

Spatial Asset Pricing: A First Step*

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Abstract

People choose where to live and how much to invest in housing. Traditionally, the first decision has been the domain of spatial economics, while the second has been analyzed in finance. Spatial asset pricing is an attempt to combine equilibrium concepts from both disciplines. In the finance context, we show how spatial decisions can be framed as an expanded portfolio problem. Within spatial economics, we identify the consequences of hedging motives for location decisions. We characterize a number of observable deviations from standard predictions in finance (e.g. the definition of the relevant market portfolio for the pricing of risk includes homeownership rates) and in spatial economics (e.g. hedging considerations and the pricing of risk affect the geographic allocation of human capital).

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1 Introduction

Two strands of literature examine the demand for real estate assets. The urban economics literature provides insights into the determinants of the allocation of households and economic activity over space and the determinants of the cross-sectional distribution of land rents. The finance literature has focused primarily on real estate assets as a single asset – housing – emphasizing some key characteristics such as the transaction costs associated with the adjustment of housing ownership and the complementarity between the consumption of housing and other commodities.

So far, these two literatures have evolved independently. The urban economics literature abstracts from the stochastic nature of returns to land over time, the investment demand for land, and any risk premium built into land prices. The finance literature abstracts from the spatial dimensions of real estate assets, the unique function of real estate as an “enabler” for human capital, and hence the endogenous determination of the cross-sectional distribution of real estate dividends, namely, rents.

Here, we explore the gains from merging the two literatures. We bring portfolio choice and asset pricing considerations to bear on location choice and the determination of real estate rents, and vice versa. The results reveal an interaction between spatial equilibrium and asset pricing, hence “spatial asset pricing.”

We design a tractable model with closed-form solutions that are comparable to well-established results in both fields. We build on the standard spatial equilibrium assumption of urban economics: Access to a location’s amenities including earning opportunities requires consuming one unit of local land. We set this spatial equilibrium problem within a CARA-normal portfolio choice and asset pricing framework.

In equilibrium, the spatial allocation of households is determined together with local rents and the volatility of rents. The spatial allocation of households also determines the weights of each location-specific real estate asset in the market portfolio that is relevant for the pricing of systematic risk. This portfolio does not include all assets in the economy; the endogenously determined quantity of real estate assets used by households for hedging purposes is not part of this portfolio. The spatial allocation of households therefore affects the pricing of all assets.

At the same time, the pricing of assets in the economy matters for the spatial allocation of households and thus the rents generated by real estate assets in each location. Therefore, the cross-sectional distribution of rents reflects not only the cross-sectional distribution of the benefits of each location but also the risk exposure of households in each location and the price of risk.

Our findings indicate it is not appropriate to summarize housing as a single aggregate asset class for the pricing of housing and other assets, as is common in the finance literature. It is also inappropriate to assume rents reflect the amenities of a location and ignore the risk exposure of local households and the contribution of the location to systematic risk, as is common in the urban economics literature.

Merging the two literatures yields a number of novel insights with regard to determinants of the allocation of households across location, the cross-sectional and time series variations of households' portfolios, and the cross-sectional variation of returns to housing.

The model. There are four classes of assets: a risk-free bond, stocks, residential properties in a number of locations, and non-transferrable human capital. As in standard asset pricing models, agents may lend and borrow at the risk-free bond rate without any constraint. Agents may also invest in stocks, defined as claims over exogenous stochastic streams of dividends. The dividend stream of residential properties, however, is determined endogenously. Residential properties provide access to a stochastic production technology that is specific to their location. An agent's human capital determines the expected level of his or her earnings at each location and the covariance of earnings with the location-specific production technology. The distribution of individual characteristics across the population is expressed in a general form.¹

Properties differ only in their location. The supply of houses is fixed in every location but one, the countryside, where the supply is unlimited. Houses can be rented at the local equilibrium market rate. They can be purchased or sold (even fractionally) at the local equilibrium price. Agents may buy residential properties in every market. They also may buy a home in their city, in which case they are homeowners.

There are no frictions on any of the asset markets; e.g., no credit constraints, no transaction costs for buying or renting, no limits to fractional ownership.

We want to obtain closed-form solutions and expressions that are comparable to standard results obtained from mean-variance asset pricing models, so we assume an overlapping generations structure with finite life and constant population size. Agents have constant-absolute risk-aversion preferences with infinite elasticity of intertemporal substitution, and both city-productivity and stock-dividends stochastic shocks are normally distributed. We do not impose any restrictions on the covariances between the stochastic processes driving stock dividends and city-specific technology shocks.

¹In most of this work, we interpret local productivity as labor-related and hence translated into labor earnings, but the model has an equivalent interpretation in terms of leisure, where productivity is understood as the ability of the agent to enjoy local amenities. We also assume that there are no spillover effects across agents; that is, the productivity of an agent depends on its location but not on who else lives in that location. Later, we show that our characterization extends to a model with generic economies of agglomeration.

Agents choose where to live at the beginning of their life. Living in a location requires consuming one unit of housing in this location. While all investment decisions can be revisited in every period, the location choice is irreversible (moving costs are infinite).²

Location security. Each location can be represented as a free “location security” composed of two parts: (1) a unit of location-specific, agent-specific human capital, which yields a stream of stochastic benefits, understood as wages or enjoyment of local amenities, and (2) a unit of local housing, which requires a stream of stochastic rent payments, and satisfies the local housing consumption requirement.

The location decision and the portfolio allocation problem of an agent can therefore be examined within the same dynamic optimization framework. Choosing a location amounts to solving an expanded portfolio problem; that is, besides choosing financial assets, each household must pick one unit of one location security. This characterization reflects the discreteness of the choice of location.

It should be obvious that when an agent moves to a city, he or she considers more than the expected level of after-rent income; agents also take into account the amount of systematic risk assumed upon choosing the combination of location-specific human capital and housing consumption. And, systematic risk depends on the equilibrium allocation of agents across locations.

Spatial allocation. In equilibrium asset prices are relevant to the spatial allocation. The reverse is true as well; the spatial allocation is relevant to asset prices. Rents are determined by the productivity of marginal residents. Marginal residents are households that are indifferent between two or more locations. We demonstrate that our model admits a set of hyper-marginal residents: households indifferent among all locations, whether city or countryside. These households are all age 1 as location choice takes place at age 1, once and for all.

That a set of hyper-marginal households exists is due to the assumption that the distribution of personal characteristics has full support. Given an agent with particular characteristics and a particular city, one can always find an agent with similar characteristics but a slightly stronger or weaker preference for that given city. This continuity argument implies that, given an equilibrium vector of prices and rents, there must be a hyper-marginal agent.³

²The assumption that people cannot move is useful for tractability of analysis, but the assumption is not necessary for results about the role of housing as a hedge. Sinai and Souleles (2009) provide evidence of the empirical relevance of this phenomenon, accounting explicitly for US household moving patterns.

³A ancillary contribution of our work is to show existence in a spatial general equilibrium model with an infinite number of types of agents. Available existence results in the literature use a different approach based on a finiteness (Grimaud and Laffont, 1989).

Determining the set of hyper-marginal households is a key step to characterize the equilibrium with stationary allocation of households across locations, linear housing rents and asset prices. We conjecture a functional form for rents and prices and we verify the validity of our guess. We build our guess by adapting standard results obtained in CARA-normal portfolio choice frameworks.

It is through the hyper-marginal residents that productivity shocks are transmitted to rents. A productivity shock in any location affects the expected utility of the hyper-marginal residents if they move to that location, and hence affects the local rent. As in equilibrium these residents are indifferent among all locations, a local rent adjustment occurs to keep them indifferent. The indifference condition of the hyper-marginal resident pins down the relative level of rents in different locations. The fact that one location (the countryside) has an unlimited supply of land determines the absolute level of rents.

The location decision of any agent is determined by comparing that agent's location-specific set of productivity parameters with that of the hyper-marginal residents. By aggregating the investment demand functions of all agents, we obtain the asset pricing formulas both for real estate in different locations and for stocks.

We are then in a position to verify that the initial conjecture about the hyper-marginal resident is correct and that this it is indeed an equilibrium. We thus prove the existence of an equilibrium where prices can be expressed as linear functions of the underlying parameters, and the allocation of households across location remains constant over time. Uniqueness can be proven in specific cases.

Portfolio choice. Because we assume households choose their location at birth, marginal residents are newborn households. As households age, we do not put any restriction on the covariance of their current income with their income at birth. A changing covariance exposes households to the risk that shocks to their income may not provide full insurance for local productivity shocks that affect their housing costs.

The optimal investment portfolio of every agent is characterized as a combination of two components: (1) An investment in local real estate that depends on the agent's exposure to local productivity shocks, and (2) a portfolio of stocks and residential properties, with identical weights across agents. The first component is a manifestation of home bias. An agent who does not own property in the city of residence is vulnerable to a combination of local productivity shocks and rent fluctuations (determined endogenously). This risk can be hedged away by an appropriate holding of local real estate. This hedging demand depends on the covariance between the agent's earnings and local equilibrium rents.

Asset pricing. All households are able to fully insure themselves with some ownership

stake in their local housing markets. Conditional on this purchase, they are all identical with regard to risk. Hence they all have the same investment demand for the remaining securities in the economy: the portfolio made up of all stocks and residential properties in the economy minus the homes held for hedging purposes. Let us call this portfolio the *adjusted market portfolio*.

Equilibrium requires that the price of all assets in the economy be such that total investment demand (beyond the hedging demand for homes) equals the adjusted market portfolio. All assets are therefore priced in this adjusted market portfolio.

The fact that the quantity of homes in each location in the adjusted market portfolio is determined endogenously, adds a channel whereby the spatial allocation of households affects the prices of all assets. The spatial allocation of households does more than determine the stochastic properties of the rents in each location; it also determines what assets are part of the portfolio that is relevant to pricing systematic risk in equilibrium.

The adjusted market portfolio includes all financial assets and housing in every location. The prices of stocks are therefore determined not only by how their dividends co-vary with those of other financial assets but also by how they co-vary with earnings of the hyper-marginal residents in each location. Note that the relevant information for asset pricing related to the presence of housing in the economy cannot be represented by a single aggregate housing good.

Implications. Our portfolio choice and asset pricing expressions specify only objects – such as prices and covariances – that are in principle observable. We can thus develop a number of empirical questions linking spatial and financial variables:

Home bias. In our model, agents face no transaction costs and can invest in real estate anywhere in the world. Still, they tend to invest a large fraction of their wealth in local real estate for the hedging reasons that we have noted. A simple numerical exercise shows that the home bias can lead to purchase of up to 40% of a housing unit. The home bias in real estate is usually attributed to transaction costs, tax distortions, or psychological components. Our model shows that more of a hedging motive may be significant and provides a simple tool to compute its effect.

Homeownership over the life-cycle. Under reasonable assumptions, our model yields a hump-shaped demand for ownership over the life-cycle. Suppose that, as agents get older, their income covaries less with the income of newcomers to their city. This implies that agents need to purchase an increasing amount of local housing for hedging purposes as they get older. Counter to this effect is the fact that as agents get older, the time left to live

is shortened and so their demand for insurance declines. We show that the combination of these two effects can yield an inverted U-shape.

Cross-sectional dispersion of housing returns. Differences in real estate returns across locations depend on differences in the average within-location covariance of the income of each resident with the income of the current and future marginal residents. For instance, in a one-company town, wages of all cohorts are highly correlated with rents; residents there do not demand local housing for hedging purposes, and prices are depressed.

Allocation of agents across cities. The allocation of agents across cities does not maximize aggregate expected production. When they choose a location, agents trade off expected net earnings opportunities (expected wage minus expected rents) against risk exposure (volatility of income minus rent). Agents therefore do not necessarily choose the location that maximizes their output. They may prefer a location with lower expected earnings minus rents if their income in that location is less correlated with rents. In such a location, the purchase of local housing provides insurance benefits. Nevertheless, they earn a risk premium on the local housing because it is priced by outsiders to whom the volatility of housing returns is a risk, not an insurance. An obvious corollary is that people are reluctant to move to a one-company town.

Asset pricing errors. We can construct and price aggregate indices of stocks and real estate assets. It is also possible to quantify the error that we make if we price stocks according to a classic beta (taking into account only the covariance with other stocks), rather than the correct beta, which is based on stocks and the portion of housing assets not demanded by local residents, and on the covariance of stock dividends with labor earnings of the marginal residents in every city.

Extensions. The analytical results we obtain for the benchmark model support two useful extensions in the modeling of locations and housing assets.

First, the benchmark model can be extended to encompass economies of agglomeration and other forms of externalities among residents. The equilibrium characterization of the benchmark model remains valid, but the presence of direct externalities amplifies the possibility of equilibrium multiplicity. If the agglomeration economies are strong enough, there will be multiple linear stationary equilibria corresponding to different allocations of talent across cities. This means we may be able to create links between real estate finance and the vast literature on agglomeration effects.

Second, the benchmark model assumes the ownership of real estate is perfectly divisible. Households are allowed to buy exactly the amount of local housing they need to perfectly

hedge their risk in income minus rent. A number of frictions are likely to lead household away from the perfect hedge investment; e.g., a preference for homeownership, preferential tax treatment of homeownership, housing transaction costs, housing property indivisibilities. Any such impediment to obtaining a perfect hedge with local housing leads households to resort to exploiting the covariance between their local risk and each of the financial assets. Agents in different location therefore purchase a different portfolio of housing and stocks. This hedging demand for stocks ends up affecting stock prices. As a case in point, we propose explicit solutions for stock prices when all households are required to own their home.⁴

The paper is organized as follows: Section 2 reviews the related literature. Section 3 sets out the model. Section 4 presents the main equilibrium characterization result, through three propositions corresponding to: portfolio allocation (Proposition 1), asset pricing (Proposition 2), and location choice (Proposition 4). Section 5 uses the main result to discuss a number of related issues. Section 6 concludes. All proofs are in the Appendix.

2 Related Literature

This is – to the best of our knowledge – the first asset pricing model where location choices, housing rents, and asset prices are endogenous.

Our work is perhaps closest in spirit to that of DeMarzo, Kaniel, and Kremer (2004). They consider an economy with multiple communities and local goods as well as a global good. In this dynamic setting, some agents (the laborers) are endowed with human capital that will be used to produce local goods in future periods, but they are currently subject to borrowing constraints. Other agents (the investors) own shares in firms that produce the global good.

This approach yields a number of powerful results. Investors care about their relative wealth in the community because they bid for scarce local goods. This generates an externality in portfolio choice, which leads to the potential presence of multiple equilibria (in the stable equilibria, investors display a strong home bias). And, if there is a behavioral bias, this externality amplifies the bias through the portfolio decisions of rational investors.

Our model differs from DeMarzo et al. (2004) in a number of important dimensions: (1) Our local good does not produce utility directly, but it enables agents to realize their human capital potential; (2) our spatial allocation is endogenous; and (3) there are no credit

⁴Other extensions yield similar predictions with regard to home bias. Suppose households enjoy more utility from the same property if they own it than if they rent it. Such an assumption implies that their investment in local housing is not driven purely by hedging considerations. Households are willing to “distort” their housing investment because of consumption benefits. It then becomes optimal to use stocks to deal with any residual risk in income minus rent not canceled with local housing investment.

constraints. We do share their goal of studying the properties of portfolio choice and asset pricing under uncertainty in the presence of community effects. As in their model, a home bias arises in equilibrium because of a hedging motive.⁵

Our contribution to the real estate finance literature, lies in endogenizing *both* price and rent in a dynamic model with multiple locations.⁶

Grossman and Laroque (1991) characterize optimal consumption and portfolio selection when households derive utility from a single durable good only and trading the durable require payment of a transaction cost. They show that CAPM holds in this environment, but CCAPM fails because consumption of housing is not a smooth function of wealth due to transaction costs.

Flavin and Nakagawa (2008) expand on the Grossman and Laroque framework by assuming that households derive utility not only from housing but also from numeraire consumption. They show that when housing asset returns do not co-vary with stock returns, the CCAPM holds. In equilibrium, all households hold a single optimal portfolio of risky financial assets. Depending on their holding of housing, households vary in how much of their wealth is invested in this portfolio but not its composition.

An extensive literature has explored the effect of housing consumption on households' life-cycle overall consumption and investment behavior. An early paper by Henderson and Ioannides (1983) considers an optimal consumption and saving problem when a household chooses whether to own or rent, and a wedge arises endogenously between the cost of renting and the cost of owning. Henderson and Ioannides show that the consumption demand for homeownership distorts households' investment decisions.

Goetzmann (1993) and Brueckner (1997) explain how this distortion affects households' portfolio choice. Flavin and Yamashita (2002) compute mean-variance optimal portfolios for homeowners using U.S. data on housing and financial asset returns.⁷ Cocco (2004) also computes optimal portfolios but in a calibrated dynamic model of household consumption and portfolio choice. Housing consumption is constrained to equal housing investment in the two papers above.

Yao and Zhang (2005) introduce discrete tenure choice (rent or own total housing con-

⁵Our results on home bias are also related to the international finance literature on the home bias puzzle (Stockman and Dellas, 1989), but we differ in our focus on real estate and in that location choice is endogenous in our model.

⁶A review of the empirical literature on the cross-sectional dispersion of housing prices is beyond the scope of this paper. For recent evidence emphasizing variations in housing price premiums see Campbell et al. (2009).

⁷Englund, Hwang, and Quigley (2002) report similar computations for Sweden, Iacoviello and Ortalo-Magné (2003) for the U.K., and LeBlanc and Lagarenne (2004) for France. Note that every one of these papers considers the stock market as a whole and so ignores the covariance between housing and specific stocks.

sumption) in a similar environment. They show the sensitivity of households' portfolio choice to tenure mode; owning a house leads households to reduce the proportion of equity investment in their net worth (a substitution effect). At the same time, households choose stocks more than bonds in their portfolio because homeownership provides insurance against equity returns and labor-income fluctuations (a diversification benefit).

All this research demonstrates that incorporating housing consumption in portfolio choice models helps reconcile theoretical predictions and cross-sectional observations. Home investment seems a key factor in explaining the very limited participation of the young in equity markets. Credit constraints play a critical role in explaining the hump-shaped we see in home ownership over the life-cycle.

Piazzesi, Schneider, and Tuzel (2007) study a consumption-based asset pricing model in which housing rents and prices are determined endogenously; the quantity of housing follows an exogenous stochastic process. Agents can invest in both housing and stocks. The focus of their analysis is on the composition risk related to fluctuations in the share of housing in the households' consumption baskets. Piazzesi et al. show that the housing share can be used to forecast excess returns of stocks – a prediction that appears to be borne out by the data.

Lustig and Van Nieuwerburgh (2007) propose a mechanism whereby the amount of housing wealth in the economy affects the ability of households to insure idiosyncratic income risk and thus shifts the market price of risky assets, housing included. In Lustig and Van Nieuwerburgh (2005), the authors present empirical evidence of the relevance of the ratio of housing wealth to human wealth for returns of stocks. We share with Piazzesi et al. (2005) and Lustig and Van Nieuwerburgh (2005, 2007) a focus on the equilibrium properties of housing rents and the risk premiums.

Lustig and Van Nieuwerburgh (2010) consider risk sharing across regions. Empirical evidence indicates that the amount of housing wealth in a region affects the sensitivity of local consumption to local income. This paper is particularly close to ours in that it considers multiple locations. Lustig and Van Nieuwerburgh, however, assume exogenous location choice and that housing supply is perfectly elastic in all locations (and hence rents depend only on aggregate shocks).

Our approach to the modeling of housing as access to a location is in the tradition of urban economics. Our location choice model follows the standard multi-cities framework of Rosen (1979) and Roback (1982), where residential properties provide access to the local labor market, and locations are differentiated by potential surplus. As in Rosen and Roback and the many more recent papers that build on this framework (e.g., Gyourko and Tracy, 1991, Kahn, 1995, Glaeser and Gyourko, 2005), we assume households face a unit housing

consumption requirement and derive utility from consumption of numeraire only.

Because we are concerned with portfolio choice in a dynamic environment, we assume households are risk-averse. Risk aversion in the face of stochastic streams of income and rent provides a motivation for ownership of local residential properties – homeownership – in our model. This approach builds on the work of Ortalo-Magné and Rady (2002), Hilber (2005), Sinai and Souleles (2005, 2009), and Davidoff (2006) and others who provide evidence of the relevance of such motivation for housing investment.

We do not here review the vast literature concerned with the determinants of housing prices. Typically in this literature, real estate prices are determined by a perfectly elastic supply function (Lustig and Van Nieuwerburgh, 2008) or by a perfectly elastic demand function (Davis and Heathcote, 2005, Davis and Ortalo-Magné, forthcoming, Gyourko, Mayer and Sinai, 2006, Kiyotaki, Michaelides and Nikolov, 2007, Van Nieuwerburgh and Weil, 2007).

3 Model

Consider an overlapping generation economy where a mass 1 of agents is born in every period. Each agent in the t -cohort is born at the beginning of period t , lives for S periods, and dies at the beginning of period $t + S$. Hence, at every time t , there are a mass S of agents alive in the economy.

3.1 Geography

There are L cities, denoted by index $l = 1, \dots, L$ and a countryside denoted by index $l = 0$. City l has an exogenously given mass of houses. Let n^l be the mass of houses per cohort that will be active on the housing market so that total supply of housing in city l equals $S \times n^l$. We assume that housing supply is scarce in cities:

$$\sum_{l=1}^L n^l < 1$$

but it is abundant once we include the countryside:

$$\sum_{l=0}^L n^l > 1.$$

Each house accommodates exactly one agent.

3.2 Production

The income of a person who lives in the countryside is (normalized to) zero. Productivity in city l follows the process

$$y_t^l = y_{t-1}^l + \tau_t^l$$

where τ_t^l is a random variable, independently and identically distributed across time.

At birth, each agent draws:

- A vector of city-specific endowment surplus, $\boldsymbol{\varepsilon} = [\varepsilon^l]^{l=1,\dots,L}$, with $\varepsilon^l \in (-\infty, \infty)$.
- A matrix of city- and age-specific insulation parameters: $\boldsymbol{\rho} = [\rho_s^l]_{s=0,\dots,S}^{l=1,\dots,L}$, with $\rho_s^l \in [0, 1]$. Assume $\rho_0^l = 0$ for all l .

The parameters $(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$ are i.i.d. across generations. Their joint distribution within a generation takes the general form $\phi(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$, with the only requirement that it should be continuous and have full support.

At time $t + s$, the income of an agent living in city l , born at time t , with parameters $(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$ is

$$y_{t,t+s}^l(\boldsymbol{\varepsilon}^l, \boldsymbol{\rho}^l) = y_{t-1}^l + \varepsilon^l + \sum_{m=0}^s (1 - \rho_m^l) \tau_{t+m}^l$$

for $s = 0, \dots, S - 1$ (note the difference between y_t^l , a city-wide variable, and $y_{t,t+s}^l(\boldsymbol{\varepsilon}^l, \boldsymbol{\rho}^l)$ an individual specific variable).

Hence, the income of each agent can be decomposed into a permanent part, which captures the initial productivity of the agent in a location and a time-dependent part, which is determined by the local productivity shocks in the city and that agent's sensitivity to the city's shocks. We call ε^l the *city-agent effect* and ρ_s^l the *shock insulation effect*.

We represent below the income earned by an agent born at time t , living in city l , for each of the first three years of life:

$$\begin{aligned} y_{t,t}^l(\boldsymbol{\varepsilon}^l, \boldsymbol{\rho}^l) &= \overbrace{\varepsilon^l}^{\text{city-agent effect}} + \overbrace{y_t^l}^{\text{city-cohort effect}} ; \\ y_{t,t+1}^l(\boldsymbol{\varepsilon}^l, \boldsymbol{\rho}^l) &= \overbrace{\varepsilon^l}^{\text{city-agent effect}} + \overbrace{y_t^l}^{\text{city-cohort effect}} + \overbrace{(1 - \rho_1^l) \tau_{t+1}^l}^{\text{year 1 innovation}} ; \\ y_{t,t+2}^l(\boldsymbol{\varepsilon}^l, \boldsymbol{\rho}^l) &= \overbrace{\varepsilon^l}^{\text{city-agent effect}} + \overbrace{y_t^l}^{\text{city-cohort effect}} + \overbrace{(1 - \rho_1^l) \tau_{t+1}^l}^{\text{year 1 innovation}} + \overbrace{(1 - \rho_2^l) \tau_{t+2}^l}^{\text{year 2 innovation}} . \end{aligned}$$

Similar formulations determine the agent's earnings until reaching age $S - 1$.⁸ At age S , the

⁸The structure of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\rho}$ could be more complex and still be amenable to analysis in our mean-variance set-up. For instance, we could say that the city-agent effect is not constant over the life of the agent but rather it follows a random walk. Also, we could assume that the extent to which a shock that occurs at age s affect future incomes depends on the age of the agent.

agent does not earn anything. It is mathematically convenient to set $\rho_S = 0$ for all agents even if it is irrelevant to the agents' earnings.

The city-agent effect, ϵ^l , is a standard object in multi-city models with heterogeneous agents. Depending on their human capital, agents face different earning opportunities in different locations.

The shock-insulation effect, ρ^l , captures two economic phenomena. First, agents may be exposed to a technological cohort-specific effect (documented by Goldin and Katz, 1998). The human capital of certain people, especially the young, may be more flexible. When a technological innovation appears, the income of certain agents will be more affected than the income of others.

Second, certain agents – like senior workers and public sector workers – may be part of an implicit labor insurance agreement. Their wages are more insulated from productivity shocks.

It is reasonable – but not necessary for the analysis – to assume that the insulation parameter, for a shock that occurs at a given age, increases with in the age of the agent: $\rho_{s+1}^l > \rho_s^l$. The two extreme cases are full insulation ($\rho_s^l = 1$) and full exposure ($\rho_s^l = 0$).⁹

For concreteness, we interpret $y_{t,t+s}^l$ as monetary income, but there is an alternative interpretation in terms of non-monetary benefits that is equivalent from a mathematical standpoint. The term $y_{t,t+s}^l$ can be viewed as a money-equivalent of the utility afforded by the amenities present in location l . The utility can be decomposed in turn into an agent-city effect (a preference for that particular location) and a shock component (perhaps an environmental or a social risk) multiplied by the agent's sensitivity to that type of shock. Of course, the model can also be interpreted as a mix of monetary and non-monetary benefits.

An agent who lives and thus produces in city l , must rent exactly one unit of housing in city l .

3.3 Housing market

The housing market is frictionless. There are no transaction costs associated with renting, buying, or selling property. There is no difference between living in an owned or a rented house.

At birth, every agent chooses in what city (or the countryside) to live. The agent cannot move afterward. Living in city l at time t entails paying the market rent, on a unit of housing, r_t^l . Rents are determined in equilibrium.

⁹We find it natural to restrict ρ_s^l to be between zero and one, but our mathematical analysis is valid even if $\rho_s^l > 1$ (the agent's productivity is negatively correlated with local shocks) and $\rho_s^l < 0$ (the agent is overexposed to local shocks).

Agents may invest in divisible shares of any city's housing stock and revise their decision at every period. Let $a_{t,t+s}^l$ denote the amount of housing of city l owned by an agent born at time t of age s .

The market price of a unit of housing in city l at time t is p_t^l . The agent revises his or her housing investment at the beginning of every period. For accounting purposes, imagine that the agent liquidates all housing assets and then buys the desired amount in each period. At the beginning of period $t + s$, the agent acquires $a_{t,t+s}^l$ units in city l at total cost $a_{t,t+s}^l p_{t+s}^l$. During period t , the agent collects rent on the housing investment for a total of $a_{t,t+s}^l r_{t+s}^l$. At the beginning of the next period, the agent liquidates the housing investment and receives $a_{t,t+s}^l p_{t+s+1}^l$. We denote $\mathbf{a}_{t,t+s}$ the vector of the agent's housing investments, $\mathbf{a}_{t,t+s} = [a_{t,t+s}^l]_{l=1,\dots,L}$.¹⁰

3.4 Stock market

Besides housing, there is another class of securities called stocks. These are claims on productive assets, that – as in regular asset pricing models – produce an exogenous stochastic stream of income. There are Sz^k units of type- k asset, with $k \in \{1, \dots, K\}$ and $z^k > 0$. A unit of stock k produces dividend d_t^k at time t . The dividend follows the stochastic process:

$$d_t^k = d_{t-1}^k + \nu_t^k$$

where ν is i.i.d. across time with probability distribution as below.

As is the case for housing, every agent can buy units of every stock and revise portfolio allocations in every period. The market price of stock k at a particular time is q_t^k . At the beginning of period $t + s$, the agent acquires $b_{t,t+s}^k$ units of stock k at total cost $b_{t,t+s}^k q_{t+s}^k$. During period $t + s$, the agent receives dividends on investment in k for a total of $b_{t,t+s}^k d_{t+s}^k$. At the beginning of the next period, the agent liquidates the stock investment and receives $b_{t,t+s}^k q_{t+s+1}^k$. We denote $\mathbf{b}_{t,t+s}$ the vector of the agent's stock investments, $\mathbf{b}_{t,t+s} = [b_{t,t+s}^k]_{k=1,\dots,K}$.

3.5 Distribution of random shocks

There are two sources of exogenous shocks in our economy: a vector $\boldsymbol{\tau}$ of local productivity shocks, and a vector $\boldsymbol{\nu}$ of dividend shocks. The shocks are independently and identically distributed over time, according to a normal distribution with mean 0 and covariance matrix $\boldsymbol{\Sigma}$: $(\boldsymbol{\tau}_t, \boldsymbol{\nu}_t) \sim N(0, \boldsymbol{\Sigma})$.

¹⁰Given the frictionless nature of the housing market, derivative securities would be superfluous. In particular, Case-Shiller home price indices for our cities (a security bought at time t which pays a price p_{t+1}^l at time $t + 1$) would be equivalent to purchasing housing for one period, net of the “rent coupon.” Given the random-walk nature of all our shocks, long-term securities are also redundant because they can be replicated by sequences of short-term investments. This includes long-term rentals or futures on real estate.

We do not impose any restriction on the correlation between local productivity shocks and dividends. One industry may be more affected by shocks in a certain market, and vice versa. We also do not impose any restriction on the correlation of productivity shocks across cities.

3.6 Consumption and savings

As our goal is to develop a closed-form expression for asset prices, we assume that agents derive CARA utility $-\exp(-\gamma w)$ from wealth at the end of their life, w , where γ is the standard risk-aversion parameter.

Agents face no credit constraints and can borrow and lend freely at discount rate $\beta \in (0, 1)$. For simplicity, we assume that agents are born with no wealth (this does not affect their decisions, given that they have CARA preferences).

3.7 Non-negativity constraints

Asset pricing models with normally distributed shocks suffer from a well-known technical problem. As the value of dividends can become negative, agents may want to dispose of assets they own. If they could, the distribution of asset values would no longer be normal, and the model would not be tractable. Hence, all models in this class assume, implicitly or explicitly, that agents cannot dispose of assets. Typically, this assumption is unrealistic because in practice both agents and firms are protected by limited liability. Instead, in the model stocks can have negative prices, and their owners must pay to get rid of them.

Our CARA-normal framework inherits this non-negativity problem. That is, productivity in a city could become negative, and house prices there may be negative.¹¹

The usual response to this criticism, which applies here as well, is that the unconstrained model should be viewed as an approximation of the model with non-negativity constraints, as long as the starting values are sufficiently far from zero.

3.8 Timing

The order of moves for an agent born at time t is as follows:

1. At birth, the agent chooses in which location l to spend the rest of his or her life.
2. At the beginning of each period $t + 0, \dots, t + S$, the agent learns the values of the random shocks for that period, ν_{t+s} and τ_{t+s} .

¹¹We assume homeowners have an obligation to rent their property (they pay a fine if it is vacant).

3. For $s = 0, \dots, S-1$, the agent revises housing and stock investments ($\mathbf{a}_{t,t+s}$ and $\mathbf{b}_{t,t+s}$), pays rent r_{t+s}^l for one unit of housing in the chosen location and collects dividends and rents on the assets owned.
4. At time $t+S$, the agent liquidates all investments ($\mathbf{a}_{t,t+S-1}$ and $\mathbf{b}_{t,t+S-1}$) and consumes all wealth before death.¹²

4 Analysis

An equilibrium is an allocation of households across cities, a vector of optimal portfolio holdings of housing and stocks for each agent, housing rents and prices for each city, and stock prices such that: (1) The location choice and portfolio holdings solve the agents' problem; (2) the housing markets (space and ownership) in each city clear; and (3) the stock markets clear.

A *stationary equilibrium* is an equilibrium where the mass of agents of a generation t who live in a given city l is the same across generations.¹³

A *linear equilibrium* is an equilibrium where stock prices, rents, and house prices can be expressed, respectively, as:

$$q_t^k = \frac{1}{1-\beta} d_t^k - \bar{q}^k \quad (1)$$

$$r_t^l = y_t^l + \bar{r}^l \quad (2)$$

$$p_t^l = \frac{1}{1-\beta} r_t^l - \bar{p}^l \quad (3)$$

where $\bar{\mathbf{q}} = [\bar{q}^k]^{k=1,\dots,K}$ and $\bar{\mathbf{p}} = [\bar{p}^l]^{l=1,\dots,L}$ are price discounts; and $\bar{\mathbf{r}} = [\bar{r}^l]^{l=1,\dots,L}$ is a rent premium to be determined in equilibrium. The rent is equal to local productivity plus a local constant. House and stock prices are equal to the discounted value of a perpetuity that pays the current rent or dividend minus an asset-specific discount.

Price discounts can also be interpreted as expected returns of zero-cost portfolios.¹⁴ Throughout the analysis we describe \bar{p}^l and \bar{q}^k as price discounts or expected returns, depending on the context.

¹²The agent does not work or pay rent in the last period of life ($t+S$) but rather consumes all wealth at the beginning of the period before death.

¹³A non-stationary equilibrium be structured as follows. As agents cannot move after they locate to city l , the stock of rented accommodation used by the t -cohort will not become available until members of the t -cohort die at the end of $t+S$. Hence, if the t -cohort is, say, overrepresented, the $t+S+1$ -cohort will be equally overrepresented. The non-stationary equilibria are characterized by cycles of length $S+1$.

¹⁴For instance, the expected return of a zero-cost one-unit portfolio invested in housing in city l (evaluated in today's dollars) is

$$E \left[\beta p_{t+1}^l - (p_t^l - r_t^l) \right] = \frac{\beta}{1-\beta} r_t^l - \beta \bar{p}^l - \frac{\beta}{1-\beta} r_t^l + \bar{p}^l = (1-\beta) \bar{p}^l.$$

Our strategy for finding equilibria is as follows. We start by conjecturing that we are in a stationary linear equilibrium. We postulate a feasible allocation of agents to cities, and we solve the portfolio problem of a generic agent living in a given city. As it turns out, solving this agent problem is enough to characterize stock prices and house prices up to a vector of city-specific constants. With this information, we compute the expected utility of every agent, conditional on city choice. We determine aggregate location demand, given any price vector by comparing expected utilities across cities.

Finally, we consider the marginal residents. We show that for every vector of city-specific constants there are a set of agents who are indifferent among all locations (the hyper-marginal residents), while all others have strict preferences. The characteristics of the hyper-marginal residents are monotonic in the vector of city-specific constants, and we can identify the hyper-marginal residents so that the mass of agents who move to each city equals the local housing supply in each city. This proves that our initial conjecture on linear prices is correct.¹⁵

As agents have CARA preferences, their lifetime utility can be decomposed into:

$$E \left[u_t^l \right] = E \left[w_t^l \right] - \gamma V \left[w_t^l \right].$$

Proposition 1 re-writes the two components of the agents utility and uses them to compute his optimal portfolio choice and his expected utility.

In what follows we focus on one agent and we drop the argument representing the agent-specific characteristics: (ε, ρ) . All proofs are in appendix.

Proposition 1 (Portfolio Allocation) *Suppose that prices and rents are given by equations (1), (2), and (3), with given \bar{r} , \bar{q} and \bar{p} . Consider any allocation of agents to cities. Consider an agent born at period t characterized by a vector ε and a matrix ρ . If this agent lives in l and chooses investment profiles $[\mathbf{a}_{t,t+s}, \mathbf{b}_{t,t+s}]_{s=0,\dots,S-1}$, the expectation and the*

¹⁵It is tempting to consider the two first parts of the analysis (portfolio choice and asset pricing) in isolation, but they are valid only if the third part is present too. If one assumes a different location model or an exogenous allocation of agents to cities, the three price processes in equations (1), (2), and (3) would be different, and Propositions 1 and 2 would no longer hold. For instance, if agents could move between cities during their lifetime, it is not clear that the rent the price in city l would depend only on productivity in city l .

We see this as both a weakness and a strength of spatial asset pricing. On the one hand, one cannot have a meaningful discussion about real estate prices in multiple locations without an underlying spatial model. On the other hand, this opens the door to a wealth of testable implications related to spatial and financial variables.

variance of the agent's end-of-life wealth can be written, respectively, as:

$$E[w_t] = \sum_{s=0}^{S-1} \beta^{s-S} \left(\varepsilon^l - \bar{r}^l + (1-\beta) \left((1-\beta^{S-s-1}) \rho_{s+1}^l \bar{p}^l + \sum_{j=1}^L \tilde{a}_{t,t+s}^j \bar{p}^j + \sum_{k=1}^K b_{t,t+s}^k \bar{q}^k \right) \right)$$

$$Var[w_t] = \frac{\beta^2}{(1-\beta)^2} \sum_{s=0}^{S-1} \beta^{2(s-S)} Var \left[\sum_{j=1}^L \tilde{a}_{t,t+s}^j \tau_{t+s+1}^j + \sum_{k=1}^K b_{t,t+s}^k \nu_{t+s+1}^k \right]$$

where

$$\tilde{a}_{t,t+s}^j = \begin{cases} a_{t,t+s}^l - (1-\beta^{S-s-1}) \rho_{s+1}^l & \text{if } j = l \\ a_{t,t+s}^j & \text{otherwise} \end{cases}.$$

The agent's optimal investment profile is given by

$$\begin{bmatrix} \tilde{\mathbf{a}}_{t,t+s} \\ \mathbf{b}_{t,t+s} \end{bmatrix} = \frac{(1-\beta)^3}{2\gamma\beta^{s+2}} \beta^S \boldsymbol{\Sigma}^{-1} \begin{bmatrix} \bar{\mathbf{p}} \\ \bar{\mathbf{q}} \end{bmatrix},$$

for $s = 0, \dots, S-1$, and expected log-utility is

$$U^l = \frac{1}{\beta^S} \sum_{s=0}^{S-1} \beta^s \left(\varepsilon^l - \bar{r}^l + (1-\beta) (1-\beta^{S-s-1}) \rho_{s+1}^l \bar{p}^l \right) + S \frac{(1-\beta)^4}{4\gamma\beta^2} \begin{bmatrix} \bar{\mathbf{p}} \\ \bar{\mathbf{q}} \end{bmatrix}' \boldsymbol{\Sigma}^{-1} \begin{bmatrix} \bar{\mathbf{p}} \\ \bar{\mathbf{q}} \end{bmatrix}.$$

Proposition 1 says that the optimal portfolio of any agent can be decomposed into:

- Demand for real estate in the city where the agent lives, $(a_{t,t+s}^l - \tilde{a}_{t,t+s}^l)$, driven by a desire to hedge shocks to disposable income due to rent fluctuations. As the price of a house is linear in the rent, a house in a certain city is a perfect hedge against rent fluctuations in that city. The hedging demand is given by $(1-\beta^{S-s-1}) \rho_{s+1}^l$. Hence, the hedging demand depends on how well the agent is insulated from local productivity shocks at time t . The hedging demand varies across agents and across time for a given agent, but it does not depend on the expected return of real estate in that city (if a city has a high return, that will be reflected in the mutual fund share only).¹⁶
- Investment in a mutual fund includes all stocks and houses in all cities, with weights $(\tilde{\mathbf{a}}, \mathbf{b})$. The mutual fund is the same for all agents. All agents within a cohort buy the same amount of mutual fund shares (but older agents buy more shares, purely because of the discount rate β). Given a vector of expected returns (which for now is still exogenous), the weights $(\tilde{\mathbf{a}}, \mathbf{b})$ that the mutual fund puts on various stocks and real estate assets are given by a standard CAPM allocation. The portfolio puts more weight on an asset if its returns are less correlated with other assets and have a higher expected value.

¹⁶Davis and Willen (2000) obtain a related result (their Proposition 1) in decomposition of the optimal portfolio of agents who face labor risk into a speculative component and a hedging component.

Now that we have solved the portfolio allocation problem for any given vector of premiums, we solve for the equilibrium expected returns. Denote any (measurable) allocation of agents to cities with the indicator function $I_{\varepsilon, \rho}^l$, which takes a value of 1 if agents with personal characteristics ε and ρ locate to city l , and zero otherwise (such that $\sum_{l=0}^L I_{\varepsilon, \rho}^l = 1$ for all ε and ρ).

Proposition 2 (Asset Pricing) *Suppose that rents are given by equation (2), with given \bar{r} . Consider any allocation of agents over space so that all cities are populated. Then, prices are given by equations (1) and (3) with discounts:*

$$\begin{bmatrix} \bar{p} \\ \bar{q} \end{bmatrix} = 2\gamma S \frac{\beta}{(1-\beta)^2 (1-\beta^S)} \Sigma \begin{bmatrix} \mathbf{n} - \mathbf{R} \\ \mathbf{z} \end{bmatrix},$$

where $\mathbf{R} = [R^1, \dots, R^l, \dots, R^L]'$ and

$$R^l = \frac{1}{S} \sum_{s=0}^{S-1} (1 - \beta^{S-s-1}) \int_{\varepsilon} \int_{\rho} I_{\varepsilon, \rho}^l \rho_{s+1}^l \phi(\varepsilon, \rho) d\varepsilon d\rho.$$

Houses and stocks are priced according to their contribution to systematic risk by a classic CAPM formula. Proposition 2 finds the correct definition of systematic risk for this model. The weights of stocks in the market portfolio correspond to the quantity of stocks available, as in the regular CAPM. The weights of real estate, however, are reduced by the total hedging demand. Namely, the weight of houses in city l is equal to the mass of homes n^l minus the integral of the hedging demand by residents of l : R^l .

To explore the pricing expressions in Proposition 2 further, define the *adjusted market portfolio* M as a portfolio allocation that includes

$$\begin{aligned} & \frac{n^l - R^l}{Q} \text{ units of housing in city } l \text{ for every city } l, \text{ and} \\ & \frac{z^k}{Q} \text{ units of stock } k \text{ for every stock } k \end{aligned}$$

where $Q = \sum_{l=1}^L (n^l - R^l) + \sum_{k=1}^K z^k$. The mutual fund that all agents buy is the adjusted market portfolio.

Denote the expectation and the variance of the adjusted market portfolio, respectively, by \bar{p}^M and $Var(M)$. Define $Cov(l, M)$ as the covariance between the return of real estate in city l and the return of M . For every stock k , define $Cov(k, M)$ similarly. Then:

Corollary 3 *The expected return of real estate in city l is given by*

$$\bar{p}^l = \frac{Cov(l, M)}{Var(M)} \bar{p}^M$$

and the expected return of stock k is

$$\bar{p}^k = \frac{Cov(k, M)}{Var(M)} \bar{p}^M.$$

The expression in the Corollary is akin to the classic CAPM pricing formula where $\frac{Cov(l, M)}{Var(M)}$ is a beta-factor for housing in city l . The main innovation lies in identification of the adjusted market portfolio, for which this formula is true.¹⁷

Propositions 1 and 2 are really intermediate results. They rest on a specific conjecture about the stochastic process that determines local market rents, described in equation (2). But rents are not primitives, and we must now check that for the location model used here the conjecture is in fact correct. It is useful to reiterate that the conjecture would in general not extend to other location models, implying that Propositions 1 and 2 are valid only if accompanied by the specific spatial allocation model that we have chosen.

Besides closing the fixed-point argument, we also need to determine the vector of rent premiums \bar{r} , and to find the vector of hedging demands \mathbf{R} .

For an agent with personal characteristics (ε, ρ) , the log-utility of locating in city l is given by U in Proposition 1, where now \bar{p} and \bar{q} are defined in terms of primitives through Proposition 2. For every (ε, ρ) , let

$$\bar{u}^l(\varepsilon, \rho) \equiv \varepsilon^l + \frac{(1-\beta)^2}{1-\beta^S} \bar{p}^l \sum_{s=1}^S (1-\beta^{S-s-1}) \rho_s^l. \quad (4)$$

with the utility of being in the countryside: $\bar{u}^0(\varepsilon, \rho) = \bar{u}^0$.¹⁸

Also let

$$\bar{U} = S \frac{(1-\beta)^4}{4\gamma\beta^2} \begin{bmatrix} \bar{p} \\ \bar{q} \end{bmatrix}' \Sigma^{-1} \begin{bmatrix} \bar{p} \\ \bar{q} \end{bmatrix}.$$

Then, we can write the utility of locating in city l as:¹⁹

$$U^l = \bar{U} + \frac{1}{\beta^S} \left(\frac{1-\beta^S}{1-\beta} (\bar{u}^l(\varepsilon, \rho) - \bar{r}^l) \right).$$

Namely, the agent's utility can be decomposed into a component that is common to all agents (and depends on investment in the mutual fund) and an agent-specific component that depends on the city-agent effect ε^l and the shock-insulation vector ρ^l that the agent faces if the choice is to locate in city l .

¹⁷For instance, if one defines the market portfolio without the $-R$ correction, such a beta representation would not be valid.

¹⁸Assuming that $\varepsilon^0 = 0$ is without loss of generality. If it was not, one could redefine all the ε as differences with ε^0 .

¹⁹To see this, note that:

$$\sum_{s=0}^{S-1} \beta^s \left(\varepsilon^l + (1-\beta) (1-\beta^{S-s}) \rho_{s+1}^l \bar{p}^l \right) = \frac{1-\beta^S}{1-\beta} \left(\varepsilon^l + \frac{(1-\beta)^2}{1-\beta^S} \bar{p}^l \sum_{s=1}^S (1-\beta^{S-s+1}) \rho_s^l \right)$$

A given agent locates in city l if and only if $U^l = \max_m U^m$. For every L -vector $\hat{\mathbf{r}}$, we can write the aggregate demand for location l as

$$\nu^l(\hat{\mathbf{r}}) = \int_{(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) : \bar{u}^l(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) - \hat{r}^l = \max_m (\bar{u}^m(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) - \hat{r}^m)} \phi(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) d(\boldsymbol{\varepsilon}, \boldsymbol{\rho}).$$

We obtain:

Proposition 4 (Location Choice) *A linear stationary equilibrium exists. In it, an agent with personal characteristics $(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$ locates in city l if and only if*

$$\bar{u}^l(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) - \bar{r}^l = \max_{j=0, \dots, L} (\bar{u}^j(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) - \bar{r}^j)$$

and $\bar{\mathbf{r}}$ is the unique value of the vector $\hat{\mathbf{r}}$ such that $\nu^l(\hat{\mathbf{r}}) = n^l$ in all cities.

The equilibrium rent in city l is

$$r_t^l = y_t^j + \bar{r}^l.$$

Proposition 4 validates the conjectures that allowed us to obtain Propositions 1 and 2. The most important step in Proposition 4 is determination of the identity of hyper-marginal residents (the agents who are indifferent among all locations including the countryside). As we argue below, a key property of our set-up is that the characteristics of the hyper-marginal residents are constant over time. The indifference conditions for these agents determine market rents. This means that the local rent processes are the same, but for a constant term, as the local productivity processes. This validates the linearity assumption for the rent process built into equation (2).

Let us retrace, at an intuitive level, the steps that lead to Proposition 4. Despite the fact that the payoff of an agent in a given city is determined by $S+1$ parameters (ε^l plus the vector $\boldsymbol{\rho}^l$), the expected utility U^l of the agent in that city can be condensed into a simple expression including $\bar{u}^l(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$. For any possible vector of rents $\hat{\mathbf{r}}$, the demand function $\nu^l(\hat{\mathbf{r}})$ establishes how many agents will live in each location.

Hence, for every vector of rent constants $\hat{\mathbf{r}}$, we identify a set of measure zero of hyper-marginal residents such that their expected utility is the same in every city and in the countryside:

$$\bar{u}^l(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) - \hat{r}^l = \bar{u}^0 \text{ for all } l.$$

Note that this correspond to multiple personal characteristic profiles: all the vectors $(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$ that yield the same $\bar{u}^l(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$. One can show that the vector of expected utilities of the hyper-marginal resident in different location is monotonic in the rent constant vector $\hat{\mathbf{r}}$. This means that the mapping can be inverted. Given the identity of the hyper-marginal

resident, there is only one vector of rents that guarantees that the hyper-marginal residents are indeed indifferent across all locations.

The assumption that the distribution of individual characteristics $\phi(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$ has full support guarantees that the demand function is continuous. As the hyper-marginal residents determine the vector of location demands, one can find a set of hyper-marginal residents that guarantees that demand equals supply in every location. This set is associated with the rent constant vector $\bar{\boldsymbol{r}}$.

In equilibrium, we have that housing demand equals housing supply on the space market in every city:

$$\boldsymbol{\nu}(\bar{\boldsymbol{r}}) = \boldsymbol{n};$$

and that the identity of the hyper-marginal residents is given by the set of values $(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$ so that, given the equilibrium rent vector, their expected utility is the same in every city and in the countryside:

$$\bar{u}^l(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) - \bar{r}^l = 0 \text{ for all } l.$$

A key feature of our location equilibrium is that the characteristics of the hyper-marginal resident are cohort-invariant. It is this feature that guarantees that the rent process is linear and that our equilibrium characterization is valid. If, for instance, agents could change city, the time-invariance property would not hold, and the rent process would not be linear. As a result, the properties of portfolio allocation and asset prices would differ.

We view this as a strength of spatial asset pricing models. The underlying geographic model – which we can potentially observe through demographic and labor data – affects equilibrium in the asset market.

The issue of uniqueness is complex. Obviously, there can be non-linear and/or non-stationary equilibria. Given an allocation of residents to cities, there is only one linear stationary equilibrium. There could be multiple spatial equilibria, however. The agent's expected utility in equation (4) includes a multiplicative term $\bar{p}^l \rho_s^l$. As Proposition 2 shows, the real estate return \bar{p}^l depends on R^l and hence on who lives in city l , which creates a non-trivial fixed-point problem. The (economically interesting) possibility remains that there are multiple allocations of residents to cities that give rise to linear stationary equilibrium.

Uniqueness can be achieved under certain functional assumptions, as some examples illustrate.

While we obtained closed-form solutions for portfolio decisions and asset premiums, Proposition 4 does not express rents in closed form. This is natural as the probability distribution over individual characteristics, $\phi(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$, is left in a general form. By making specific assumptions over personal characteristics and geography, one can obtain closed-form expressions for all variables, as the following example illustrates.

Assume that:

- Agents in each cohort draw city-specific endowments ε from a uniform distribution defined over $[0, 1]^L$.
- At each age, all agents face the same city-specific insulation parameter $[\tilde{\rho}_s^l]_{s=0, \dots, S}^{l=1, \dots, L}$.
- All cities are of the same size: $n^l = \frac{1}{L}N$ for every l , with $N \in (0, 1)$.

Proposition 5 *An agent with human capital ε locates in city l if: (1) $\varepsilon^l = \max_m \varepsilon^m$; and (2) $\varepsilon^l \geq (1 - N)^{\frac{1}{L}}$. The equilibrium rent in city l is*

$$r^l = (1 - N)^{\frac{1}{L}} + \frac{(1 - \beta)^2}{1 - \beta^S} \bar{p}^l \sum_{s=1}^S (1 - \beta^{S-s-1}) \tilde{\rho}_s^l.$$

If there are only two cities ($L = 2$), we can provide a two-dimensional representation of the equilibrium allocation. If, for instance, we assume that $n^1 = n^2 = \frac{1}{3}$ (and hence $n^0 \geq \frac{1}{3}$), we have the situation depicted in Figure 1. The agents who locate in the countryside are those with a low ε^1 and a low ε^2 (the bottom left square region). Those who locate in city 1 have $\varepsilon^1 \geq (1 - N)^{\frac{1}{2}}$ and $\varepsilon^1 \geq \varepsilon^2$ (bottom right trapezoid). Those who locate in city 2 have $\varepsilon^2 \geq (1 - N)^{\frac{1}{2}}$ and $\varepsilon^2 \geq \varepsilon^1$. The marginal resident is found at the intersection of those three regions.

We can also see what happens when cities have different sizes. The general characterization is more complex than the one in Proposition 5, but one can work out examples. For instance, if $n^1 = \frac{2}{9}$ and $n^2 = \frac{4}{9}$ (and a measure $\frac{1}{3}$ of agents still locate in the countryside), the allocation is depicted in Figure 2. The city-1 region is now smaller and the city-2 region is larger. The hyper-marginal residents are now to the southeast of the hyper-marginal residents of Figure 1.

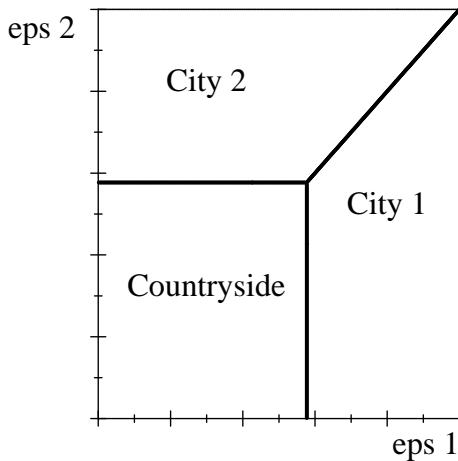


Figure 1

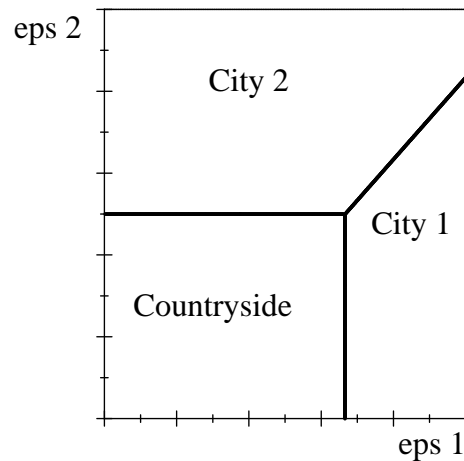


Figure 2

Another simple case assumes there is only one city, and agents differ on two dimensions: ϵ and ρ . Suppose both types are uniformly (and independently) distributed on $[0, 1]$. This equilibrium allocation is represented in Figure 3.

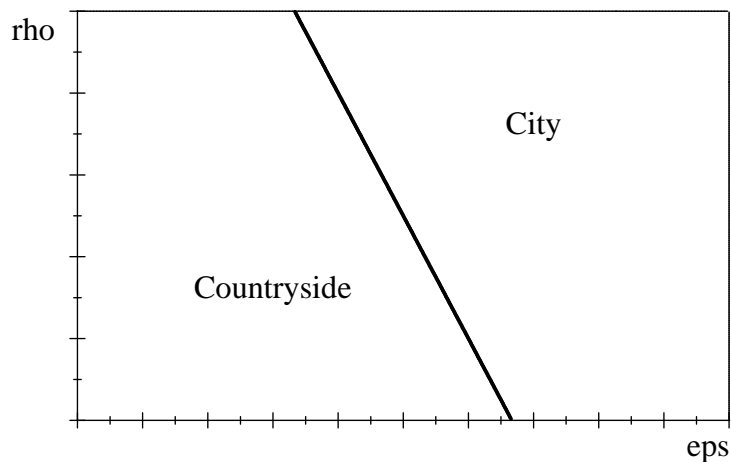


Figure 3

In Figure 3, there are values for ϵ so that agents with a low ρ locate in the countryside, and agents with a high ρ locate in the city. Agents with a high ρ are more insulated from city-level technology shocks. They buy more housing for hedging purposes than agents with a low ρ . They earn the risk premium on their housing investment although for them it provides insurance. This benefit that comes from living in the city is not available to agents with a low ρ who purchase less housing for hedging purposes.

5 Discussion

Our spatial asset pricing model yields a rich set of implications linking spatial and financial variables. We first discuss cross-sectional and life-cycle implications, and then talent allocation across cities. We explore the pricing of portfolios of stocks and portfolios of real estate. We conclude with a short discussion of how the model can be extended to include economies of agglomeration and frictions in the housing market.

5.1 Returns on housing across cities

Our model yields predictions about the cross-sectional differences in real estate returns (Proposition 2 and Corollary 3). To get some understanding of those predictions, consider a simple benchmark. Assume that shocks across cities are uncorrelated, and suppose there

are no stocks. Let $Var(\tau^l) = \sigma_l^2$. Proposition 2 yields

$$\bar{p}^l = 2\gamma \frac{\beta}{(1-\beta)^2(1-\beta^S)} \sigma_l^2 (n^l - R^l).$$

The expected return of housing in city l is an increasing function of the variance of shocks in that city and of the outstanding real estate stock $n^l - R^l$. In turn, the latter is a declining function of the average housing demand for hedging purposes, R^l , in that city. The value of R^l is determined in equilibrium.

If a location specializes in an industry and thus offers low shock-insulation parameters. All residents, whether old or young, are affected by industry productivity shocks in the same way. The residents have a low demand for housing for hedging purposes. The city's homeownership rate is low, and so are prices. The opposite, a city centered around an industry with high shock-insulation parameters – perhaps a high-tech industry where older workers struggle to keep up with innovation or a highly protected sector, where older workers face implicit insurance – will display a high hedging demand for housing, high homeownership rates, and high housing prices compared to rents.

5.2 Home ownership over the life-cycle

The model yields intertemporal predictions on individual home ownership rates. We know from Proposition 1 that housing demand for hedging purposes depends on the shock-insulation parameter, which in turn varies with age. The hedging demand by someone at age s anticipating a shock-insulation parameter the following period of ρ_{s+1}^l is

$$D_s^l = (1 - \beta^{S-s-1}) \rho_{s+1}^l.$$

Suppose the parameters ρ_s^l are determined by a differentiable function $g(s)$ defined over the positive real line. Suppose that the covariance of an agent's earnings with the earnings of the young marginal newcomers in that city declines with age; i.e., $g'(s) < 0$. Then, abusing notation for ease of exposition, we write the change in hedging demand for local home ownership with age as follows:

$$\frac{d}{ds} D_s^l = (1 - \beta^{S-s-1}) g'(s) + \log \beta \cdot \beta^{S-s-1} g(s)$$

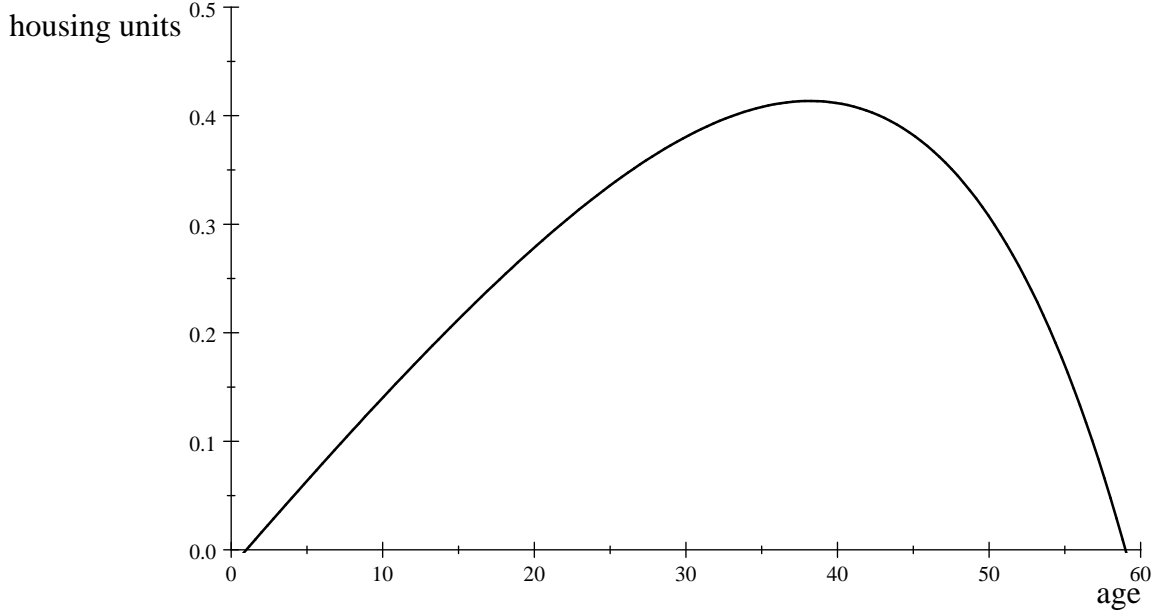
The first term represents the increasing insurance demand as the agent gets older due to ρ declining with age. The second term is the effect of the declining number of periods of life as the agent gets older. The first term increases with age; the second term is declines with age.

Next we ask under what conditions do the two effects generate a hump-shaped pattern for homeownership by looking at the second derivative:

$$\frac{d^2}{ds^2} D_s^l = (1 - \beta^{S-s-1}) g''(s) + 2 \log \beta \cdot \beta^{S-s-1} g'(s) - (\log \beta)^2 \cdot \beta^{S-s-1} g(s)$$

A sufficient condition for this expression to be negative is that $g''(s)$ be negative.

Assume that the shock-insulation parameter can be written as $\rho_s^l = k \frac{s-1}{S-1}$, with $k \in [0, 1]$ (implying $\rho_1^l = 0$ and ρ_s^l linearly increasing with age). If $\beta = 0.95$, $S = 60$, and $k = 1$, the hedging demand over the life-cycle is plotted in Figure 4.



This result offers another explanation – complementary to credit constraints – for why home-ownership rates should be lower for younger people. Younger households do not need much insurance against rent shocks because their earnings provides such insurance. As they get older, earnings provide less insurance, and their hedging demand for home ownership increases. Against this force is the fact that as an agent gets older, there are fewer remaining periods of life, reducing the demand for insurance; this last point is made by Sinai and Souleles (2005) who provide evidence of its empirical relevance.

5.3 Talent allocation

Does our market equilibrium have the potential to attain productive inefficiency?

Let us begin by defining and characterizing productive efficiency. The economy's total product at time t is

$$Y_t = \sum_{l=1}^L \int_{(\varepsilon, \rho): \bar{u}^l(\varepsilon, \rho) - \bar{r}^l = \max_m (\bar{u}^m(\varepsilon, \rho) - \bar{r}^m)} y_{t,t+s}^l(\varepsilon^l, \rho^l) \phi(\varepsilon, \rho) d(\varepsilon, \rho).$$

Suppose a planner wishes to maximize the expected discounted sum of future total products:

$$Y = \sum_{s=0}^{\infty} \beta^s E[Y_{t+s}].$$

We begin by characterizing the solution of the production maximization problem:

Proposition 6 *The allocation of agents to cities that maximizes Y depends only on ε , not on ρ . An agent with ε locates in city l if $\varepsilon^l - \bar{\varepsilon}^l = \max_m \varepsilon^m - \bar{\varepsilon}^m$, where $\bar{\varepsilon}$ is the unique vector that guarantees that the mass of agents in every city equals the housing supply.*

We can show that productive efficiency is typically not achieved, except in very special circumstances:

Proposition 7 *Exactly one of the following statements is true:*

- (1) *For all cities, $\bar{p}^l = 0$.*
- (2) *The linear stationary equilibrium does not maximize Y .*

Proposition 7 says that productive efficiency is reached if and only if the expected return on real estate is zero in every city. In that case, insurance against rent risk is available at a cost of zero instead of a negative cost if the return is positive. Agents base their location decisions exclusively on ε , and output is maximized.

Expected returns on real estate are zero when: (1) The covariance matrix Σ is such that there is no systematic risk; (2) the local productivity shocks are uncorrelated and the number of cities goes to infinity (there is still systematic risk coming from stocks). Outside these restrictive conditions, the distribution of ρ matters for location choices, and the equilibrium allocation does not maximize expected product.

Of course, productive inefficiency does not imply overall inefficiency. Our market equilibrium may be constrained-efficient, given the insurance options available in the model. Full insurance is offered only if local labor shocks – and hence local house prices – are uncorrelated with systematic risk. Beyond that special case, local real estate prices carry systematic risk, and location choices are affected by the desire of agents to cash in on risk premiums.²⁰

To reinforce the point of Proposition 7, we fully solve an example in closed-form. For ease of exposition, we let $S = 2$, and restrict the stock market to a single stock. We assume agents enjoy a constant insulation parameter ρ over life. Each cohort is equally divided into two types of agent: Type 0 agents have no insulation ($\rho = 0$), and type 1 agents have full insulation, $\rho = 1$.²¹ The distribution of agent-city match parameter is independent of agent type, ε , and uniform over the unit interval.

²⁰Proving welfare theorems in our case is difficult because the allocation space includes a discrete variable, the allocation of agents to cities.

²¹This example is not, strictly speaking, included in our model because it violates the assumption that the distribution of types is continuous and has full support. It demonstrates that the full-support assumption is sufficient but not necessary for equilibrium existence.

An agent (ε, ρ) locates in the city if and only if:

$$\frac{1}{\beta^2} \sum_{s=0}^1 \beta^s (\varepsilon - \bar{r} + (1 - \beta) (1 - \beta^{1-s}) \rho \bar{p}) \geq 0.$$

The marginal city dwellers of type 0, $\hat{\varepsilon}^0$, and type 1, $\hat{\varepsilon}^1$, satisfy:

$$\begin{cases} \hat{\varepsilon}^0 = \bar{r} \\ \hat{\varepsilon}^1 = \bar{r} - \frac{(1-\beta)^2 \bar{p}}{1+\beta}. \end{cases}$$

The market clearing condition on the spatial market is $\frac{(1-\hat{\varepsilon}^1+1-\hat{\varepsilon}^0)}{2} = n$, which yields a solution for the rent premium as a function of the housing price discount:

$$\bar{r} = 1 - n + \frac{(1 - \beta)^2}{2(1 + \beta)} \bar{p}.$$

The asset market clearing conditions are

$$\begin{aligned} \begin{bmatrix} 2n \\ 2z \end{bmatrix} &= \left(1 + \frac{1}{\beta}\right) \frac{(1 - \beta)^3}{2\gamma} \Sigma^{-1} \begin{bmatrix} \bar{p} \\ \bar{q} \end{bmatrix} + \begin{bmatrix} (1 - \beta) \left(n - \frac{1}{2} + \frac{\bar{r}}{2}\right) \\ 0 \end{bmatrix}, \\ \left(1 + \frac{1}{\beta}\right) \frac{(1 - \beta)^3}{2\gamma} \Sigma^{-1} \begin{bmatrix} \bar{p} \\ \bar{q} \end{bmatrix} &= \begin{bmatrix} 2n \\ 2z \end{bmatrix} - \begin{bmatrix} (1 - \beta) \left(n - \frac{1}{2} + \frac{\bar{r}}{2}\right) \\ 0 \end{bmatrix}, \\ \begin{bmatrix} \bar{p} \\ \bar{q} \end{bmatrix} &= \frac{2\gamma}{\left(1 + \frac{1}{\beta}\right) (1 - \beta)^3} \Sigma \begin{bmatrix} 2n - (1 - \beta) \left(n - \frac{1}{2} + \frac{\bar{r}}{2}\right) \\ 2z \end{bmatrix}. \end{aligned}$$

Let $\Sigma = \begin{bmatrix} \sigma_h^2 & \sigma_{hs} \\ \sigma_{hs} & \sigma_s^2 \end{bmatrix}$. We now have

$$\bar{p} = \frac{2\gamma}{\left(1 + \frac{1}{\beta}\right) (1 - \beta)^3} \left(\left(2n - (1 - \beta) \left(n - \frac{1}{2} + \frac{\bar{r}}{2} \right) \right) \sigma_h^2 + 2z\sigma_{hs} \right)$$

which yields

$$\bar{p} = \frac{4\gamma\beta(1 + \beta)}{(1 - \beta)^3 \left(2(1 + \beta)^2 + \gamma\beta \right)} \frac{\left(\left(2 - \frac{(1-\beta)}{2} \right) n\sigma_h^2 + 2z\sigma_{hs} \right)}{\sigma_h^2}$$

The last equation provides a solution for \bar{p} as a function of parameters only. It is then easy to obtain \bar{r} and \bar{q} from the equations above. In particular,

$$\bar{q} = \frac{2\gamma}{\left(1 + \frac{1}{\beta}\right) (1 - \beta)^3} \left(\left(2n - (1 - \beta) \left(n - \frac{1}{2} + \frac{\bar{r}}{2} \right) \right) \sigma_{hs} + 2z\sigma_s^2 \right).$$

We therefore obtain a full characterization of the equilibrium.

Assume numerical values $\beta = n = z = -\sigma_{hs} = 0.5$ and $\sigma_h = \sigma_s = \gamma = 1$; the equilibrium solution is $\bar{r} = \hat{\varepsilon}^0 = 0.65$; $\hat{\varepsilon}^1 = 0.35$; $\bar{p} = 1.8$; and $\bar{q} = 3.1$. Maximizing output would have required $\hat{\varepsilon}^0 = \hat{\varepsilon}^1 = 0.5$; i.e., not enough type 0 agents locate in the city (agents with $\varepsilon^0 \in [0.5, 0.65]$ are in the countryside instead of the city) and too many type 1 agents locate in the city (agents with $\varepsilon^1 \in [0.35, 0.5]$ are in the city instead of the countryside).

5.4 Housing and stock indices

As in the CAPM, one can price any portfolio with respect to the market. In this model, the relevant market is defined by the adjusted market portfolio M , discussed in Corollary 3.

We can price a housing-only index with weights $\frac{n-R}{\mathbf{1} \cdot [n-R]}$ (called H) and a stock-only index with weights $\frac{z}{\mathbf{1} \cdot z}$ (called S). We have:

$$\begin{aligned}\bar{p}^H &= \frac{Cov(H, M)}{Var(M)} \bar{p}^M, \\ \bar{p}^S &= \frac{Cov(S, M)}{Var(M)} \bar{p}^M.\end{aligned}$$

Note that H can be interpreted as an index tracking the market portfolio of REITs. It is the housing demand vector that is the same for all agents. It includes all houses that are not owned by local residents for hedging purposes. The result in Corollary 8 is immediate (by putting together the two return expressions above):

Corollary 8 *The relative returns of the housing index and the stock index are given by*

$$\bar{p}^H = \frac{Cov(H, M)}{Cov(S, M)} \bar{p}^S.$$

The corollary implies that, ceteris paribus, the difference between real estate returns and stock returns is related to home-ownership rates. The higher the percentage of residential property owned by local residents, the lower the returns on real estate.

Our model can also be used for predictions on stock returns. Often, the return of a stock is computed according to a CAPM formula that takes into account stocks only. Namely, the return of stock k is assumed to be

$$\tilde{q}^k = \frac{Cov(k, S)}{Var(S)} \bar{p}^S.$$

In our setting, this expression is of course incorrect, because it does not take into account the presence of housing. The correct expression is $\bar{q}^k = \frac{Cov(k, M)}{Var(M)} \bar{p}^M$. The ratio between the wrong expression and the correct one is

$$\frac{\tilde{q}^k}{\bar{q}^k} = \frac{Cov(k, S)}{Cov(k, M)} \frac{Var(M)}{Var(S)} \frac{\bar{p}^S}{\bar{p}^M}.$$

5.5 Economies of agglomeration

So far we have assumed no production externalities (or amenity externalities, if one embraces the amenity interpretation of our model). The model can be easily extended to incorporate externalities. Most results still hold, except possibly uniqueness.

Assume that the income of an agent if he or she locates in l is now given by

$$y_{t,t+s}^l(\varepsilon^l, \boldsymbol{\rho}^l) = y_{t-1}^l + \varepsilon^l(\mathbf{E}^l) + \sum_{m=0}^s (1 - \rho_m^l) \tau_{t+m}^l,$$

where \mathbf{E}^l is the collection of ε^l of other agents living in city l .

It is easy to see that Propositions 1 and 2 hold as stated. Proposition 4 can be restated as follows. For every $(\varepsilon, \boldsymbol{\rho}, \mathbf{E}^l)$, let

$$\bar{u}^l(\varepsilon, \boldsymbol{\rho}, \mathbf{E}^l) = \varepsilon^l(\mathbf{E}^l) + \frac{(1 - \beta)^2}{1 - \beta^S} \bar{p}^l \sum_{s=1}^S (1 - \beta^{S-s-1}) \rho_s^l.$$

As before, an agent locates in city l if and only if $U^l = \max_m U^m$.

An allocation of agents to cities is described by $\mathbf{E} = (\mathbf{E}^1, \dots, \mathbf{E}^L)$. Hold \mathbf{E} constant. For every L -vector \hat{r} , the aggregate demand for location l is

$$\nu^l(\hat{r}, \mathbf{E}) = \int_{(\varepsilon, \boldsymbol{\rho}) : \bar{u}^l(\varepsilon, \boldsymbol{\rho}, \mathbf{E}^l) - \hat{r}^l = \max_m (\bar{u}^m(\varepsilon, \boldsymbol{\rho}, \mathbf{E}^m) - \hat{r}^m)} \phi(\varepsilon, \boldsymbol{\rho}) d(\varepsilon, \boldsymbol{\rho}).$$

Proposition 9 *An allocation \mathbf{E} is part of a linear stationary equilibrium if and only if:*
(1) *for all $(\varepsilon, \boldsymbol{\rho})$, an agent with personal characteristics $(\varepsilon, \boldsymbol{\rho})$ locates in city l if and only if*

$$\bar{u}^l(\varepsilon, \boldsymbol{\rho}, \mathbf{E}^l) - \hat{r}^l = \max_m (\bar{u}^m(\varepsilon, \boldsymbol{\rho}, \mathbf{E}^m) - \hat{r}^m)$$

and (2) $\bar{\mathbf{r}}$ is the unique value of the vector $\hat{\mathbf{r}}$ such that $\nu^l(\hat{\mathbf{r}}, \mathbf{E}) = n^l$ in all cities.

Thus, the equilibrium characterization part of Proposition 4 is still valid. Existence of an equilibrium will depend on the properties of the functions $\varepsilon^l(\cdot)$. Moreover, as is well known, agglomeration economies tend to lead to multiple equilibria.

5.6 Ownership only

In our frictionless model, there are no intrinsic advantages to owning or renting, and households are free to own any divisible amount of their home. Consider instead the extreme case where renting is impossible. An agent can move to city l only if he or she buys one house there. In this world, all houses are owned by residents, and all residents own exactly one house. Agents can still invest in stocks.

Note that the covariance matrix can be written as

$$\Sigma = \begin{bmatrix} \Sigma_{\tau\tau} & \Sigma_{\tau\nu} \\ \Sigma_{\nu\tau} & \Sigma_{\nu\nu} \end{bmatrix}.$$

We first characterize the optimal portfolio allocation:

Proposition 10 *Given a vector of stock premiums \bar{q} , the optimal portfolio allocation for an agent with parameters (ε, ρ) is*

$$\begin{bmatrix} b_{t,t+s}^1 \\ \vdots \\ b_{t,t+s}^K \end{bmatrix} = \Sigma_{\nu\nu}^{-1} \left(\frac{1}{H} \begin{bmatrix} \bar{q}^1 \\ \vdots \\ \bar{q}^K \end{bmatrix} - \begin{bmatrix} \text{cov}(\nu^1, \tau^l) \\ \vdots \\ \text{cov}(\nu^k, \tau^l) \end{bmatrix} \omega_s(\rho_{s+1}^l) \right)$$

where $H = 2\gamma \frac{\beta^{s-S+2}}{(1-\beta)^3}$ and

$$\omega_s(\rho_{s+1}^l) = (1 - \rho_{s+1}^l) (1 - \beta^{S-s-1}) + \beta^{S-s-1}.$$

The expected utility of an agent with parameters (ε, ρ) if he or she locates in city l can be expressed as

$$U^l(\varepsilon, \rho) = \kappa_0 + \kappa_1 \bar{p}^l + \kappa_2 \varepsilon^l + \sum_{s=1}^S \xi_s \rho_s^l + \varsigma_s (\rho_s^l)^2,$$

where $\kappa_0, \kappa_1, \kappa_2, \xi_s$, and ς_s do not depend on (ε, ρ) or on \bar{p}^l , and $\kappa_1 > 0, \kappa_2 > 0$.

The optimal portfolio allocation is different from the allocation in the frictionless case. Agents can no longer choose their real estate investment. They resort to stocks to insure against the risk created by local productivity shocks. The extent to which stocks are helpful in providing insurance depends on the covariance matrix Σ .

The amount of stock k that a certain agent demand is determined by two components:

- A classic speculative element (the same as in Proposition 1).
- A hedging element, which is a function of $-\text{cov}(\nu^k, \tau^l) \omega(\rho_{s+1}^l)$, where $\omega_s(\rho_{s+1}^l)$ is a measure of hedging demand and $-\text{cov}(\nu^k, \tau^l)$ determines the value of stock k as a hedge for homes in city l . If dividend shocks are positively correlated with local productivity shocks, the hedging demand is negative.

Proposition 11 characterizes asset pricing in the ownership only economy:

Proposition 11 *For a given allocation of agents to cities, the excess return on stocks is given by*

$$\bar{q} = H \Sigma_{\nu\nu} \mathbf{z} + \frac{H}{S} \Sigma_{\nu\tau} \mathbf{\Omega},$$

where $\mathbf{\Omega} = [\Omega^1 \ \dots \ \Omega^L]'$ and

$$\Omega^l = \sum_{s=0}^{S-1} \int_{\hat{l}(\varepsilon, \rho)=l} \omega_s(\rho_{s+1}^l) \phi(\varepsilon, \rho) d(\varepsilon, \rho).$$

Our asset pricing characterization now refers only to stocks. As real estate investment is fully determined by location decisions, nothing can be said about house prices until location decisions are discussed. Stock prices have two components: a classic beta-pricing element, $H\Sigma_{\nu\nu}\mathbf{z}$, and an additional part that depends on their use for hedging against local productivity risk. This hedging component is proportional to $\Sigma_{\nu\tau}\mathbf{\Omega}$.

To understand the hedging component of the stock price, note that $\mathbf{\Omega}$ is a vector of aggregate hedging demands, one for every city. The total hedging demand $\mathbf{\Omega}^l$ in city l depends on the size of the city and how low the average shock-insulation parameter is for residents of that city. The price of stock k depends on how its dividend shocks covary with productivity shocks in all cities, weighted by the total hedging demand in every city.

To discuss the optimal location, let

$$\bar{u}^l(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) = \frac{1}{\kappa_1} \left(\kappa_2 \varepsilon^l + \sum_{s=1}^S \xi_s \rho_s^l + \varsigma_s (\rho_s^l)^2 \right).$$

For every L -vector $\hat{\mathbf{p}}$, we can write the aggregate demand for location l as

$$\nu^l(\hat{\mathbf{p}}) = \int_{\bar{u}^l(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) - \bar{p}^l = \max_{j=0, \dots, L} (\bar{u}^j(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) - \bar{p}^j)} \phi(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) d(\boldsymbol{\varepsilon}, \boldsymbol{\rho}).$$

Then, we have

Proposition 12 *There is a linear stationary equilibrium. An agent with characteristics $(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$ locates in city l if and only if*

$$\bar{u}^l(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) + \bar{p}^l = \max_{j=0, \dots, L} (\bar{u}^j(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) + \bar{p}^j)$$

and $\bar{\mathbf{p}}$ is the unique value of the vector $\hat{\mathbf{p}}$ such that $\nu^l(\hat{\mathbf{p}}) = n^l$ in all cities.

The equilibrium price in city l is

$$p_t^l = \frac{1}{1 - \beta} y_t^j - \bar{p}^l.$$

As in the frictionless case, the equilibrium housing price is ultimately determined by the preferences of the hyper-marginal residents. As before, the expected utility of an agent who locates in city l depends only on the value of parameters for city l (i.e. ε^l and ρ_s^l , for all s).²² As in Proposition 4, there is a unique price vector for which aggregate demand equals aggregate supply.

²²However, now the expected utility of an agent who locates in city l takes a different form (quadratic in ρ_s^l).

6 Conclusion

Choosing where to live amounts to choosing a zero-net price combination of a positive stream of dividends (income, access to local amenities) and a negative stream of dividends (housing costs). With this insight in mind, we approach the pricing of residential properties with a model that combines a standard spatial equilibrium framework with a standard portfolio choice and asset pricing framework. Housing rents are determined by market clearing in the space market; every home in a populated city. The pricing of homes in each city and the pricing of all other assets in the economy are determined by market clearing in the asset market; all assets are held by investors.

Our model highlights significant interactions between the space market and the asset market. For one, the location choice of households depends not only on expected income minus rent (as in standard spatial equilibrium models) but also on the risk premium embedded in the price of local homes and the risk each household faces as measured by the covariance of its income with that of other city residents.

The pricing of assets depends on the location choices of the households. Who lives where determines (1) the expected rents for residential properties everywhere, (2) their volatility and covariance with other assets, and (3) the weight of residential properties from each location in the adjusted market portfolio that is relevant for the pricing of all assets in the economy.

Our results show the cost of ignoring the spatial nature of the economy when one studies the allocation of households over space and the pricing of all assets. The theory also generates new empirical questions with regard to the cross-sectional and time series variations of households' portfolios and returns to housing, and invites further refinements to standard practices in the pricing of financial assets.

This represents just a first step toward a theory of spatial asset pricing. Our goal has been to obtain a simple, tractable set-up to illustrate the links between location decisions on the one hand and investment decisions and asset prices on the other. Future research can explore, analytically or numerically, richer models of spatial asset pricing. Clearly, it would be interesting to move beyond CARA utility functions. It would also be useful to study real estate prices when moving costs are finite. Finally, it would be useful to allow for an elastic housing supply, perhaps even one that is determined endogenously through the political process (see Ortalo-Magné and Prat, 2009, for a first step in this direction).

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Appendix

Proof of Proposition 1

The cash flow at period $t + s$ for agent born at t , living in city l is

$$\begin{aligned} v_{t,t+s} &= y_{t,t+s}^l - r_{t+s}^l \\ &\quad - \sum_j \left((p_{t+s}^j - r_{t+s}^j) a_{t,t+s}^j - p_{t+s}^j a_{t,t+s-1}^j \right) \\ &\quad - \sum_k \left((q_{t+s}^k - d_{t+s}^k) b_{t,t+s}^k - q_{t+s}^k b_{t,t+s-1}^k \right) \end{aligned}$$

for $s = 0, \dots, S - 1$ and

$$v_{t,t+S} = \sum_j p_{t+S}^j a_{t,t+S-1}^j + \sum_k q_{t+S}^k b_{t,t+S-1}^k.$$

The end-of-life wealth of an agent born in t (evaluated at the beginning of his or her life) is:

$$w_t = \frac{1}{\beta^S} \sum_{s=0}^S \beta^s v_{t,t+s}$$

Plug in the income process and the linear prices:

$$\begin{aligned} w_t &= \frac{1}{\beta^S} \sum_{s=0}^{S-1} \beta^s \left(y_{t-1}^l + \varepsilon^l + \sum_{m=0}^s (1 - \rho_m^l) \tau_{t+m}^l - y_{t-1}^l - \sum_{m=0}^s \tau_{t+m}^l - \bar{r}^l \right) \\ &\quad + \frac{1}{\beta^S} \sum_j \sum_{s=0}^{S-1} \beta^s a_{t,t+s}^j \left(\frac{\beta}{1-\beta} \tau_{t+s+1}^j + (1-\beta) \bar{p}^j \right) \\ &\quad + \frac{1}{\beta^S} \sum_k \sum_{s=0}^{S-1} \beta^s b_{t,t+s}^k \left(\frac{\beta}{1-\beta} \nu_{t+s+1}^k + (1-\beta) \bar{q}^k \right) \\ &= \frac{1}{\beta^S} \sum_{s=0}^{S-1} \beta^s (\varepsilon^l - \bar{r}^l) - \frac{1}{\beta^S} \sum_{s=0}^{S-1} \beta^s \sum_{m=0}^s \rho_m^l \tau_{t+m}^l \\ &\quad + \frac{1}{\beta^S} \sum_j \sum_{s=0}^{S-1} \beta^s a_{t,t+s}^j \left(\frac{\beta}{1-\beta} \tau_{t+s+1}^j + (1-\beta) \bar{p}^j \right) \\ &\quad + \frac{1}{\beta^S} \sum_k \sum_{s=0}^{S-1} \beta^s b_{t,t+s}^k \left(\frac{\beta}{1-\beta} \nu_{t+s+1}^k + (1-\beta) \bar{q}^k \right) \end{aligned}$$

because $\rho_0^l = 0$. Note that

$$\begin{aligned}
\sum_{s=0}^{S-1} \beta^s \sum_{m=0}^s \rho_m^l \tau_{t+m}^l &= \sum_{s=0}^{S-1} \rho_s^l \tau_{t+s}^l \sum_{m=s}^{S-1} \beta^m = \sum_{s=0}^{S-1} \rho_s^l \tau_{t+s}^l \beta^s \sum_{m=0}^{S-1-s} \beta^m \\
&= \sum_{s=0}^{S-1} \rho_s^l \tau_{t+s}^l \frac{\beta^s (1 - \beta^{S-s})}{1 - \beta} \\
&= \sum_{s=1}^{S-1} \rho_s^l \tau_{t+s}^l \frac{\beta^s (1 - \beta^{S-s})}{1 - \beta} \text{ because } \rho_0 = 0 \\
&= \sum_{s=0}^{S-2} \beta^s (1 - \beta^{S-s-1}) \rho_{s+1}^l \frac{\beta}{1 - \beta} \tau_{t+s+1}^l \\
&= \sum_{s=0}^{S-1} \beta^s (1 - \beta^{S-s-1}) \rho_{s+1}^l \frac{\beta}{1 - \beta} \tau_{t+s+1}^l \text{ because } \rho_S \equiv 0
\end{aligned}$$

Then,

$$\begin{aligned}
w_t &= \sum_{s=0}^{S-1} \beta^{s-S} \left((\varepsilon^l - \bar{r}^l) - (1 - \beta^{S-s-1}) \rho_{s+1}^l \frac{\beta}{1 - \beta} \tau_{t+s+1}^l \right) \\
&\quad + \sum_{s=0}^{S-1} \beta^{s-S} \sum_j a_{t,t+s}^j \left(\frac{\beta}{1 - \beta} \tau_{t+s+1}^j + (1 - \beta) \bar{p}^j \right) + \sum_{s=0}^{S-1} \beta^{s-S} \sum_k b_{t,t+s}^k \left(\frac{\beta}{1 - \beta} \nu_{t+s+1}^k + (1 - \beta) \bar{q}^k \right)
\end{aligned}$$

Hence,

$$\begin{aligned}
E[w_t] &= \sum_{s=0}^{S-1} \beta^{s-S} \left(\varepsilon^l - \bar{r}^l + (1 - \beta) \left((1 - \beta^{S-s-1}) \rho_{s+1}^l \bar{p}^l + \sum_{j=1}^L \tilde{a}_{t,t+s}^j \bar{p}^j + \sum_{k=1}^K b_{t,t+s}^k \bar{q}^k \right) \right) \\
Var[w_t] &= \frac{\beta^2}{(1 - \beta)^2} \sum_{s=0}^{S-1} \beta^{2(s-S)} Var \left[\sum_{j=1}^L \tilde{a}_{t,t+s}^j \tau_{t+s+1}^j + \sum_{k=1}^K b_{t,t+s}^k \nu_{t+s+1}^k \right]
\end{aligned}$$

where $\tilde{a}_{t,t+s}^l = a_{t,t+s}^l - (1 - \beta^{S-s-1}) \rho_{s+1}^l$ and $\tilde{a}_{t,t+s}^j = a_{t,t+s}^j$ for all $j \neq l$.

In a matrix form, this is rewritten as

$$\begin{aligned}
E[w_t] &= \sum_{s=0}^{S-1} \beta^{s-S} \left(\varepsilon^l - \bar{r}^l + (1 - \beta) \left((1 - \beta^{S-s-1}) \rho_{s+1}^l \bar{p}^l + \begin{bmatrix} \tilde{\mathbf{a}}_{t,t+s} \\ \mathbf{b}_{t,t+s} \end{bmatrix}' \begin{bmatrix} \bar{\mathbf{p}} \\ \bar{\mathbf{q}} \end{bmatrix} \right) \right) \\
Var[w_t] &= \frac{\beta^2}{(1 - \beta)^2} \sum_{s=0}^{S-1} \beta^{2(s-S)} \begin{bmatrix} \tilde{\mathbf{a}}_{t,t+s} \\ \mathbf{b}_{t,t+s} \end{bmatrix}' \Sigma \begin{bmatrix} \tilde{\mathbf{a}}_{t,t+s} \\ \mathbf{b}_{t,t+s} \end{bmatrix}
\end{aligned}$$

The first-order conditions yield

$$\begin{bmatrix} \tilde{\mathbf{a}}_{t,t+s} \\ \mathbf{b}_{t,t+s} \end{bmatrix} = \frac{(1 - \beta)^3}{2\gamma\beta^{s+2}} \beta^S \Sigma^{-1} \begin{bmatrix} \bar{\mathbf{p}} \\ \bar{\mathbf{q}} \end{bmatrix},$$

Plug back into the utility function:

$$\begin{aligned}
U &= \frac{1}{\beta^S} \sum_{s=0}^{S-1} \beta^s (\varepsilon^l - \bar{r}^l) + \frac{1}{\beta^S} \sum_{s=0}^{S-1} \beta^s (1 - \beta^{S-s-1}) \rho_{s+1}^l (1 - \beta) \bar{p}^l \\
&\quad + \frac{1}{\beta^S} \sum_{s=0}^{S-1} \beta^s (1 - \beta) \left(\frac{(1 - \beta)^3}{2\gamma\beta^{s+2}} \beta^S \left[\begin{array}{c} \bar{p} \\ \bar{q} \end{array} \right]' \Sigma^{-1} \left[\begin{array}{c} \bar{p} \\ \bar{q} \end{array} \right] \right) \\
&\quad - \gamma \frac{1}{\beta^{2S}} \frac{\beta^2}{(1 - \beta)^2} \sum_{s=0}^{S-1} \beta^{2s} \frac{(1 - \beta)^3}{2\gamma\beta^{s+2}} \frac{(1 - \beta)^3}{2\gamma\beta^{s+2}} \beta^{2S} \left[\begin{array}{c} \bar{p} \\ \bar{q} \end{array} \right]' \Sigma^{-1} \Sigma \Sigma^{-1} \left[\begin{array}{c} \bar{p} \\ \bar{q} \end{array} \right] \\
&= \frac{1}{\beta^S} \sum_{s=0}^{S-1} \beta^s (\varepsilon^l - \bar{r}^l + (1 - \beta) (1 - \beta^{S-s-1}) \rho_{s+1}^l \bar{p}^l) + S \frac{(1 - \beta)^4}{4\gamma\beta^2} \left[\begin{array}{c} \bar{p} \\ \bar{q} \end{array} \right]' \Sigma^{-1} \left[\begin{array}{c} \bar{p} \\ \bar{q} \end{array} \right]
\end{aligned}$$

■

Proof of Proposition 2

The demands for assets beyond a hedging motive can be written as

$$\left[\begin{array}{c} \tilde{\mathbf{a}}_{t-s,t} \\ \mathbf{b}_{t-s,t} \end{array} \right] = \frac{(1 - \beta)^3}{2\gamma\beta^{s+2}} \beta^S \Sigma^{-1} \left[\begin{array}{c} \bar{p} \\ \bar{q} \end{array} \right]$$

for $s = 0, \dots, S - 1$. Since all agents have the same portfolio and there is a measure one of agents in each cohort, the aggregate portfolio demand for assets (excluding the hedging motive), is

$$\sum_{s=0}^{S-1} \left[\begin{array}{c} \tilde{\mathbf{a}}_{t-s,t} \\ \mathbf{b}_{t-s,t} \end{array} \right] = \left(1 + \frac{1}{\beta} + \frac{1}{\beta^2} + \dots + \frac{1}{\beta^{S-1}} \right) \left[\begin{array}{c} \tilde{\mathbf{a}}_{t,t} \\ \mathbf{b}_{t,t} \end{array} \right] = \frac{1 - \beta^S}{(1 - \beta) \beta^{S-1}} \left[\begin{array}{c} \tilde{\mathbf{a}}_{t,t} \\ \mathbf{b}_{t,t} \end{array} \right].$$

The housing demand in city l by people with age s due to the hedging motive is

$$(1 - \beta^{S-s-1}) \int_{\varepsilon} \int_{\rho} I_{\varepsilon, \rho}^l \rho_{s+1}^l \phi(\varepsilon, \rho) d\varepsilon d\rho.$$

It is then easy to see that the total housing demand in city l due to the hedging motive is $S R^l$, where R^l is defined as in the proposition.

The supply of houses minus the hedging demand in every city is $S(\mathbf{n} - \mathbf{R})$. The housing market clearing condition is therefore

$$\frac{1 - \beta^S}{(1 - \beta) \beta^{S-1}} \tilde{\mathbf{a}}_{t,t} = S(\mathbf{n} - \mathbf{R}).$$

Hence

$$\tilde{\mathbf{a}}_{t,t} = S \frac{(1 - \beta) \beta^{S-1}}{(1 - \beta^S)} (\mathbf{n} - \mathbf{R})$$

and by analogy

$$\mathbf{b}_{t,t} = S \frac{(1 - \beta) \beta^{S-1}}{(1 - \beta^S)} \mathbf{z}.$$

Plugging in the demand function yields a solution to the housing and stock risk premiums:

$$\begin{aligned}
S \frac{(1 - \beta) \beta^{S-1}}{(1 - \beta^S)} \left[\begin{array}{c} (\mathbf{n} - \mathbf{R}) \\ \mathbf{z} \end{array} \right] &= \frac{(1 - \beta)^3}{2\gamma\beta^2} \beta^S \Sigma^{-1} \left[\begin{array}{c} \bar{p} \\ \bar{q} \end{array} \right] \\
2\gamma S \frac{\beta}{(1 - \beta)^2 (1 - \beta^S)} \Sigma \left[\begin{array}{c} \mathbf{n} - \mathbf{R} \\ \mathbf{z} \end{array} \right] &= \left[\begin{array}{c} \bar{p} \\ \bar{q} \end{array} \right].
\end{aligned}$$

■

Proof of Corollary 3

Note that

$$\begin{aligned} Cov(l, M) &= \frac{1}{Q} \left(\sum_{m=1}^L (n^m - R^m) Cov(\tau^l, \tau^m) + \sum_{k=1}^K z^k Cov(\tau^l, \nu^k) \right) \\ Var(M) &= \frac{1}{Q^2} \begin{bmatrix} \mathbf{n} - \mathbf{R} \\ \mathbf{z} \end{bmatrix}' \Sigma \begin{bmatrix} \mathbf{n} - \mathbf{R} \\ \mathbf{z} \end{bmatrix} \end{aligned}$$

The expected return of a zero-cost market portfolio containing one unit of M is given by

$$\begin{aligned} \bar{p}^M &= \frac{\sum_{l=1}^L (n^l - R^l) \bar{p}^l + \sum_{k=1}^K z^k \bar{q}^k}{\sum_{l=1}^L (n^l - R^l) + \sum_{k=1}^K z^k} = \frac{1}{Q} \begin{bmatrix} \mathbf{n} - \mathbf{R} \\ \mathbf{z} \end{bmatrix}' \begin{bmatrix} \bar{\mathbf{p}} \\ \bar{\mathbf{q}} \end{bmatrix} \\ &= 2\gamma \frac{\beta}{(1-\beta)^2 (1-\beta^S)} S \frac{1}{Q} \begin{bmatrix} \mathbf{n} - \mathbf{R} \\ \mathbf{z} \end{bmatrix}' \Sigma \begin{bmatrix} \mathbf{n} - \mathbf{R} \\ \mathbf{z} \end{bmatrix} = 2\gamma \frac{\beta}{(1-\beta)^2 (1-\beta^S)} SQVar(M). \end{aligned}$$

Similarly

$$\begin{aligned} \bar{p}^l &= 2\gamma \frac{\beta}{(1-\beta)^2 (1-\beta^S)} S \left(\sum_{m=1}^L (n^m - R^m) Cov(\tau^l, \tau^m) + \sum_{k=1}^K z^k Cov(\tau^l, \nu^k) \right) \\ &= 2\gamma \frac{\beta}{(1-\beta)^2 (1-\beta^S)} SQCov(l, M) \end{aligned}$$

Hence, we can write

$$\frac{\bar{p}^l}{\bar{p}^M} = \frac{Cov(l, M)}{Var(M)}.$$

The proof for k follows similar lines and is omitted. ■

Proof of Proposition 4

It is immediate to see that a solution to $\boldsymbol{\nu}(\hat{\mathbf{r}}) = \mathbf{n}$ constitutes a linear stationary equilibrium; no agent wants to change location, by definition $r_t^l = y_t^j + \bar{r}^l$, and the conditions for Propositions 1 and 2 are satisfied.

To prove existence, note that $\nu^l(\hat{\mathbf{r}})$ is continuous in $\hat{\mathbf{r}}$ and that $\lim_{\hat{r}^l \rightarrow -\infty} \nu^l(\hat{\mathbf{r}}) = 1$ and $\lim_{\hat{r}^l \rightarrow \infty} \nu^l(\hat{\mathbf{r}}) = 0$.

To prove that prices and rents are uniquely determined *given an allocation of agents to cities*, suppose that the system $\boldsymbol{\nu}(\hat{\mathbf{r}}) = \mathbf{n}$ has two distinct solutions $\bar{\mathbf{r}}$ and $\bar{\mathbf{r}}'$. Assume without loss of generality that there are a non-empty set of cities \tilde{L} for which $(\bar{r}')^l < \bar{r}^l$. The set of agents who locate in a city in \tilde{L} is given by

$$\left\{ (\boldsymbol{\varepsilon}, \boldsymbol{\rho}) : \max_{l \in \tilde{L}} (\bar{u}^l(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) - \bar{r}^l) \geq \max_{j \notin \tilde{L}} (\bar{u}^j(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) - \bar{r}^j) \right\}$$

Note, however, that this set must become strictly larger when $\bar{\mathbf{r}}$ is replaced by $\bar{\mathbf{r}}'$, because all elements $\bar{u}^l(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) - \bar{r}^l$ on one side become strictly larger and all elements $\bar{u}^j(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) - \bar{r}^j$ on the other side do not become larger. Hence, more agents will want to locate in cities in \tilde{L} , but this is impossible as the mass of agents who locate in \tilde{L} must sum up to $\sum_{l \in \tilde{L}} n^l$ in both solutions. ■

Proof of Proposition 5

Note that

$$\bar{u}^l = \varepsilon^l + \frac{1-\beta}{1-\beta^S} \left(1 - (S+1)\beta^S + S\beta^{S+1}\right) \bar{p}^l \bar{\rho}_{s+1}^l.$$

As $\bar{\rho}_{s+1}^l$ are the same for all agents and the $\tilde{\varepsilon}$ are uniformly distributed, we write

$$\nu^l(\hat{r}) = \int_{\tilde{\varepsilon}: \tilde{\varepsilon}^l - \hat{r}^l = \max_m (\tilde{\varepsilon}^m - \hat{r}^m)} d\tilde{\varepsilon}$$

This problem is symmetric in l . Hence, the unique solution to $\nu^l(\hat{r}) = \frac{1}{L}N$ for $l = 1, \dots, L$ must be symmetric in l , namely $\bar{r}^l = \bar{r}$. This means that in every period, the mass of young agent who locates in the countryside is $\nu^0(\hat{r}) = \bar{r}^L$. This implies $\bar{r} = (1-N)^{\frac{1}{L}}$. According to equation (4), the equilibrium rent is given by

$$r^l = \bar{r} + \frac{(1-\beta)^2}{1-\beta^S} \bar{p}^l \sum_{s=1}^S \left(1 - \beta^{S-s-1}\right) \bar{\rho}_s^l.$$

■

Proof of Proposition 6

Consider any allocation of agents to cities. Suppose an agent with $(\varepsilon^l, \varepsilon^m)$ is allocated to city l and another agent with $((\varepsilon^l)', (\varepsilon^m)')$ is allocated to m . Swapping agents does not increase total expected production if and only if

$$\varepsilon^l - (\varepsilon^l)' \geq \varepsilon^m - (\varepsilon^m)'$$

If this holds true for every agent, one can find a unique vector $\bar{\varepsilon}$ such that the condition in the statement is satisfied. ■

Proof of Proposition 7

According to Proposition 4, in a linear stationary equilibrium agents are assigned to cities according to

$$\bar{u}^l(\varepsilon, \rho) = \varepsilon^l + \frac{(1-\beta)^2}{1-\beta^S} \bar{p}^l \sum_{s=1}^S \left(1 - \beta^{S-s-1}\right) \rho_s^l.$$

Suppose that an agent with a certain (ε, ρ) locates in city l . The next preferred city is m , and the utility difference between the two cities is given by

$$D = \bar{u}^l(\varepsilon, \rho) - \bar{u}^m(\varepsilon, \rho),$$

where D is sufficiently low. Consider another agent with (ε', ρ') that is identical to (ε, ρ) except that $(\varepsilon')^l = \varepsilon^l + \delta$ and $\sum_{s=1}^S \left(1 - \beta^{S-s-1}\right) (\rho')^l_s = \sum_{s=1}^S \left(1 - \beta^{S-s-1}\right) \rho_s^l - \alpha$. Given a positive \bar{p}^l , it is always possible to find α and δ such that $\bar{u}^l(\varepsilon', \rho') < \bar{u}^m(\varepsilon', \rho')$. By the assumption that ϕ has full support, agents with (ε, ρ) and (ε', ρ') exist. The sum of expected outputs of the two agents would be higher if the agents switch cities.

The only time this cannot be done is when \bar{p}^l is the same for all cities. In that case, it is easy to see that agents choose location in order to maximize ε^l . ■

Proof of Proposition 9

The first part is immediate. If \mathbf{E} is an allocation and prices are linear, then every agent is using $\bar{u}^l(\varepsilon, \rho, \mathbf{E}^l) - \bar{r}^l$ as a criterion to locate and rents must equate demand and supply. The argument for the uniqueness of \bar{r}^l (given \mathbf{E}) is unchanged from the proof of Proposition 4. \blacksquare

Proof of Proposition 10

Consider an agent born in period t with parameters (ε, ρ) who locates in city l . His or her wealth at the end of life is

$$w_t = \sum_{s=0}^{S-1} \beta^{s-S} \left(y_{t-1}^l + \varepsilon^l + \sum_{m=0}^s (1 - \rho_m^l) \tau_{t+m}^l + \sum_k b_{t,t+s}^k \left(\frac{\beta}{1-\beta} \nu_{t+s+1}^j + (1-\beta) \bar{q}^k \right) \right) - \frac{1}{\beta^S} p_t^l + p_{t+S}^l.$$

Conjecture $p_t^l = \frac{1}{1-\beta} y_t^l - \bar{p}^l$. This implies $p_{t+S}^l = \frac{1}{1-\beta} y_{t+S}^l - \bar{p}^l = \frac{1}{1-\beta} (y_t^l + \tau_{t+1} + \dots + \tau_{t+S-1} + \tau_{t+S}) - \bar{p}^l = p_t^l + \frac{1}{1-\beta} \left(\sum_{s=0}^{S-1} \tau_{t+s+1} \right)$. Replacing in the above equation yields

$$\begin{aligned} w_t &= \sum_{s=0}^{S-1} \beta^{s-S} \left(y_{t-1}^l + \varepsilon^l + \sum_{m=0}^s (1 - \rho_m^l) \tau_{t+m}^l + \sum_k b_{t,t+s}^k \left(\frac{\beta}{1-\beta} \nu_{t+s+1}^j + (1-\beta) \bar{q}^k \right) \right) \\ &\quad + \left(1 - \frac{1}{\beta^S} \right) p_t^l + \sum_{s=1}^S \frac{1}{1-\beta} \tau_{t+s}^l \\ w_t &= \sum_{s=0}^{S-1} \beta^{s-S} \left(y_{t-1}^l + \varepsilon^l + \sum_k b_{t,t+s}^k \left(\frac{\beta}{1-\beta} \nu_{t+s+1}^j + (1-\beta) \bar{q}^k \right) \right) + \sum_{s=0}^{S-1} \beta^{s-S} \sum_{m=0}^s (1 - \rho_m^l) \tau_{t+m}^l \\ &\quad + \left(1 - \frac{1}{\beta^S} \right) p_t^l + \sum_{s=0}^{S-1} \frac{1}{1-\beta} \tau_{t+s+1}^l. \end{aligned}$$

Note

$$\begin{aligned}
\sum_{s=0}^{S-1} \beta^{s-S} \sum_{m=0}^s (1 - \rho_m^l) \tau_{t+m}^l &= \frac{1}{\beta^S} \sum_{s=0}^{S-1} (1 - \rho_s^l) \tau_{t+s}^l \sum_{m=s}^{S-1} \beta^m \\
&= \frac{1}{\beta^S} \sum_{s=0}^{S-1} (1 - \rho_s^l) \tau_{t+s}^l \beta^s \frac{1 - \beta^{S-s}}{1 - \beta} \\
&= \sum_{s=0}^{S-1} (1 - \rho_s^l) \frac{\beta^{s-S} - 1}{1 - \beta} \tau_{t+s}^l \\
&= \sum_{s=1}^S (1 - \rho_s^l) \frac{\beta^{s-S} - 1}{1 - \beta} \tau_{t+s}^l + (1 - \rho_0^l) \frac{\beta^{0-S} - 1}{1 - \beta} \tau_t^l \\
&\quad - (1 - \rho_S^l) \frac{\beta^{S-S} - 1}{1 - \beta} \tau_{t+S}^l \\
&= \sum_{s=0}^{S-1} (1 - \rho_{s+1}^l) \frac{\beta^{s-S+1} - 1}{1 - \beta} \tau_{t+s+1}^l + (1 - \rho_0^l) \frac{\beta^{-S} - 1}{1 - \beta} \tau_t^l \\
&\quad - (1 - \rho_S^l) \frac{\beta^{S-S} - 1}{1 - \beta} \tau_{t+S}^l \\
&= \sum_{s=0}^{S-1} \beta^{s-S} \frac{\beta}{1 - \beta} (1 - \beta^{S-s-1}) (1 - \rho_{s+1}^l) \tau_{t+s+1}^l + \frac{\beta^{-S} - 1}{1 - \beta} \tau_t^l.
\end{aligned}$$

We have therefore

$$\begin{aligned}
w_t &= \sum_{s=0}^{S-1} \beta^{s-S} \left(y_{t-1}^l + \varepsilon^l + \sum_k b_{t,t+s}^k \left(\frac{\beta}{1 - \beta} \nu_{t+s+1}^j + (1 - \beta) \bar{q}^k \right) \right) \\
&\quad + \sum_{s=0}^{S-1} \beta^{s-S} \frac{\beta}{1 - \beta} (1 - \beta^{S-s-1}) (1 - \rho_{s+1}^l) \tau_{t+s+1}^l \\
&\quad + \frac{\beta^{-S} - 1}{1 - \beta} \tau_t^l + \left(1 - \frac{1}{\beta^S} \right) p_t^l + \sum_{s=0}^{S-1} \frac{1}{1 - \beta} \tau_{t+s+1}^l \\
w_t &= \sum_{s=0}^{S-1} \beta^{s-S} \left(y_{t-1}^l + \varepsilon^l + \sum_k b_{t,t+s}^k \left(\frac{\beta}{1 - \beta} \nu_{t+s+1}^j + (1 - \beta) \bar{q}^k \right) \right) \\
&\quad + \sum_{s=0}^{S-1} \beta^{s-S} \left(\frac{\beta}{1 - \beta} (1 - \beta^{S-s-1}) (1 - \rho_{s+1}^l) + \frac{\beta}{1 - \beta} \beta^{S-s-1} \right) \tau_{t+s+1}^l \\
&\quad + \frac{\beta^{-S} - 1}{1 - \beta} \tau_t^l + \left(1 - \frac{1}{\beta^S} \right) p_t^l \\
w_t &= \sum_{s=0}^{S-1} \beta^{s-S} \left(y_{t-1}^l + \varepsilon^l + \left((1 - \rho_{s+1}^l) (1 - \beta^{S-s-1}) + \beta^{S-s-1} \right) \frac{\beta}{1 - \beta} \tau_{t+s+1}^l \right) \\
&\quad + \sum_k b_{t,t+s}^k \left(\frac{\beta}{1 - \beta} \nu_{t+s+1}^j + (1 - \beta) \bar{q}^k \right) \\
&\quad + \frac{\beta^{-S} - 1}{1 - \beta} \tau_t^l + \left(1 - \frac{1}{\beta^S} \right) p_t^l.
\end{aligned}$$

Note $\frac{\beta^{-s}-1}{1-\beta}\tau_t^l + \left(1 - \frac{1}{\beta^s}\right)p_t^l = \frac{1}{\beta^s}\frac{1-\beta^s}{1-\beta}\tau_t^l - \frac{1}{\beta^s}\frac{1-\beta^s}{1-\beta}(1-\beta)p_t^l = \sum_{s=0}^{S-1}\beta^{s-S}(\tau_t^l - (1-\beta)p_t^l)$. Using $p_t^l = \frac{1}{1-\beta}(y_{t-1} + \tau_t) - \bar{p}^l$ this yields

$$\begin{aligned} \frac{\beta^{-S}-1}{1-\beta}\tau_t^l + \left(1 - \frac{1}{\beta^S}\right)p_t^l &= \sum_{s=0}^{S-1}\beta^{s-S}(\tau_t^l - (1-\beta)p_t^l) \\ &= \sum_{s=0}^{S-1}\beta^{s-S}\left(\tau_t^l - (1-\beta)\left(\frac{1}{1-\beta}(y_{t-1} + \tau_t^l) - \bar{p}^l\right)\right) \\ &= \sum_{s=0}^{S-1}\beta^{s-S}(-y_{t-1} + (1-\beta)\bar{p}^l). \end{aligned}$$

So we now have

$$\begin{aligned} w_t &= \sum_{s=0}^{S-1}\beta^{s-S}\left(y_{t-1}^l + \varepsilon^l - y_{t-1} + (1-\beta)\bar{p}^l + \left((1-\rho_{s+1}^l)(1-\beta^{S-s-1}) + \beta^{S-s-1}\right)\frac{\beta}{1-\beta}\tau_{t+s+1}^l\right. \\ &\quad \left. + \sum_k b_{t,t+s}^k\left(\frac{\beta}{1-\beta}\nu_{t+s+1}^j + (1-\beta)\bar{q}^k\right)\right) \\ w_t &= \sum_{s=0}^{S-1}\beta^{s-S}\left(\varepsilon^l + (1-\beta)\bar{p}^l + \left((1-\rho_{s+1}^l)(1-\beta^{S-s-1}) + \beta^{S-s-1}\right)\frac{\beta}{1-\beta}\tau_{t+s+1}^l\right. \\ &\quad \left. + \sum_k b_{t,t+s}^k\left(\frac{\beta}{1-\beta}\nu_{t+s+1}^j + (1-\beta)\bar{q}^k\right)\right). \end{aligned}$$

Then

$$\begin{aligned} E[w_t] &= \sum_{s=0}^{S-1}\beta^{s-S}\left(\varepsilon^l + (1-\beta)\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ b_{t,t+s}^1 \\ \vdots \\ b_{t,t+s}^K \end{bmatrix}' \begin{bmatrix} \bar{p}^1 \\ \vdots \\ \bar{p}^l \\ \vdots \\ \bar{p}^L \\ \bar{q}^1 \\ \vdots \\ \bar{q}^K \end{bmatrix}\right), \\ \text{Var}[w_t] &= \frac{\beta^2}{(1-\beta)^2}\sum_{s=0}^{S-1}\beta^{2(s-S)}\begin{bmatrix} 0 \\ \vdots \\ \omega_s(\rho_{s+1}^l) \\ \vdots \\ 0 \\ b_{t,t+s}^1 \\ \vdots \\ b_{t,t+s}^K \end{bmatrix}' \Sigma \begin{bmatrix} 0 \\ \vdots \\ \omega_s(\rho_{s+1}^l) \\ \vdots \\ 0 \\ b_{t,t+s}^1 \\ \vdots \\ b_{t,t+s}^K \end{bmatrix}, \end{aligned}$$

where

$$\omega_s(\rho_{s+1}^l) = (1-\rho_{s+1}^l)(1-\beta^{S-s-1}) + \beta^{S-s-1}.$$

The first-order condition for the optimal stock investment is

$$\begin{bmatrix} \bar{q}^1 \\ \vdots \\ \bar{q}^K \end{bmatrix} = H \begin{bmatrix} \Sigma_{\nu\tau} & \Sigma_{\nu\nu} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ \omega_s (\rho_{s+1}^l) \\ \vdots \\ 0 \\ b_{t,t+s}^1 \\ \vdots \\ b_{t,t+s}^K \end{bmatrix} = H \begin{bmatrix} cov(\nu^1, \tau^l) \\ \vdots \\ cov(\nu^k, \tau^l) \end{bmatrix} \omega_s (\rho_{s+1}^l) + H \Sigma_{\nu\nu} \begin{bmatrix} b_{t,t+s}^1 \\ \vdots \\ b_{t,t+s}^K \end{bmatrix},$$

where

$$H = 2\gamma \frac{\beta^{s-S+2}}{(1-\beta)^3}.$$

The vector of individual demands for stocks is

$$\begin{bmatrix} b_{t,t+s}^1 \\ \vdots \\ b_{t,t+s}^K \end{bmatrix} = \frac{\Sigma_{\nu\nu}^{-1}}{H} \begin{bmatrix} \bar{q}^1 \\ \vdots \\ \bar{q}^K \end{bmatrix} - \Sigma_{\nu\nu}^{-1} \begin{bmatrix} cov(\nu^1, \tau^l) \\ \vdots \\ cov(\nu^k, \tau^l) \end{bmatrix} \omega_s (\rho_{s+1}^l).$$

Let

$$\mathbf{h}^l = \begin{bmatrix} cov(\nu^1, \tau^l) \\ \vdots \\ cov(\nu^k, \tau^l) \end{bmatrix}.$$

The expectation and the variance of final wealth are given respectively by

$$\begin{aligned} E[w_t] &= \sum_{s=0}^{S-1} \beta^{s-S} (\varepsilon^l + (1-\beta) (\bar{p}^l + \mathbf{b}' \bar{\mathbf{q}})) \\ &= \sum_{s=0}^{S-1} \beta^{s-S} \left(\varepsilon^l + (1-\beta) \left(\bar{p}^l + \left(\frac{\Sigma_{\nu\nu}^{-1}}{H} \bar{\mathbf{q}} - \Sigma_{\nu\nu}^{-1} \begin{bmatrix} cov(\nu^1, \tau^l) \\ \vdots \\ cov(\nu^k, \tau^l) \end{bmatrix} \omega_s \right)' \bar{\mathbf{q}} \right) \right) \\ &= \sum_{s=0}^{S-1} \beta^{s-S} \left(\varepsilon^l + (1-\beta) \left(\bar{p}^l + \bar{\mathbf{q}}' \frac{\Sigma_{\nu\nu}^{-1}}{H} \bar{\mathbf{q}} - \bar{\mathbf{q}}' \Sigma_{\nu\nu}^{-1} \mathbf{h}^l \omega_s \right) \right), \end{aligned}$$

and

$$\begin{aligned} Var[w_t] &= \frac{\beta^2}{(1-\beta)^2} \sum_{s=0}^{S-1} \beta^{2(s-S)} \left(Var(\tau^l) \omega_s^2 + \mathbf{b}' \begin{bmatrix} cov(\nu^1, \tau^l) \\ \vdots \\ cov(\nu^k, \tau^l) \end{bmatrix} \omega_s + \mathbf{b}' \Sigma_{\nu\nu} \mathbf{b} \right) \\ &= \frac{\beta^2}{(1-\beta)^2} \sum_{s=0}^{S-1} \beta^{2(s-S)} \left(Var(\tau^l) \omega_s^2 + \left(\frac{\Sigma_{\nu\nu}^{-1}}{H} \bar{\mathbf{q}} - \Sigma_{\nu\nu}^{-1} \mathbf{h}^l \omega_s \right)' \mathbf{h}^l \omega_s \right. \\ &\quad \left. + \left(\frac{\Sigma_{\nu\nu}^{-1}}{H} \bar{\mathbf{q}} - \Sigma_{\nu\nu}^{-1} \mathbf{h}^l \omega_s \right)' \Sigma_{\nu\nu} \left(\frac{\Sigma_{\nu\nu}^{-1}}{H} \bar{\mathbf{q}} - \Sigma_{\nu\nu}^{-1} \mathbf{h}^l \omega_s \right) \right) \\ &= \frac{\beta^2}{(1-\beta)^2} \sum_{s=0}^{S-1} \beta^{2(s-S)} \left(Var(\tau^l) \omega_s^2 + \frac{1}{H} (\mathbf{h}^l)' \Sigma_{\nu\nu}^{-1} \bar{\mathbf{q}} \omega_s - (\mathbf{h}^l)' \Sigma_{\nu\nu}^{-1} \mathbf{h}^l \omega_s^2 \right. \\ &\quad \left. + \frac{1}{H^2} \bar{\mathbf{q}}' \Sigma_{\nu\nu}^{-1} \bar{\mathbf{q}} - 2 \bar{\mathbf{q}}' \Sigma_{\nu\nu}^{-1} \mathbf{h}^l \omega_s + (\mathbf{h}^l)' \Sigma_{\nu\nu}^{-1} \mathbf{h}^l \omega_s^2 \right). \end{aligned}$$

Note that

$$\begin{aligned}\kappa_1 &= \sum_{s=0}^{S-1} \beta^{s-S} (1 - \beta) = \frac{1 - \beta^S}{\beta^S} > 0 \\ \kappa_2 &= \sum_{s=0}^{S-1} \beta^{s-S} = \frac{1 - \beta^S}{\beta^S (1 - \beta)} > 0\end{aligned}$$

■

Proof of Proposition 11

The market clearing condition is

$$\begin{aligned}S \begin{bmatrix} z^1 \\ \vdots \\ z^K \end{bmatrix} &= \sum_{s=0}^{S-1} \sum_{l=0}^L \int_{\hat{l}(\boldsymbol{\varepsilon}, \boldsymbol{\rho})=l} \begin{bmatrix} b_{t,t+s}^1 \\ \vdots \\ b_{t,t+s}^K \end{bmatrix} \phi(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) d(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) \\ &= \sum_{s=0}^{S-1} \sum_{l=0}^L \int_{\hat{l}(\boldsymbol{\varepsilon}, \boldsymbol{\rho})=l} \left(\frac{\Sigma_{\nu\nu}^{-1}}{H} \begin{bmatrix} \bar{q}^1 \\ \vdots \\ \bar{q}^K \end{bmatrix} - \Sigma_{\nu\nu}^{-1} \begin{bmatrix} \text{cov}(\nu^1, \tau^l) \\ \vdots \\ \text{cov}(\nu^K, \tau^l) \end{bmatrix} \omega(\rho_{s+1}^l) \right) \phi(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) d(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) \\ &= S \frac{\Sigma_{\nu\nu}^{-1}}{H} \begin{bmatrix} \bar{q}^1 \\ \vdots \\ \bar{q}^K \end{bmatrix} - \sum_{l=0}^L \Sigma_{\nu\nu}^{-1} \begin{bmatrix} \text{cov}(\nu^1, \tau^l) \\ \vdots \\ \text{cov}(\nu^K, \tau^l) \end{bmatrix} \sum_{s=0}^{S-1} \int_{\hat{l}(\boldsymbol{\varepsilon}, \boldsymbol{\rho})=l} \omega(\rho_{s+1}^l) \phi(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) d(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) \\ &= S \frac{\Sigma_{\nu\nu}^{-1}}{H} \begin{bmatrix} \bar{q}^1 \\ \vdots \\ \bar{q}^K \end{bmatrix} - \Sigma_{\nu\nu}^{-1} \sum_{l=0}^L \begin{bmatrix} \text{cov}(\nu^1, \tau^l) \\ \vdots \\ \text{cov}(\nu^K, \tau^l) \end{bmatrix} \Omega^l\end{aligned}$$

where

$$\Omega^l = \sum_{s=0}^{S-1} \int_{\hat{l}(\boldsymbol{\varepsilon}, \boldsymbol{\rho})=l} \omega(\rho_{s+1}^l) \phi(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) d(\boldsymbol{\varepsilon}, \boldsymbol{\rho}).$$

Then,

$$\begin{aligned}\begin{bmatrix} \bar{q}^1 \\ \vdots \\ \bar{q}^K \end{bmatrix} &= H \Sigma_{\nu\nu} \begin{bmatrix} z^1 \\ \vdots \\ z^K \end{bmatrix} + \frac{H}{S} \sum_{l=0}^L \begin{bmatrix} \text{cov}(\nu^1, \tau^l) \\ \vdots \\ \text{cov}(\nu^K, \tau^l) \end{bmatrix} \Omega^l \\ &= H \Sigma_{\nu\nu} \mathbf{z} + \frac{H}{S} \Sigma_{\nu\tau} \boldsymbol{\Omega},\end{aligned}$$

where

$$\boldsymbol{\Omega} = [\Omega^1 \quad \dots \quad \Omega^L]'.$$

■

Proof of Proposition 12

Given a vector of housing premiums $\bar{\mathbf{p}}$, and agent with $(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$ locates in l if

$$U^l = \max_{j=0, \dots, L} U^j$$

but this is equivalent to

$$\kappa_1 \bar{p}^l + \kappa_2 \varepsilon^l + \sum_{s=1}^S \xi_s \rho_s^l + \varsigma_s (\rho_s^l)^2 = \max_{j=0, \dots, L} \left(\kappa_1 \bar{p}^j + \kappa_2 \varepsilon^j + \sum_{s=1}^S \xi_s \rho_s^j + \varsigma_s (\rho_s^j)^2 \right)$$

or

$$\frac{1}{\kappa_1} \left(\kappa_2 \varepsilon^l + \sum_{s=1}^S \xi_s \rho_s^l + \varsigma_s (\rho_s^l)^2 \right) + \bar{p}^l = \max_{j=0, \dots, L} \left(\frac{1}{\kappa_1} \left(\kappa_2 \varepsilon^j + \sum_{s=1}^S \xi_s \rho_s^j + \varsigma_s (\rho_s^j)^2 \right) + \bar{p}^j \right).$$

The rest of the proof is similar to the proof of Proposition 4, and it is omitted. Note that in our conjecture we treat \bar{r}^l and \bar{p}^l asymmetrically, one with a positive sign, the other with a negative sign. This explains the difference in signs for each of these two terms between Proposition 4 and the Proposition 12. ■