ co-investments in deals, has only recently begun.

Section II introduces two main problems that research has encountered in measuring PE risk and returns. The first problem is the statistical problem that arises because PE returns are only observed infrequently, typically with well-performing funds being overrepresented in the data. This makes it difficult to estimate standard measures of risk and return, such as alphas and betas. Even after addressing this problem, a second problem arises about how to best interpret the resulting estimates. Standard asset-pricing models are derived under assumptions that are appropriate for traditional financial markets – with transparent, liquid, and low-friction transactions. These assumptions are problematic for PE investments, and the estimated alphas and betas may need to be adjusted to provide meaningful measures of risk and return in the PE context. One way of interpreting the risks and returns of PE investments, especially for illiquidity risk, is for an investor to consider PE from an investor-specific asset allocation context.

Section III summarizes the existing literature on the optimal allocation of PE in portfolios consisting of public liquid public equity and illiquid PE. A new generation of asset allocation models considers these issues as the first generation of asset allocation approaches assumed that assets can be rebalanced without cost at any time. The literatures on asset allocation incorporating transactions costs, which are very high for PE investment, and search frictions, since counterparties are often
hard to find for transferring PE investments, lead to some stark recommendations on optimal holdings of illiquid PE assets.

In Section IV, we survey the literature on PE contracts with a special emphasis on fees and opaqueness. Most PE investments are made through intermediaries. Current PE investment vehicles cannot disentangle factor returns unique to the PE asset class from manager skill. Furthermore, commonly used contracts may exacerbate, rather than alleviate agency issues.

Finally, Section V concludes by summarizing our recommendations of best practices for institutional investors when assessing and benchmarking PE returns, allocating funds between PE and liquid markets, and setting limited partnership agreements.

II. Private Equity Risk and Return

II. A. Evaluating Private Equity Risk and Return

To understand the problems of evaluating PE investments, and to fix notation and terminology, it is useful to start from the standard model of risk and return. For standard liquid financial assets, risk and return are often measured by the $\alpha$ and $\beta$ coefficients from a one-factor linear regression (the Expected Return Regression),

$$E[R_i(t)] - R_f(t) = \alpha + \beta[R_m(t) - R_f(t)]$$

In this equation $E[R_i(t)]$ is the expected return earned by the investor from period $t-1$ to period $t$, $R_f(t)$ is the risk-free rate over the period from $t-1$ to $t$, and $R_m(t)$ is the return on the market portfolio. Specifically, the return earned on a financial asset from time $t-1$ to $t$ is defined as

$$R(t) = \frac{P(t) + D(t)}{P(t-1)}$$
Here $D(t)$ is the dividend (or any cash flow more generally) paid out at time $t$, and $P(t)$ is the market price quoted at time $t$, immediately after the payment of the dividend.

For standard traded assets, the Expected Return Regression is straightforward to estimate by regressing the asset’s observed returns on the market returns over the same period. The one-factor specification given above, which includes just the excess market return factor (MKT) in the bracket on the right-hand side of the equation, can be extended to multifactor specifications by including more factors, typically the value factor (HML), the size factor (SMB), and possibly a momentum factor (UMD) in the spirit of Fama and French (1993).

Under appropriate assumptions about investors’ preferences, such as CRRA or mean-variance utility, and assumptions about the market environment, such as the absence of transaction costs, short-sales constraints, and the ability of investors to continuously trade and rebalance, the CAPM model specifies that each asset’s expected return is given by the Expected Return Regression with an alpha equal to zero. This important result has several implications: It implies that the appropriate measure of risk is $\beta$, which measures the correlation between the return on the asset and return on the overall market (Systematic Risk). Furthermore, the Systematic Risk is the only type of risk that is priced in equilibrium. Idiosyncratic risk is not priced, because it can be diversified away by investors. The Expected Return Regression further implies that an asset’s expected return increases linearly in $\beta$. Finally, it implies that in equilibrium, without arbitrage, $\alpha$ should be zero. A positive $\alpha$ can then be interpreted as an abnormal positive return.

Following this logic, the standard approach to evaluating risks and returns of financial assets proceeds in two steps: First, $\alpha$ and $\beta$ are estimated using the Expected Return Regression. Second, invoking the CAPM model, the estimated $\alpha$ coefficient is interpreted as the abnormal return, and the estimated $\beta$ is interpreted as the risk.
For PE investments problems arise at both steps: First, privately held companies do not have market values, by definition, and hence the returns earned from investing in these companies are only observed infrequently, making it difficult to estimate the Expected Return Regression. Moreover, privately held companies with better performance tend to be overrepresented in the data, creating a sample selection problem that may lead to an upward bias in the estimated $\alpha$ and a downward bias in the $\beta$. Second, even after obtaining estimates of $\alpha$ and $\beta$ for PE investments, it is not obvious that these coefficients can be interpreted as measures of risk and returns. The assumptions of liquid and transparent markets underlying the CAPM model are far from the realities of PE investing, and the estimated coefficients may have to be adjusted in various ways – e.g., to account for the cost of illiquidity, idiosyncratic risk, persistence, funding risk, etc. – to reflect the actual risks and returns faced by the LP investors.

II. B. Estimating Alphas and Betas of PE Investments

The lack of regularly quoted market prices for PE investments means that PE returns are only infrequently observed, making it difficult to estimate the Expected Return Regression directly. The academic literature has approached this problem using several alternative data sources, primarily data with performance and valuations of company-level investments and data with fund-level cash flows between LPs and GPs. The result and limitations of these approaches are discussed next.

II. B. 1. Company-Level Data

Two primary studies using company-level data are Cochrane (2005) and Korteweg and Sorensen (2010). Both studies focus on VC investments. There is no comparable study of buyout (BO) investments, primarily due to the lack of data with deal-level cash-flow information for buyout investments.
Following Cochrane (2005) and Kortweg and Sorensen (2010), there are two main advantages of estimating risk and return from company-level data relative to studies using fund-level data (discussed below): First, there are many more companies than funds, and companies have well-defined industries and types (early-stage, late-stage, etc.), leading to greater statistical power and a more nuanced differentiation of the risks and returns across industries, investment types, and time periods. Second, returns are well defined. Absent intermediate payouts, company-level data directly lead to the return earned over the life of the investment (specifically, \( P(t-1) \) is the initial investment, \( D(t) \) is the subsequent realized cash flow, earned when the investment is exited, and \( P(t) \) is the market value after the exit, typically zero).

Company-level data have three main disadvantages: First, company-level cash-flows do not subtract management fees and carried interest paid by the LPs to the GPs, hence estimated risks and returns reflect the total (gross of fees) risks and returns of the investments, not those earned by an LP (which are net of fees). Inferring the returns after fees to LPs requires additional assumptions (see Metrick and Yasuda (2010)). We comment further on investor fees in Section IV. Moreover, company-level data require continuous-time specifications and must correct for selection bias, as discussed next.

*Continuous-Time Specifications*

A technical complication with company-level data is that the returns are measured over different lengths of time. The standard (discrete-time) CAPM model, with non-zero alpha, does not compound (it defines all returns as one-period returns, where a period may last for an arbitrary amount of time – a day, a month, a quarter, etc. – but all returns must be defined over periods of equal length).

This is a standard problem in empirical finance, and it is typically addressed by using the continuous-time version of the CAPM model, which does compound over time and hence can used to compare risks and returns of investments of different
durations. In the continuous-time CAPM model, the Expected-Return Regression is restated in log-returns (Continuously Compounded Returns) as follows

\[ E[\ln(R_t(t))] - \ln(R_f(t)) = \delta + \beta(\ln(R_m(t)) - \ln(R_f(t))] \]

Consequently, Cochrane (2005) and Korteweg and Sorensen (2010) estimate the continuous-time version of the model. One complication arises, however, because the intercept of the continuous-time specification can no longer be interpreted as the abnormal arithmetic return, as in the standard discrete-time version of the model. Instead, the abnormal return is calculated by adding one-half times the square of the estimated volatility, as follows

\[ \alpha = \delta + \frac{1}{2} \sigma^2. \]

Volatility, however, is very high as Cochrane (2005) and Kortweg and Sorensen (2011) point out, and the arithmetic alphas calculated using this adjustment become correspondingly high. For example, Cochrane (2005) reports an annual volatility around 90%, resulting in an estimated alpha of 32% annually, which appears very high compared to studies using fund-level data (see below).

**Selection Bias**

Studies using company-level data face a sample-selection problem. To illustrate, VC investments are structured over multiple rounds, and better-performing companies tend to raise more rounds of financings. Hence, data sets with valuations of individual VC rounds are dominated by these better-performing companies. Moreover, distressed companies are usually not formally liquidated, and are often left as shell companies without economic value (sometimes called “zombies”). This introduces another selection problem for the empirical analysis. When observing old companies without new financing rounds or exits, these companies may be alive and doing well or they may be distressed zombies, in which case it is unclear when the write-off of the company’s value should be recorded.
To illustrate the magnitude of the selection problem consider Figure 1 (from Korteweg and Sorensen (2010)). The universe of returns is illustrated by all dots. The data, however, only contains the observed good returns above the $x$-axis (drawn in black). Worse returns (shaded in gray) are unobserved. Since only the black dots are observed in the data, a simple estimation of the Expected Return Regression gives an estimate of alpha that is biased upwards, an estimate of beta that is biased downwards, and a total volatility that is too low. Hence, standard statistical analysis that does not correct for this selection bias will result in an overly optimistic view of the risk and return of these investments.

![Figure 1: Illustration of selection bias](image)

Models to account for such selection bias were first developed by Heckman (1979). Cochrane (2005) estimates the first selection model on VC data and finds that the effect of selection bias is indeed large. The selection correction reduces the intercept of the log-market model, denoted $\delta$ above, from 92% to -7.1%. Cochrane also highlights the difficult of translating this intercept into an abnormal return. Korteweg and Sorensen (2010) estimate a more flexible and robust version of Cochrane’s model. They also find that selection over-states the risk and return trade-off of VC investments. Without selection, the estimate of $\delta$ is -0.0159 per month while taking into account the selection bias reduces this estimate to -0.0563 per month.
In the continuous-time model, the estimated $\beta$ coefficient, without any adjustments, can still be interpreted as a measure of the systematic risk in the CAPM model. Cochrane finds a slope of 0.6-1.9 for the systematic risk. This figure seems low, however, since it includes estimates at the individual industry levels of, for example, -0.1 for retail investments. Korteweg and Sorensen (2010) report beta estimates that are substantially higher than those by Cochrane, in the range of 2.6-2.8, which may be more reasonable for young startups funded by VC investors.

Korteweg and Sorensen also find substantial variation over time. They estimate alphas over the periods 1987-93, 1994-2000, and 2001-05, and find that the alphas in the early period were positive but modest, the alphas in the late 1990s were very high, but the alphas in the 2000s have been negative, consistent with patterns found by studies using fund-level data.

II. B. 2. Advantages of Fund-Level Data

A recent survey by Harris, Jenkinson, and Kaplan (2011) summarizes the academic studies using fund-level data from various data providers.¹ There are several advantages to fund-level data: First, fund-level data reflect actual LP returns, net of fees, resulting in estimates of the risks and returns actually realized by the LPs. Additionally, the sample selection problem is smaller, since the performance of companies that ultimately never produce any returns for the investing funds (zombies) is eventually reflected in the fund-level cash flows. Other sample selection problems may arise, however. Fund-level performance is typically self-reported, and better performing funds may be more likely to report their performance (as suggested by Phalippou and Gottschalg (2008), although Stucke (2011) finds that reported returns by Venture Economics have understated actual historical returns).² Still, these selection problems are likely smaller than the more substantial problems that arise for company-level data. Finally, since funds have

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¹ These studies include Ljungqvist and Richardson (2003), Kaplan and Schoar (2005), Phalippou and Gottschalg (2008), and Robinson and Sensoy (2011).
² Anecdotal evidence from Harris, Jenkinson, and Kaplan (2011) suggests that this bias made Venture Economics more attractive for benchmarking GP performance.
similar lifetimes (typically ten years), the Expected Return Equation can be estimated directly, avoiding the problems with the continuous-time log-return specification used for company-level data.

The main disadvantage of fund-level data, however, is that it is unclear how to measure the "return." Absent quoted market prices, a stream of cash flows does not map into the definition of a return. Several alternatives have been proposed, but none of these are rooted in an economic valuation model and they all have various limitations, as discussed next.

II. C. Fund-Performance Measures

Fund-level data typically contain a cash flow stream between the GP and LPs. Absent quoted market prices, these cash flows cannot be directly translated into returns. Calculating period-by-period returns requires assessing the market values of the PE investment (P(t) in the return calculation) at intermediate periods. Such market values are typically unavailable, and reported NAVs are noisy substitutes for these values (for example, it has been customary to value investments in individual companies at cost until the company experienced a material change in the circumstances, which clearly does not capture smaller ongoing changes in the prospects and market values of these companies). Given the absence of regularly quoted returns, several alternative measures have been proposed. However, none of these measures define a return, as defined in the CAPM model, and their relationships to standard asset pricing models are somewhat tenuous.

II. C. 1. Internal Rate of Return (IRR)

A natural starting point is to interpret the internal rate of return (IRR) of the cash flows between the LP and GP as a return earned over the life of the fund. Denoting the cash flow at time \( t \) as \( CF(t) \), and separating those into the capital calls paid by the
LP to the GP, denoted $Call(t)$, and the distributions of capital from the GP back to the LP, denoted $Dist(t)$, the IRR is defined as follows:

$$PV = \sum \frac{CF(t)}{(1 + IRR)^t} = \sum \frac{Dist(t) - Call(t)}{(1 + IRR)^t} = 0$$

\[
\Rightarrow \frac{\left(\sum \frac{Dist(t)}{(1 + IRR)^t}\right)}{\left(\sum \frac{Call(t)}{(1 + IRR)^t}\right)} = 1
\]

Ljungqvist and Richardson (2003) investigate cash-flow data from a large LP investing in funds raised in 1981-1993 (19 VC funds and 54 BO funds). They report average fund IRRs (net of fees), combining PE and VC investments, for 1981-1993, of 19.81%, while the average S&P/500 return is 14.1%, suggesting that PE investments outperform the market.

Kaplan and Schoar (2005) use fund-level quarterly performance measures from Venture Economics covering for 1,090 VC and BO funds, of which 746 funds were fully or mostly liquidated at the time of the study. Kaplan and Schoar find VC and BO returns slightly below those of the S&P/500 index on an equal-weighted basis (value-weighted VC funds perform slightly better than the index) using their sample of fully liquidated funds. The value-weighted IRR equals 13%.\(^3\) Extending the sample to mature but not liquidated funds raises the IRR for VC to 30% but leaves it unchanged at 13% for BOs, resulting in an overall average IRR of 18%.\(^4\)

\(^3\) As pointed out by Phalippou and Gottschalg (2009), it is not obvious how to value-weight PE funds. One possibility is to weight by total committed capital, but funds vary in their investment speed, and worse performing funds may invest more slowly, introducing a downward bias in value-weighted performance estimates.

\(^4\) The final reported NAV of funds that are not fully liquidated is treated as a final cash flow in the calculation. Phalippou and Gottschalg (2009) argue that interim NAVs may exaggerate the actual values, leading to upward-biased performance estimates. In contrast, Stucke (2011) argues that the NAVs are substantially below actual economic value, using Venture Economics data. Kaplan and Schoar (2005) and Harris, Jenkinson, and Kaplan (2011) use reported NAVs as stated.

The recent summary by Harris, Jenkinson, and Kaplan (2011) compares performance measures for different data sources. For BO funds, they report weighted average IRRs of 12.3-16.9%. For VC funds, the weighted average IRRs are 11.7-19.3%. Across time periods, BO funds have had more stable performance, with weighted average IRRs of 15.1-22.0% in the 1980s, 11.8-19.3% in the 1990s, and 5.8-12.8% in the 2000s. VC fund performance has more volatile over time, with weighted average IRRs ranging from 8.6 to 18.7% in the 1980s, 22.9 to 38.6% in the 1990s, and -4.9 to 1.6% in the 2000s.

Overall these figures reveal substantial uncertainty about even this most basic performance measure across the various studies and data sources. Moreover, the IRR is a problematic measure of economic performance. The IRR calculation implicitly assumes that invested and returned capital can be reinvested at the IRR rate. This means that if a fund makes a small investment with a rapid large positive return, this single investment can largely define the IRR for the entire fund, regardless of the performance of subsequent investments.5

II. C. 2. Total-Value-To-Paid-In Capital (TVPI)

An alternative performance measure that is less susceptible to manipulation than the IRR is the total-value-to-paid-in capital (TVPI) multiple. This multiple is calculated as the total amount of capital returned to the LP investors (net of fees) divided by the total amount invested (including fees). This calculation is performed without adjusting for the time value of money. Whereas the IRR is calculated under the implicit assumption that capital can be reinvested at the IRR rate, the TVPI is calculated under the implicit assumption that the capital can be reinvested at a zero rate. Formally, the TVPI is defined as

5 Phalippou (2011) suggests that GPs may actively manage their investments to inflate fund IRRs.
Harris, Jenkinson, and Kaplan (2011) report weighted average TVPIs of 1.76-2.30 for buyout investors, and 2.19-2.46 for VCs. This multiple has varied substantially over time. For BO funds, the reported multiple was 2.72-4.05 in the 1980s, 1.61-2.07 in the 1990s, and 1.29-1.51 in the 2000s. For VC funds, the reported multiple was 2.31-2.58 in the 1980s, 3.13-3.38 in the 1990s, and 1.06-1.09 in the 2000s. In other words, recent VC funds have, on average, simply returned invested capital, albeit after a ten-year investment period.

II. C. 3. Public Market Equivalent (PME)

Both IRR and TVPI measure absolute performance, but it is even important to also understand the performance relative to the general market, and the third measure, the public market equivalent (PME), is a relative performance measure.

The PME is calculated as the ratio of the discounted value of the LP’s inflows divided by the discounted value of outflows, with the discounting performed using realized market returns.

\[
PM_{E} = \frac{\left( \sum \frac{Dist(t)}{(1 + R_m(t))} \right)}{\left( \sum \frac{Call(t)}{(1 + R_m(t))} \right)}
\]

As noted by Kaplan and Schoar (2005), under the assumption that PE investments have the same risk as the general market (a beta equal to one), a PME greater than one is equivalent to a positive economic return for the LPs. While generally reasonable, this interpretation may still be misleading because the economic risk of distributions (the numerator in the PME) is likely substantially greater than the economic risk of capital calls (including management fees, which resemble a risk-free liability). Using a lower discount rate for capital calls would inflate the denominator and reduce the PME. Hence, more carefully accounting for these
different risks suggests that the PME (calculated the traditional way) may have to exceed one by some amount before LPs actually earn a positive economic return.\footnote{Additionally, as a technical point, the CAPM model prescribes that the discounting should be performed using expected returns, not realized returns as in the PME. Using the realized returns distorts the calculation (according to Jensen's inequality). The magnitude of this distortion is unclear, but most likely modest.}

Kaplan and Schoar (2005) find average equal-weighted PMEs of 0.96. Value-weighted, the VC PME is 1.21 and the BO one is 0.93. Moreover, they report that larger funds have higher PMEs, although when funds become very large performance declines. For BO funds those with higher sequence numbers have higher PMEs. Phalippou and Gottschalg (2008) use 852 funds to calculate a PME of 1.01, although this figure decreases to 0.88 after various adjustments.

Looking across databases, Harris, Jenkinson and Kaplan (2011) report weighted average PMEs of 1.16-1.27 for BO funds, and 1.02-1.45 for VC funds. BO PMEs have varied from 1.03-1.11 in the 1980s, to 1.17-1.34 in the 1990s, and 1.25-1.29 in the 2000s. For VC, the reported PMEs are 0.90-1.08 in the 1980s, to 1.99-2.12 in the 1990s, and 0.84-0.95 in the 2000s. Relative to the market, the 1990s were the VC decade, and the 2000s have been the BO decade.

II. C. 4. Risk Measures

In contrast to company-level data, fund-level are poorly suited for estimating the risk of PE investing. Few, if any, academic studies attempt to use fund-level data to do so. Instead, Ljungqvist and Richardson (2003) estimate risk by assigning each portfolio company to one of 48 broad industry groups and use the corresponding average beta for publicly traded companies in the same industry. They report that the corresponding beta for publicly traded companies is 1.08 for buyouts and 1.12 for venture capital investments. Note that these betas do not adjust for the higher (lower) leverage used for buyouts (venture capital) investments. Assigning betas, they find a 5-6% premium, which they interpret as the illiquidity premium of venture capital investments.
Kaplan and Schoar state that they “believe it is possible that the systematic risk of LBO funds exceeds 1 because these funds invest in highly levered companies.” They regress IRRs on S&P/500 returns, and find a coefficient of 1.23 for VC funds and 0.41 for BO funds. A (levered) beta of 0.41 seems unreasonably low, however.

Hence, based on the existing evidence from studies using fund-level data, it seems too early for a precise assessment of how the risk of PE investing compares to the risk of investing in publicly traded equities even in terms of these most basic metrics.

II. C. 5. Persistence and Predictability

A PE firm typically raises a sequence of partially overlapping funds (a fund family). Kaplan and Schoar (2005), Phalippou and Gottschlag (2009), and several other studies find evidence of performance persistence of these funds. Good performance of a previous fund predicts good performance of subsequent funds in the same fund family. This is interpreted as evidence that GPs vary in their skills and abilities to pick investments and manage the companies. Estimates suggest that a performance increase of 1.0% for a fund is associated with around 0.5% greater performance for the subsequent fund in the same family, measured either in terms of PME or IRR. For more distant funds, the persistence declines.

PE performance can also be predicted by other PE firm or fund characteristics along with macro economic variables. It has been suggested that greater fund size may be related to smaller returns. Though, Kaplan and Schoar (2005) and Harris, Jenkinson, and Kaplan (2011) find no relation between size and performance for BO funds. For venture capital funds, Harris, Jenkinson, and Kaplan (2011) report a positive relation between size and performance after controlling for vintage years.

Due to data limitations, studies that document predictability in PE returns conduct statistical analysis in sample, rather than on an out-of-sample basis. In Kaplan and Schoar, for example, PE funds in the “top quartile” do well, but these funds are...
identified ex post. Within a fund family, funds often have lifetimes of 10 years but overlap to some extent. In-sample analysis uses the ultimate performance of a previous fund to predict the performance of a subsequent fund, even if this subsequent fund is raised before the ultimate performance of the previous funds is fully realized. The studies employ various robustness checks, such as using intermediate NAVs instead of ultimate performance or using the performance of funds several generations ago to predict future performance to mitigate this concern. Still, some recent research, such as Hochberg, Ljungvist and Vissing-Jorgensen (2010) find weaker evidence of persistence using only information available when the new fund is raised.

II. D. Summary

Measuring PE risk and returns is difficult because of the infrequent observations of fund or company values and selection bias. Studies using company-level data which account for selection bias find high alphas for PE investments only during the late 1990s, but negative alphas post-2000. The positive alpha estimates are hard to interpret in terms of arithmetic returns, however, because of very high volatility. Estimates of PE betas vary substantially, ranging as high as 3.6 for venture capital investments. Studies using fund-level data have fewer selection problems, but still suffer from the fact that no direct PE returns are observed. Unlike standard return measures, fund-level IRRs, TVPI, and PME measures can all be misleading and should be used with caution to infer PE performance. In terms of raw performance, in the words of Harris, Jenkinson, and Kaplan (2011) "it seems likely that buyout funds have outperformed public markets in the 1980s, 1990s, and 2000s." However, due to the uncertainty about the risk of private equity investments, it is not yet possible to say whether this outperformance is sufficient to compensate investors for the risk of these investments and whether the investments outperform on a risk-adjusted basis. Finally, there is evidence of persistence of PE fund returns and some, albeit weaker and less consistent, evidence that characteristics like fund size and past capital raisings predict PE fund returns.
III. Asset Allocations to Private Equity

Having discussed the measurement of PE returns, we now consider optimal allocations to PE. This requires, of course, a suitable risk-return trade-off for PE investments as well as correlations of PE returns with other assets in the investor’s opportunity set. As Section II points out, measuring these inputs for PE for use in an optimization problem requires special considerations. We take as given these inputs and focus on the dimension of illiquidity risk of PE and how to incorporate illiquidity PE risk into an optimal asset allocation framework. There have been several approaches to handling illiquidity risk in asset allocation, all of which have relevance in dealing with PE allocation. To put into context these contributions, we start with the case of asset allocation without frictions.

III. A. Frictionless Asset Allocation

The seminal contributions of Merton (1969, 1971) and Samuelson (1969) characterize the optimal asset allocation of an investor with CRRA utility investing in a risk-free asset (with constant risk-free rate) and a set of risky assets. The CRRA utility function with risk aversion \( \gamma \) is given by

\[
U(W) = \frac{W^{1-\gamma}}{1-\gamma}.
\]

CRRA utility is homogeneous of degree one, which means that exactly the same portfolio weights arise whether $10 million of wealth is being managed or $1 billion. This makes the utility function ideal for institutional asset management.

Assume the risky assets are jointly log-normally distributed. Under the case of iid returns, the vector of optimal holdings, \( w \), of the risky assets are given by

\[
w = \frac{1}{\gamma} \Sigma^{-1}(\mu - r),
\]
where \( \Sigma \) is the covariance matrix of the risky asset returns, \( \mu \) is the vector of expected returns of the risky assets, and \( r_f \) is the risk-free rate. This is also the portfolio held by an investor with mean-variance utility optimizing over a discrete, one-period horizon.

There are two key features of this solution that bear further comment. First, the Merton-Samuelson solution is a dynamic solution that involves continuous rebalancing. That is, although the portfolio weights, \( w \), are constant, the investor’s policy is always to continuously sell assets that have risen in value and to buy assets that have fallen in value in such a way as to maintain constant weights. Clearly, the discrete nature of PE investment and the inability to trade it frequently mean that allocations to PE should not be done with the standard Merton model.

Second, the cost of employing a non-optimal strategy, for example, not holding a particular asset which should be held in an optimal portfolio, can be compared to the optimal strategy and the cost of holding the non-optimal portfolio depends on the investor’s risk aversion. That is, the cost of bearing non-optimal weights is dependent on the investor’s risk preferences. The costs are computed using utility certainty equivalents: the certainty equivalent cost is how much an investor must be compensated in dollars per initial wealth to take a non-optimal strategy but have the same utility as the optimal strategy. A relevant cost, which the subsequent literature explores, is how much an investor should be compensated for the inability to trade assets like PE for certain periods of time or to be compensated for being forced to pay a cost whenever an asset is traded.

III. B. Asset Allocation with Transactions Costs

Investing in PE incurs large transactions costs in initially finding an appropriate PE manager and conducting appropriate due diligence. Then, there are potentially large discounts to the recorded asset values that may be taken in transferring ownership of a PE stake in illiquid secondary markets. Since Constantinides (1986), a large literature has extended the Merton setup to incorporate transactions costs.
Constantinides considers the case of one risk-free and one risky asset. When there are proportional transactions costs, so that whenever the holdings of the risky asset increase (or decrease) by $v$, the holding of the riskless asset decreases by $(1+k)v$. When there are trading costs, the investor now trades infrequently. Constantinides shows that the optimal trading strategy is to trade whenever the risky asset position hits upper and lower bounds, $\underline{w}$ and $\overline{w}$, respectively. These bounds straddle the optimal Merton solution where there are no frictions. The holdings of risky to risk-free assets, $y/x$, satisfy

$$\underline{w} \leq \frac{y}{x} \leq \overline{w},$$

so that when $y/x$ lies within the interval $[\underline{w}, \overline{w}]$ there is no trade and when $y/x$ hits the boundaries on either side, the investor buys and sells appropriate amounts of the risky asset to bring the portfolio back to the Merton solution.

The no-trade interval, $\overline{w} - \underline{w}$, increases with the transactions costs, $k$, and the volatility of the risky asset. Transactions costs to sell PE portfolios in secondary markets can be extremely steep. When Harvard endowment tried to sell its PE investments in 2008, potential buyers were requiring discounts to book value of more than 50%.

Even for transactions costs of 10%, Constantinides computes no-trade intervals greater than 0.25 around an optimal holding of 0.26 for a risky asset with a volatility of 35% per annum. Thus, PE investors should expect to rebalance PE holdings very infrequently.

The certainty equivalent cost to holding a risky asset with large transactions costs is small for modest transactions costs, at approximately 0.2% for proportional transactions costs of 1%, but can be substantial for large transactions costs—which is the more relevant range of transactions costs for PE investments. For

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transactions costs of 15% or more, the required premium to bring the investor to the same level of utility as the frictionless Merton case is more than 5% per annum.

The literature has extended this framework to multiple assets (see, for example, Liu (2004)) and different types of rebalancing bands. Leland (1996) and Donohue and Yip (2003) suggest rebalancing to the edge of a band rather than to a target within a band. Others, like Pliska and Suzuki (2004) and Brown, Ozik, and Scholtz (2007) advocate extensions to two sets of bands, where different forms of trading are done at the inner band with more drastic rebalancing done at the outer band. In all these extensions, the intuition is the same: PE investments should be expected to be rebalanced very infrequently, and the rebalancing bands will be very wide. The case of transactions costs when returns are predictable is considered by Garleanu and Pedersen (2010). A related study is Longstaff (2001), who allows investors to trade continuously, but only with bounded variation so there are upper and lower bounds on the number of shares which can be traded every period. This makes Longstaff’s model similar to a time-varying transactions cost.

A major shortcoming of this literature is that it assumes that trade in assets is always possible, albeit at a cost. This is not true for PE—over a short horizon, there may be no opportunity to find a buyer and even if a buyer is found, there is not enough time, relative to the investor’s desired short horizon to raise capital, to go through legal and accounting procedures to transfer ownership. An important friction for PE investors in secondary markets is the search process in finding an appropriate buyer. There may be no opportunity to trade, even if desired, at considerable discounts. This case is what the next literature considers.

III. C. Asset Allocation with Search Frictions

As PE investments do not trade on a centralized exchange, an important part of rebalancing a PE portfolio is finding a counterparty in over-the-counter markets. Or, if money is spun off from existing PE investments, new or existing PE funds must be found to invest in. This entails a search process, incurring opportunity and search
costs, as well as a bargaining process, which reflects investors’ needs for immediate trade. The latter is captured by a transactions cost, as modeled in the previous section. The former requires a trading process that captures the discrete nature of trading opportunities.

Since Diamond (1982), search-based frictions have been modeled by Poisson arrival processes. Agents find counterparties with an intensity $\lambda$, and conditional on the arrival of the Poisson process, agents can trade and rebalance. This produces intervals where no rebalancing is possible for illiquid assets and the times when rebalancing are possible are stochastic. This notion of illiquidity is that there are times where it is not possible to trade, at any price, an illiquid asset. These particular types of stochastic rebalancing opportunities are attractive for modeling PE in another way: the exit in PE vehicles is often uncertain. Although a PE vehicle may have a stated horizon, say of 10 years, the return of cash from the underlying deals may cause large amounts of capital to be returned before the stated horizon, or in many cases the horizon is extended to maximize profitability of the underlying investments (or to maximize the collection of fees by GPs).

A number of authors have used this search technology to consider the impact of illiquidity (search) frictions in various over-the-counter markets, such as Duffie, Garleanu and Pedersen (2005, 2009). While these are important advances for showing the effect of illiquidity risk on asset prices, they are less useful for deriving asset allocation advice on optimal PE holdings. Duffie, Garleanu and Pedersen (2005, 2009) consider only risk-neutral and CARA utility cases and restrict asset holdings to be 0 or 1. Garleanu (2009) and Lagos and Rocheteau (2009) allow for unrestricted portfolio choice, but Garleanu considers only CARA utility and Lagos and Rocheteau focus on showing the existence of equilibrium with search frictions rather than on any practical calibrations. Neither study considers asset allocation with both liquid and illiquid assets.

III. D. Asset Allocation with Stochastic Non-Traded Periods
Ang, Papanikolaou, and Westerfield (2011) [APW] solve an asset allocation problem with liquid securities, corresponding to traded equity markets which can be traded at any time, and illiquid securities, which can be interpreted as a PE portfolio. The investor has CRRA utility with an infinite horizon and can only trade the illiquid security when a liquidity event occurs, which is the arrival of a Poisson process with intensity \( \lambda \). In this framework, the special case of Merton with continuous rebalancing is given by \( \lambda \to \infty \). As \( \lambda \) decreases to zero, the opportunities to rebalance the illiquid asset become more and more infrequent. The mean time between rebalancing opportunities is \( 1/\lambda \). Thus, \( \lambda \) indexes a range of illiquidity outcomes.

The inability to trade for stochastic periods introduces a new source of risk that the investor cannot hedge. This illiquidity risk induces large effects on optimal allocation relative to the Merton case. APW show that illiquidity risk affects the mix of liquid and illiquid securities even when the liquid and illiquid returns are uncorrelated and the investor has log utility.

The most important result that APW derive is that the presence of illiquidity risk induces time-varying, endogenous risk aversion. The intuition is that there are two levels of wealth that are relevant for the investor: total wealth, which is the same effect as the standard Merton problem where the risk is that if total wealth goes to zero the agent cannot consume, and liquid wealth. The agent can only consume out of liquid wealth. Thus, with illiquid and liquid assets, the investor also cares about the risk of liquid wealth going to zero. This can be interpreted as a solvency condition: an agent could be wealthy but if this wealth is tied up all in illiquid assets, the agent cannot consume. Although the CRRA agent has constant relative risk aversion, the effective risk aversion—the local curvature of how the agent trades off liquid and illiquid risk in her portfolio—is affected by the solvency ratio of the ratio of liquid to illiquid wealth. This solvency ratio also becomes a state variable that determines optimal asset allocation and consumption. This illiquidity risk causes
the optimal holdings of even the liquid asset to be lower than the optimal holding of liquid assets in a pure Merton setting.

APW derive five findings that are important considerations for investing in PE:

1. Illiquidity risk induces marked reductions in the optimal holdings of assets compared to the Merton case. Under APW’s calibrations for the same risk aversion as a 60% risky asset holding (and 40% risk-free holding) in the Merton case, introducing an average rebalancing period of once a year reduces the risky asset holding to 37%. When the average rebalancing period is once every five years, the optimal allocation is just 11%. Thus, PE, which is highly illiquid, should be held in modest amounts (if at all) in investors’ portfolios.

2. In the presence of infrequent trading, the fraction of wealth held in the illiquid asset can vary substantially and is very right skewed. That is, suppose the optimal holding to illiquid assets is 0.2 when rebalancing can take place. Then the investor should expect the range of illiquid holdings to vary from 0.15 to 0.35 during non-rebalancing times. Because of the skew, the average holdings to the illiquid asset will be higher than the optimal rebalancing point, at say 0.25. Thus, when an illiquid PE portfolio is rebalanced, the optimal rebalancing point is to a holding much lower than the average holding.

3. The consumption policy (or payout policy) with illiquid assets must be lower than the Merton payout policy with only liquid assets. Intuitively, holding illiquid assets means that there is additional solvency risk that liquid wealth goes to zero and consumption cannot be funded. Thus, payouts of funds holding illiquid assets should be lower than the case when these assets all are fully traded.
4. The presence of illiquidity risk means that an investor will not fully take advantage of opportunities that might look like close to an “arbitrage”, for example, where correlations to the liquid and illiquid returns are nearly plus or minus one. Traditional mean-variance optimizers without constraints would produce weights close to plus or minus infinity in these two assets. This does not happen when one asset is illiquid because taking advantage of this apparent arbitrage involves a strategy that causes the investor’s liquid wealth to drop to zero with positive probability. Thus, near-arbitrage conditions when there is illiquidity risk are not exploited like the Merton setting.

5. Finally, the certainty equivalent reward required for bearing illiquidity risk is large. APW report that when the liquid and illiquid returns are lowly correlated and the illiquid portfolio can be rebalanced, on average, once every five years (which is a typical turnover of many PE portfolios), the liquidity premium is over 4%. For rebalancing once a year, on average, the illiquidity premium is approximately 1%. These numbers can be used as hurdle rates for investors considering investing in PE.

A number of authors including Dai, Li, and Liu (2008), Longstaff (2009), De Roon, Guo, and Ter Horst (2009), and Ang and Bollen (2010) also consider asset allocation where the illiquid asset cannot be traded over certain periods. However, in these studies, the period of non-trading is deterministic. In contrast, the APW framework has stochastic and recurring periods of illiquidity. Deterministic non-trading periods are probably more appropriate for hedge fund investments where lock-ups have known expirations. PE investing is more open ended and has random, and infrequent, opportunities to rebalance.

APW still miss a number of practical considerations that the future literature should address. The most important one is that in the Merton setting into which APW introduce illiquidity, there are no cash distributions; all risky asset returns (both liquid and illiquid) are capital gains. PE investments require cashflow management
of capital calls and distributions. Some ad-hoc simulations have been conducted by some industry analysts on this issue, like Siegel (2008) and Leibowitz and Bova (2009), but without explicitly solving optimal portfolios with illiquidity risk. An extension of APW to incorporate cashflow streams could address this.

III. E. Summary

The inability to continuously rebalance PE positions, potentially even by paying transactions costs, makes optimal holdings of illiquid PE investments very different from the standard Merton framework which assumes no illiquidity risk. Since transactions costs in rebalancing PE portfolios are very large, in both entering new PE positions and selling existing PE positions, PE positions should be expected to be rebalanced very infrequently and investors should set very wide rebalancing bands. In asset allocation models where illiquid assets like PE can only be traded upon the arrival of a (stochastically occurring) liquidity event, illiquidity risk markedly reduces the holdings of illiquid assets compared to the standard Merton model. For example, an asset which could be traded continuously in the Merton setting that is held with a 60% optimal weight would have an optimal holding of less than 10% if it could be rebalanced only once every ten years, on average. The certainty equivalent reward, or equivalently the hurdle rate, for bearing illiquidity risk is large. For a typical PE investment that can be traded only once in ten years, on average, the illiquidity premium is well above 4%.

IV. Intermediary Issues in Private Equity

Most commonly, asset owners make PE investments as an LP in a fund where investment decisions are made by fund managers acting as GPs. This arrangement raises potential agency issues. One characteristic of PE investment is that the investment decisions arising from such management considerations and the related agency issues become intrinsically intertwined with PE performance. In public
equity markets, factor returns and active management can mostly be separated due to the existence of investable index strategies.

While the agency problem is central for PE investments, there is only a small literature on optimal delegated portfolio management (see the good surveys by the BIS (2003) and Stracca (2006)). There is, however, a large literature on agency issues in standard corporate finance settings (see, for example, the textbook by Salanie (1997) and Bolton and Dewatripont (2005)). Delegated portfolio management is different from standard agency problems because the “action” chosen is generally observed (the investments made by the GP), but the set of actions is unknown (the full set of deals available to the GP). In contrast, in standard moral hazard problems the “action” is unobservable, but the set of potential actions is usually known. Thus, little is known about the optimal delegated portfolio contract, and the literature has few, if any, specific conclusions or prescriptions about what form the optimal PE contract between LPs and GPs should take.

PE investing is further complicated by having two levels of principal-agent relations rather than just a single one: a level between the LPs (principal) and GPs (agent) and another level between the GPs as fund managers (principal) and its underlying portfolio of companies (agent). Both levels rely on strong direct monetary incentives. Apart from these monetary incentives, however, the relation between LPs and GPs is one with limited information, poor monitoring, rigid fee structures, and the inability to withdraw capital, or directly control managers. These features may exacerbate tensions between the LPs and GPs and exacerbate, rather than alleviate, agency issues. On the other hand, the distance between the LP and GP may allow GPs to invest and manage companies more freely.

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8 There are other technical reasons that make the delegated optimal portfolio management problem challenging. The agent (fund manager) can control the both the mean, which is the response to the signal by buying a good stock, but also the variance, through leverage. In a typical agency problem the agent controls only the mean (occasionally the variance), but not both. In continuous time, which is often used to solve agency problems, diffusion dynamics are effectively observable at high enough frequencies.
The other principal-agent relation between the fund and its portfolio companies is one with strong governance, transparent information flows, good incentives for monitoring, and a high alignment of interests between owners and management (Jensen, 1989). There is strong evidence that PE funds add significant value, on average, to the companies in their portfolio. This literature is surveyed by Kaplan and Stromberg (2009).

The interactions between these two layers of principal-agent problems have not been fully explored. It is not inconceivable, though, that mitigating the principal-agent problems at the LP-GP level would come at the cost of increasing the problems at the fund-company level. For example, greater transparency about the management of individual portfolio companies may in turn lead GPs to manage these companies with an eye towards managing short-term earnings expectations and satisfying public expectations more broadly, a concern for publicly traded companies, rather than simply managing companies to maximize their total value.

IV. B. Private Equity Contracts

Because PE is, by its nature, private, it is difficult to perform systematic large-sample studies of contractual features and see how they relate to performance. Gompers and Lerner (1999), Litvak (2009), and Metrick and Yasuda (2010) examine small samples of various PE contracts. Several tentative conclusions emerge:

1. PE contracts are largely standardized. An often-quoted fee arrangement is a management fee of 2% and a carry of 20%. There is some variation in the numbers (e.g., management fees tend to vary between 1-2.5% and carried interest varies between 20-35%), but the general structure is widely used. Additionally, a substantial part of the GPs compensation may be in the form of transaction fees. PE fees are high.

2. There is some variation in the specific provisions governing the calculation and timing of the fees and carried interest. For example, a management fee could be flat (on committed capital), declining over the life of the fund, a
(time-varying but deterministic) combination of committed and managed capital, or even an absolute amount.

3. Fixed fee and performance components are not substitutes but complements. That is, funds tend to raise both the fixed fee and variable fee components (and the other compensation components) together. Fund size tends to positively correlated with fees, yet as Kaplan and Schoar (2005) and many others find, is negatively correlated with performance. Ignoring the endogeneity of fees and effort expended by the GP, these two facts imply that LPs are worse off in larger funds, which have both relatively poorer gross and net performance.

4. There is a debate about the performance sensitivity of PE compensation. Metrick and Yasuda (2010) find that close to one-half the present value of GP compensation are from management fees rather than carried interest and find this to be true for both VC and BO funds. However, Chung et al. (2011) point out that a substantial amount of GPs' performance pay arises through the continuation value of raising future funds, which are highly sensitive to current performance.

5. PE contracts are complex documents. Litvak (2009), however, finds little relation between opaqueness and total compensation.

The management fees charged by private equity and venture capital funds are high. According to Metrick and Yasuda (2010) such fees consume at least one-fifth of gross PE returns. Metrick and Yasuda found that out of every $100 invested with a VC fund, an average of $23 is paid to the GPs in the form of carry and management fees. For BOs, the mean of the carry and management fees comes to $18 per $100.

The high fees charged by GPs point to the fact that if an institutional investor wishing to allocate to PE can do this in-house, then there are substantial savings available. Of course, attracting talent and running an in-house PE shop presents a different set of agency issues than out-sourcing to PE funds with GPs. Despite the pessimistic view of returns of PE investments to LPs in Section II, the high PE fees
implies that if asset owners can come close to capturing gross returns, PE becomes much more attractive.

While opacity per se does not seem to be related to total compensation and returns, opacity has other important knock-on effects for other aspects of an asset owner’s larger portfolio. Complexity and non-transparency can increase agency problems and make risk management more difficult. The leverage involved in many BO funds can be more expensive, and is often harder to monitor, than leverage done directly by the asset owner.

V. Conclusions

To summarize our findings and recommendations for PE investing:

1. In measuring PE risk and returns, selection bias – the phenomenon that valuations, or company deals, tend to be observed only when underlying returns are good – is a major problem. Naïve analysis without accounting for sample selection bias overstates average returns, understates risk measures like betas, and results in volatility estimates that are too low. Studies taking into account sample selection reduce alpha, or average return, estimates by more than half. Company-level volatility estimates of PE returns are extremely high.

Recommendation: For investors with current PE portfolios where company-level data is available, measurement of PE returns must use econometric techniques to handle selection bias. Investors should substantially adjust downwards expected return estimates and adjust upwards risk estimates of “raw” PE returns.

2. The most commonly used fund-performance measure, the IRR, is highly problematic and does not represent true underlying returns. Even this most basic measure has substantial variation across various studies and data
sources. Other fund-level performance statistics show that since the 2000s, PE has not out-performed public market equities and in many cases has significantly under-performed.

Recommendation: For PE investments post-2000, expect performance in line or below that, on average, of public equity market returns.

3. Models of asset allocation that take into account transactions costs, which are large for PE, and illiquidity risk, which is substantial for PE, recommend modest holdings of PE. In these models, rebalancing will be infrequent, so wide swings in the holdings of PE should be expected, and the holdings of illiquid PE will be much lower than predicted by asset allocation models assuming that all assets can be rebalanced when desired.

Recommendation: When determining optimal PE allocations, asset allocation models must take into account the inability to rebalance PE positions. There should be generally modest allocations to illiquid PE investments.

4. Current PE vehicles have substantial agency issues which public equity vehicles do not. While there is heterogeneity in PE contracts, PE fees are very large and consume at least one-fifth of gross PE returns. Incentive fees account for less than one-third of GP compensation.

Recommendation: If any of the fees paid to externally-managed PE funds with GPs can be brought back in-house to institutional asset owners, then if quality in PE investments can be maintained, there will be substantial savings to asset owners.
Bibliography


Ang, Andrew, Dimitri Papanikolaou and Mark Westerfield (2011) "Portfolio Choice with Illiquid Asset," working paper


Dai, Li and Liu (2008) wp

De Roon, Guo and Ter Horst (2009) wp


Harris, Robert, Tim Jenkinson, and Steven N. Kaplan (2011) “Private Equity Performance: What Do We Know?” working paper


Longstaff (2009) forthcoming AER


