Risk Taking and Optimal Contracts for Money Managers*

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Abstract

We develop a general model of delegated portfolio management, with the feature that the agent can control the riskiness of the portfolio. This represents a departure from the existing literature on agency theory in that moral hazard is not only effort exertion but also risk taking behavior. The principal's problem now involves an incentive-compatibility constraint on risk, which we characterize. Under general conditions, we show that the optimal contract is simply a bonus contract: the agent is paid a fixed sum if the portfolio return is above a threshold. We derive a criterion to decide whether the optimal contract induces excessive or insufficient risk. If a deviation from efficient risk taking causes a large (small) reduction in the expected return of the portfolio, the optimal contract induces excessive (insufficient) risk. In other words, the cheaper it is to play with risk, the *less* risk the agent takes.

Keywords: money managers, risk, moral hazard, limited liability.

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1 Introduction

A money manager receives funds from investors who desire to invest their money in financial markets but do not have the time or the knowledge to do it personally. In such a situation, the money manager chooses an investment strategy on behalf of the clients. While investors or the law can dictate some characteristics of the investment strategy, the manager is usually left with a degree of freedom in determining the composition of the portfolio. In particular, the money manager has some control over the riskiness of the portfolio.

One can expect money managers to adjust the risk level in order to maximize their (implicit or explicit) compensation, which gives rise to an agency model in which the agent controls risk.[1] This problem is linked to two lines of work: the microeconomic literature on moral hazard and the finance literature on delegated portfolio management. Unfortunately, neither of the two lines gives a satisfying answer on this point.

The vast literature on moral hazard (See Salanié (1997) or Hart and Holmstrom (1987) for surveys) has mostly focused on the problem of a principal who wants to induce an agent to exert the 'right' amount of effort. The possibility of the agent controlling the riskiness of the outcome is excluded by the monotone likelihood ratio assumption (Milgrom 1981), which guarantees that the effect that different actions have on the expected value of the outcome dominates all other effects.[2]

The literature on delegated portfolio management includes, among others, Cohen and Starks (1988), Grinblatt and Titman (1989), Stoughton (1993), Heinkel and Stoughton (1994), Goetzmann, Ingersoll, and Ross (1997), and Das and Sundaram (1998). In contrast with the moral hazard literature, these authors attach great importance to the agent's incentives for risk-taking. However, they do not consider full-fledged principal-agent models. Rather, they restrict attention to particular classes of contracts, such as piecewise linear. Instead, our model does not put any limitation on the shape of the contracts the investor can use.

In the present paper, we model the interaction between a risk neutral investor and a risk neutral money manager who can refuse negative compensations ex post. This limited liability assumption prevents the investor from selling the investment return to the manager in exchange for its expected

[1] For empirical studies of how fund managers adjust risk in response to incentive considerations, See Brown, Harlow and Starks (1996) and Chevalier and Ellison (1997).

[2] There are some exceptions, which are reviewed in the Related Literature section.

value, which would be first-best.

The money manager makes two choices: effort and risk. Effort can be seen as time and resources put into collecting and processing information on investment opportunities. A high level of effort allows the manager to access a set of feasible portfolios which is better in a first-order stochastic dominance sense. The other dimension of the manager's choice is the riskiness of the portfolio. He can select an element of the feasible set of portfolios. There is an efficient level of risk. Higher or lower levels reduce the investor's expected return.

While large institutional investors may monitor the behavior of their money managers, most smaller investors do not have the time or the knowledge to perform the monitoring and do not observe the distribution of the portfolio the agent chooses but only the realized return on the portfolio. Hence, the contract between the investor and the agent can only be based on the observed return.[3] The agent is also assumed to be able to sabotage the portfolio ex post. This means that the principal is not able to offer contracts in which compensation is nondecreasing in the observed return (Innes 1990).

In such a framework, the investor maximizes her expected net return subject to an incentive-compatibility constraint on effort, which is a familiar concept in moral hazard, and to an incentive-compatibility constraint on risk, which is introduced here. This paper analyzes the properties of optimal contracts, with a particular emphasis on two questions. First, does the optimal contract take the following very simple shape: the agent receives a fixed sum if the observed return is above a certain threshold? This schedule is called a *bonus contract* and appears to be widely used in practice. Second, do optimal contracts induce the agent to take the efficient level of risk? In case of negative answer, is the agent's behavior too conservative or too reckless?

Results depend crucially on the ability of a low-effort agent to control risk. To isolate this effect, we first look at the special case in which an agent who does not exert effort can only access riskless portfolios. Under mild conditions, there exists a bonus contract that leaves no rent to the agent and induces him to exert high effort and to take the efficient level of risk.

In the more general case in which a low-effort agent accesses also risky portfolios, the principal is forced to leave a rent to the agent and, generically, the optimal contract does not induce efficient

[3] Alternatively, one could assume that the investor observes the composition of the portfolio but she does not know as precisely as the agent how that affects the mean and the variance of the portfolio return (if she did, she would not hire the agent in the first place). Unless the principal can use a two-stage mechanism à la Bhattacharya and Pfleiderer (1985), this more general framework would yield similar result.

risk taking. The principal cannot separate revenue maximization and rent minimization. Even if attacked through the first-order approach (Mirrlees 1975, Grossman and Hart 1983, Rogerson 1985, and Jewitt 1988), the principal's problem appears very hard. However, we show that, under the validity of the first-order approach, the set of optimal contracts contains a bonus contract, which has a simple characterization. The threshold above which the agent receives the bonus is determined by a tradeoff between being as close as possible to the efficient portfolio and reducing the agent's rent, which can be expressed as a function of the difference between the hazard rate of the high-effort agent and that of the low-effort agent. Given the threshold, the amount of the bonus is computed to make the agent indifferent between low and high effort.

Given that the optimal contract does not induce efficient risk taking, does it make the agent too conservative or too reckless? In the asymptotic case in which the effect of effort (and its cost) goes to zero, this question can be answered precisely, through a condition that depends only on the properties of the hazard rate and the curvature of the efficient frontier of the set of feasible portfolios, both evaluated at the efficient risk level.

The intuition is as follows. The bonus threshold and the agent's risk behavior are linked in a monotonic way. A higher threshold induces a higher risk level, both for a low-effort agent and a high-effort agent. Then, risk taking is excessive if and only if the bonus threshold is above the threshold that induces efficient risk taking.

In the determination of the bonus threshold, three forces are at work. The *efficiency effect* makes the principal reluctant to deviate from the threshold that generates the efficient level of risk. We can ignore this first effect because we are interested in determining the direction of the deviation from the efficient portfolio but not its magnitude. The other two effects come from the principal's desire to reduce rent. If the principal moves the threshold, the agent's expected rent changes for two reasons: because, holding risk taking behavior constant, a threshold change affects the probability of making the bonus (*direct compensation effect*) and because, net of the first effect, a threshold change modifies the agent's risk taking behavior, which in turn affects the probability of making the bonus (*indirect compensation effect*). The direct compensation effect leads the principal to offer a contract with a higher threshold, because increasing the threshold decreases the probability that the low-effort agent is above the threshold more (in relative terms) than it decreases the probability that the high-effort agent is above the threshold. In other words, it hurts the bad agent more than the good agent. Instead, the indirect compensation effect militates in favor of a lower threshold. Reducing the threshold lowers risk taking on the part of both the low-effort and the high-effort agent. This makes it easier for the principal to "spot" the low-effort agent and allows her to reduce the rent left to the high-effort agent.

The relative strength of the two compensation effects depends on how quickly expected return deteriorates if the agent chooses to deviate from the efficient risk level, that is, on the curvature of the efficient frontier of the set of feasible portfolios. If a deviation is punished with a big drop in expected return, then the direct effect is greater than the indirect effect, and the principal offers a contract that generates excessive risk. The opposite is true if the expected return is not very sensitive to risk.

In conclusion, if "it is cheap for the agent to play with risk," because for instance the agent can access high-powered financial instruments that allow for a good deal of hedging as well as reckless bets, then we should expect the principal to set a low bonus threshold and we should observe an inefficiently conservative behavior on the part of the agent. On the contrary, agents who operate in a more rigid environment – perhaps selecting among real projects – will be given tougher goals and will take too much risk in equilibrium.

The organization of the paper is as follows. The next subsection reviews the literature. Section 2 presents the model. Section 3 looks at the special case in which the low-effort agent can only access riskless portfolios. Section 4 studies the general case. Section 5 concludes.

Related Literature

Bhattacharya and Pfleiderer (1985) were the first to study delegated portfolio management in a principal-agent framework. In their model, an investor faces a large number of agents who vary in their forecasting ability. The first problem for the investor is to screen agents. The problem is made more difficult by the assumption that better forecasters have higher opportunity costs. Once an agent is hired he observes a private signal, and the second problem of the principal is that of eliciting the agent's private signal in order to make the right portfolio decision. Bhattacharya and Pfleiderer show the existence of an optimal contract in which agents truthfully report their forecasting ability and their private signal. Compensation is a concave function of return, increasing if return is below the mean and decreasing if it is above the mean. Our work differs from Bhattacharya and Pfleiderer because it is a hidden action model rather than a hidden information model. In their model, the principal is able to verify the level of risk taken by the agent, while in ours she is not. Another difference is that we assume that the agent can sabotage the ex post return. This forces the principal to offer only nondecreasing contracts. In contrast, Bhattacharya and Pfleiderer's optimal contract is nonmonotonic.[4]

Some authors have considered moral hazard problems in the presence of the limited liability constraint. Sappington (1983) does it for a model of hidden information, while Innes (1990) does it for hidden action and is therefore more closely related to our work. Innes assumes the Monotone Likelihood Ratio Condition, which in our paper is clearly violated because the agent controls risk. It is interesting to compare our results with his. He shows that the optimal contract (subject to the monotonicity constraint) is a debt contract whereby the principal receives the whole return up to a certain level and the residual belongs to the agent. In our framework a debt contract is clearly suboptimal because it gives the agent an incentive to take inefficiently high levels of risk. Indeed, we prove that any contract that is convex (or concave) in portfolio return is not first-best.

Gollier, Koehl, and Rochet (1997) consider the problem of a risk-averse decision maker with limited liability. The decision maker chooses the size of a risky project. The distribution function of returns is left in a general form. The authors show that the level of risk chosen by the decision-maker is always higher under limited liability than under full liability. They also provide comparative static results on the role of the decison-maker initial wealth. While we keep the same level of generality as Gollier, Koehl, and Rochet, our model is clearly different because it is developed in a principal-agent framework.

The most closely related work is Diamond (1998). Like us, he studies a hidden action moral hazard problem in which the agent controls both effort and the distribution of the outcome. He asks whether, as the cost of effort shrinks relative to the payoffs, the optimal contract converges to the linear contract. The answer is positive if the control space of the agent has full dimensionality (i.e. if the principal has less degrees of freedom in setting the incentives than the agent has degrees of freedom in responding), but not otherwise. Dimensionality formalizes the important intuition that, if the agent has several ways to manipulate the outcome, the principal should offer the simplest possible compensation scheme, that is, the linear contract. There are some important differences between our framework and Diamond's. First, while he considers only three possible outcomes, our model encompasses a continuum of outcomes. Therefore, it allows for a much richer set of possible contracts and it is more suited to study financial

[4]Bhattacharya and Pfleiderer do not make a limited liability assumption. In their optimal contract the agent can incur unbounded losses. However, it is easy to modify the optimal contract in order to allow for a limited liability clause. Thus, limited liability does not seem to be a crucial difference between our work and theirs.

intermediation. Second, by using an explicit parameterization of risk, we are able to analyze the sign of inefficiencies in risk taking.

Also Hellwig (1994) and Biais and Casamatta (1999) study moral hazard with both effort and risk. However, Hellwig makes the important assumption that the result of the investment is binary: with a certain probability the investment yields a positive return; otherwise, the investment yields zero. Biais and Casamatta assume that there three possible outcomes and only two levels of risk. While these setups seem appropriate for the entrepreneur-financier relation, they do not provide an accurate description of the fund manager-investor relation in which the manager chooses a portfolio among a very large number of feasible ones and the return is an intrinsically continuous variable.[5]

2 The Model

There are two risk neutral individuals: an investor (or principal) and a money manager (or agent). The principal delegates the selection of a portfolio of risky financial assets to the agent.

We define a portfolio of financial assets as a probability distribution on the monetary return $x \in [\underline{x}, \bar{x}]$, where $\underline{x}$ could be $-\infty$ and $\bar{x}$ could be $+\infty$. The *set of feasible portfolios* is the family of distribution functions $\mathcal{F}$. We assume that an element of $\mathcal{F}$ is uniquely identified by its mean μ and a risk measure $r \in [0, \infty)$. The risk measure need not correspond to variance. A typical element of $\mathcal{F}$ is $f(\cdot|\mu, r)$. For simplicity, $f(\cdot|\mu, r)$ is assumed twice continuously differentiable for all μ and r.

We assume that, if $\mu'' > \mu'$, $f(\cdot|\mu'', r)$ dominates $f(\cdot|\mu', r)$ in a first-order stochastic dominance (FOSD) sense. If $r'' > r'$, $f(\cdot|\mu, r')$ dominates $f(\cdot|\mu, r'')$ in the second-order stochastic sense (SOSD). Thus, if two assets have the same mean, the one with the lower r is less risky than the other. Our model represents a clear departure from previous models of delegated investment in that the set of feasible portfolios is not completely ordered with respect to first-order stochastic dominance.[6]

Let $(\mu, r) \in A$ where $A = \{\mu, r | r \geq 0, -\infty < \mu \leq m(r)\}$. The function $m(\cdot)$ is twice differentiable, strictly concave and has a maximum at $\mu^* = m(r^*)$ with r^* strictly positive. Moreover, $\lim_{r \to 0} m'(r) =$

[5] This paper assumes that there is only one money manager. In the presence of multiple managers, the contract offered to one manager could depend also on the performance of the other managers (e.g. return is evaluated relative to average return). See Goriaev, Palomino, and Prat (2001) and the references therein for games with competing fund managers.

[6] Most models of moral hazard consider only two outcomes, in which case all the actions are ordered by first-order stochastic dominance. The models that consider more than one outcome usually assume the Monotone Likelihood Ratio property, which implies FOSD. See for instance Grossman and Hart (1983) or Innes (1990).

∞ and $\lim_{r\to\infty} m'(r) = -\infty$.

To interpret A, Figure 1 is useful. A represents the set of feasible risky portfolios and it is bounded above by the curve $\mu = m(r)$, which can be viewed as the efficient portfolio frontier. Each point on the frontier represents the maximum expected return that can be achieved given a certain level of risk. In a typical textbook, only the increasing part of $m(\cdot)$ is depicted. That is because a risk-neutral or risk-averse investor who selects her portfolio without using an agent would never choose portfolios to the right of r^*, as they are dominated in both a first- and second-order stochastic sense by the portfolio with μ^* and r^*. However, as we will see, a money manager with the 'wrong' incentive scheme might want to choose a portfolio to the right of r^*.[7]

[Insert Figure 1 here]

It is also assumed that the agent cannot shortsell, or has limited shortselling power. With unlimited shortselling, the principal-agent problem may not have a solution, because the agent could want to choose unbounded levels of risk.

The agent also chooses effort $e \in \{0, 1\}$. If he exerts high effort $e = 1$, he pays a monetary cost $c > 0$ and accesses the set of feasible portfolios $\mathcal{F}$ described above. If he selects $e = 0$, he can only invest in a risk-free asset with return $x_0 \in (\underline{x}, \mu^*)$. The assumption is made here to simplify the analysis but it will be removed in Section 4 where it is assumed that also the low-effort agent can choose risky portfolios.

The principal does not observe whether the agent spent c nor the portfolio he selected. Therefore, the compensation contract the principal offers to the agent can only depend on the realized outcome x. Let $b(x)$ denote such a contract. If the agent accepts the offered contract then his opportunity cost of working for this principal is normalized at zero. We make the following assumptions about the set of feasible contract and the return reported by the agent.

Assumption 1 $b(x) \geq 0$ *for* $x \in [\underline{x}, \bar{x}]$.

[7] One may object that, from a standard portfolio theory perspective, there cannot exist assets that dominate other assets on both mean and risk. This objection is correct only if all assets carry only systemic risk. A portfolio to the right of μ^* can exist if, for instance, there is one asset with a low mean, a low systemic risk, and a high idiosyncratic risk. In order to game the fee, the agent may want to select such a portfolio. This situation may capture some of the recent financial crashes, in which the money managers accumulated nonsystemic risk.

Assumption 2 *The agent can sabotage x, that is, given an actual return x, he can report any return $x' \in [\underline{x}, x]$.*

Assumption 1 implies that the agent has limited liability (normalized at zero). Assumption 2 is identical to that made by Innes (1990) and means that the agent can report a performance lower than the actual one.

A direct consequence of Assumption 2 is that the optimal contract must be nondecreasing in x. Otherwise, if the agent finds that x falls on a decreasing section of $b(x)$, he can increase his compensation by reducing x. In the rest of the paper we will consider only nondecreasing contracts.[8]

Another implicit assumption has been made: the agent cannot artificially inflate x by adding money from his own pocket. The agent may want to do that if the contract schedule displays a positive jump and he realizes that the return is just below the discontinuity point. The assumption that the agent cannot overreport is not at odds with the assumption just made that he can freely underreport. Sabotage requires destroying wealth, which in real situations could be achieved by incurring unnecessary expenses or by selling below market value. Overreporting is more complicated because it requires a monetary transfer from the agent into the portfolio, which in reality would be easy to detect and prevent. Moreover, consistently with Assumption 1, the agent is likely to be wealth constrained.

Given the assumption of risk-neutrality, the efficient portfolio is simply (μ^*, r^*). The model could be readily extended to risk-averse players. It is easy to check that the agent only chooses portfolios that lie on the efficient frontier $\mu = m(r)$.

Lemma 1 *For any $b(\cdot)$, the agent maximizes $E[b(x)|\mu, r]$ by choosing μ and r such that $\mu = m(r)$.*

Proof. Immediate from $b(\cdot)$ being nondecreasing and first-order stochastic dominance. ■

To summarize, the timing of the principal-agent relationship is: (1) The principal proposes a contract $b(\cdot)$ to the agent; (2) If the agent accepts, he receives a unitary sum to manage for one period; (3) The agent chooses whether or not to spend c. If he does not spend c, he invests in the risk-free asset with return x_0; (4) If the agent spends c, he chooses $(\mu, r) \in A$. x is realized according to $f(x|\mu, r)$; (5) The principal pays $b(x)$ to the agent and keeps $x - b(x)$.

[8] A discussion on the reasons for excluding nonmonotonic contracts is found in Innes (1990).

3 Simplified Environment

We now analyze the model under the assumption – discussed above – that an agent who does not exert effort can only buy riskless portfolios. The next section will deal with the general case.

A contract b is *optimal* if it maximizes the expected net payoff of the principal $E[x - b(x)|m(r), r]$ subject to the agent's choice of effort and risk given b. The best that the principal can hope for is a contract that induces the agent to choose the efficient risk r^* and leaves him no rent. We call such a contract *first-best* because actions and payoffs are the same that would arise if the principal could contract directly on actions (r and e). If a first-best contract exists it must be optimal.

Definition 1 *A contract b is first-best if:*
(i) The agent's participation constraint choice binds

$$E[b(x)|\mu^*, r^*] = c; \tag{1}$$

(ii) The incentive-compatibility constraint on effort is satisfied

$$E[b(x)|\mu^*, r^*] - c \geq b(x_0); \tag{2}$$

and (iii) The incentive-compatibility constraint on risk is satisfied

$$E[b(x)|\mu^*, r^*] \geq E[b(x)|m(r), r] \qquad \text{for any } r \geq 0. \tag{3}$$

As $b(x_0) \geq 0$, (i) is satisfied only if (ii) is binding and $b(x_0) = 0$. A linear contract $b(x) = B + Ax$ (with $A \geq 0$) always satisfies (3). To satisfy (1) and (2), it must be that $B + A\mu^* = c$, implying a negative B. But this means that the limited liability of the agent is violated with positive probability (except in the trivial case in which $\underline{x} = x_0$). Thus, there is no first-best linear contract.

3.1 Necessary conditions for the existence of a first-best contract

Let $f_r(x|\mu, r) \equiv \frac{\partial}{\partial r} f(x|\mu, r)$ and $f_\mu(x|\mu, r) \equiv \frac{\partial}{\partial \mu} f(x|\mu, r)$. The following lemma provides a simple condition that first-best contracts must satisfy:

Lemma 2 *If, given contract $b(\cdot)$, the agent chooses $r = r^*$, then*

$$\int_{\underline{x}}^{\bar{x}} b(x) f_r(x|\mu^*, r^*) dx = 0. \tag{4}$$

Proof. Recall that $f(x|\mu, r)$ is continuous and differentiable in μ and r for any fixed x. Therefore, $E[b(x)|\mu, r]$ is continuous and differentiable in μ and r. For given $b(\cdot)$ and $m(\cdot)$, the agent sets $\hat{r}$ such that

$$\frac{\partial}{\partial\mu}E[b(x)|m(\hat{r}),\hat{r}]m'(\hat{r}) + \frac{\partial}{\partial r}E[b(x)|m(\hat{r}),\hat{r}] = 0. \tag{5}$$

If $\hat{r} = r^*$, then $m'(\hat{r}) = 0$, and (5) can be rewritten as (4). ■

Lemma 2 allows us to exclude two types of contract from the class of first-best contracts:[9]

Proposition 1 *If $b(\cdot)$ is twice differentiable and globally strictly convex or globally strictly concave on a bounded interval $[\underline{x}, \bar{x}]$, then it cannot implement the first best portfolio (μ^*, r^*).*[10]

Proof. Let us assume that $b(\cdot)$ is globally strictly convex. By integrating per parts twice,

$$\begin{aligned}
\int_{\underline{x}}^{\bar{x}} b(x)f_r(x|\mu^*,r^*)dx &= b(\bar{x})F_r(\bar{x}|\mu^*,r^*) - b(\underline{x})F_r(\underline{x}|\mu^*,r^*) - \int_{\underline{x}}^{\bar{x}} b'(x)F_r(x|\mu^*,r^*)dx \\
&= -\int_{\underline{x}}^{\bar{x}} b'(x)F_r(x|\mu^*,r^*)dx \\
&= \left[-b'(x)\int_{\underline{x}}^{x} F_r(t|\mu^*,r^*)dt\right]_{\underline{x}}^{\bar{x}} + \int_{\underline{x}}^{\bar{x}} b''(x)\left(\int_{\underline{x}}^{x} F_r(t|\mu^*,r^*)dt\right)dx \\
&= -b'(\bar{x})\int_{\underline{x}}^{\bar{x}} F_r(x|\mu^*,r^*)dx + \int_{\underline{x}}^{\bar{x}} b''(x)\left(\int_{\underline{x}}^{x} F_r(t|\mu^*,r^*)dt\right)dx,
\end{aligned}$$

where the second equality is due to the fact that $F_r(\bar{x}|\mu^*, r^*) = F_r(\underline{x}|\mu^*, r^*) = 0$ (because $F(\bar{x}|\mu, r) = 1$ and $F(\underline{x}|\mu, r) = 0$ for every μ and r). However,

$$\int_{\underline{x}}^{\bar{x}} F_r(x|\mu^*,r^*)dx = \bar{x}F_r(\bar{x}|\mu^*,r^*) - \underline{x}F_r(\underline{x}|\mu^*,r^*) - \int_{\underline{x}}^{\bar{x}} xf_r(x|\mu^*,r^*)dx = 0,$$

because $\int_{\underline{x}}^{\bar{x}} xf_r(x|\mu^*, r^*)dx = \frac{\partial}{\partial r}E(x|\mu^*, r^*) = 0$.

Moreover, if $b''(\cdot) > 0$ for all x, then $\int_{\underline{x}}^{\bar{x}} b''(x) \int_{\underline{x}}^{x} F_r(t|\mu, r)dtdx > 0$ for any $x \in (\underline{x}, \bar{x})$ by the definition of second-order stochastic dominance. Thus, we have proven that $\int_{\underline{x}}^{\bar{x}} b(x)f_r(x|\mu^*, r^*)dx > 0$. A convex contract cannot be optimal. The proof for the concave case is identical. ■

With a concave or convex contract, the incentive-compatibility constraint on risk is violated. This result is hardly surprising. With a convex contract, the agent is rewarded for high returns more than he is punished for low returns and he has an incentive to increase risk above the efficient level. The opposite holds for a concave contract.

[9] Notice that Lemma 2 and Proposition 1 do not depend on the assumption of limited liability. That is why it is not absurd to consider the possibility of concave contracts.

[10] The restriction to bounded intervals is made to avoid the possibility of infinitely high or low payments.

3.2 First-best bonus contracts

A *bonus contract* is defined by a threshold $\bar{b} \in [\underline{x}, \bar{x}]$ and a bonus amount $B > 0$:

$$b(x) = \begin{cases} B & \text{if } x \geq \bar{b} \\ 0 & \text{if } x < \bar{b} \end{cases}$$

As bonus contracts are neither everywhere concave nor convex, they are candidates for first best. We shall now give conditions under which a first-best bonus contract exists.

Assumption 3 *(i) $\mathcal{F}$ is such that, for any μ and any $r' \neq r''$, $F(x|\mu^*, r')$ and $F(x|\mu^*, r'')$ cross exactly once on $(\underline{x}, \bar{x})$. (ii) Let $\tilde{x}(r', r'')$ denote the point at which they cross. $\tilde{x}(r', r'')$ is nondecreasing in both r' and r''. (iii) $x_0 < \lim_{r' \to r^*, r'' \to r^*} \tilde{x}(r', r'')$.*

Part (i) is a technical assumption. For a given μ, $F(x|\mu, r')$ and $F(x|\mu, r'')$ must cross at least once. However, they may cross more than once. We assume they cross exactly once. Part (ii) is central to the results. The x at which two cumulative distributions cross must be nondecreasing in the r's of the two distributions. We will discuss (ii) in more detail later. Part (iii) of the assumption requires that the rate of return of the riskless portfolio is not too high.[11]

Proposition 2 *If Assumption 3 is satisfied, there exists a first-best contract, which takes the form of a bonus contract with threshold $\bar{b} = \lim_{r' \to r^*, r'' \to r^*} \tilde{x}(r', r'')$.*

Proof. By Assumption 3(i) and second-order stochastic dominance, given $r' < r''$,

$$F(x|\mu^*, r') \geq (\leq) F(x|\mu^*, r'') \text{ if } x \geq (\leq) \tilde{x}(r', r'').$$

Let $k = \lim_{r' \to r^*, r'' \to r^*} \tilde{x}(r', r'')$. If $r < r^*$, Assumption 3(ii) implies $k \geq \tilde{x}(r, r^*)$ and hence $F(k|\mu^*, r) \geq F(k|\mu^*, r^*)$. Similarly, if $r > r^*$, $k \leq \tilde{x}(r^*, r)$ and, again, $F(k|\mu^*, r) \geq F(k|\mu^*, r^*)$. Thus, for any r,

$$F(k|\mu^*, r^*) \leq F(k|\mu^*, r) \leq F(k|m(r), r),$$

where the second inequality is due to first-order stochastic dominance.

[11] Part (i) of the assumption is similar to the notion of mean preserving dispersal discussed by Müller (1998). A distribution $G(y)$ is a mean preserving t-dispersal of distribution $H(y)$ if they have the same expected value and if there exists a t such that $G(y) \geq (\leq) H(y)$ if $y < (>) t$.

If the principal offers a bonus contract with threshold $\bar{b} = k$, an agent with $e = 1$ should minimize $F(k|m(r), r)$, which is achieved by choosing $r = r^*$. By setting $B = \frac{c}{1-F(k|\mu^*,r^*)}$, the agent's expected compensation from choosing $e = 1$ and $r = r^*$ is exactly c, while by Assumption 3(iii) the compensation from choosing $e = 0$ is 0. The incentive-compatibility constraint on effort holds as an equality. This bonus contract is first-best. ■

With a bonus contract, the agent maximizes the probability of being above the threshold. If there exists a $k \in (\underline{x}, \bar{x})$ such that $F(k|\mu^*, r^*)$ is lower than $F(k|m(r), r)$ for any other r, the principal can induce the agent to choose r^* by offering a bonus contract with threshold $\bar{b} = k$. The principal can then leave no rent to the agent by letting $B = \frac{c}{1-F(k|\mu^*,r^*)}$.

Assumption 3 provides a sufficient condition for the existence of a k such that $F(k|\mu^*, r^*) = \min_r F(k|\mu^*, r)$. By FOSD, this implies that there is a point k such that $F(k|\mu^*, r^*) = \min_r F(k|m(r), r)$.

Part (ii) of the assumption has a simple geometric interpretation. Figure 2 depicts three cumulative distributions $F(x|m(r), r)$ corresponding to three levels of r: low (r_1), medium (r_2), and high (r_3). The efficient r is the medium one (it yields $\mu = 5.2$ while the other two yield only $\mu = 5$). Also assume that $x_0 = 0$, so Assumption 3(iii) is satisfied.[12]

[Insert Figure 2 here]

In the interval $[4.5, 5.833]$ the cumulative distribution corresponding to r_2 is lower than the other two. If the principal offers a bonus contract with threshold $\bar{b} \in [4.5, 5.833]$, the agent chooses r_2. Note that in this example parts (i) and (ii) of Assumption 3 are satisfied. Cumulative distributions cross only once and the crossing points are ordered in the "right" way: $\tilde{x}(r_1, r_2) < \tilde{x}(r_1, r_3) < \tilde{x}(r_2, r_3)$.[13]

[12] This example uses density functions:

$$f(x|m(r_1), r_1) = \begin{cases} .5 & \text{if } 4 \le x \le 6 \\ 0 & \text{otherwise} \end{cases}$$

$$f(x|m(r_2), r_2) = \begin{cases} .25 & \text{if } 3.2 \le x \le 7.2 \\ 0 & \text{otherwise} \end{cases}$$

$$f(x|m(r_3), r_3) = \begin{cases} .1 & \text{if } 0 \le x \le 10 \\ 0 & \text{otherwise} \end{cases}$$

[13] There are two caveats about the figure. First, it depicts a finite number of risk levels while the model assumes a continuum. But clearly one could add more levels and still respect Assumption 3. Second, Assumption 3 refers to

What happens when Assumption 3(ii) fails? Suppose that the cumulative distributions look as in Figure 3.[14] Again, the expected return is 5 for low risk and high risk, and 5.2 for medium risk, which is the efficient choice. However, now the intersections between the cumulative functions are not monotonic in r. The crossing between low risk and medium risk is higher than the crossing between high and medium. The kinked CDF representing medium risk is always above at least one of the straight CDF's representing inefficient levels of risk. As there is no x for which r_2 minimizes F, there is no bonus contract that induces the agent to choose r_2.

[Insert Figure 3 here]

However, one can find a non-bonus contract that is first-best, like a two-step contract with

$$b(x) = \begin{cases} 0 & \text{if} \quad x \in [0,3) \\ B_1 & \text{if} \quad x \in [3,6) \\ B_2 & \text{if} \quad x \in [6,10] \end{cases},$$

where B_1 and B_2 satisfy the IC constraints on risk

$$B_1 + 0.3B_2 \geq \max\{0.7B_1 + 0.4B_2, B_1\},$$

and the binding IC contraint on effort $B_1 + 0.3B_2 = c$ (for instance, $2B_1 = B_2 = 1.25c$ would work).

A special case is when the family of distribution functions $\mathcal{F}$ contains only distributions that are symmetric around the mean (like the family of normal distributions):

Assumption 4 *Let $\underline{x} = -\infty$ and $\bar{x} = \infty$. For all feasible μ and r and for all $z \in \Re$, $f(\mu - z|\mu, r) = f(\mu + z|\mu, r)$.*

$F(\cdot|\mu^*, r)$ while the figure depicts $F(\cdot|m(r), r)$. The monotonicity condition on the the crossings of $F(|\mu^*, r)$ is sufficient for the existence of a first-best bonus contract but does not imply monotonicity on the crossings of $F(|m(r), r)$. Thus, the figure is approximately correct only if r_1 and r_3 are close to r_2.

[14] The present example uses the same distributions for r_1 and r_3 of the previous example. The density function for r_2 is

$$f(x|m(r_2), r_2) = \begin{cases} .3 & \text{if } 3 \leq x \leq 5 \\ .1 & \text{if } 5 \leq x \leq 9 \\ 0 & \text{otherwise} \end{cases}.$$

Assumption 4 implies (i) and (ii) of Assumption 3 because in a symmetric distribution, all cumulative functions with mean μ^* cross at μ^*. Also, in a symmetric distribution, Assumption 3(iii) reduces to $\mu^* \geq x_0$. Then, Proposition 2 rewrites as:

Corollary 1 *If Assumption 4 is satisfied and $\mu^* \geq x_0$, then a bonus contract with $\bar{b} = \mu^*$ is first-best.*

Some remarks are in order. First, these results would not be valid if at least one of the two parties were not risk-neutral. With an insurance motive on either side, the optimal contact tends to be smoother than the bonus contract.

Second, there are distributions that satisfy Assumption 3 but not 4. An example is the lognormal. To see this, recall that if $\log x$ is normally distributed with mean θ and variance σ^2, then x is lognormally distributed with mean $\mu = e^{\theta + \frac{\sigma^2}{2}}$ and variance $r^2 = e^{2\theta + \sigma^2}\left(e^{\sigma^2} - 1\right)$. Take two lognormals generated from parameters (θ_1, σ_1) and (θ_2, σ_2) with $\sigma_1 \neq \sigma_2$. Assume that $\theta_1 + \frac{\sigma_1^2}{2} = \theta_2 + \frac{\sigma_2^2}{2} = \log \mu$. The two cumulative distributions cross at k such that

$$\frac{\log k - \log \mu + \frac{\sigma_1^2}{2}}{\sigma_1} = \frac{\log k - \log \mu + \frac{\sigma_2^2}{2}}{\sigma_2},$$

implying $\log k = \log \mu + \frac{\sigma_1 \sigma_2}{2}$. Then, k is increasing in σ_1 and σ_2 and therefore in r_1 and r_2.

Third, there are other first-best contracts that are not bonus contracts. If Assumption 4 holds, sufficient conditions for a non-decreasing contract to be first-best are that: *(i)* $b(x_0) = 0$; and *(ii)* $b(\mu^* - z) = B - b(\mu^* + z)$ for all $z \in \Re$.

4 The General Case

In the two previous sections, it was assumed that an agent who does not exert effort can only access a riskless asset. This assumption is unrealistic in financial markets because the agent has always the option of creating a risky portfolio by buying assets at random. This section extends the analysis to the general case in which also the low-effort agent can play with risk.

We first set out the problem formally and show that, under a certain condition, the set of optimal contracts contains a bonus contract. We then ask whether the optimal bonus contract generates risk taking below or above the efficient level.

4.1 Optimality of bonus contracts

If $e = 1$, the agent accesses the set of feasible portfolios described in Section 2. If $e = 0$, the agent chooses a risk level r_0, which translates into an expected value $\mu_0 = m_0(r_0)$. Let $g(x|\mu_0, r_0)$ be the density function of x given that the agent selects $e = 0$ and r_0. Let $G(x|\mu_0, r_0)$ be the corresponding cumulative distribution function. The functions g, G, and m_0 have the same regularity properties of f, F, and m. We also assume that, for any r, $F(x|m(r), r)$ strictly dominates $G(x|m_0(r), r)$ in a first-order stochastic sense. This implies that $\max_{r_0} m_0(r_0) < m(r^*)$. All F and G have the same support $[\underline{x}, \bar{x}]$. As before, we assume that in the optimal solution it is always in the interest of the principal that the agent exerts effort $e = 1$.

In this richer setup we should not expect optimal contracts to be first best. As we shall see, optimal contracts typically induce inefficient risk taking and leave a positive rent to the agent. The principal's problem is

$$\max_{b,\hat{r},\hat{r}_0,e} m(\hat{r}) - \int_{\underline{x}}^{\bar{x}} b(x) f(x|m(\hat{r}), \hat{r}) dx \tag{6}$$

subject to the IC constraint on effort, the IC constraint on risk for the high-effort type, the IC constraint on effort for the low-effort type, and the nonnegativity and monotonicity constraints:[15]

$$\int_{\underline{x}}^{\bar{x}} b(x) f(x|m(\hat{r}), \hat{r}) dx - \int_{\underline{x}}^{\bar{x}} b(x) g(x|m_0(\hat{r}_0), \hat{r}_0) dx \geq c; \tag{7}$$

$$\hat{r} \in \arg\max_r \int_{\underline{x}}^{\bar{x}} b(x) f(x|m(r), r) dx; \tag{8}$$

$$\hat{r}_0 \in \arg\max_r \int_{\underline{x}}^{\bar{x}} b(x) g(x|m_0(r), r) dx; \tag{9}$$

$$b(x) \geq 0, b(x') \geq b(x) \qquad \forall x, \forall x' > x. \tag{10}$$

The principal's problem can be attacked through the so-called *first-order approach* (Mirrlees 1975, Grossman and Hart 1983, Rogerson 1985, and Jewitt 1988). First, instead of (8) and (9), one writes

[15] The incentive compatibilty constraints should be on the action of the agent, which can be written as (e, r, r_0). One should compare the expected utility of the agent in $(1, \hat{r}, \hat{r}_0)$ with that of any other (e, r, r_0). These comparisons reduce to (7), (8), and (9).

the IC constraints on risk as first-order conditions:

$$\left.\frac{d}{dr}\int_{\underline{x}}^{\bar{x}} b(x)f(x|m(r),r)dx\right|_{r=\hat{r}} = 0; \tag{11}$$

$$\left.\frac{d}{dr}\int_{\underline{x}}^{\bar{x}} b(x)g(x|m_0(r),r)dx\right|_{r=\hat{r}_0} = 0. \tag{12}$$

Second, one finds the first-order conditions to the constrained maximization problem defined by (6) subject to (7), (10), (11), and (12). If the solution (assuming there is one) of the system of first-order conditions is also a solution to the original principal's problem, then we say that the first-order approach is valid.[16]

The first-order approach generates an optimal control problem on the function b, which is still a hard problem. Fortunately, we can show that the set of optimal contracts always contains a bonus contract. This simplifies dramatically the problem because a bonus contract is identified by just two variables. The proposition also supplies a characterization of the optimal bonus contract:

Proposition 3 *If the first-order approach is valid, the set of optimal contracts contains a bonus contract* $(\bar{b}, B)$*. The bonus contract satisfies*

$$\left.\frac{d}{dr}F(\bar{b}|m(r),r)\right|_{r=\hat{r}} = 0 \qquad \left.\frac{d}{dr}G(\bar{b}|m_0(r_0),r_0)\right|_{r_0=\hat{r}_0} = 0; \tag{13}$$

$$m'(\hat{r})\frac{d\hat{r}}{d\bar{b}} = K\left(h^f(\bar{b}|m(\hat{r}),\hat{r}) - h^g(\bar{b}|m_0(\hat{r}_0),\hat{r}_0)\right)c; \tag{14}$$

$$B = \frac{c}{G\left(\bar{b}|m_0(\hat{r}_0),\hat{r}_0\right) - F\left(\bar{b}|m(\hat{r}),\hat{r}\right)}; \tag{15}$$

where

$$h^f(x|\mu,r) \equiv \frac{f(x|\mu,r)}{1-F(x|\mu,r)} \qquad h^g(x|\mu,r) \equiv \frac{g(x|\mu,r)}{1-G(x|\mu,r)};$$

$$\frac{d\hat{r}}{d\bar{b}} = -\frac{f_\mu(\bar{b}|m(\hat{r}),\hat{r})m'(\hat{r}) + f_r(\bar{b}|m(\hat{r}),\hat{r})}{F_{\mu\mu}(\bar{b}|m(\hat{r}),\hat{r})\left(m'(\hat{r})\right)^2 + 2F_{\mu r}(\bar{b}|m(\hat{r}),\hat{r})m'(\hat{r}) + F_\mu(\bar{b}|m(\hat{r}),\hat{r})m''(\hat{r}) + F_{rr}(\bar{b}|m(\hat{r}),\hat{r})}; \tag{16}$$

[16] General sufficient conditions for the validity of the first-order approach are discussed by Rogerson (1985) and Jewitt (1988). Unfortunately, they do not apply here because they include the monotone likelihood ratio property. An extension of those conditions is outside the scope of the paper. However, the lack of *general* sufficient conditions need not prevent the approach to be valid and useful in many particular problems. A possible method is to apply the first-order approach first and then check that the solution thus found is indeed a global maximum.

$$K = \frac{\left(1 - G(\bar{b}|m_0(\hat{r}_0), \hat{r}_0)\right)\left(1 - F(\bar{b}|m(\hat{r}), \hat{r})\right)}{\left(G\left(\bar{b}|m_0(\hat{r}_0), \hat{r}_0\right) - F\left(\bar{b}|m(\hat{r}), \hat{r}\right)\right)^2} > 0.$$

Proof. See Appendix. ■

The first part of the proposition says that either the first-order approach is not valid (in which case, the hope of finding an analytical solution is slim) or the principal can focus without loss of generality on bonus contracts. The proof proceeds by showing that, if a bonus contract satisfies the first-order conditions for optimality within the class of bonus contracts, then it also satisfies the first-order conditions for optimality for unrestricted contracts.[17]

The intuition is simple. In the classical principal-agent problem without risk taking (see Hart and Holmstrom 1987), if both parties are risk-neutral the principal offers a high compensation for only one outcome and zero for all the others (which creates an openness problem in the case in which the set of outcomes is infinite, as it is here). The outcome with the high compensation is the one that maximizes the likelihood ratio that the outcome comes from a high-effort rather than a low-effort agent. The optimal contract is then a "bang-bang" solution: for almost all outcomes, compensation is zero and for one outcome it is as high as possible. There are two differences between the classical principal-agent problem and the present problem: risk taking and the constraint that contracts must be monotonic non-decreasing. Risk taking complicates the analysis but does not change the essentially linear nature of the problem. The second difference leads to a re-writing of the problem in terms of increments in compensation rather than absolute values. But the gist of the result is the same: the solution has a bang-bang character, meaning that only one of the increments in $b(x)$ is strictly positive, that is, the contract is a bonus contract (and the openness problem goes away because the probability of getting positive compensation does not go to zero).

The intuition may fail if the risk choice of the agent is not continuous in the contract he is offered because the solution could be at the discontinuity point. This is why the assumption that the first-order approach is valid must be made.[18]

The second part of the proposition offers a straightforward way of finding the optimal contract.

[17] The proposition does not say that the set of optimal contracts contains only bonus contracts. However, from the proof one can see that situations in which also more complicated contracts satisfy the conditions for the first-order approach are nongeneric. See Footnote 22 on page 25 for details.

[18] Given that Propositions 2 and 3 are both about first-best bonus contracts, it is natural to ask what the connection is between the first-order approach and Assumption 3. Take the environment of Section 4 and let g degenerate to a distribution that puts probability 1 on x_0 independently of r_0. Then, we have the environment of Section 3.

The conditions in (13) give the risk levels $\hat{r}$ and $\hat{r}_0$ as implicit functions of the threshold $\bar{b}$. Condition (15) sets the amount of the bonus B so that the IC on effort is binding. Condition (14) is the most interesting. It finds the optimal threshold $\bar{b}$ by equating $\frac{d}{db}E[x]$ on the left-hand side with $\frac{d}{db}E[b(x)]$ on the right. The marginal effect of the threshold on $E[x]$ is easy to interpret (although its sign will remain ambiguous until the next subsection). To understand the right-hand side, note that, provided that the incentive-compatibility constraint on effort is binding, the expected compensation is

$$Y\left(\bar{b}\right) = B\left(1 - F(\bar{b}|\hat{r}(\bar{b}))\right) = \frac{1 - F(\bar{b}|\hat{r}(\bar{b}))}{G\left(\bar{b}|\hat{r}_0(\bar{b})\right) - F\left(\bar{b}|\hat{r}(\bar{b})\right)}c = c + \frac{1 - G(\bar{b}|\hat{r}_0(\bar{b}))}{\left(1 - F(\bar{b}|\hat{r}(\bar{b}))\right) - \left(1 - G(\bar{b}|\hat{r}_0(\bar{b}))\right)}c. \quad (17)$$

Total compensation is made up of the refund of the cost of effort c plus a rent component that is equal to the ratio of the probability that the good type gets the bonus and the probability that the good type gets the bonus while the bad type does not. In the extreme case when the low type never gets the bonus (and the good type gets it with positive probability), the rent is zero. In the other extreme case when both types make it with the same probability, the rent goes to infinity.

A well-known tool for the analysis of moral hazard is the likelihood ratio between the distribution generated by high effort and that generated by low effort (Hart and Holmstrom 1987). Usually, this refers to density functions, but here, because of the restriction that contracts must be monotonic, it applies to cumulative distributions. Reducing the agent's rent is equivalent to maximizing the likelihood ratio between the event that the high effort makes the bonus and the event that the low effort agent makes the bonus: $\frac{1-F(\bar{b}|\hat{r}(\bar{b}))}{1-G(\bar{b}|\hat{r}_0(\bar{b}))}$.

The derivative of this ratio with respect to $\bar{b}$ turns out to be an increasing function of the difference between the hazard rate of the low-effort type at $x = \bar{b}$ (i.e., $h^g(\bar{b}|m_0(\hat{r}_0), \hat{r}_0)$) and the hazard rate of the high-effort type again at $x = \bar{b}$ (i.e., $h^f(\bar{b}|m(\hat{r}), \hat{r})$). This is because the hazard rate at $x = \bar{b}$ measures the relative reduction in compensation. The principal wants to increase the bonus threshold if this hurts, in relative terms, the low-effort agent more than the high-effort one.

A condition for the validity of the first-order approach is that the first-order condition for the high-effort agent (11) has a unique solution. With multiple solutions, we do not know what r the agent chooses. For every x the function $F(x|m(r), r)$ must have a U-shape with a unique local minimum $\hat{r}$. This is where Assumption 3 comes to play. Reconsider Figure 3: the optimal contract is not a bonus contract. In a neighborhood of $x = 5$, $F(x|m(r_2), r_2) > F(x|m(r_1), r_1)$ and $F(x|m(r_2), r_2) > F(x|m(r_3), r_3)$. There are two local minima: r_1 and r_3. The function is not U-shaped. If we take an F that for some values of r behaves like in Figure 3, we get that: (1) Assumption 3(ii) is violated; (2) the first-order approach is not valid.

Notice instead that, if F is like in Figure 2, for every x, $F(x|m(r), r)$ is U-shaped and has a unique minimum.

4.2 Risk properties of optimal bonus contracts

It is easy to see that in a generic setup the optimal bonus contract does not generate efficient risk taking. In order to induce $\hat{r} = r^*$, it must be that $h^f(\bar{b}|m(r^*), r^*) = h^g(\bar{b}|m_0(\hat{r}_0), \hat{r}_0)$, but it would be a pure coincidence if the threshold that induces the high-effort agent to choose r^* is also the threshold that induces the low-effort agent to choose the $\hat{r}_0$ that equates the two hazard rates. The goal of this subsection is to put a sign on the direction of the inefficiency.

But first, we have to be able to sign $\frac{d\hat{r}}{d\bar{b}}$. Intuitively, one expects the agent's choice of risk to be increasing in the threshold because a higher target rewards a fatter upper tail. However, one could find functional forms for which $\frac{d\hat{r}}{d\bar{b}} < 0$ over some intervals. In order to ensure monotonicity, we need a technical condition:

Assumption 5 *For every x, μ, and r, $f_r(x|\mu, r) < \frac{f_\mu(x|\mu,r)F_r(x|\mu,r)}{F_\mu(x|\mu,r)}$.*

Second-order stochastic dominance imposes no restriction on f_r for a particular x. From (16), we see that $\frac{d\hat{r}}{d\bar{b}}$ evaluated at a particular x depends on f_r and f_μ at that x. Assumption 5 puts an upper bound on $f_r(x|\mu, r)$.[19]

Proposition 4 *Under Assumption 5, $\frac{d\hat{r}}{d\bar{b}} > 0$ for every $\bar{b}$.*

Proof. See Appendix. ■

An immediate consequence of Propositions 3 and 4 and of the concavity of m is:

Corollary 2 *Under Assumption 5, $\hat{r} > (<)(=)r^*$ if and only if $h^g(\bar{b}|m_0(\hat{r}_0), \hat{r}_0) > (<)(=)h^f(\bar{b}|m(\hat{r}), \hat{r})$.*

The principal tolerates an inefficiently high (low) risk only if a marginal increase (decrease) in risk damages the low-effort agent more than the high-effort agent.

We are now ready to ask whether the optimal contract deviates in the direction of too much or too little risk taking. To answer this question, we look at the asymptotic case in which the effect of effort on the portfolio distribution goes to zero.

[19] Assumption 5 is satisfied, for example, when x is normally distributed:

$$\frac{f_\mu(x|\mu,r)F_r(x|\mu,r)}{F_\mu(x|\mu,r)} - f_r(x|\mu,r) = \frac{1}{\sqrt{2\pi r^2}} \exp\left(-\frac{(x-\mu)^2}{2r^2}\right) > 0.$$

We first make several technical assumptions. Let $\underline{x} = -\infty$ and $\bar{x} = \infty$. We assume that the feasible portfolio set of the low-effort agent is just a shifted version of that of the high-effort agent: for every x and r,

$$G(x|m_0(r), r) = F(x + s|m(r), r),$$

where s is a positive constant. This implies that, for every r, $m_0(r) = m(r) - s$. Also, $h^g(x|\mu, r) = h^g(x + s|\mu, r)$ (in what follows we drop the superscript f).

Also, f is symmetric around the mean, namely, for every x, μ, and r, $f(\mu - x|\mu, r) = f(\mu + x|\mu, r)$. Finally, $f_r(x|m(r^*), r^*) = 0$ for at most two values of x. Then an increase in r is guaranteed to increase the density of the tails and reduce it in the middle. Assumption 5 is kept as well. All the assumptions made so far are satisfied, for example, by the normal distribution. As usual, subscripts denote partial derivatives. In particular,

$$h_x(x|\mu, r) = \frac{\partial}{\partial x} h(x|\mu, r) \qquad \text{and} \qquad h_r(x|\mu, r) = \frac{\partial}{\partial r} h(x|\mu, r).$$

Given f, the principal's problem is now defined by two nonnegative parameters: c and s. Assume that c and s are such that the principal wants to induce $e = 1$. Let $\bar{b}^*(c, s)$ be the optimal bonus threshold given c and s, and let $\hat{r}\left(\bar{b}^*(c, s)\right)$ be the associated risk.

Proposition 5 *There exists $\varepsilon > 0$ such that for all $s \in (0, \varepsilon)$ there exists $\delta(s) > 0$ such that for all $c \in (0, \delta(s))$, $\hat{r}\left(\bar{b}^*(c, s)\right) \geq (\leq) r^*$ if and only if*

$$h_x(\mu^*|\mu^*, r^*) \geq (\leq) h_r(\mu^*|\mu^*, r^*) \frac{f_r(\mu^*|\mu^*, r^*)}{m''(r^*) F_\mu(\mu^*|\mu^*, r^*)}. \tag{18}$$

Both sides of the inequality are positive.

Proof. See Appendix. ■

In a symmetric distribution, a threshold $\bar{b} = \mu^*$ induces the agent to choose the efficient risk level. By Proposition 4, $\bar{b}^*(c, s) \geq \mu^*$ if and only if $\hat{r}(c, s) \geq r^*$. So Proposition 5 supplies a condition under which the optimal bonus threshold is above/below the mean and risk is above/below the efficient level.

The proposition identifies three forces that determine the optimal $\hat{r}$: the efficiency effect, the direct compensation effect, and the indirect compensation effect. All three effects go to zero in the asymptotic case but their signs are determined unambiguously for s positive but small.

The efficiency effect is the simplest and it is captured by the left-hand side of (14). A change in the bonus threshold affects the principal's expected return through the agent's choice of risk. The principal wants $\hat{r}$ to be as close as possible to r^*, which in the symmetric case means choosing $\bar{b}^* = \mu^*$. The efficiency effect determines how much the principal is willing to deviate from $\hat{r} = r^*$ but it says nothing about whether the principal prefers $\hat{r} > r^*$ or $\hat{r} < r^*$. This is why this effect does not appear in Proposition 5, which is only about the sign of the deviation from r^*.

The other two effects relate to the principal's desire to keep the agent's rent low, which, as we saw above, is equivalent to maximizing $\frac{1-F(\bar{b}|\hat{r}(\bar{b}))}{1-G(\bar{b}|\hat{r}_0(\bar{b}))}$. The threshold $\bar{b}$ affects the expected compensation through a change in the probability of making the bonus (direct effect) and indirectly through a change in the risk choice of the agent (indirect effect). The direct effect is captured by the left-hand side of inequality (18), that is, the derivative of the hazard rate with respect to the outcome. In a symmetric distribution, $h_x(\mu^*|\mu^*, r^*) \geq 0$ (for the normal distribution, h_x is always positive). The direct effect militates in favor of higher risk taking. With an increasing hazard rate, the ratio $\frac{1-F(x|r)}{1-F(x+s|r)}$ is increasing in x. If the high-effort agent and the low-effort agent choose the same level of risk r, an increase in the threshold $\bar{b}^*$ increases the ratio between the expected rewards of the two agents, which in turn reduces the rent left to the high-effort agent.

Figures 4 and 5 illustrate the direct effect for a normal distribution. In Figure 4 the bonus threshold is set at the efficient level $\bar{b} = \mu^*$, while in Figure 5 it has been shifted to the right. Each graph depicts the density function for the low-effort agent and that for the high effort agent. The expected compensation of the agent is proportional to the area of the tail of the distribution to the right of the threshold. As the picture shows, an increase in $\bar{b}$ reduces the tails for both the high-effort and the low-effort agent, but the reduction of the latter is larger in relative terms.[20]

[Insert Figures 4 and 5 here]

If the agent could not play with risk, the analysis would stop here and the principal would set a large threshold $\bar{b}$ (mitigated only by the efficiency effect). However, the endogeneity of risk creates an opposite force, the indirect compensation effect, which corresponds to the following mental exercise.

[20] Figures 4 and 5, as well as Figures 6–9, are trying to illustrate an asymptotic phenomenon but they are drawn on the basis of a noninfintesimal s. Therefore, they should be taken as approximations.

Suppose the principal could control $\hat{r}$ and $\hat{r}_0$ without actually having to move the threshold away from the efficient level $\bar{b} = \mu^*$. In which direction would the principal want to move $\hat{r}$ and $\hat{r}_0$? The answer in the asymptotic case is: always downwards. The intuition is that, if the agent chooses a conservative portfolio, the outcome x has a low variance and it is easy to tell whether the agent exerted high effort. Formally, the indirect effect is captured by the right-hand side of inequality (18). It is easy to check that, mostly because of symmetry, each of the four terms is negative.

To illustrate the indirect effect, consider Figures 6 through 9, where the bonus threshold is held constant at the efficient level but risk changes. In Figures 6 and 7 (the thicker line represents a density function corresponding to a lower risk), the high effort agent decreases $\hat{r}$. As the agent was at the efficient level μ^* and, in the asymptotic case, the second-order effect on $m(r)$ can be ignored, symmetry implies that a change in $\hat{r}$ does not affect the probability of making the bonus, which is one half. Things change in Figures 8 and 9, in which a decrease in $\hat{r}_0$ is shown to damage the low-effort agent because the probability that he gets the bonus decreases.

[Insert Figures 6, 7, 8, and 9 here]

In general, we cannot say whether the direct or the indirect effect prevails. The answer has to do with the ability of the agent to play with risk. Holding F fixed, the cost incurred for deviating from the optimal r depends on the curvature of m at the efficient level, that is, $-m''(r^*)$. As (18) shows, there exists a curvature ρ, such that the direct compensation effect prevails if and only if $-m''(r^*) > \rho$.

This leads to a simple, but perhaps counterintuitive, implication: everything else equal, if it is cheap for the agent to play with risk – i.e. the curvature ρ is low – the principal should set a threshold below μ^*. The agent has then a high probability of making the bonus and he plays conservatively. Thus, if principals offer optimal contracts, situations in which the agent can easily manipulate risk should be associated with safe bonusses and excessively conservative behavior. For instance, an agent who can trade any stock has more betting and hedging opportunities – and therefore a lower ρ – than one who can only trade domestic stocks. This model predicts that the agent whose trade is restricted will face a higher bonus threshold and will take more risk in equilibrium.

When return is normally distributed: $h_x(\mu^*|\mu^*, r^*) = \frac{2}{\pi (r^*)^2}$; $h_r(\mu^*|\mu^*, r^*) = -\frac{\sqrt{\frac{2}{\pi}}}{(r^*)^2}$; $f_r(\mu^*|\mu^*, r^*) =$

$-\frac{1}{(r^*)^2\sqrt{2\pi}}$; and $F_\mu(\mu^*|\mu^*, r^*) = -f(\mu^*|\mu^*, r^*) = -\frac{1}{r^*\sqrt{2\pi}}$. Inequality (18) becomes

$$-m''(r^*) \geq \sqrt{\frac{\pi}{2(r^*)^2}}.$$

For instance, if $m(r) = -\frac{1}{2}\gamma(1-r)^2$ and $r^* = 1$, the optimal contract generates inefficiently high risk if and only if $\gamma > \sqrt{\frac{\pi}{2}}$.

5 Conclusions

This work is a first step toward a general theory of moral hazard with risk taking. The main substantive question we have tackled is whether the optimal contract generates risk taking above or below first best. Future research should extend the analysis to two other problems that are of theoretical interest and practical importance.

First, in our model effort is a binary variable and the principal always wants to induce high effort. One could instead assume that effort, like risk, is a continuous variable and that in general the principal wants an interior solution. It would be interesting to know if optimal contracts generate the efficient level of effort. If there are distortions, one should investigate whether they are linked to distortions in risk taking, e.g. optimal contracts that generate inefficiently low risk taking also generate inefficiently low effort.

Second, the present model is static. More realistically, a money manager receives his bonus based on results at the end of the year but has daily control of his portfolio. The manager can then condition his current portfolio on past performance, which creates an additional opportunity for gaming the fee. For instance, an agent with a low interim result is tempted to look for high-risk investments – a strategy known as "gambling for resurrection" – while an agent with a good interim performance would want to play conservatively in the later part of the year (these effects are observed empirically by Chevalier and Ellison (1997) and Brown, Harlow and Starks 1996)). It would be interesting to know what the optimal contract is in such a model. One may conjecture that highly nonlinear schemes like bonus contracts give the agent too much incentive to play with risk, and are less likely to emerge in a dynamic setup.[21]

[21] A previous version of this paper, available from the authors on request, presents some preliminary results on the multi-period case.

Appendix

Proof of Proposition 3

In this proof, we use the following notation:

$$\begin{aligned}\tilde{F}(x|r) &= F(x|m(r),r) \qquad \tilde{f}(x|r) = f(x|m(r),r);\\ \tilde{G}(x|r) &= G(x|m(r),r) \qquad \tilde{g}(x|r) = g(x|m(r),r).\end{aligned}$$

As before, partial derivatives are denoted with subscripts (but note that, for instance, $\tilde{F}_r(x|r)$ is different from $F_r(x|m(r),r)$).

The proposition is proven through a discretization of the original continuous problem. Suppose the principal is restricted to use contracts that have no more than n steps, where n is a positive integer (the figure represents a 6-step contract defined on $[\underline{x},\bar{x}] = [0,10]$). Being nondecreasing, $b(\cdot)$ is Riemann-integrable. As n tends to infinity, any contract $b(\cdot)$ is approximated by a sequence of n-step contracts.

[Insert Figure 10 here]

Formally, an n-step contract is defined by n thresholds and bonus levels $\{b_i, B_i\}_{i=1,\ldots,n}$, with b_i and B_i nondecreasing in i and the convention that $b_{n+1} = \bar{x}$. Note that the first step may or may not coincide with $\underline{x}$ (in the figure it does not). If n is held fixed and the principal is restricted to using n-step contracts, the principal's problem defined by (6), (7), (10), (11), and (12) specializes to:

$$\max_{b,\hat{r},\hat{r}_0,e} m(\hat{r}) - \sum_{i=1}^{n} B_i \left(\tilde{F}(b_{i+1}|\hat{r}) - \tilde{F}(b_i|\hat{r})\right)$$

subject to:

$$\sum_{i=1}^{n} B_i \left(\tilde{F}(b_{i+1}|\hat{r}) - \tilde{F}(b_i|\hat{r})\right) - \sum_{i=1}^{n} B_i \left(\tilde{G}(b_{i+1}|\hat{r}) - \tilde{G}(b_i|\hat{r})\right) \geq c;$$

$$\sum_{i=1}^{n} B_i \left(\tilde{F}_r(b_{i+1}|\hat{r}) - \tilde{F}_r(b_i|\hat{r})\right) = 0;$$

$$\sum_{i=1}^{n} B_i \left(\tilde{G}_r(b_{i+1}|\hat{r}_0) - \tilde{G}_r(b_i|\hat{r}_0)\right) = 0;$$

$$b_1 \geq 0, B_1 \geq 0 \qquad \text{and} \qquad b_{i+1} \geq b_i, B_{i+1} \geq B_i \qquad \forall i.$$

For $i = 1, ..., n$, let $\beta_i = B_i - B_{i-1}$ (with the convention that $B_0 = 0$). Then, the n-step problem rewrites as:

$$\max_{b,\beta,\hat{r},\hat{r}_0} m(\hat{r}) - \sum_i \beta_i \left(1 - \tilde{F}(b_i|\hat{r})\right) \tag{19}$$

subject to

$$c \leq \sum_i \beta_i \left(\tilde{G}\left(b_i|\hat{r}_0\right) - \tilde{F}\left(b_i|\hat{r}\right)\right) \tag{20}$$

$$\sum_i \beta_i \tilde{F}_r\left(b_i|\hat{r}\right) = 0 \tag{21}$$

$$\sum_i \beta_i \tilde{G}_r\left(b_i|\hat{r}_0\right) = 0 \tag{22}$$

$$\beta_i \geq 0 \quad \forall i \tag{23}$$

The constraint that b_i is nondecreasing can be dropped. If the condition is violated, there exists a reindexing of i such that the constraint is satisfied but everything else is the same.

We are now ready to apply the first-order approach:

Lemma 3 *The first-order approach conditions for the n-step problem are: (23), (21), (22) and:*

$$-m'(\hat{r})\frac{\tilde{f}_r(b_j|\hat{r})}{\sum_i \beta_i \tilde{F}_{rr}(b_i|\hat{r})} + \frac{\tilde{g}(b_j|\hat{r}_0)Y - \tilde{f}(b_j|\hat{r})Y_0}{Y - Y_0} = 0 \quad \forall j; \tag{24}$$

$$-m'(\hat{r})\frac{\tilde{F}_r(b_j|\hat{r})}{\sum_i \beta_i \tilde{F}_{rr}(b_i|\hat{r})} - \frac{\left(1-\tilde{G}(b_j|\hat{r}_0)\right)Y - \left(1-\tilde{F}(b_j|\hat{r})\right)Y_0}{Y - Y_0} = 0 \quad \forall j; \tag{25}$$

$$Y - Y_0 = c; \tag{26}$$

where $Y = \sum_i \beta_i \left(1 - \tilde{F}(b_i|\hat{r})\right)$ *and* $Y_0 = \sum_i \beta_i (1 - \tilde{G}(b_i|\hat{r}_0))$.[22]

Proof of Lemma 3: The lagrangian is

$$\begin{aligned} L &= m(\hat{r}) - \sum_i \beta_i \left(1 - \tilde{F}(b_i|\hat{r})\right) + \lambda \sum_i \beta_i \tilde{F}_r(b_i|\hat{r}) \\ &\quad + \lambda_0 \sum_i \beta_i \tilde{G}_r(b_i|\hat{r}_0) - \gamma \left(\sum_i \beta_i \left(\tilde{G}\left(b_i|\hat{r}_0\right) - \tilde{F}\left(b_i|\hat{r}\right)\right) - c\right), \end{aligned}$$

[22] Consider the system of $2n$ equations formed by (24) and (25). The vector β appears only through $\hat{r}$, $\hat{r}_0$, $\sum_i \beta_i \tilde{F}_{rr}(b_i|\hat{r})$, Y, and Y_0. However, Y_0 is fixed given Y. This means that (24), (25) can be rewritten as a system with $n+4$ variables (the b's, $\sum_i \beta_i \tilde{F}_{rr}(b_i|\hat{r})$, Y, $\hat{r}$ and $\hat{r}_0$). Thus, if $n \geq 3$, the system has more equations than variables and generically it has no solution (besides the "special" solution that is discussed in Lemma 3).

yielding first-order conditions: (21), (22), (26) (it is immediate to check that (20) is binding), and

$$-1+\tilde{F}(b_j|\hat{r})+\lambda\tilde{F}_r(b_j|\hat{r})+\lambda_0\tilde{G}_r(b_j|\hat{r}_0)-\gamma\left(\tilde{G}\left(b_j|\hat{r}_0\right)-\tilde{F}\left(b_j|\hat{r}\right)\right)=0 \quad \forall j; \tag{27}$$

$$\tilde{f}(b_j|\hat{r})+\lambda\tilde{f}_r(b_j|\hat{r})+\lambda_0\tilde{g}_r(b_j|\hat{r}_0)-\gamma\left(\tilde{g}\left(b_j|\hat{r}_0\right)-\tilde{f}\left(b_j|\hat{r}\right)\right)=0 \quad \forall j; \tag{28}$$

$$m'(\hat{r})+\lambda\sum_i\beta_i\tilde{F}_{rr}(b_i|\hat{r})=0; \tag{29}$$

$$\lambda_0\sum_i\beta_i\tilde{G}_{rr}(b_i|\hat{r}_0)=0; \tag{30}$$

where the fact that all β's are positive has been used for (28), and (21) and (22) have been used to get (29) and (30).

From (30), $\lambda_0=0$ (a change in $\hat{r}_0$ has no first-order effect on $\sum_i\beta_i\tilde{G}_r(b_i|\hat{r}_0)$, which is the only channel through which it affects the objective function).

Pre-multiplying (27) by β_i, summing over i, and substituting (21) and (22), yields

$$\gamma=-\frac{\sum_i\beta_i\left(1-\tilde{F}(b_i|\hat{r})\right)}{\sum_i\beta_i\left(\tilde{G}\left(b_i|\hat{r}_0\right)-\tilde{F}\left(b_i|\hat{r}\right)\right)}.$$

Hence,

$$\begin{aligned}
&1-\tilde{F}(b_j|\hat{r})+\gamma\left(\tilde{G}\left(b_j|\hat{r}_0\right)-\tilde{F}\left(b_j|\hat{r}\right)\right)\\
=&\ \frac{\left(1-\tilde{F}(b_j|\hat{r})\right)\left(\sum_i\beta_i\left(\tilde{G}\left(b_i|\hat{r}_0\right)-\tilde{F}\left(b_i|\hat{r}\right)\right)\right)-\left(\sum_i\beta_i\left(1-\tilde{F}(b_i|\hat{r})\right)\right)\left(\tilde{G}\left(b_j|\hat{r}_0\right)-\tilde{F}\left(b_j|\hat{r}\right)\right)}{\sum_i\beta_i\left(\tilde{G}\left(b_i|\hat{r}_0\right)-\tilde{F}\left(b_i|\hat{r}\right)\right)}\\
=&\ \frac{\left(1-\tilde{G}\left(b_j|\hat{r}_0\right)\right)\sum_i\beta_i\left(1-\tilde{F}(b_i|\hat{r})\right)-\left(1-\tilde{F}(b_j|\hat{r})\right)\sum_i\beta_i\left(1-\tilde{G}\left(b_i|\hat{r}_0\right)\right)}{\sum_i\beta_i\left(\tilde{G}\left(b_i|\hat{r}_0\right)-\tilde{F}\left(b_i|\hat{r}\right)\right)}.
\end{aligned}$$

With this and substituting $\lambda=-\frac{m'(\hat{r})}{\sum_i\beta_i\tilde{F}_{rr}(b_i|\hat{r})}$, (27) rewrites as (25). Similarly,

$$\begin{aligned}
&\tilde{f}(b_j|\hat{r})-\gamma\left(\tilde{g}\left(b_j|\hat{r}_0\right)-\tilde{f}\left(b_j|\hat{r}\right)\right)\\
=&\ \frac{\left(\sum_i\beta_i\left(\tilde{G}\left(b_i|\hat{r}_0\right)-\tilde{F}\left(b_i|\hat{r}\right)\right)\right)\tilde{f}(b_j|\hat{r})+\left(\sum_i\beta_i\left(1-\tilde{F}(b_i|\hat{r})\right)\right)\left(\tilde{g}\left(b_j|\hat{r}_0\right)-\tilde{f}\left(b_j|\hat{r}\right)\right)}{\sum_i\beta_i\left(\tilde{G}\left(b_i|\hat{r}_0\right)-\tilde{F}\left(b_i|\hat{r}\right)\right)}\\
=&\ \frac{\left(\sum_i\beta_i\left(1-\tilde{F}(b_i|\hat{r})\right)\right)\tilde{g}\left(b_j|\hat{r}_0\right)-\left(\sum_i\beta_i\left(1-\tilde{G}\left(b_i|\hat{r}_0\right)\right)\right)\tilde{f}(b_j|\hat{r})}{\sum_i\beta_i\left(\tilde{G}\left(b_i|\hat{r}_0\right)-\tilde{F}\left(b_i|\hat{r}\right)\right)},
\end{aligned}$$

and (28) rewrites as (24). This completes the proof of the lemma.

The n-step contract problem with $n = 1$ is the *bonus contract problem*, namely (letting $\bar{b} = b_1$ and $\bar{\beta} = \beta_1$):

$$\max_{\bar{b},\bar{\beta},\hat{r},\hat{r}_0} m(\hat{r}) - \bar{\beta}\left(1 - \tilde{F}(\bar{b}|\hat{r})\right)$$

subject to

$$\begin{aligned} c &\leq \bar{\beta}\left(\tilde{G}\left(\bar{b}|\hat{r}_0\right) - \tilde{F}\left(\bar{b}|\hat{r}\right)\right) \\ \tilde{F}_r(\bar{b}|\hat{r}) = 0 &\qquad \tilde{G}_r(\bar{b}|\hat{r}_0) = 0 \\ \bar{\beta} &\geq 0 \end{aligned} \tag{31}$$

From Lemma 3, the first-order conditions for the bonus contract problem are:

$$\begin{aligned} \tilde{F}_r(\bar{b}|\hat{r}) &= 0 \\ \tilde{G}_r(\bar{b}|\hat{r}_0) &= 0 \end{aligned}$$

$$-m'(\hat{r})\frac{\tilde{f}_r(\bar{b}|\hat{r})}{\bar{\beta}\tilde{F}_{rr}(\bar{b}|\hat{r})} + \frac{\tilde{g}(\bar{b}|\hat{r}_0)\left(1 - \tilde{F}(\bar{b}|\hat{r})\right) - \tilde{f}(\bar{b}|\hat{r})\left(1 - \tilde{G}(\bar{b}|\hat{r}_0)\right)}{\tilde{G}\left(\bar{b}|\hat{r}_0\right) - \tilde{F}\left(\bar{b}|\hat{r}\right)} = 0 \tag{32}$$

$$\bar{\beta} = \frac{c}{\tilde{G}\left(\bar{b}|\hat{r}_0\right) - \tilde{F}\left(\bar{b}|\hat{r}\right)}$$

Condition (25) is automatically satisfied for any $\bar{b}$ because $\tilde{F}_r(\bar{b}|\hat{r}) = 0$, $1-\tilde{F}(\bar{b}|\hat{r}) = Y$, and $1-\tilde{G}(\bar{b}|\hat{r}_0) = Y_0$.

A crucial fact about conditions (24) and (25) is that, for each j, they can be written as

$$\xi(b_j, \hat{r}, \hat{r}_0, b, \beta) = 0 \quad \text{and} \quad \chi(b_j, \hat{r}, \hat{r}_0, b, \beta) = 0,$$

where ξ and χ are some functions that do not depend on j. This observation provides intuition for the following result:

Lemma 4 *If $\bar{b}$ and $\bar{\beta}$ satisfy the first-order conditions for the bonus contract problem, then, for any n, $b_1 = ... = b_n = \bar{b}$ and $\beta_1 = ... = \beta_n = \frac{1}{n}\bar{\beta}$ satisfy the first-order conditions for the n-step contract problem.*

Proof of the lemma: Let $b_1 = ... = b_n = \bar{b}$ and $\beta_1 = ... = \beta_n = \frac{1}{n}\bar{\beta}$ and substitute into the conditions listed in Proposition 3. The n conditions in (25) become identical and they are automatically satisfied. Each of the n conditions in (24) reduces to (32). The other conditions are immediate to check.

Lemma 4 holds for every n. Letting n go to infinity, we have the desired result.

We now complete the characterization of the optimal bonus contract. Substituting the binding IC constraint on effort into (32) (in place of the β that multiplies $\tilde{F}_{rr}(\bar{b}|\hat{r})$),

$$-m'(\hat{r})\frac{\tilde{f}_r(\bar{b}|\hat{r})}{\tilde{F}_{rr}(\bar{b}|\hat{r})} + \frac{\tilde{g}(\bar{b}|\hat{r}_0)\left(1-\tilde{F}(\bar{b}|\hat{r})\right) - \tilde{f}(\bar{b}|\hat{r})\left(1-\tilde{G}(\bar{b}|\hat{r}_0)\right)}{\left(\tilde{G}\left(\bar{b}|\hat{r}_0\right) - \tilde{F}\left(\bar{b}|\hat{r}\right)\right)^2}c = 0$$

which rearranges as

$$-m'(\hat{r})\frac{\tilde{f}_r(\bar{b}|\hat{r})}{\tilde{F}_{rr}(\bar{b}|\hat{r})} - \frac{(1-\tilde{G}(\bar{b}|\hat{r}_0))\left(1-\tilde{F}(\bar{b}|\hat{r})\right)}{\left(\tilde{G}\left(\bar{b}|\hat{r}_0\right) - \tilde{F}\left(\bar{b}|\hat{r}\right)\right)^2}\left(\frac{\tilde{f}(\bar{b}|\hat{r})}{1-\tilde{F}(\bar{b}|\hat{r})} - \frac{\tilde{g}(\bar{b}|\hat{r}_0)}{1-\tilde{G}(\bar{b}|\hat{r}_0)}\right)c = 0.$$

Applying the implicit function theorem on the IC constraint on risk for the high-effort agent, yields $\frac{d\hat{r}}{d\bar{b}} = -\frac{\tilde{f}_r(\bar{b}|\hat{r})}{\tilde{F}_{rr}(\bar{b}|\hat{r})}$, which gives (14).

Finally, we express $\frac{d\hat{r}}{d\bar{b}}$ in terms of f and F rather than $\tilde{f}$ and $\tilde{F}$:

$$\begin{aligned}\tilde{F}_{rr}(x|r) &= \frac{d}{dr}\left(F_\mu(x|m(r),r)m'(r) + F_r(x|m(r),r)\right) \qquad (33)\\ &= F_{\mu\mu}(x|m(r),r)\left(m'(r)\right)^2 + 2F_{\mu r}(x|m(r),r)m'(r) + F_\mu(x|m(r),r)m''(r) + F_{rr}(x|m(r),r);\end{aligned}$$

$$\tilde{f}_r(x|r) = f_\mu(x|m(r),r)m'(r) + f_r(x|m(r),r). \qquad (34)$$

Proof of Proposition 4

We are trying to determine the sign of the right-hand side of (16). The denominator corresponds to $\frac{d^2}{dr^2}F(\bar{b}|m(r),r)\Big|_{r=\hat{r}}$, or $-\frac{d^2}{dr^2}\left(1-F(\bar{b}|m(r),r)\right)\Big|_{r=\hat{r}}$, which is positive by assumption if $\hat{r}$ is an optimal choice for the high-effort agent. By substituting the first-order condition (13)

$$\frac{d}{dr}F(\bar{b}|m(\hat{r}),\hat{r})\Big|_{r=\hat{r}} = F_\mu(\bar{b}|m(\hat{r}),\hat{r})m'(\hat{r}) + F_r(\bar{b}|m(\hat{r}),\hat{r}) = 0.$$

into the numerator of (16), we obtain

$$f_\mu(\bar{b}|m(\hat{r}),\hat{r})m'(\hat{r}) + f_r(\bar{b}|m(\hat{r}),\hat{r}) = -\frac{f_\mu(\bar{b}|m(\hat{r}),\hat{r})F_r(\bar{b}|m(\hat{r}),\hat{r})}{F_\mu(\bar{b}|m(\hat{r}),\hat{r})} + f_r(\bar{b}|m(\hat{r}),\hat{r}),$$

which is negative by Assumption 5. The right-hand side of (16) is positive.

Proof of Proposition 5

For $y \in \Re \cup \{-\infty\} \cup \{\infty\}$, let sign$(y)$ be -1 if $y < 0$, 0 if $y = 0$, and 1 if $y > 0$.

Claim 1 *Suppose that p and q are real functions that are continuous on $(0, \varepsilon)$ for some positive ε. Also, for every $y \in (0, \varepsilon)$, $q(y) > 0$. If $\lim_{y \to 0^+} p(y)$ exists and is different from zero, then*

$$\lim_{y \to 0^+} \text{sign}\left(\frac{p(y)}{q(y)}\right) = \text{sign}\left(\lim_{y \to 0^+} p(y)\right).$$

Proof. For every $y \in (0, \varepsilon)$, sign$\left(\frac{p(y)}{q(y)}\right) = \text{sign}(p(y))$. If $\lim_{y \to 0^+} p(y)$ exists and is different from zero, then there exists a $\delta > 0$ such that, for all $y \in (0, \delta)$, $p(y)$ has the same sign of $\lim_{y \to 0^+} p(y)$. ■

Assume that

$$h_x(\mu^*|\mu^*, r^*) \neq h_r(\mu^*|\mu^*, r^*) \frac{f_r(\mu^*|\mu^*, r^*)}{m''(r^*) F_\mu(\mu^*|\mu^*, r^*)} \tag{35}$$

In the case of equality, the statement of Proposition 5 is always true. We shall prove that, provided (35) holds,

$$\lim_{s \to 0} \lim_{c \to 0} \text{sign}(\hat{r} - r^*) = \text{sign}\left(h_x(\mu^*|\mu^*, r^*) - h_r(\mu^*|\mu^*, r^*) \frac{f_r(\mu^*|\mu^*, r^*)}{m''(r^*) F_\mu(\mu^*|\mu^*, r^*)}\right).$$

For a given c and s, the principal may want to induce $e = 0$ or $e = 1$. For the purpose of the proof, we assume that the principal *must* offer a contract that induces $e = 1$. For any s, there exists a c such that the principal finds it worthwhile to induce the agent to choose $e = 1$. Then, at the limit (given the order we take limits), the principal wants to induce $e = 1$.

The optimal bonus contract that induces $e = 1$ satisfies the conditions in Proposition 3. Let $\bar{b}^*(c, s)$ denote the optimal $\bar{b}$ given s and c. Let $\hat{r}\left(\bar{b}\right)$ and $\hat{r}_0\left(\bar{b}, s\right)$ denote the agent's risk choices given any threshold $\bar{b}$ and s (arguments of $\bar{b}^*$, $\hat{r}$, and $\hat{r}_0$ are sometimes suppressed whenever doing so does not create confusion).

Claim 2 *For any $s > 0$, $\lim_{c \to 0} \hat{r}\left(\bar{b}^*\right) = r^*$ and $\lim_{c \to 0} \bar{b}^* = \mu^*$ Moreover, $\lim_{s \to 0} \lim_{c \to 0} \hat{r}_0\left(\bar{b}^*\right) = r^*$.*

Proof. Given the symmetry assumption, for any $s > 0$, if $\bar{b}^* = \mu^*$ the agent chooses $\hat{r} = r^*$. As $c \to 0$, the expected cost of compensation (17) goes to zero. In the limit, the optimal contract is efficient: any deviation from efficiency will cause an infinitesimal change in compensation but a first-order loss in $m(\hat{r})$. Thus, $\lim_{c \to 0} \hat{r}\left(\bar{b}^*, s\right) = r^*$ and $\lim_{c \to 0} \bar{b}^* = \mu^*$. For the second part, just note that $\lim_{c \to 0} \hat{r}_0\left(\bar{b}^*\right) = \hat{r}_0(\mu^*, s)$ and $\lim_{s \to 0} \hat{r}_0(\mu^*, s) = \hat{r}(\mu^*) = r^*$. ■

Claim 3 *For any $s > 0$, if $\lim_{c\to 0} sign\left(\frac{\partial}{\partial c}\bar{b}^*(c,s)\right)$ exists and is different from zero,*

$$\lim_{c\to 0} sign\left(\hat{r}\left(\bar{b}^*(c,s)\right) - r^*\right) = \lim_{c\to 0} sign\left(\bar{b}^*(c,s) - \mu^*\right) = \lim_{c\to 0} sign\left(\frac{\partial}{\partial c}\bar{b}^*(c,s)\right).$$

Proof. Given that the first-order approach is valid, it is easy to see from Proposition 3 that, for every positive c and s, $\bar{b}^*$ is continuous and differentiable in c. Then,

$$\bar{b}^*(c,s) - \mu^* = \int_0^c \frac{\partial}{\partial \tilde{c}}\bar{b}^*(\tilde{c},s)\, d\tilde{c}.$$

If $\lim_{c\to 0}$ sign $\frac{\partial}{\partial c}\bar{b}^*(c,s) \neq 0$, the second equality in the claim is proven. The first equality is due to Proposition 4. ■

Claim 4 *Given any positive s, if $h(\mu^* + s|m(\hat{r}_0(\mu^*)), \hat{r}_0(\mu^*)) \neq h(\mu^*|\mu^*, r^*)$, then*

$$\lim_{c\to 0}\ sign\ \left(\frac{\partial}{\partial c}\bar{b}^*(c,s)\right) = sign\ \left(h(\mu^* + s|m(\hat{r}_0(\mu^*,s)), \hat{r}_0(\mu^*,s)) - h(\mu^*|\mu^*, r^*)\right).$$

Proof. For any $\bar{b}$ (which need not be optimal), c, and s, define

$$W(\bar{b},c,s) = m'\left(\hat{r}\left(\bar{b}\right)\right)\frac{d\hat{r}}{d\bar{b}} - K\left(h\left(\bar{b}|m\left(\hat{r}\left(\bar{b}\right)\right), \hat{r}\left(\bar{b}\right)\right) - h\left(\bar{b}+s|m\left(\hat{r}_0\left(\bar{b},s\right)\right), \hat{r}_0\left(\bar{b},s\right)\right)\right)c, \tag{36}$$

where $\frac{d\hat{r}}{db}$ is as in (16). From (14), $W(\bar{b}^*(c,s), c, s) = 0$. By the implicit function theorem, for any $c, s > 0$,

$$\frac{\partial}{\partial c}\bar{b}^*(c,s) = -\frac{\left.\frac{\partial W}{\partial c}\right|_{\bar{b}=\bar{b}^*}}{\left.\frac{\partial W}{\partial \bar{b}}\right|_{\bar{b}=\bar{b}^*}}.$$

It is easy to see that $\frac{\partial W}{\partial c}$ and $\frac{\partial W}{\partial \bar{b}}$ exist and are continuous for c and s positive. As $\bar{b}^*$ is by assumption an optimum, $\left.\frac{\partial W}{\partial \bar{b}}\right|_{\bar{b}=\bar{b}^*} < 0$. By Claim 1, if $\lim_{c\to 0}$ sign $\left(\left.\frac{\partial W}{\partial c}\right|_{\bar{b}=\bar{b}^*}\right)$ exists and is different from zero, then

$$\lim_{c\to 0}\ \text{sign}\ \left(\frac{\partial}{\partial c}\bar{b}^*(c,s)\right) = \text{sign}\ \left(\lim_{c\to 0}\left.\frac{\partial W}{\partial c}\right|_{\bar{b}=\bar{b}^*}\right).$$

From (36),

$$\frac{\partial W}{\partial c} = K\left(h\left(\bar{b}+s|m\left(\hat{r}_0\left(\bar{b},s\right)\right), \hat{r}_0\left(\bar{b},s\right)\right) - h\left(\bar{b}|m\left(\hat{r}\left(\bar{b}\right)\right), \hat{r}\left(\bar{b}\right)\right)\right).$$

Using Claim 2, we see that

$$\lim_{c\to 0}\left.\frac{\partial W}{\partial c}\right|_{\bar{b}=\bar{b}^*} = K\left(h(\mu^* + s|m(\hat{r}_0(\mu^*,s)), \hat{r}_0(\mu^*,s)) - h(\mu^*|\mu^*, r^*)\right).$$

As $K > 0$, the claim is proven. ■

Claim 5 *If (35), then*

$$\begin{aligned}
&\lim_{s\to 0} sign\left(h(\mu^* + s|m\left(\hat{r}_0\left(\mu^*, s\right)\right), \hat{r}_0\left(\mu^*, s\right)) - h(\mu^*|\mu^*, r^*)\right) \\
= \; & sign\left(h_x(\mu^*|\mu^*, r^*) - h_r(\mu^*|\mu^*, r^*)\frac{f_r(\mu^*|\mu^*, r^*)}{m''\left(r^*\right) F_\mu(\mu^*|\mu^*, r^*)}\right).
\end{aligned}$$

Proof. As h is continuous and differentiable,

$$\begin{aligned}
&\lim_{s\to 0} sign\left(h(\mu^* + s|m\left(\hat{r}_0\left(\mu^*, s\right)\right), \hat{r}_0\left(\mu^*, s\right)) - h(\mu^*|\mu^*, r^*)\right) \\
= \; & sign\left(\frac{\partial}{\partial s} h(\mu^* + s|m\left(\hat{r}_0\left(\mu^*, s\right)\right), \hat{r}_0\left(\mu^*, s\right))\Big|_{s=0}\right) \\
= \; & sign\left(h_x(\mu^*|\mu^*, r^*) - h_r(\mu^*|\mu^*, r^*)\left.\frac{\partial \hat{r}_0}{\partial s}\right|_{\bar{b}=\mu^*, s=0}\right).
\end{aligned}$$

In turn,

$$\begin{aligned}
\left.\frac{\partial \hat{r}_0}{\partial s}\right|_{\bar{b}=\bar{b}^*, s=0} &= -\frac{\frac{\partial}{\partial s} F_r(\bar{b} + s|m(\hat{r}_0), \hat{r}_0)}{\frac{\partial}{\partial \hat{r}_0} F_r(\bar{b} + s|m(\hat{r}_0), \hat{r}_0)} \\
= \; & -\frac{f_r(\mu^*|\mu^*, r^*)}{F_{\mu\mu}(\mu^*|\mu^*, r^*)\left(m'(r^*)\right)^2 + 2F_{\mu r}(\mu^*|\mu^*, r^*)m'(r^*) + F_\mu(\mu^*|\mu^*, r^*)m''(r^*) + F_{rr}(\mu^*|\mu^*, r^*)} \\
= \; & -\frac{f_r(\mu^*|\mu^*, r^*)}{F_\mu(\mu^*|\mu^*, r^*)m''(\hat{r})},
\end{aligned}$$

where the last equality is due to $m'(r^*) = 0$ and $F_{rr}(\mu^*|\mu^*, r^*) = 0$ (if F is symmetric, $F(\mu^*|\mu^*, r)$ is always equal to $\frac{1}{2}$ and therefore constant in r). ■

Claim 6 *Both sides of inequality (18) are positive*

Proof. By symmetry, $F(\mu^*|\mu^*, r^*) = \frac{1}{2}$, and

$$h_x(\mu^*|\mu^*, r^*) = \frac{f_x(\mu^*|\mu^*, r^*)\frac{1}{2} + \left(f(\mu^*|\mu^*, r^*)\right)^2}{\left(\frac{1}{2}\right)^2} = 4\left(f(\mu^*|\mu^*, r^*)\right)^2 > 0,$$

where the second equality is due to the fact that $f_x(\mu^*|\mu^*, r^*) = 0$ because f is differentiable and symmetric.

Also,

$$h_r(\mu^*|\mu^*, r^*) = \frac{f_r(\mu^*|\mu^*, r^*)\frac{1}{2} + \left(F_r(\mu^*|\mu^*, r^*)\right)^2}{\left(\frac{1}{2}\right)^2} = 2f_r(\mu^*|\mu^*, r^*) < 0,$$

because SOSD, symmetry, and the assumption that $f_r(x|m(r^*), r^*) = 0$ for at most two values of x imply that $f_r(\mu^*|\mu^*, r^*) < 0$.

Finally, $F_\mu(\mu^*|\mu^*, r^*) < 0$ because of FOSD and $m''(r^*) < 0$ by definition. ■

Combining Claims 3 and 4, we have that for every positive s, if $h(\mu^* + s|m(\hat{r}_0(\mu^*)), \hat{r}_0(\mu^*)) \neq h(\mu^*|\mu^*, r^*)$, then

$$\lim_{c\to 0} sign\left(\hat{r}\left(\bar{b}^*(c,s)\right) - r^*\right) = \text{ sign }\left(h(\mu^* + s|m(\hat{r}_0(\mu^*, s)), \hat{r}_0(\mu^*, s)) - h(\mu^*|\mu^*, r^*)\right)$$

Taking the limit on s, by Claim 5, we have that, if (35),

$$\lim_{s\to 0}\lim_{c\to 0} sign\left(\hat{r}\left(\bar{b}^*(c,s)\right) - r^*\right) = \text{ sign }\left(h_x(\mu^*|\mu^*, r^*) - h_r(\mu^*|\mu^*, r^*)\frac{f_r(\mu^*|\mu^*, r^*)}{m''(r^*)F_\mu(\mu^*|\mu^*, r^*)}\right).$$

Claim 6 completes the proof.

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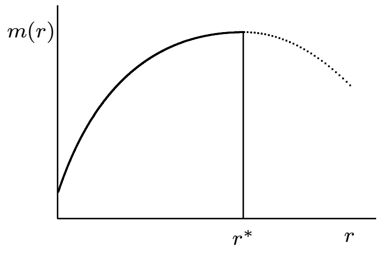


Figure 1: Figure 1

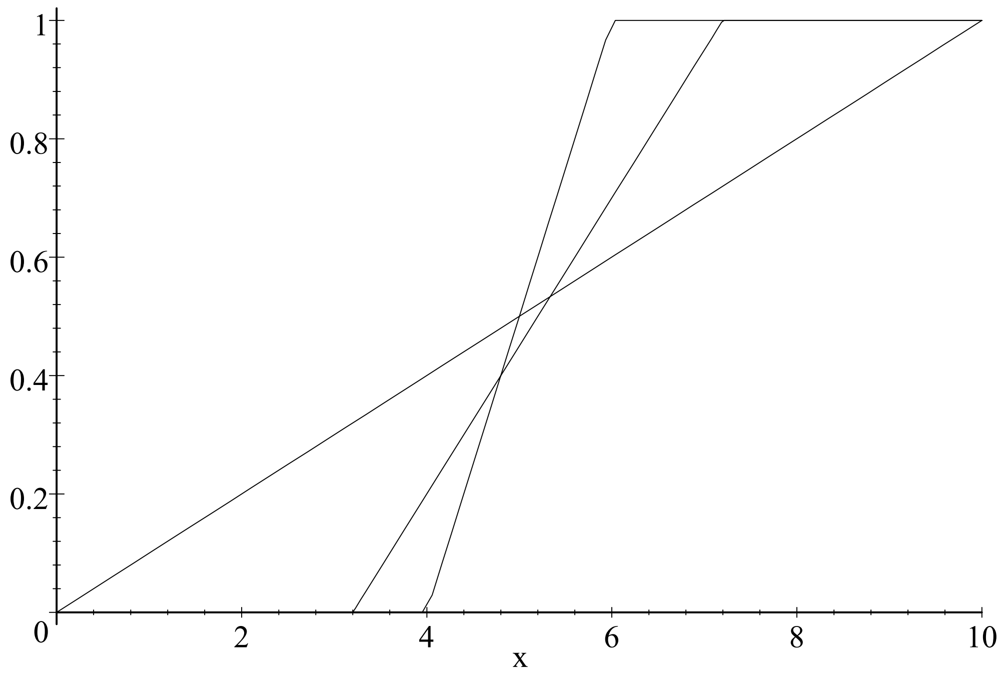


Figure 2

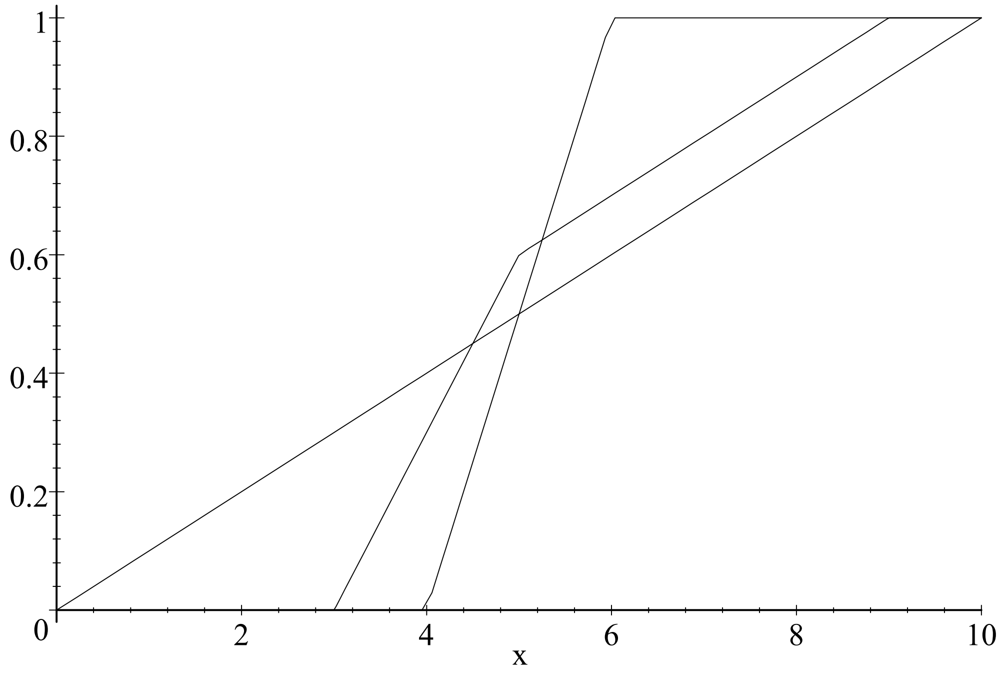


Figure 3

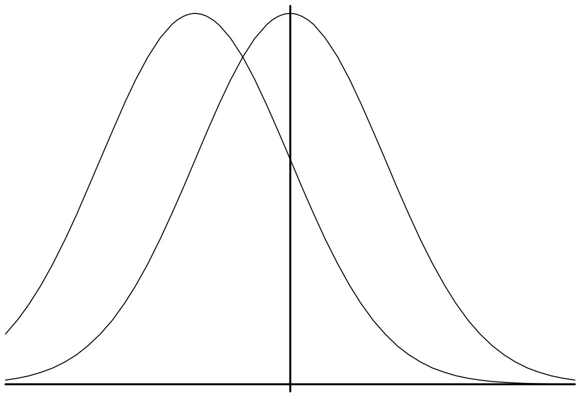


Figure 4

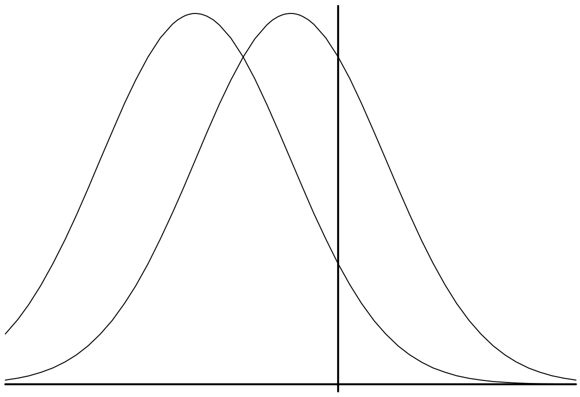


Figure 5

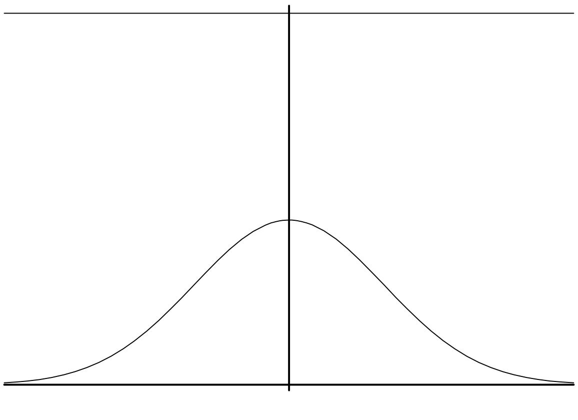
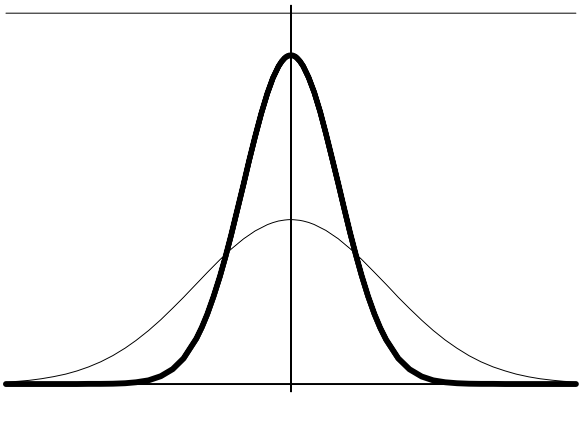


Figure 6



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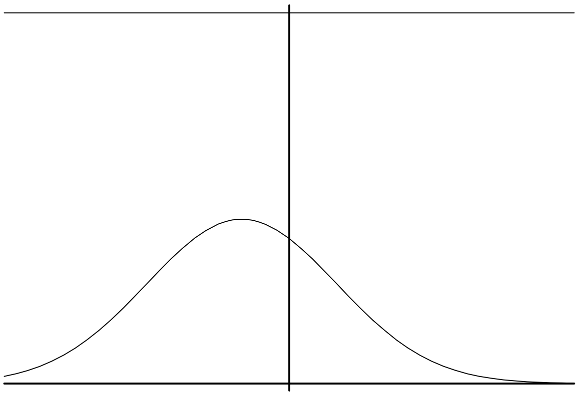

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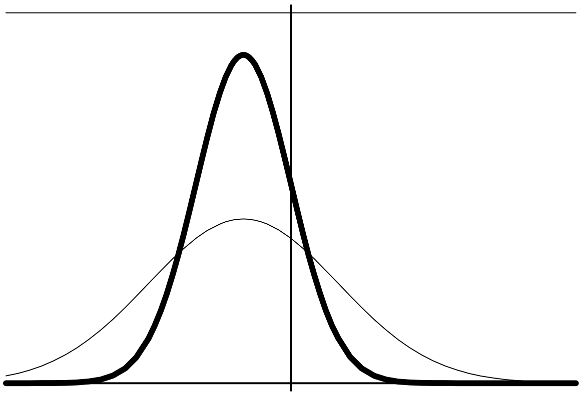

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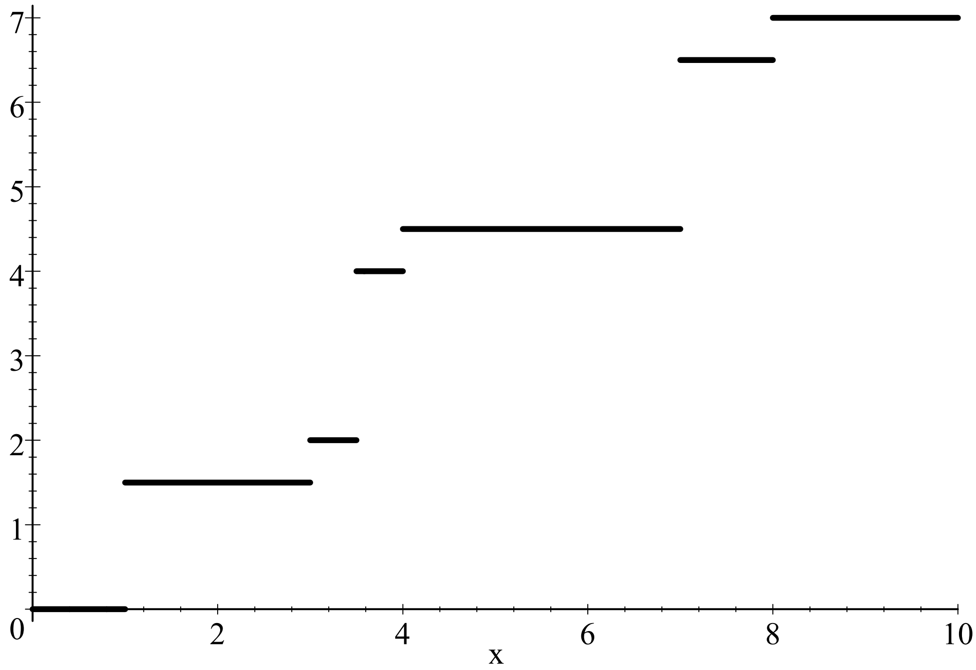

Figure 10