

Thompson Sampling for the MNL-Bandit

author names withheld

Editor: Under Review for COLT 2017

Abstract

We consider a sequential subset selection problem under parameter uncertainty, where at each time step, the decision maker selects a subset of cardinality K from N possible items (arms), and observes a (bandit) feedback in the form of the index of one of the items in said subset, or none. Each item in the index set is ascribed a certain value (reward), and the feedback is governed by a Multinomial Logit (MNL) choice model whose parameters are a priori unknown. The objective of the decision maker is to maximize the expected cumulative rewards over a finite horizon T , or alternatively, minimize the regret relative to an oracle that knows the MNL parameters. We refer to this as the MNL-Bandit problem. This problem is representative of a larger family of exploration-exploitation problems that involve a combinatorial objective, and arise in several important application domains. We present an approach to adapt Thompson Sampling to this problem and show that it achieves near-optimal regret as well as attractive numerical performance.

Keywords: Thompson Sampling, Exploration-Exploitation, Multinomial Logit Choice Model

1. Introduction and Problem Formulation

Background. In the traditional stochastic multi-armed Bandit (MAB) problem, the decision maker selects one of, say, N arms in each round and receives feedback in the form of a noisy reward characteristic of that arm. Regret minimizing strategies are typically based on the principle of optimism in the face of uncertainty, a prime example of which are the family of upper confidence bound policies (UCB), which allow the player to learn the identity of the best arm through sequential experimentation, while concurrently not spending “too much” of the sampling efforts on the sub-optimal arms. In this paper we consider a combinatorial variant of this problem where in each time step the player selects a bundle of K arms, after which s/he gets to see the reward associated with one of the arms in that bundle, or observing no reward at all. One can think of the “no reward” as the result of augmenting each bundle with a further index that belongs to a “null arm” that cannot be directly chosen but can be manifest as a feedback; this structure will be further motivated shortly. The identity of the arm within the bundle that yields the reward observation (or the “null” arm that yields no observation) is determined by means of a probability distribution on the index set of cardinality $K + 1$ (the K arms plus the “null” arm). In this paper the distribution is specified by means of a multinomial logit model (MNL); hence the name MNL-Bandit.

A possible interpretation of this MNL-Bandit problem is as follows. A decision maker is faced with the problem of determining which subset (of at most cardinality K) of N items to present to users that arrive sequentially, where user preferences for said items are unknown. Each user either selects one of the items s/he are offered or selects none (the “null arm” option described above). Every item presents some reward which is item-specific. Based on the observations of items users have selected, the decision maker needs to ascertain the composition of the “best” bundle,” which involves balancing an exploration over bundles to learn the users’ preferences, while simultaneously

exploiting the bundles that exhibit good reward. (The exact mathematical formulation is given below.) A significant challenge here is the combinatorial nature of the problem just described, as the space of possible subsets of cardinality K is exponentially large, and for reasonable sized time horizons cannot be efficiently explored.

The problem as stated above is not new, but there is surprisingly little antecedent literature on it; the review below will expound on its history and related strands of work. It arises in many real-world instances, perhaps most notably in display-based online advertising. Here the publisher has to select a set of advertisements to display to users. Due to competing ads, the click rates for an individual ad depends on the overall subset of ads to be displayed; this is referred to as a *substitution* effect. For example, consider a user presented with two similar vacation packages from two different sources. The user’s likelihood of clicking on one of the ads in this scenario, would most likely differ from the situation where one of the ads is presented as a standalone. Because every advertisement is valued differently from the publisher’s perspective, the set of ads selected for display has a significant impact on revenues. A similar problem arises in online retail settings, where the retailer need to select a subset (assortment) of products to offer. Here demand for a specific product is influenced by the assortment of products offered. To capture these substitution effects, choice models are often used to specify user preferences in the form of a probability distribution over items in a subset.

The MNL-Bandit is a natural way to cast the exploration-exploitation problem discussed above into a well studied machine learning paradigm, and allows to more easily adapt algorithmic ideas developed in that setting. In particular, this paper focuses on a Thompson Sampling (TS) approach to the MNL-Bandit problem. This is primarily motivated by the attractive empirical properties that have been observed over a stream of recent papers in the context of TS versus more traditional approaches such as upper confidence bound policies (UCB). For the MNL-Bandit this has further importance given the combinatorial nature of the dynamic optimization problem one is attempting to solve. One of the main contributions of the present paper is in highlighting the salient features of TS that need to be adapted or customized to facilitate the design of an algorithm in the MNL-Bandit, and to elucidate their role in proving regret-optimality for this variant of TS. To the best of our knowledge some of these ideas are new in the TS-context, and can hopefully extend its scope to combinatorial-type problems that will go beyond the MNL-Bandit.

Problem Formulation. To formally state our problem, consider an option space containing N distinct elements, indexed by $1, 2, \dots, N$ and their values denoted by r_1, \dots, r_N , with r mnemonic for reward, though we will also use the term *revenue* in this context. Since the user need not necessarily choose any of the options presented, we model this “outside alternative” as an additional item denoted with an index of “0” which augments the index set. We assume that for any offer set, $S \subset \{1, \dots, N\}$, the user will be selecting only one of the offered alternatives or item 0, and this selection is given by a Multinomial Logit (MNL) choice model. Under this model, the probability that a user chooses item $i \in S$ is given by,

$$p_i(S) = \begin{cases} \frac{v_i}{v_0 + \sum_{j \in S} v_j}, & \text{if } i \in S \cup \{0\} \\ 0, & \text{otherwise,} \end{cases} \quad (1)$$

where v_i is a parameter of the MNL model corresponding to item i . Without loss of generality, we can assume that $v_0 = 1$. (The focus on MNL is due to its prevalent use in the context of modeling substitution effects, and its tractability; see further discussion in related work.)

Given the above, the expected revenue corresponding to the offer set S , $R(S)$ is given by

$$R(S, \mathbf{v}) = \sum_{i \in S} r_i p_i(S) = \sum_{i \in S} \frac{r_i v_i}{1 + \sum_{j \in S} v_j}. \quad (2)$$

and the corresponding *static* optimization problem, i.e., when the parameter vector $\mathbf{v} = (v_0, \dots, v_N)$ and henceforth, $p_i(S)$ is known a priori, is given by,

$$\max \left\{ R(S, \mathbf{v}) \mid |S| \leq K \right\}. \quad (3)$$

The cardinality constraints specified above, arise naturally in many applications. Specifically, a publisher/retailer is constrained by the space for advertisements/products and has to limit the number of ads/products that can be displayed.

Consider a time horizon T , where a subset of items can be offered at time periods $t = 1, \dots, T$. Let S^* be the *offline optimal offer set* for (3) under full information, namely, when the values of $p_i(S)$, as given by (1), are known a priori. In the MNL-Bandit, the decision maker does not know the values of $p_i(S)$ and can only make sequential offer set decisions, S_1, \dots, S_T , at times $1, \dots, T$, respectively. The objective is to design an algorithm that selects a (non-anticipating) sequence of offer sets in a path-dependent manner (namely, based on past choices and observed responses) to maximize cumulative expected revenues over the said horizon, or alternatively, minimize the *regret* defined as

$$\text{Reg}(T, \mathbf{v}) = \mathbb{E} \left[\sum_{t=1}^T R(S^*, \mathbf{v}) - R(S_t, \mathbf{v}) \right], \quad (4)$$

where $R(S, \mathbf{v})$ is the expected revenue when the offer set is S , and is as defined in (2). Here we make explicit the dependence of regret on the time horizon T and the parameter vector \mathbf{v} of the MNL model that determines the user preferences and choices.

Outline. We review related literature and describe our contributions in Section 2. In Section 3, we present our adaptations of the Thompson Sampling algorithm for the MNL-bandit, and in Section 4, we prove our main result that our algorithm achieves an $\tilde{O}(\sqrt{NT} \log TK)$ regret upper bound. Section 5 demonstrates the empirical efficiency of our algorithm design.

2. Related Work and Overview of Contribution

A basic pillar in the MNL-Bandit problem is the MNL choice model, originally introduced (independently) by Luce (1959) and Plackett (1975); see also Train (2003); McFadden (1978); Ben-Akiva and Lerman (1985) for further discussion and survey of other commonly used choice models. This model is by far the most widely used choice model insofar as capturing substitution effects that are a significant element in our problem. Initial motivation for this traces to online retail, where a retailer has to decide on a subset of items to offer from a universe of substitutable products for display. In this context, Rusmevichientong et al. (2010) and Sauré and Zeevi (2013) were the first two papers we are aware of, to consider a dynamic learning problem, in particular, focusing on minimizing regret under the MNL choice model. Both papers develop an “explore first and exploit later” approach. Assuming knowledge of the “gap” between the optimal and the next-best assortment, they show an asymptotic $O(N \log T)$ regret bound. (This assumption is akin to the “separated arm” case in the MAB setting.) It is worth noting that the algorithms developed in those papers require a priori knowledge of this gap as a tuning input, which makes the algorithms parameter dependent. In a more recent paper, Agrawal et al. (2016) show how to exploit specific characteristics of the MNL

model to develop a policy based on the principle of “optimism under uncertainty” (UCB-like algorithm, see [Auer et al. \(2002\)](#)) which does not rely on the a priori knowledge of this gap or separation information and achieves a worst-case regret bound of $O(\sqrt{NT \log T})$. A regret lower bound of $\Omega(\sqrt{NT/K})$ for this problem is also presented in this work.

It is widely recognized that UCB-type algorithms that optimize the worst case regret typically tend to spend “too much time” in the exploration phase, resulting in poor performance in practice (regret-optimality bounds notwithstanding). To that end, several studies ([Olivier and Li \(2011\)](#), [Graepel et al. \(2010\)](#), [May et al. \(2012\)](#)) have demonstrated that TS significantly outperforms the state of the art methods in practice. Despite being easy to implement and often empirically superior, TS based algorithms are hard to analyze and theoretical work on TS is limited. To the best of our knowledge, [Agrawal and Goyal \(2013a\)](#) is the first work to provide a finite time worst-case regret bounds for the MAB problem that are independent of problem parameters.

A naive translation of the MNL-bandit problem to an MAB-type setting would create $\binom{N}{K}$ “arms” (one for each offer set of size K). For an “arm” corresponding to subset S , the reward is given by $R(S)$ (3). Managing this exponentially large arm space is prohibitive for obvious reasons. Popular extensions of MAB for “large scale” problems include the linear bandit (e.g., [Auer \(2003\)](#), [Rusmevichientong and Tsitsiklis \(2010\)](#)) for which [Agrawal and Goyal \(2013b\)](#) present a TS-based algorithm and provide finite time regret bounds. However, these approaches do not apply directly to our problem, since the revenue corresponding to each offered set is not linear in problem parameters. Moreover, for the regret bounds in those settings to be attractive, the dimension d of parameters should be small, this dimension would be N here. [Gopalan et al. \(2014\)](#) consider a variant of MAB where one can play a subset of arms in each round and the expected reward is a function of rewards of the arms played. This setting is similar to the MNL-bandit, though the regret bounds they develop are dependent on the instance parameters as well as the number of possible actions which can be large in our combinatorial problem setting. Moreover, the computational tractability of updating the posterior and computing the optimal action set is not immediately clear.

Our Contributions. In this work, relying on structural properties of the MNL model, we develop a TS approach that is computationally efficient and yet achieves parameter independent (optimal in order) regret bounds. Specifically, we present a computationally efficient TS algorithm for the MNL-bandit which uses a prior distribution on the parameters of the MNL model such that the posterior update under the MNL-bandit feedback is tractable. A key ingredient in our approach is a two moment approximation of the posterior and the ability to judiciously correlate samples, which is done by embedding the two-moment approximation in a normal family. It is shown that our algorithm achieves a worst-case (prior-free) regret bound of $O(\sqrt{NT} \log TK)$ under a mild assumption that $v_0 \geq v_i$ for all i (more on the practicality of this assumption later in the text); the bound is non-asymptotic, the “big oh” notation is used for brevity. This regret bound is independent of the parameters of the MNL choice model and hence holds uniformly over all problem instances. The regret is comparable to the existing upper bound of $O(\sqrt{NT \log T})$ and the lower bound of $\Omega(\sqrt{NT/K})$ provided by [Agrawal et al. \(2016\)](#) under the same assumption, yet the numerical results demonstrate that our Thompson Sampling based approach significantly outperforms the UCB-based approach of [Agrawal et al. \(2016\)](#). The methods developed in this paper highlight some of the key challenges involved in adapting the TS approach to the MNL-bandit, and present a blueprint to address these issues that we hope will be more broadly applicable, and form the basis for further work in the intersection of combinatorial optimization and machine learning.

3. Algorithm

In this section, we describe our posterior sampling (aka Thompson Sampling) based algorithm for the MNL-bandit problem. The basic structure of Thompson Sampling involves maintaining a posterior on the unknown problem parameters, which is updated every time new feedback is obtained. In the beginning of every round, a sample set of parameters is generated from the current posterior distribution, and the algorithm chooses the best option according to these sample parameters. Due to its combinatorial nature, designing an algorithm in this framework for the MNL-bandit problem involves several new challenges as we describe below, along with our algorithm design choices to address them.

3.1. Challenges and key ideas

Conjugate priors for the MNL parameters. In the MNL-bandit problem, there is one unknown parameter v_i associated with each item. To adapt the TS algorithm for the classical MAB problem, here we would need to maintain a joint posterior for (v_1, \dots, v_N) . However, updating such a joint posterior is non-trivial since the feedback observed in every round is the choice made by the user among the offered set of items S , and the observed choice provides a sample from multinomial choice probability $\frac{v_i}{1 + \sum_{j \in S} v_j}$, which clearly depends on the subset S offered in that round. In particular, even if we initialize with an independent prior from a nice analytical family such as multivariate Gaussian, the posterior distribution after observing the MNL choice feedback can have a complex description.

Another possibility is to maintain a posterior each for revenue function of $\binom{N}{K}$ possible assortments, where the posterior for the set S is updated only when that set is offered. However, due to the exponential number of possible offered sets, such an approach would learn very slowly and result in regret exponential in K , in addition to being computationally inefficient.

One of key ideas utilized in our algorithm design is that of repeated offering of assortments in a way that allows us to efficiently maintain independent conjugate (Beta) priors for parameters v_i for each i . Details of the resulting TS algorithm are presented in Algorithm 1 in Section 3.2.

Posterior approximation and Correlated sampling. Algorithm 1 samples the posterior distribution for each parameter *independently* in each round. However, this algorithm presents unique challenges in theoretical analysis. A worst case regret analysis of Thompson Sampling based algorithms for MAB typically proceeds by showing that the best arm is optimistic at least once every few steps, in the sense that its sampled parameter is better than its true parameter. Such a proof approach for our combinatorial problem requires that every few steps, all the K items in the optimal offer set have sampled parameters that are better than their true counterparts. This makes the probability of being optimistic exponentially small in K .

We address this challenge by employing *correlated sampling* across items. To implement correlated sampling, we find it useful to approximate the Beta posterior by a Gaussian distribution with approximately the same mean and variance as the Beta distribution; what was referred to in the introduction as a two-moment approximation. This allows us to generate correlated samples from the N Gaussian distributions as linear transforms of a single standard Gaussian. Under such correlated sampling, the probability of all K optimal items to be simultaneously optimistic is a constant, as opposed to being exponentially small (in K) in the case of independent samples. However, such correlated sampling reduces the overall variance of the maximum of N samples severely, thus reducing

exploration. We boost the variance by taking K samples instead of a single sample of the standard Gaussian. The resulting variant of Thompson Sampling algorithm is presented in Algorithm 2 in Section 3.3. We prove near-optimal regret bound for this algorithm in Section 4.

To the best of our knowledge the idea of correlated sampling for combinatorial arms is novel, and potentially useful for further extensions to other combinatorial problems. In fact, by reducing sampling variance, correlated sampling may lead to better empirical performance, and may even compensate for the boosting due to multiple samples. In Section 5, we present some preliminary numerical simulation results, which illustrate this intuition.

3.2. Warmup: A TS algorithm with independent conjugate Beta priors

In this first version of our Thompson sampling algorithm, we maintain a Beta posterior distribution for each item $i = 1, \dots, N$, which is updated as we observe users' choice of items from the offered subsets. A key challenge here is to design priors that can be efficiently updated on observing user choice feedback, in order to obtain increasingly accurate estimates of parameters $\{v_i\}$. To address this, we use a technique introduced in Agrawal et al. (2016). The idea is to offer a set S multiple times; in particular, a chosen S is offered repeatedly until an “outside option” is picked (in the motivating application discussed earlier, this corresponds displaying the same subset of ads until we observe a user who does not click on any of the displayed ads). Proceeding in this manner, the average number of times an item i is selected provides an unbiased estimate of parameter v_i . Moreover, the number of times an item i is selected is also independent of the displayed set and is a geometric distribution with success probability $1/(1 + v_i)$ and mean v_i . Precise statements are provided in Lemma 1. This observation is used as the basis for our epoch based algorithmic structure and our choice of prior/posterior, as a conjugate to this geometric distribution.

Epoch based offerings: Our algorithm proceeds in epochs $\ell = 1, 2, \dots$. An epoch is a group of consecutive time steps, where a set S_ℓ is offered repeatedly until the outside option is picked in response to offering S_ℓ . The set S_ℓ to be offered in an epoch ℓ is picked in the beginning of the epoch based on the sampled parameters from the current posterior distribution; the construction of these posteriors and choice of S_ℓ is described in the next paragraph. We denote the group of time steps in an epoch as \mathcal{E}_ℓ , which includes the time step at which an outside option was preferred.

Construction of conjugate prior/posterior: Suppose that the prior distribution for parameter v_i for $i \in S_\ell$ in the beginning of an epoch ℓ is same as that of $X_i = \frac{1}{\text{Beta}(n_i, V_i)} - 1$, In Lemma 2, we show that after observing the geometric variable $\tilde{v}_{i,\ell} = m$: the number of picks of item i in epoch ℓ , the posterior distribution of v_i is same as that of, $X'_i = \frac{1}{\text{Beta}(n_i+1, V_i+m)} - 1$. Therefore, we use the distribution of $\frac{1}{\text{Beta}(1,1)} - 1$ as the starting prior for v_i , and then, in the beginning of epoch ℓ , the posterior is distributed as $\frac{1}{\text{Beta}(n_i(\ell), V_i(\ell))} - 1$, with $n_i(\ell)$ being the number of epochs the item i has been offered before epoch ℓ (as part of an assortment), and $V_i(\ell)$ being the number of times it was picked by the user. This posterior distribution has expected value $\frac{V_i(\ell)+1}{n_i(\ell)-3}$ and variance close to $\frac{v_i(1+v_i)}{n_i(\ell)}$ (refer to Lemma 3) so that the samples for parameter from this distribution will approach v_i , as the number of epochs $n_i(\ell)$ where item i is offered becomes large.

Selection of subset to be offered: To choose the subset to be offered in epoch ℓ , the algorithm samples a set of parameters $\mu_1(\ell), \dots, \mu_N(\ell)$ independently from the current posteriors and finds the set that maximizes the expected revenue as per the sampled parameters. In particular, the set S_ℓ

to be offered in epoch ℓ is chosen as:

$$S_\ell := \arg \max_{S \in \mathcal{S}} R(S, \boldsymbol{\mu}(\ell)) \quad (5)$$

There are efficient polynomial time algorithms available to solve this optimization problem (e.g., refer to [Davis et al. \(2013\)](#), [Avadhanula et al. \(2016\)](#), [Rusmevichientong et al. \(2010\)](#)).

The details of our procedure are provided in Algorithm 1.

Algorithm 1 A TS algorithm for MNL-bandit with Independent Beta priors

Initialization: For each item $i = 1, \dots, N$, $V_i = 1$, $n_i = 1$ for all .

$t = 1$, keeps track of the time steps

$\ell = 1$, keeps count of total number of epochs

while $t \leq T$ **do**

(a) (*Posterior Sampling*) For each item $i = 1, \dots, N$, sample $\theta_i(\ell)$ from the $\text{Beta}(n_i, V_i)$ and compute $\mu_i(\ell) = \frac{1}{\theta_i(\ell)} - 1$

(b) (*Subset Selection*) Compute $S_\ell = \underset{S \in \mathcal{S}}{\operatorname{argmax}} R(S, \boldsymbol{\mu}(\ell)) = \frac{\sum_{i \in S} r_i \mu_i(\ell)}{1 + \sum_{j \in S} \mu_j(\ell)}$

(c) (*Epoch-based offering*)

repeat

 Offer the set S_ℓ , and observe the user choice \mathbf{c}_t ;

 Update $\mathcal{E}_\ell = \mathcal{E}_\ell \cup t$, time indices corresponding to epoch ℓ ; $t = t + 1$

until $\mathbf{c}_t = 0$;

(d) (*Posterior update*)

 for each item $i \in S_\ell$, compute $\tilde{v}_{i,\ell} = \sum_{t \in \mathcal{E}_\ell} \mathbb{I}(\mathbf{c}_t = i)$, no. of picks of item i in epoch ℓ .

 Update $V_i = V_i + \tilde{v}_{i,\ell}$, $n_i = n_i + 1$, $\ell = \ell + 1$.

end

The following lemmas provide important building blocks for our construction. Their proofs have been deferred to the appendix.

Lemma 1 (Agrawal et al. (2016)) *Let $\tilde{v}_{i,\ell}$ be the number of times an item $i \in S_\ell$ is picked when the set S_ℓ is offered repeatedly until no-click (outside option is picked). Then, $\tilde{v}_{i,\ell}, \forall \ell, i$ are i.i.d geometrical random variables with success probability $\frac{1}{1+v_i}$, and expected value v_i .*

Lemma 2 (Conjugate Priors) *For any $\alpha > 3, \beta > 0$, let $X_{\alpha,\beta} = \frac{1}{\text{Beta}(\alpha,\beta)} - 1$ and $f_{\alpha,\beta}$ be a probability distribution of the random variable $X_{\alpha,\beta}$. If v_i is distributed as $f_{\alpha,\beta}$ and $\tilde{v}_{i,\ell}$ is a geometric random variable with success probability $\frac{1}{v_i+1}$, then we have,*

$$\mathbb{P}\left(v_i \mid \tilde{v}_{i,\ell} = m\right) = f_{\alpha+1,\beta+m}(v_i).$$

Lemma 3 (Moments of the Posterior Distribution) *If X is a random variable distributed as $\text{Beta}(\alpha, \beta)$, then*

$$\mathbb{E}\left(\frac{1}{X} - 1\right) = \frac{\beta}{\alpha-3}, \quad \text{and} \quad \text{Var}\left(\frac{1}{X} - 1\right) = \frac{\frac{\beta}{\alpha-3}\left(\frac{\beta}{\alpha-3} + 1\right)}{\alpha-4}.$$

3.3. A TS algorithm with posterior approximation and correlated sampling

Motivated by the challenges in theoretical analysis of Algorithm 1 described earlier, in this section we design a variant, Algorithm 2, with the main changes being the introduction of a posterior approximation by means of a Gaussian distribution, correlated sampling, and taking multiple samples (“variance boosting”). We describe each of these below.

Posterior approximation: We approximate the posterior distributions used in Algorithm 1 for the MNL parameters v_i , by Gaussian distributions with approximately the same mean and variance (refer to Lemma 3). In particular, let

$$\hat{v}_i(\ell) := \frac{V_i(\ell)}{n_i(\ell)}, \quad \hat{\sigma}_i(\ell) := \sqrt{\frac{50\hat{v}_i(\ell)(\hat{v}_i(\ell)+1)}{n_i(\ell)}} + 75\frac{\sqrt{\log TK}}{n_i(\ell)},$$

then the posterior distribution used for item i in the beginning of epoch ℓ is

$$\mathcal{N}(\hat{v}_i(\ell), \hat{\sigma}_i(\ell)^2).$$

Correlated sampling: Given the posterior approximation by Gaussian distributions, we correlate the samples by using a common standard Gaussian sample and constructing our posterior samples as an appropriate transform of this common standard Gaussian sample. This allows us to generate sample parameters for $i = 1, \dots, N$ that are either simultaneously high or simultaneously low, thereby, boosting the probability that the sample parameters for *all* the K items in the best assortment are optimistic.

Multiple (K) samples: The correlated sampling decreases the joint variance of the sample set. In order to boost this joint variance and ensure sufficient exploration, we generate multiple sets of samples. In particular, in the beginning of an epoch ℓ , we generate K independent samples from the standard normal distribution, $\theta^{(j)} \sim \mathcal{N}(0, 1), j = 1, \dots, K$. And then, the j^{th} sample set is generated as:

$$\mu_i^{(j)}(\ell) = \hat{v}_i(\ell) + \theta^{(j)}\hat{\sigma}_i(\ell), \quad i = 1, \dots, N,$$

and we use the highest valued samples

$$\mu_i(\ell) = \max_{j=1, \dots, K} \mu_i^{(j)}(\ell), \forall i,$$

to decide the assortment to offer in epoch ℓ ,

$$S_\ell = \arg \max_{S \in \mathcal{S}} R(S, \boldsymbol{\mu}(\ell))$$

We summarize the steps in Algorithm 2. Here, we also have an “initial exploration period,” where for every item i , we offer a set containing only i until the user selects the outside option.

Intuitively, while the second moment approximation by Gaussian distribution and multiple samples in Algorithm 2 may make posterior converge slower and increase exploration, the correlated sampling may compensate for these effects by reducing the variance of the maximum of N samples, and therefore reducing the overall exploration effort. In Section 5, we illustrate some of these insights through some preliminary numerical simulations, where correlated sampling in fact shows significantly better performance compared to independent sampling, and posterior approximation by Gaussian distribution has little effect.

Algorithm 2 A TS algorithm for MNL-bandit with Gaussian approximation and correlated sampling

Initialization: $t = 0, \ell = 0, n_i = 0$ for all $i = 1, \dots, N$.

for each item, $i = 1, \dots, N$ **do**

Display item i to users until the user selects the “outside option”. Let $\tilde{v}_{i,1}$ be the number of times item i was offered. Update: $\hat{v}_i = \tilde{v}_{i,1} - 1, t = t + \tilde{v}_{i,1}, \ell = \ell + 1$ and $n_i = n_i + 1$.

end

while $t \leq T$ **do**

(a) (*Correlated sampling*) **for** $j = 1, \dots, K$ **do**

Sample $\theta^{(j)}(\ell)$ from the distribution $\mathcal{N}(0, 1)$.

For each item $i \leq N$, compute $\mu_i^{(j)}(\ell) = \hat{v}_i + \theta^{(j)}(\ell) \cdot \left(\sqrt{\frac{50\hat{v}_i(\hat{v}_i+1)}{n_i}} + \frac{75\sqrt{\log TK}}{n_i} \right)$.

end

For each item $i \leq N$, compute $\mu_i(\ell) = \max_{j=1, \dots, K} \mu_i^{(j)}(\ell)$

(b) (*Subset selection*) Same as step (b) of Algorithm 1.

(c) (*Epoch-based offering*) Same as step (c) of Algorithm 1.

(d) (*Posterior update*) Same as step (d) of Algorithm 1.

end

4. Regret Analysis

We prove an upper bound on the regret of Algorithm 2 for the MNL-bandit problem, under the following assumption.

Assumption 1 For every item $i \in \{1, \dots, N\}$, the MNL parameter v_i satisfies $v_i \leq v_0 = 1$.

This assumption is equivalent to the outside option being more preferable to any other item. This assumption holds for many applications like display advertising, where users do not click on any of the displayed ads more often than not. Our main theoretical result is the following upper bound on the regret of Algorithm 2.

Theorem 1 For any instance $\mathbf{v} = (v_0, \dots, v_N)$ of the MNL-bandit problem with N products, $r_i \in [0, 1]$, and satisfying Assumption 1, the regret of Algorithm 2 in time T is bounded as,

$$\text{Reg}(T, \mathbf{v}) \leq C_1 \sqrt{NT} \log TK + C_2 N \log^2 TK,$$

where C_1 and C_2 are absolute constants (independent of problem parameters).

4.1. Proof Sketch

We break down the expression for total regret

$$\text{Reg}(T, \mathbf{v}) := \mathbb{E} \left[\sum_{t=1}^T R(S^*, \mathbf{v}) - R(S_t, \mathbf{v}) \right],$$

into regret per epoch, and rewrite it as follows:

$$\text{Reg}(T, \mathbf{v}) = \underbrace{\mathbb{E} \left[\sum_{\ell=1}^L |\mathcal{E}_\ell| (R(S^*, \mathbf{v}) - R(S_\ell, \mu(\ell))) \right]}_{\text{Reg}_1(T, \mathbf{v})} + \underbrace{\mathbb{E} \left[\sum_{\ell=1}^L |\mathcal{E}_\ell| (R(S_\ell, \mu(\ell)) - R(S_\ell, \mathbf{v})) \right]}_{\text{Reg}_2(T, \mathbf{v})},$$

where $|\mathcal{E}_\ell|$ is the number of time steps in epoch ℓ , and S_ℓ is the set repeatedly offered by our algorithm in epoch ℓ . Then, we bound the two terms: $\text{Reg}_1(T, \mathbf{v})$ and $\text{Reg}_2(T, \mathbf{v})$ separately.

The first term $\text{Reg}_1(T, \mathbf{v})$ is essentially the difference between the optimal revenue of the true instance and the optimal revenue of the sampled instance (since S_ℓ was chosen by the algorithm to be an optimal K -sized subset for the sampled instance). Therefore, this term would contribute no regret if the sampled instances were always optimistic. Unlike optimism under uncertainty approaches like UCB, this property is not ensured by our Thompson Sampling based algorithm. To bound this term, we utilize anti-concentration properties of the posterior, as well as the dependence between samples for different items, in order to prove that at least one of our K sampled instances is optimistic often enough.

The second term $\text{Reg}_2(T, \mathbf{v})$ captures the difference in the revenue of the offered set S_ℓ when evaluated on sampled parameters vs. true parameters. This is bounded by utilizing the concentration properties of our posterior distributions. It involves showing that for the sets that are played often, the posterior will converge quickly, so that revenue on the sampled parameters will be close to that on the true parameters.

Some further details are provided below. A complete proof is provided in Appendix C.

Bounding the first term $\text{Reg}_1(T, \mathbf{v})$. Firstly, by our assumption $v_0 \geq v_i, \forall i$, the outside option is picked at least as often as any particular item i , and therefore, it is not difficult to see that the expected value of epoch length $|\mathcal{E}_\ell|$ is bounded by $K + 1$, so that $\text{Reg}_1(T, \mathbf{v})$ is bounded as

$$(K + 1) \mathbb{E} \left(\sum_{\ell=1}^L R(S^*, \mathbf{v}) - R(S_\ell, \mu(\ell)) \right)$$

Let us call an epoch ℓ optimistic if the sampled parameters $\mu(\ell)$ are such that the optimal set S^* has at least as much revenue on the sampled parameters as on the true parameters, i.e., if $R(S^*, \mu(\ell)) \geq R(S^*, \mathbf{v})$. Then, clearly, such epochs don't contribute to this term:

$$R(S^*, \mathbf{v}) - R(S_\ell, \mu(\ell)) \leq R(S^*, \mu(\ell)) - R(S_\ell, \mu(\ell)) \leq 0. \text{ (by optimality of } S_\ell)$$

To bound the contribution of the remaining epochs, we first bound the number of consecutive epochs between two optimistic epochs, by analyzing the probability of being optimistic. Intuitively, using anti-concentration properties of Gaussian distribution, we have that with a constant probability a sampled parameter $\mu_i^j(\ell)$ for any item i will exceed the posterior mean by a few standard deviations. Now, since our Gaussian posterior's mean is equal to the unbiased estimate $\hat{v}_{i,\ell}$, and its standard deviation is close to the expected deviation of estimate $\hat{v}_{i,\ell}$ from the true parameter v_i , we can conclude that any sampled parameter $\mu_i^j(\ell)$ will be optimistic with a constant probability, i.e., $\mu_i^j(\ell) \geq v_i$. However, for an epoch to be optimistic, sampled parameters for *all* the items in S^* may need to be optimistic. This is where the correlated sampling feature of our algorithm is crucially utilized. Using the dependence structure between samples for different items in S^* , and variance boosting provided by the sampling of K independent sample sets, we prove an upper bound of roughly $1/K$ on the number of consecutive epochs between two optimistic epochs. The precise lemma is as follows, which forms one of the primary technical components of our proof:

Lemma 1 (Spacing of optimistic epochs) *Let $\mathcal{E}^{\text{An}}(\tau)$ be the group of consecutive epochs between an optimistic epoch τ and the next optimistic epoch τ' . Then, for any $p \in [1, 2]$, we have,*

$$\mathbb{E}^{1/p} [|\mathcal{E}^{\text{An}}(\tau)|^p] \leq \frac{e^{12}}{K} + 30^{1/p}.$$

A proof for above lemma is provided in Appendix C.1. Next, we bound the individual contribution of any ‘‘non-optimistic’’ epoch ℓ (i.e., any epoch in $\mathcal{E}^{\text{An}}(\tau)$) by relating it to the closest optimistic epoch τ before it. By definition of an optimistic epoch,

$$R(S^*, \mathbf{v}) - R(S_\ell, \mu(\ell)) \leq R(S_\tau, \mu(\tau)) - R(S_\ell, \mu(\ell))$$

and by the choice of S_ℓ as the revenue maximizing set for the sampled parameters $\mu(\ell)$:

$$R(S_\tau, \mu(\tau)) - R(S_\ell, \mu(\ell)) \leq R(S_\tau, \mu(\tau)) - R(S_\tau, \mu(\ell))$$

What remains to bound is the difference in the revenue of the set S_τ for two different sample parameters: $\mu(\tau)$ and $\mu(\ell)$. Over time, as the posterior distributions concentrate around their means, which in turn concentrate around the true parameters, this difference becomes smaller. In fact, using Lipschitz property of $R(S_\tau, \cdot)$, $|R(S_\tau, \mu(\tau)) - R(S_\tau, \mu(\ell))|$ can be bounded by $\tilde{O}(\hat{\sigma}_i(\tau)) + \tilde{O}(\hat{\sigma}_i(\ell)) \leq \tilde{O}(\hat{\sigma}_i(\tau))$ (refer to Lemma 7 in the appendix), where $\hat{\sigma}_i(\tau)$ was defined as the standard deviation of the posterior distribution in the beginning of epoch τ , which is larger than $\hat{\sigma}_i(\ell)$, and roughly equal to the deviation of posterior mean from the true parameter v_i .

To summarize, since between two optimistic epochs τ and τ' , there are $O(1/K)$ non-optimistic epochs, and each of their contribution to $\text{Reg}_1(T, \mathbf{v})$ is bounded by some multiples of $\hat{\sigma}_i(\tau)$, this term can be bounded roughly as:

$$\text{Reg}_1(T, \mathbf{v}) \lesssim (K+1)\mathbb{E} \left[\sum_{\tau \text{ optimistic}} \frac{1}{K} \sum_{i \in S_\tau} \hat{\sigma}_i(\tau) \right] \lesssim \mathbb{E} \left[\sum_{\ell=1}^L \sum_{i \in S_\ell} \hat{\sigma}_i(\ell) \right].$$

A bound of $\tilde{O}(\sqrt{NT})$ on the sum of these deviations can be derived, which will also be useful for bounding the second term, as discussed next.

Bounding the second term $\text{Reg}_2(T, \mathbf{v})$.

Noting that the expected epoch length when set S_ℓ is offered is $1 + V(S_\ell)$, where $V(S_\ell) = 1 + \sum_{j \in S_\ell} v_j$, $\text{Reg}_2(T, \mathbf{v})$ can be reformulated as

$$\text{Reg}_2(T, \mathbf{v}) = \mathbb{E} \left[\sum_{\ell=1}^L (1 + V(S_\ell)) (R(S_\ell, \mu(\ell)) - R(S_\ell, \mathbf{v})) \right],$$

Again, as discussed above, using Lipschitz property of revenue function, this can be bounded in terms of posterior standard deviation (refer to Lemma 7)

$$\text{Reg}_2(T, \mathbf{v}) \lesssim O\left(\mathbb{E} \left[\sum_{\ell=1}^L \sum_{i \in S_\ell} \hat{\sigma}_i(\ell) \right]\right)$$

Overall, the above analysis on Reg_1 and Reg_2 implies roughly the following bound on regret

$$O\left(\sum_{\ell=1}^L \sum_{i \in S_\ell} \hat{\sigma}_i(\ell)\right) = O\left(\sum_{\ell=1}^L \sum_{i \in S_\ell} \sqrt{\frac{v_i}{n_i(\ell)}} + \frac{1}{n_i(\ell)}\right) \log TK \leq O\left(\sum_{i=1}^N \log TK \sqrt{v_i n_i}\right)$$

where n_i is total number of times i was offered in time T . Then, utilizing the bound of T on the expected number of total picks, i.e., $\sum_{i=1}^N v_i n_i \leq T$, and doing a worst scenario analysis, we obtain a bound of $\tilde{O}(\sqrt{NT})$ on $\text{Reg}(T, \mathbf{v})$.

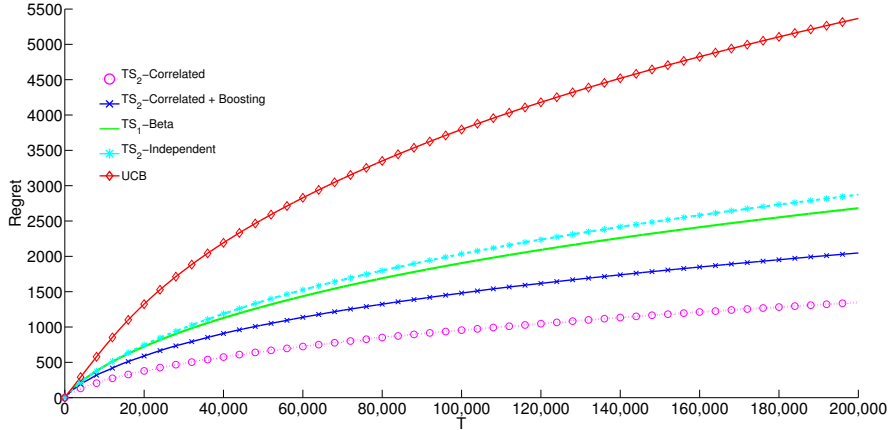


Figure 1: Regret growth with T for various heuristics on a randomly generated MNL-bandit instance with $N = 1000$, $K = 10$.

5. Empirical study

In this section, we analyze the various design components of our Thompson Sampling approach through numerical simulations. The aim is to isolate and understand the effect of individual features of our algorithm design like Beta posteriors vs. Gaussian approximation, independent sampling vs. correlated sampling, and single sample vs. multiple samples, on the practical performance.

We simulate an instance of MNL-bandit problem with $N = 1000$, $K = 10$ and $T = 2 \times 10^5$, and the MNL parameters $\{v_i\}_{i=1,\dots,N}$ generated randomly from $\text{Unif}[0, 1]$. And, we compute the average regret based on 50 independent simulations over the randomly generated instance. In Figure 1, we report performance of successive variants of TS: *i*) the basic version of TS with independent Beta priors, as described in Algorithm 1, referred to as TS₁-Beta, *ii*) Gaussian posterior approximation with independent sampling, referred to as TS₂-Independent, *iii*) Gaussian posterior approximation with correlated sampling, referred to as TS₂-Correlated, and finally, *iv*) Gaussian posterior approximation with correlated sampling and boosting by using multiple (K) samples, referred to as TS₂-Correlated + Boosting, which is essentially the version with all the features of Algorithm 2. For comparison, we also present the performance of UCB approach in Agrawal et al. (2016). We repeated this experiment on several randomly generated instances and a similar performance was observed.

The performance of all the variants of TS is observed to be better than the UCB approach in our experiments, which is consistent with the other empirical evidence in the literature.

Among the TS variants, the performance of TS₁-Beta, i.e., the basic version with independent beta priors (essentially Algorithm 1) is quite similar to TS₂-Independent, the version with independent Gaussian (approximate) posteriors; indicating that the effect of posterior approximation is minor. The performance of TS₂-Correlated, where we generated correlated samples from the Gaussian distributions, is significantly better than all the other variants of the algorithm. This is consistent with our remark earlier that to adapt the Thompson sampling approach of the classical MAB problem to our setting, ideally we would like to maintain a joint prior over the parameters $\{v_i\}_{i=1,\dots,N}$ and update it to a joint posterior on observing the bandit feedback. However, since this can be quite challenging and intractable, we used independent priors over the parameters. The

superior performance of TS_2 -Correlated demonstrates the potential benefits of considering a joint (correlated) prior/posterior in such settings with combinatorial arms. Finally, we observe that the performance of TS_2 -Correlated + Boosting, where an additional “variance boosting” is provided through K independent samples, is worse than TS_2 -Correlated as expected, but still significantly better than the independent Beta posterior version TS_1 -Beta. Therefore, significant improvements in performance due to correlated sampling feature of Algorithm 2 compensate for the slight deterioration caused by boosting.

References

- S. Agrawal, V. Avadhanula, V. Goyal, and A. Zeevi. 2016. A Near-Optimal Exploration-Exploitation Approach for Assortment Selection. *ACM conference on Economics and Computation (EC)* (2016).
- S. Agrawal and N. Goyal. 2013a. Further Optimal Regret Bounds for Thompson Sampling.. In *Aistats*. 99–107.
- S. Agrawal and N. Goyal. 2013b. Thompson Sampling for Contextual Bandits with Linear Payoffs.. In *ICML*. 127–135.
- P. Auer. 2003. Using Confidence Bounds for Exploitation-exploration Trade-offs. *J. Mach. Learn. Res.* (2003).
- P. Auer, N. Cesa-Bianchi, and P. Fischer. 2002. Finite-time Analysis of the Multiarmed Bandit Problem. *Machine Learning* (2002).
- V. Avadhanula, J. Bhandari, V. Goyal, and A. Zeevi. 2016. On the tightness of an LP relaxation for rational optimization and its applications. *Operations Research Letters* 44, 5 (2016), 612–617.
- M. Ben-Akiva and S. Lerman. 1985. *Discrete choice analysis: theory and application to travel demand*. Vol. 9. MIT press.
- J. Davis, G. Gallego, and H. Topaloglu. 2013. Assortment planning under the multinomial logit model with totally unimodular constraint structures. *Technical Report* (2013).
- A. Gopalan, S. Mannor, and Y. Mansour. 2014. Thompson Sampling for Complex Online Problems.. In *ICML*, Vol. 14. 100–108.
- T. Graepel, J. Q. Candela, T. Borchert, and R. Herbrich. 2010. Web-scale bayesian click-through rate prediction for sponsored search advertising in microsoft’s bing search engine. In *Proceedings of the 27th international conference on machine learning (ICML)*. 13–20.
- R.D. Luce. 1959. *Individual choice behavior: A theoretical analysis*. Wiley.
- B. C. May, N. Korda, A. Lee, and D. S. Leslie. 2012. Optimistic Bayesian sampling in contextual-bandit problems. *Journal of Machine Learning Research* 13, Jun (2012), 2069–2106.
- D. McFadden. 1978. *Modelling the choice of residential location*. Institute of Transportation Studies, University of California.
- C. Olivier and L. Li. 2011. An Empirical Evaluation of Thompson Sampling. In *Advances in Neural Information Processing Systems* 24. 2249–2257.
- R. L. Plackett. 1975. The Analysis of Permutations. *Journal of the Royal Statistical Society. Series C (Applied Statistics)* (1975).

- P. Rusmevichientong, Z. M. Shen, and D.B. Shmoys. 2010. Dynamic assortment optimization with a multinomial logit choice model and capacity constraint. *Operations research* (2010).
- P. Rusmevichientong and J. N. Tsitsiklis. 2010. Linearly Parameterized Bandits. *Math. Oper. Res.* 35, 2 (2010), 395–411.
- D. Sauré and A. Zeevi. 2013. Optimal Dynamic Assortment Planning with Demand Learning. *Manufacturing & Service Operations Management* (2013).
- K. Train. 2003. *Discrete Choice Methods with Simulation*. Cambridge University Press.

Appendix A. Unbiased Estimate $\tilde{v}_{i,\ell}$ and Conjugate priors

Some of the results in this section adapted from Agrawal et al. (2016), but we provide the proofs again for the sake of completeness.

We first prove that the estimate obtained from epoch based offerings, $\tilde{v}_{i,\ell}$ in Algorithm 1 is unbiased estimate and is distributed geometrically with probability of success $\frac{1}{1+v_i}$. Specifically, we have the following result.

Lemma 1 (Agrawal et al. (2016)) *Let $\tilde{v}_{i,\ell}$ be the number of times an item $i \in S_\ell$ is picked when the set S_ℓ is offered repeatedly until no-click (outside option is picked). Then, $\tilde{v}_{i,\ell}, \forall \ell, i$ are i.i.d geometrical random variables with success probability $\frac{1}{1+v_i}$, and expected value v_i .*

Proof We prove the result by computing the moment generating function, from which we can establish that $\tilde{v}_{i,\ell}$ is a geometric random variable with parameter $\frac{1}{1+v_i}$. Thereby also establishing that $\tilde{v}_{i,\ell}$ are unbiased estimators of v_i . Specifically, we show the following result.

The moment generating function of estimate conditioned on S_ℓ, \hat{v}_i , is given by,

$$\mathbb{E} \left(e^{\theta \tilde{v}_{i,\ell}} \middle| S_\ell \right) = \frac{1}{1 - v_i(e^\theta - 1)}, \text{ for all } \theta \leq \log \frac{1 + v_i}{v_i}, \text{ for all } i = 1, \dots, N.$$

We focus on proving the above result. From (1), we have that probability of no purchase event when assortment S_ℓ is offered is given by

$$p_0(S_\ell) = \frac{1}{1 + \sum_{j \in S_\ell} v_j}.$$

Let n_ℓ be the total number of offerings in epoch ℓ before a no purchased occurred, i.e., $n_\ell = |\mathcal{E}_\ell| - 1$. Therefore, n_ℓ is a geometric random variable with probability of success $p_0(S_\ell)$. And, given any fixed value of n_ℓ , $\tilde{v}_{i,\ell}$ is a binomial random variable with n_ℓ trials and probability of success given by

$$q_i(S_\ell) = \frac{v_i}{\sum_{j \in S_\ell} v_j}.$$

In the calculations below, for brevity we use p_0 and q_i respectively to denote $p_0(S_\ell)$ and $q_i(S_\ell)$. Hence, we have

$$\mathbb{E} \left(e^{\theta \tilde{v}_{i,\ell}} \right) = E_{n_\ell} \left\{ \mathbb{E} \left(e^{\theta \tilde{v}_{i,\ell}} \middle| n_\ell \right) \right\}.$$

Since the moment generating function for a binomial random variable with parameters n, p is $(pe^\theta + 1 - p)^n$, we have

$$\mathbb{E} \left(e^{\theta \tilde{v}_{i,\ell}} \middle| n_\ell \right) = E_{n_\ell} \left\{ \left(q_i e^\theta + 1 - q_i \right)^{n_\ell} \right\}.$$

For any α , such that, $\alpha(1-p) < 1$ n is a geometric random variable with parameter p , we have

$$\mathbb{E}(\alpha^n) = \frac{p}{1 - \alpha(1-p)}.$$

Note that for all $\theta < \log \frac{1+v_i}{v_i}$, we have $(q_i e^\theta + (1-q_i))(1-p_0) = (1-p_0) + p_0 v_i (e^\theta - 1) < 1$. Therefore, we have $\mathbb{E}(e^{\theta \tilde{v}_{i,\ell}}) = \frac{1}{1 - v_i(e^\theta - 1)}$ for all $\theta < \log \frac{1+v_i}{v_i}$. \square

Building on this result. We will prove Lemma 2 that helped construct Algorithm 1. Recall Lemma 2

Lemma 2 (Conjugate Priors) For any $\alpha > 3, \beta > 0$, let $X_{\alpha,\beta} = \frac{1}{\text{Beta}(\alpha,\beta)} - 1$ and $f_{\alpha,\beta}$ be a probability distribution of the random variable $X_{\alpha,\beta}$. If v_i is distributed as $f_{\alpha,\beta}$ and $\tilde{v}_{i,\ell}$ is a geometric random variable with success probability $\frac{1}{v_i+1}$, then we have,

$$\mathbb{P}(v_i | \tilde{v}_{i,\ell} = m) = f_{\alpha+1, \beta+m}(v_i).$$

Proof . The proof of the lemma follows from the following result on the probability density function of the random variable $X_{\alpha,\beta}$. Specifically, we have for any $x > 0$

$$f_{\alpha,\beta}(x) = \frac{1}{B(\alpha-3, \beta)} \left(\frac{1}{1+x} \right)^{\alpha-1} \left(\frac{x}{x+1} \right)^{\beta-1}, \quad (6)$$

where $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ and $\Gamma(a)$ is the gamma function. Since we assume that the parameter v_i 's prior distribution is same as that of $X_{\alpha,\beta}$, we have from (6) and Lemma 1,

$$\mathbb{P}(v_i | \tilde{v}_{i,\ell} = m) \propto \left(\frac{1}{1+v_i} \right)^{\alpha+1} \left(\frac{x}{x+1} \right)^{\beta+m}. \square$$

Given the pdf of the posterior in (6), it is possible to compute the mean and variance of the posterior distribution. We show that they have simple closed form expressions. Recall Lemma 3.

Lemma 3 (Moments of the Posterior Distribution) If X is a random variable distributed as $\text{Beta}(\alpha, \beta)$, then

$$\mathbb{E}\left(\frac{1}{X} - 1\right) = \frac{\beta}{\alpha-3}, \quad \text{and} \quad \text{Var}\left(\frac{1}{X} - 1\right) = \frac{\frac{\beta}{\alpha-3}(\frac{\beta}{\alpha-3}+1)}{\alpha-4}.$$

Proof . We prove the result by relating the mean of the posterior to the mean of the Beta distribution. Let $\hat{X} = \frac{1}{X} - 1$ From (6), we have

$$\mathbb{E}(\hat{X}) = \frac{1}{B(\alpha, \beta)} \int_0^\infty x \left(\frac{1}{1+x} \right)^{\alpha-1} \left(\frac{x}{x+1} \right)^{\beta-1} dx,$$

Substituting $y = \frac{1}{1+x}$, we have

$$\mathbb{E}(\hat{X}) = \frac{1}{B(\alpha-2, \beta)} \int_0^1 y^{\alpha-4} y^\beta dx = \frac{B(\alpha-3, \beta+1)}{B(\alpha-2, \beta)} = \frac{\beta}{\alpha-3}.$$

Similarly, we can derive the expression for the $\text{Var}(\hat{X})$.

Appendix B. Bounds on the deviation of MNL Expected Revenue

Here, we prove a Lipschitz kind of bound on the deviation of function $R(S, \mathbf{v})$ with change in the parameter \mathbf{v} .

Lemma 4 For any $\mathbf{v}, \mathbf{w} \in \mathcal{R}^n$ and S^* such that $S^* = \operatorname{argmax}_{S:|S|\leq K} R(S, \mathbf{v})$, we have,

$$R(S^*, \mathbf{v}) - R(S^*, \mathbf{w}) \leq \min \left\{ \frac{\sum_{i \in S^*} |v_i - w_i|}{1 + \sum_{j \in S^*} w_j}, \frac{\sum_{i \in S^*} |v_i - w_i|}{1 + \sum_{j \in S^*} v_j} \right\}$$

Proof Define sets $\mathcal{I}(S^*)$ and $\mathcal{D}(S^*)$ as

$$\begin{aligned} \mathcal{I}(S^*) &= \{i | i \in S^* \text{ and } v_i \geq w_i\} \\ \mathcal{D}(S^*) &= \{i | i \in S^* \text{ and } v_i < w_i\}, \end{aligned}$$

and vector \mathbf{u} as,

$$u_i = \begin{cases} w_i & \text{if } i \in \mathcal{D}(S^*), \\ v_i & \text{otherwise.} \end{cases}$$

By construction of \mathbf{u} , we have $u_i \geq v_i$ and $u_i \geq w_i$ for all i . Therefore from lemma 5, we have

$$\begin{aligned} R(S^*, \mathbf{v}) - R(S^*, \mathbf{w}) &\leq R(S^*, \mathbf{u}) - R(S^*, \mathbf{w}) \\ &\leq \frac{\sum_{i \in S^*} r_i u_i}{1 + \sum_{j \in S^*} u_j} - \frac{\sum_{i \in S^*} r_i w_i}{1 + \sum_{j \in S^*} u_j} \\ &\leq \frac{\sum_{i \in S^*} (u_i - w_i)}{1 + \sum_{j \in S^*} u_j} \end{aligned}$$

The result follows from the fact that $u_i \geq v_i$ and $u_i \geq w_i$ for all $i \in S^*$. \square

Following a similar proof, we can also establish the following result.

Lemma 5 (Agrawal et al. (2016)) Assume $0 \leq v_i \leq w_i$ for all $i = 1, \dots, n$. Suppose S^* is an optimal assortment when the MNL are parameters are given by \mathbf{v} , i.e.

$$S^* = \operatorname{argmax}_{S:|S|\leq K} R(S, \mathbf{v}).$$

Then, $R(S^*, \mathbf{w}) \geq R(S^*, \mathbf{v})$.

Lemma 6 For any $\ell \leq T$ and $i \in \{1, \dots, N\}$, we have for any $r > 0$,

$$\mathbb{P} \left(|\theta_i(\ell) - \hat{v}_i(\ell)| > 4\hat{\sigma}_i(\ell) \sqrt{\log r K} \right) \leq \frac{1}{r^4 K^3},$$

where $\hat{\sigma}_i(\ell) = \sqrt{\frac{50\hat{v}_i(\hat{v}_i+1)}{n_i}} + \frac{75\sqrt{\log TK}}{n_i}$.

Proof Hence, we have

$$\begin{aligned}
 \mathbb{P} \left\{ |\theta_i(\ell) - \hat{v}_i(\ell)| \mid \hat{v}_i(\ell) \right\} &= \mathbb{P} \left(\bigcup_{j=1}^K \left\{ \theta^j(\ell) > 2\sqrt{\log mk} \right\} \right) \\
 &\leq \sum_{j=1}^K \mathbb{P} \left(\theta^j(\ell) > 4\sqrt{\log rK} \right) \\
 &\leq \frac{1}{r^4 K^3}.
 \end{aligned} \tag{7}$$

where the first inequality follows from the union bound and the second inequality follows from the following bound derived in For a Gaussian distributed random variable Z with mean m and variance σ^2 , for any z ,

$$\frac{1}{4\sqrt{\pi}} \cdot e^{-7z^2/2} < \Pr(|Z - m| > z\sigma) \leq \frac{1}{2} e^{-z^2/2}. \square$$

From Lemma 4, Lemma 6 and Lemma 10, we have the following result.

Lemma 7 For any epoch ℓ , if $S_\ell = \operatorname{argmax}_{S:|S|\leq K} R(S, \boldsymbol{\mu}(\ell))$

$$\mathbb{E} \left\{ \left(1 + \sum_{j \in S_\ell} v_j \right) [R(S_\ell, \boldsymbol{\mu}(\ell)) - R(S_\ell, \mathbf{v})] \right\} \leq \mathbb{E} \left[C_1 \sum_{i \in S_\ell} \sqrt{\frac{v_i \log TK}{n_i(\ell)}} + C_2 \frac{\log TK}{n_i(\ell)}, \right]$$

We defer the proof of the above result to next section, where we also specify the exact constants C_1 and C_2 .

Appendix C. Proof of Theorem 1

$$\begin{aligned}
 \operatorname{Reg}(T, \mathbf{v}) &:= \mathbb{E} \left[\sum_{\ell=1}^L |\mathcal{E}_\ell^{\text{Al}}| (R(S^*, \mathbf{v}) - R(S_\ell, \mathbf{v})) \right] \\
 &= \mathbb{E} \left[\sum_{\ell=1}^L |\mathcal{E}_\ell^{\text{Al}}| (R(S^*, \mathbf{v}) - R(S_\ell, \boldsymbol{\theta}(\ell))) \right] + \mathbb{E} \left[\sum_{\ell=1}^L |\mathcal{E}_\ell^{\text{Al}}| (R(S_\ell, \boldsymbol{\theta}(\ell)) - R(S_\ell, \mathbf{v})) \right]
 \end{aligned} \tag{8}$$

We will complete the proof by bounding the two terms in (8).

Notations. For the sake of brevity, we introduce some notations.

- For any assortment S , $V(S) \triangleq \sum_{i \in S} v_i$
- For any $\ell, \tau \leq L$, define ΔR_ℓ and $\Delta R_{\ell, \tau}$ in the following manner

$$\begin{aligned}
 \Delta R_\ell &\triangleq (1 + V(S_\ell)) [R(S_\ell, \boldsymbol{\theta}(\ell)) - R(S_\ell, \mathbf{v})] \\
 \Delta R_{\ell, \tau} &\triangleq (1 + V(S_\tau)) [R(S_\ell, \boldsymbol{\theta}(\ell)) - R(S_\ell, \boldsymbol{\theta}(\tau))]
 \end{aligned}$$

- Let $Reg_1(T, \mathbf{v})$ and $Reg_2(T, \mathbf{v})$ denote the first and second terms in (8) respectively, i.e.

$$Reg_1(T, \mathbf{v}) = \mathbb{E} \left[\sum_{\ell=1}^L |\mathcal{E}_\ell^{\text{Al}}| (R(S^*, \mathbf{v}) - R(S_\ell, \boldsymbol{\theta}(\ell))) \right]$$

$$Reg_2(T, \mathbf{v}) = \mathbb{E} \left[\sum_{\ell=1}^L |\mathcal{E}_\ell^{\text{Al}}| (R(S_\ell, \boldsymbol{\theta}(\ell)) - R(S_\ell, \mathbf{v})) \right]$$

- Let \mathcal{A}_0 denote the complete set Ω and for all $\ell = 1, \dots, L$, define events \mathcal{A}_ℓ as

$$\mathcal{A}_\ell = \left\{ |\hat{v}_i(\ell) - v_i| \geq \sqrt{\frac{24v_i \log(\ell+1)}{n_i(\ell)}} + \frac{48 \log(\ell+1)}{n_i(\ell)} \text{ for some } i = 1, \dots, N \right\}$$

where $\hat{\sigma}_i(\ell) = \sqrt{\frac{50\hat{v}_i(\hat{v}_i+1)}{n_i}} + \frac{75\sqrt{\log TK}}{n_i}$.

-

$$\begin{aligned} \mathcal{T} &= \{\ell : \mu_i(\ell) \geq v_i \text{ for all } i \in S^*\}, \\ \text{succ}(\ell) &= \min\{\bar{\ell} \in \mathcal{T} : \bar{\ell} > \ell\} \\ \mathcal{E}^{\text{An}}(\ell) &= \{\tau : \tau \in (\ell, \text{succ}(\ell))\} \text{ for all } \ell \in \mathcal{T}. \end{aligned} \tag{9}$$

Here \mathcal{T} is the set of “optimistic” epoch indices, i.e. when value of $\mu_i(\ell)$ is higher than the value of v_i for all products i in the optimal offer set S^* and $\text{succ}(\ell)$ denote the successive epoch index after ℓ that is optimistic. $\mathcal{E}^{\text{An}}(\ell)$ be the set of epoch indices’s between an optimistic epoch, $\ell \in \mathcal{T}$ and the successive epoch. We will refer to $\mathcal{E}^{\text{An}}(\ell)$ as the “analysis epoch” starting at ℓ . To avoid confusion, we will refer to the epoch in which a selected offer set is offered until an outside option is preferred as “algorithmic epoch”. Note that the analysis epoch can contain one or more algorithmic epochs.

We first focus on bounding $Reg_2(T, \mathbf{v})$.

Bounding $Reg_2(T, \mathbf{v})$: We have,

$$\mathbb{E} \left[|\mathcal{E}_\ell^{\text{Al}}| (R(S_\ell, \boldsymbol{\theta}(\ell)) - R(S_\ell, \mathbf{v})) \right] = \mathbb{E} \left[\mathbb{E} \left(|\mathcal{E}_\ell^{\text{Al}}| \middle| S_\ell \right) (R(S_\ell, \boldsymbol{\theta}(\ell)) - R(S_\ell, \mathbf{v})) \right],$$

and conditioned on the event $S_\ell = S$, the length of the ℓ^{th} epoch, $|\mathcal{E}^{\text{Al}}|$ is a geometric random variable with probability of success $p_0(S_\ell)$, where

$$p_0(S_\ell) = \frac{1}{1 + \sum_{j \in S_\ell} v_j}.$$

Therefore, it follows that

$$\mathbb{E} \left(|\mathcal{E}^{\text{Al}}| \middle| S_\ell = S \right) = 1 + V(S). \tag{10}$$

Hence the second term in (8) can be reformulated as

$$Reg_2(T, \mathbf{v}) = \mathbb{E} \left\{ \sum_{\ell=1}^L \Delta R_\ell \right\} \tag{11}$$

Noting that \mathcal{A}_ℓ is a “low probability” event, we analyze the regret in two scenarios, one when \mathcal{A}_ℓ is true and another when \mathcal{A}_ℓ^c is true. More specifically,

$$\mathbb{E}(\Delta R_\ell) = \mathbb{E}[\Delta R_\ell \cdot \mathbb{1}(\mathcal{A}_{\ell-1}) + \Delta R_\ell \cdot \mathbb{1}(\mathcal{A}_{\ell-1}^c)]$$

Using the fact that $R(S_\ell, \boldsymbol{\theta}(\ell))$ and $R(S_\ell, \mathbf{v})$ are both bounded by one and $V(S_\ell) \leq K$, we have

$$\mathbb{E}(\Delta R_\ell) \leq (K + 1)\mathbb{P}(\mathcal{A}_{\ell-1}) + \mathbb{E}[\Delta R_\ell \cdot \mathbb{1}(\mathcal{A}_{\ell-1}^c)].$$

Substituting $m = 2$ in lemma 10, we obtain that $\mathbb{P}(\mathcal{A}_{\ell-1}) \leq \frac{1}{\ell^2}$. Therefore, it follows that,

$$\mathbb{E}\{\Delta R_\ell\} \leq \frac{K + 1}{\ell^2} + \mathbb{E}[\Delta R_\ell \cdot \mathbb{1}(\mathcal{A}_{\ell-1}^c)]. \quad (12)$$

In lemma 4, we show that

$$R(S_\ell, \boldsymbol{\theta}(\ell)) - R(S_\ell, \mathbf{v}) \leq \frac{\sum_{i \in S_\ell} |\theta_i(\ell) - v_i|}{1 + \sum_{j \in S_\ell} v_j}.$$

Therefore, from (10) it follows that,

$$\mathbb{E}[\Delta R_\ell \cdot \mathbb{1}(\mathcal{A}_{\ell-1}^c)] \leq \mathbb{E}\left[\sum_{i \in S_\ell} |\theta_i(\ell) - v_i| \cdot \mathbb{1}(\mathcal{A}_{\ell-1}^c)\right].$$

From triangle inequality, we have

$$\mathbb{E}[\Delta R_\ell \cdot \mathbb{1}(\mathcal{A}_{\ell-1}^c)] \leq \mathbb{E}\left[\sum_{i \in S_\ell} |\theta_i(\ell) - \hat{v}_i(\ell)| \cdot \mathbb{1}(\mathcal{A}_{\ell-1}^c)\right] + \mathbb{E}\left[\sum_{i \in S_\ell} |\hat{v}_i(\ell) - v_i| \cdot \mathbb{1}(\mathcal{A}_{\ell-1}^c)\right],$$

and from the definition of the event $\mathcal{A}_{\ell-1}^c$, we have,

$$\mathbb{E}[\Delta R_\ell \cdot \mathbb{1}(\mathcal{A}_{\ell-1}^c)] \leq \mathbb{E}\left[\sum_{i \in S_\ell} |\theta_i(\ell) - \hat{v}_i(\ell)|\right] + \mathbb{E}\left[\sqrt{\frac{24v_i \log(\ell + 1)}{n_i(\ell)}} + \frac{48 \log(\ell + 1)}{n_i(\ell)}\right]. \quad (13)$$

We will now focus on bounding the first term in (13). In lemma 6, we show that for any $r > 0$ and $i = 1, \dots, N$, we have,

$$\mathbb{P}\left(|\theta_i(\ell) - \hat{v}_i(\ell)| > 4\hat{\sigma}_i(\ell)\sqrt{\log rK}\right) \leq \frac{1}{r^4 K^3},$$

where $\hat{\sigma}_i(\ell) = \sqrt{\frac{50\hat{v}_i(\hat{v}_i+1)}{n_i}} + \frac{75\sqrt{\log TK}}{n_i}$. Since $S_\ell \subset \{1, \dots, N\}$, we have for any $i \in S_\ell$ and $r > 0$, we have

$$\mathbb{P}\left(|\theta_i(\ell) - \hat{v}_i(\ell)| > 4\hat{\sigma}_i(\ell)\sqrt{\log rK}\right) \leq \frac{n}{r^4 K^3}. \quad (14)$$

Since $|\theta_i(\ell) - \hat{v}_i(\ell)|$ is a non-negative random variable, we have

$$\begin{aligned}
\mathbb{E}(|\theta_i(\ell) - \hat{v}_i(\ell)|) &= \int_0^\infty \mathbb{P}\{|\theta_i(\ell) - \hat{v}_i(\ell)| \geq x\} dx \\
&= \int_0^{4\hat{\sigma}_i(\ell)\sqrt{\log TK}} \mathbb{P}\{|\theta_i(\ell) - \hat{v}_i(\ell)| \geq x\} dx + \int_{4\hat{\sigma}_i(\ell)\sqrt{\log TK}}^\infty \mathbb{P}\{|\theta_i(\ell) - \hat{v}_i(\ell)| \geq x\} dx \\
&\leq 4\hat{\sigma}_i(\ell)\sqrt{\log TK} + \sum_{r=T}^\infty \int_{4\hat{\sigma}_i(\ell)\sqrt{\log rK}}^{4\hat{\sigma}_i(\ell)\sqrt{\log(r+1)K}} \mathbb{P}\{Y \geq x\} dx \\
&\stackrel{a}{\leq} 4\hat{\sigma}_i(\ell)\sqrt{\log TK} + \sum_{r=T}^\infty \frac{n\sqrt{\log(rK+1)} - n\sqrt{\log rK}}{r^4 K^3} \\
&\leq 4\hat{\sigma}_i(\ell)\sqrt{\log TK} \text{ any } T \geq n,
\end{aligned} \tag{15}$$

where the inequality (a) follows from (14). From (11), (12), (13) and lemma 10, we have,

$$\text{Reg}_2(T, \mathbf{v}) \leq C_1 \mathbb{E} \left(\sum_{\ell=1}^L \sum_{i \in S_\ell} \sqrt{\frac{v_i \log TK}{n_i(\ell)}} \right) + C_2 \mathbb{E} \left(\sum_{\ell=1}^L \sum_{i \in S_\ell} \frac{\log TK}{n_i(\ell)} \right),$$

where C_1 and C_2 are absolute constants. If T_i denote the total number of epochs product i is offered, then we have,

$$\begin{aligned}
\text{Reg}_2(T, \mathbf{v}) &\stackrel{(a)}{\leq} C_2 N \log^2 TK + C_1 \mathbb{E} \left(\sum_{i=1}^n \sqrt{v_i T_i \log TK} \right) \\
&\stackrel{(b)}{\leq} C_2 N \log^2 TK + C_1 \sum_{i=1}^N \sqrt{v_i \log(TK) \mathbb{E}(T_i)}
\end{aligned} \tag{16}$$

Inequality (a) follows from the observation that $L \leq T$, $T_i \leq T$, $\sum_{n_i(\ell)=1}^{T_i} \frac{1}{\sqrt{n_i(\ell)}} \leq \sqrt{T_i}$ and

$\sum_{n_i(\ell)=1}^{T_i} \frac{1}{n_i(\ell)} \leq \log T_i$, while Inequality (b) follows from Jensen's inequality.

For any realization of L , $\mathcal{E}_\ell^{\text{Al}}$, T_i , and S_ℓ in Algorithm 1, we have the following relation $\sum_{\ell=1}^L |\mathcal{E}_\ell^{\text{Al}}| \leq T$. Hence, we have $\mathbb{E} \left(\sum_{\ell=1}^L |\mathcal{E}_\ell^{\text{Al}}| \right) \leq T$. Let \mathcal{S} denote the filtration corresponding to the offered assortments S_1, \dots, S_L , then by law of total expectation, we have,

$$\begin{aligned}
\mathbb{E} \left(\sum_{\ell=1}^L |\mathcal{E}_\ell^{\text{Al}}| \right) &= \mathbb{E} \left\{ \sum_{\ell=1}^L E_{\mathcal{S}} \left(|\mathcal{E}_\ell^{\text{Al}}| \right) \right\} = \mathbb{E} \left\{ \sum_{\ell=1}^L 1 + \sum_{i \in S_\ell} v_i \right\} \\
&= \mathbb{E} \left\{ L + \sum_{i=1}^n v_i T_i \right\} = \mathbb{E}\{L\} + \sum_{i=1}^n v_i \mathbb{E}(T_i).
\end{aligned}$$

Therefore, it follows that

$$\sum v_i \mathbb{E}(T_i) \leq T.$$

To obtain the worst case upper bound, we maximize the bound in equation (16) subject to the above condition and hence, we have

$$Reg_2(T, \mathbf{v}) \leq C_1 \sqrt{NT \log TK} + C_2 N \log^2 TK. \quad (17)$$

We will now focus on the first term in (8).

Bounding $Reg_1(T, \mathbf{v})$: Note that for any ℓ , by design we have $R(S_\ell, \boldsymbol{\theta}(\ell)) \geq R(S^*, \boldsymbol{\theta}(\ell))$ and in lemma 5, we show that for any $\ell \in \mathcal{T}$, we have $R(S^*, \boldsymbol{\theta}(\ell)) \geq R(S^*, \mathbf{v})$. Therefore, we have

$$\mathbb{E} \left[\sum_{\ell=1}^L |\mathcal{E}_\ell^{\text{AI}}| (R(S^*, \mathbf{v}) - R(S_\ell, \boldsymbol{\theta}(\ell))) \right] \leq \mathbb{E} \left[\sum_{\ell=1}^L \mathbb{1}(\ell \in \mathcal{T}) \cdot \sum_{\tau \in \mathcal{E}^{\text{An}}(\ell)} |\mathcal{E}_\tau^{\text{AI}}| (R(S_\ell, \boldsymbol{\theta}(\ell)) - R(S_\tau, \boldsymbol{\theta}(\tau))) \right]$$

Observe that by design for any t , $R(S_\tau, \boldsymbol{\theta}(\tau)) \geq R(S, \boldsymbol{\theta}(\tau))$ for any assortment S . Therefore, we have for any τ , we have $R(S_\tau, \boldsymbol{\theta}(\tau)) \geq R(S_\ell, \boldsymbol{\theta}(\tau))$. From (10) we have,

$$Reg_1(T, \mathbf{v}) \leq \mathbb{E} \left[\sum_{\ell=1}^L \mathbb{1}(\ell \in \mathcal{T}) \cdot \sum_{\tau \in \mathcal{E}^{\text{An}}(\ell)} \Delta R_{\ell, \tau} \right] \quad (18)$$

Following the approach of bounding $Reg_2(T, \mathbf{v})$, we analyze the first term, $Reg_1(T, \mathbf{v})$ in two scenarios, one when \mathcal{A}_ℓ is true and another when \mathcal{A}_ℓ^c is true. More specifically,

$$\mathbb{E} \left(\sum_{\tau \in \mathcal{E}^{\text{An}}(\ell)} \Delta R_{\ell, \tau} \right) = \mathbb{E} \left[\sum_{\tau \in \mathcal{E}^{\text{An}}(\ell)} \Delta R_{\ell, \tau} \cdot \mathbb{1}(\mathcal{A}_{\ell-1}) + \Delta R_{\ell, \tau} \cdot \mathbb{1}(\mathcal{A}_{\ell-1}^c) \right]$$

Adding and subtracting $R(S_\ell, \mathbf{v})$, from triangle inequality and lemma 4, we obtain

$$R(S_\ell, \boldsymbol{\theta}(\ell)) - R(S_\ell, \boldsymbol{\theta}(\tau)) \leq \frac{\sum_{i \in S_\ell} |\theta_i(\ell) - v_i| + |\theta_i(\tau) - v_i|}{1 + V(S_\ell)}.$$

Using the fact that $R(S_\ell, \boldsymbol{\theta}(\ell))$ and $R(S_\ell, \boldsymbol{\theta}(\tau))$ are both bounded by one and $V(S_\tau) \leq K$, we have

$$\mathbb{E} \left(\sum_{\tau \in \mathcal{E}^{\text{An}}(\ell)} \Delta R_{\ell, \tau} \right) \leq (K+1) \mathbb{E} \left[|\mathcal{E}^{\text{An}}(\ell)| \cdot \mathbb{1}(\mathcal{A}_{\ell-1}) + \sum_{\tau \in \mathcal{E}^{\text{An}}(\ell)} \frac{\sum_{i \in S_\ell} |\theta_i(\ell) - v_i| + |\theta_i(\tau) - v_i|}{1 + V(S_\ell)} \cdot \mathbb{1}(\mathcal{A}_{\ell-1}^c) \right]. \quad (19)$$

Noting that $|\mathcal{E}^{\text{An}}(\ell)| \leq T$ and following the approach of Bounding $Reg_1(T, \mathbf{v})$, specifically along the lines of (11), (12), (13) and (15), we can show that

$$\mathbb{E} \left[\sum_{\tau \in \mathcal{E}^{\text{An}}(\ell)} \frac{\sum_{i \in S_\ell} |\theta_i(\ell) - v_i| + |\theta_i(\tau) - v_i|}{1 + V(S_\ell)} \cdot \mathbb{1}(\mathcal{A}_{\ell-1}^c) \right] \leq \mathbb{E} \left[\frac{|\mathcal{E}^{\text{An}}(\ell)|}{1 + V(S_\ell)} \sum_{i \in S_\ell} \left(C_1 \sqrt{\frac{v_i \log TK}{n_i(\ell)}} + C_2 \frac{\log TK}{n_i(\ell)} \right) \right],$$

where C_1 and C_2 are constants. Hence, from (18) and (19), we have

$$\frac{Reg_1(T, \mathbf{v})}{K+1} \leq \mathbb{E} \left[\sum_{\ell=1}^L |\mathcal{E}^{\text{An}}(\ell)| \cdot \mathbb{1}(\mathcal{A}_{\ell-1}) + \frac{|\mathcal{E}^{\text{An}}(\ell)|}{1 + V(S_\ell)} \sum_{i \in S_\ell} \left(C_1 \sqrt{\frac{v_i \log TK}{n_i(\ell)}} + C_2 \frac{\log TK}{n_i(\ell)} \right) \right], \quad (20)$$

We bound each of term in the above expression to complete the proof. We have by Cauchy-Schwartz inequality,

$$\mathbb{E} \left[|\mathcal{E}^{\text{An}}(\ell)| \cdot \mathbb{1}(\mathcal{A}_{\ell-1}) \right] \leq \mathbb{E}^{1/2} \left(|\mathcal{E}^{\text{An}}(\ell)|^2 \right) \cdot \mathbb{P}^{1/2}(\mathcal{A}_{\ell-1}).$$

Therefore, from lemma 8 and lemma 10, we have

$$\mathbb{E} \left[\sum_{\ell=1}^L |\mathcal{E}^{\text{An}}(\ell)| \cdot I(\mathcal{A}_{\ell-1}) \right] \leq \frac{e^{13}}{K}. \quad (21)$$

Now we bound the second term in (20). For notational brevity, let

$$\begin{aligned} \delta_i(\ell) &= \frac{C_1}{1 + V(S_\ell)} \sum_{i \in S_\ell} \sqrt{\frac{v_i \log TK}{n_i(\ell)}} \\ \Delta_i(\ell) &= \frac{C_2}{1 + V(S_\ell)} \sum_{i \in S_\ell} \frac{\log TK}{n_i(\ell)} \end{aligned}$$

From Cauchy-Schwartz inequality, we have

$$\sum_{\ell=1}^L \frac{|\mathcal{E}^{\text{An}}(\ell)|}{1 + V(S_\ell)} \sum_{i \in S_\ell} \left(C_1 \sqrt{\frac{v_i \log TK}{n_i(\ell)}} + C_2 \frac{\log TK}{n_i(\ell)} \right) \leq \left(\sum_{\ell=1}^L |\mathcal{E}^{\text{An}}(\ell)|^2 \right)^{1/2} \cdot \left[\left(\sum_{\ell=1}^L \delta_i^2(\ell) \right)^{1/2} + \left(\sum_{\ell=1}^L \Delta_i^2(\ell) \right)^{1/2} \right] \quad (22)$$

Again applying Cauchy-Schwartz, we have

$$\begin{aligned} \delta_i^2(\ell) &\leq \frac{C_1^2 V(S_\ell)}{(1 + V(S_\ell))^2} \cdot \sum_{i \in S_\ell} \frac{\log TK}{n_i(\ell)} \\ &\leq C_1^2 \sum_{i \in S_\ell} \frac{\log TK}{n_i(\ell)}. \end{aligned}$$

Let T_i denote the total number of epochs product i is offered, then we have,

$$\sum_{\ell=1}^L \sum_{i \in S_\ell} \frac{\log TK}{n_i(\ell)} = \sum_{i=1}^N \sum_{n_i(\ell)=1}^{T_i} \frac{\log TK}{n_i(\ell)} \leq N \log TK \log T.$$

From lemma 8 and preceding two equations, it follows that

$$\mathbb{E} \left[\left(\sum_{\ell=1}^L |\mathcal{E}^{\text{An}}(\ell)|^2 \right)^{1/2} \cdot \left(\sum_{\ell=1}^L \delta_i^2(\ell) \right)^{1/2} \right] \leq \frac{C_1 e^{13} \sqrt{NT} \log TK}{K}$$

Noting that $\sum_{\ell=1}^L \Delta_i^2(\ell) \leq C_2 \log^2 TK$, we have from lemma 8

$$\mathbb{E} \left[\left(\sum_{\ell=1}^L |\mathcal{E}^{\text{An}}(\ell)|^2 \right)^{1/2} \cdot \left(\sum_{\ell=1}^L \delta_i^2(\ell) \right)^{1/2} \right] \leq \frac{C_2 e^{13} \sqrt{T} \log TK}{K}$$

Hence, from the preceding two results and from (20), from (21) and (22), we have

$$\text{Reg}_1(T, \mathbf{v}) \leq C \sqrt{NT} \log TK, \quad (23)$$

where C is a constant. The result follows from (23) and (17).

C.1. Bounding the analysis epoch length

Here, we prove that the expected length (and the second moment) of the analysis epoch (see 9) is bounded by a constant. Specifically, we have the following result.

Lemma 8 *For any $p \in [1, 2]$, we have,*

$$\mathbb{E}^{1/p} \left[\left| \mathcal{E}^{\text{An}}(\tau) \right|^p \right] \leq \frac{e^{12}}{K} + 30^{1/p}, \text{ for all } \tau \in \mathcal{T},$$

where $\mathcal{E}^{\text{An}}(\tau)$ is defined as in (9).

Proof For notational brevity, we introduce some notation.

Notation.

- $n_i(\ell)$ denote the number of epochs product i has been offered until epoch ℓ (including epoch ℓ) in Algorithm 2.

- Let $\hat{v}_i(\ell)$ denote the value of \hat{v}_i after epoch ℓ .

-

$$r = \lfloor (q+1)^{1/p} \rfloor,$$

$$z = \sqrt{\log(rK+1)},$$

and for each $i = 1, \dots, N$,

$$\hat{\sigma}_i(\ell) = 4\sqrt{\frac{m\hat{v}_i(\ell)(\hat{v}_i(\ell)+1)}{n_i(\ell)}} + \frac{24m\sqrt{\log T}}{n_i(\ell)}.$$

- Define events,

$$\begin{aligned} A_\ell &= \{\theta_i(\ell) \geq \hat{v}_i(\ell) + z\hat{\sigma}_i(\ell) \text{ for all } i \in S^*\} \\ B_\ell &= \{\hat{v}_i(\ell) + z\hat{\sigma}_i(\ell) \geq v_i \text{ for all } i \in S^*\} \\ \mathcal{B}_\tau &= \bigcap_{\ell=\tau+1}^{\tau+r} B_\ell \end{aligned} \tag{24}$$

We have,

$$\mathbb{P}\{|\mathcal{E}(\tau)|^p < q+1\} = \mathbb{P}\{|\mathcal{E}(\tau)| \leq r\}$$

By definition, length of the analysis epoch, $\mathcal{E}^{\text{An}}(\tau)$ less than r , implies that one of the algorithm epochs from $\tau+1, \dots, \tau+r$ is optimistic. Hence, we have,

$$\begin{aligned} \mathbb{P}\{|\mathcal{E}(\tau)| < r\} &= \mathbb{P}\left(\left\{ \{\mu_i(\ell) \geq v_i \text{ for all } i \in S^*\} \text{ for some } \ell \in (\tau, \tau+r) \right\}\right), \\ &\geq \mathbb{P}\left(\left\{ \{\mu_i(\ell) \geq \hat{v}_i(\ell) + z\hat{\sigma}_i(\ell) \geq v_i \text{ for all } i \in S^*\} \text{ for some } \ell \in (\tau, \tau+r] \right\}\right). \end{aligned}$$

From (24), we have,

$$\begin{aligned} \mathbb{P}\{|\mathcal{E}(\tau)| < r\} &\geq \mathbb{P}\left(\bigcup_{\ell=\tau+1}^{\tau+r} A_\ell \cap B_\ell\right) \\ &= 1 - \mathbb{P}\left(\bigcap_{\ell=\tau+1}^{\tau+r} A_\ell^c \cup B_\ell^c\right). \end{aligned} \tag{25}$$

We will now focus on the term, $\mathbb{P}\left(\bigcap_{\ell=\tau+1}^{\tau+r} A_\ell^c \cup B_\ell^c\right)$,

$$\begin{aligned} \mathbb{P}\left(\bigcap_{\ell=\tau+1}^{\tau+r} A_\ell^c \cup B_\ell^c\right) &= \mathbb{P}\left(\left\{\bigcap_{\ell=\tau+1}^{\tau+r} A_\ell^c \cup B_\ell^c\right\} \cap \mathcal{B}_\tau\right) + \mathbb{P}\left(\left\{\bigcap_{\ell=\tau+1}^{\tau+r} A_\ell^c \cup B_\ell^c\right\} \cap \mathcal{B}_\tau^c\right) \\ &\leq \mathbb{P}\left(\bigcap_{\ell=\tau+1}^{\tau+r} A_\ell^c\right) + \mathbb{P}(\mathcal{B}_\tau^c), \\ &\leq \mathbb{P}\left(\bigcap_{\ell=\tau+1}^{\tau+r} A_\ell^c\right) + \sum_{\ell=\tau+1}^{\tau+r} \mathbb{P}(B_\ell^c), \end{aligned} \quad (26)$$

where the inequality follows from union bound. Note that,

$$\begin{aligned} \mathbb{P}(B_\ell^c) &= \mathbb{P}\left(\bigcup_{i \in S^*} \{\hat{v}_i(\ell) + z\hat{\sigma}_i(\ell) < v_i\}\right) \\ &\leq \sum_{i \in S^*} \mathbb{P}(\hat{v}_i(\ell) + z\hat{\sigma}_i(\ell) < v_i). \end{aligned} \quad (27)$$

Since r is trivially less than T , we have $rK + 1 \leq TK$, we have $\sqrt{\log(r+1) \cdot \log TK} \geq \log(rK + 1)$. Substituting $m = 3.1$ in lemma 10, we obtain,

$$\mathbb{P}(\hat{v}_i(\ell) + z\hat{\sigma}_i(\ell) < v_i) \leq \frac{1}{(rK)^{3.1}}. \quad (28)$$

From (27) and (28), we obtain,

$$\begin{aligned} \mathbb{P}(B_\ell^c) &\leq \frac{1}{r^{3.1} K^{2.1}}, \text{ and} \\ \sum_{\ell=\tau+1}^{\tau+r} \mathbb{P}(B_\ell^c) &\leq \frac{1}{(rK)^{2.1}}. \end{aligned} \quad (29)$$

We will now use the tail bounds for Gaussian random variables to bound the probability $\mathbb{P}(A_\ell^c)$. For any Gaussian random variable, Z with mean μ and standard deviation σ , we have,

$$\Pr(Z > \mu + x\sigma) \geq \frac{1}{\sqrt{2\pi}} \frac{x}{x^2 + 1} e^{-x^2/2}.$$

Note that by construction of $\mu_i(\ell)$ in Algorithm 2. We have,

$$\mathbb{P}\left(\bigcap_{\ell=\tau+1}^{\tau+r} A_\ell^c\right) = \mathbb{P}\left(\theta^{(j)}(\ell) \leq z \text{ for all } \ell \in (\tau, \tau+r) \text{ and for all } j = 1, \dots, K\right)$$

Since $\theta^{(j)}(\ell)$, $j = 1, \dots, K$, $\ell = \tau+1, \dots, \tau+r$ are independently sampled from the distribution, $\mathcal{N}(0, 1)$, we have

$$\begin{aligned} \mathbb{P} \left\{ \bigcap_{\ell=\tau+1}^{\tau+r} A_\ell^c \right\} &\leq \left[1 - \left(\frac{1}{\sqrt{2\pi}} \frac{\sqrt{\log(rK+1)}}{\log(rK+1)+1} \cdot \frac{1}{\sqrt{rK+1}} \right) \right]^{rK} \\ &\leq \exp \left(-\frac{r^{1/2}}{\sqrt{2\pi}} \frac{2\sqrt{\log(rK+1)}}{4\log(rK+1)+1} \right) \\ &\leq \frac{1}{(rK)^{2.2}} \text{ for any } r \geq \frac{e^{12}}{K}. \end{aligned} \quad (30)$$

From (25), (26), (29) and (30), we have that,

$$\mathbb{P} \{ |\mathcal{E}(\tau)| < r \} \geq 1 - \frac{1}{(rK)^{2.1}} - \frac{1}{(rK)^{2.2}} \text{ for any } r \geq \frac{e^{12}}{K}.$$

From definition $r \geq (q+1)^{1/p} - 1$, we obtain

$$\mathbb{P} \{ |\mathcal{E}(\tau)|^p < q+1 \} \geq 1 - \frac{1}{(q+1)^{2.1/p} - 1} - \frac{1}{(q+1)^{2.2/p} - 1} \text{ for any } q \geq \left(\frac{e^{12}}{K} + 1 \right)^p.$$

Therefore, we have,

$$\begin{aligned} \mathbb{E} [|\mathcal{E}(\tau)|^p] &= \sum_{q=0}^{\infty} \mathbb{P} \{ |\mathcal{E}(\tau)|^p \geq q \} \\ &\leq \left(\frac{e^{12}}{K} + 1 \right)^p + \sum_{q=\frac{e^{12p}}{K^p}}^{\infty} \mathbb{P} \{ |\mathcal{E}(\tau)|^p \geq q \} \\ &\leq e^{12p} + \sum_{q=\frac{e^{12p}}{K^p}}^{\infty} \frac{1}{q^{2.1/p}} + \frac{1}{q^{2.2/p}} \leq \left(\frac{e^{12}}{K} + 1 \right)^p + 30. \end{aligned}$$

The result follows from the above inequality.

C.2. Some concentration bounds

In this section, we prove bounds on how fast our estimate \hat{v}_i converges to the true mean. For the rest of this section, we assume that $\hat{v}_i(\ell)$ and $n_i(\ell)$ are the values of \hat{v}_i and n_i in Algorithm 2 before the beginning of epoch ℓ . The concentration bounds we prove in the section are similar to Chernoff bounds, but for the fact that $n_i(\ell)$ is a random variable and $\hat{v}_i(\ell)$ is the mean of random number of i.i.d samples. Hence, we use a self-normalized martingale technique to derive concentration bounds. Specifically, we have

Lemma 9 *Let δ_i , $i = 1, \dots, N$ be arbitrary random variables. If $v_i \leq 1$, for all $i = 1, \dots, N$, then we have, for all $i = 1, \dots, N$,*

1.

$$Pr(\hat{v}_i(\ell) > (1 + \delta_i)v_i) \leq \mathbb{E}^{\frac{1}{2}} \left[\exp \left(-\frac{v_i \delta_i^2 n_i(\ell)}{2(1 + \delta_i)(1 + v_i)^2} \right) \right],$$

and

2.

$$Pr(\hat{v}_i(\ell) < (1 - \delta_i)v_i) \leq \mathbb{E}^{\frac{1}{2}} \left[\exp \left(-\frac{v_i \delta_i^2 n_i(\ell)}{6(1 + v_i)^2} \left(3 - \frac{2\delta_i v_i}{1 + v_i} \right) \right) \right]$$

Proof Proof. Fix i . We have

$$\hat{v}_i(\ell) = \sum_{\tau=1}^{\ell} \tilde{v}_{i,\tau} \mathbb{1}(i \in S_{\tau}).$$

Therefore, bounding $Pr(\hat{v}_i(\ell) > (1 + \delta_i)v_i)$ and $Pr(\hat{v}_i(\ell) < (1 - \delta_i)v_i)$ is equivalent to bounding $Pr\left(\sum_{\tau=1}^{\ell} \tilde{v}_{i,\tau} \mathbb{1}(i \in S_{\tau}) > (1 + \delta_i)v_i n_i(\ell)\right)$ and $Pr\left(\sum_{\tau=1}^{\ell} \tilde{v}_{i,\tau} \mathbb{1}(i \in S_{\tau}) < (1 - \delta_i)v_i n_i(\ell)\right)$. We will bound the first term and then follow a similar approach for bounding the second term to complete the proof.

Bounding $Pr(\hat{v}_i(\ell) > (1 + \delta_i)v_i)$:

From Markov Inequality, we have for any $\lambda > 0$,

$$\begin{aligned} Pr\left(\sum_{\tau=1}^{\ell} \tilde{v}_{i,\tau} \mathbb{1}(i \in S_{\tau}) > (1 + \delta_i)v_i n_i(\ell)\right) &= Pr\left\{\exp\left(\lambda \sum_{\tau=1}^{\ell} \tilde{v}_{i,\tau} \mathbb{1}(i \in S_{\tau})\right) > \exp(\lambda(1 + \delta_i)v_i n_i(\ell))\right\}, \\ &= Pr\left\{\exp\left(\lambda \sum_{\tau=1}^{\ell} \tilde{v}_{i,\tau} \mathbb{1}(i \in S_{\tau}) - \lambda(1 + \delta_i)v_i n_i(\ell)\right) > 1\right\}, \\ &\leq \mathbb{E}\left[\exp\left(\lambda \sum_{\tau=1}^{\ell} \tilde{v}_{i,\tau} \mathbb{1}(i \in S_{\tau}) - \lambda(1 + \delta_i)v_i n_i(\ell)\right)\right]. \end{aligned} \tag{31}$$

For notational brevity, denote $f(\lambda, v_i)$ by the function,

$$f(\lambda, v_i) = -\frac{\log(1 - v_i(e^{2\lambda} - 1))}{2}.$$

We have,

$$\begin{aligned} &\mathbb{E}\left[\exp\left(\lambda \sum_{\tau=1}^{\ell} \tilde{v}_{i,\tau} \mathbb{1}(i \in S_{\tau}) - \lambda(1 + \delta_i)v_i n_i(\ell)\right)\right] \\ &= \mathbb{E}\left[\exp\left(\sum_{\tau=1}^{\ell} (\lambda \tilde{v}_{i,\tau} - f(\lambda, v_i)) \cdot \mathbb{1}(i \in S_{\tau})\right) \cdot \exp\left(-\lambda(1 + \delta_i)v_i(1 - f(\lambda, v_i))n_i(\ell)\right)\right], \\ &\leq \mathbb{E}^{\frac{1}{2}}\left[\exp\left(\sum_{\tau=1}^{\ell} (2\lambda \tilde{v}_{i,\tau} - 2f(\lambda, v_i)) \cdot \mathbb{1}(i \in S_{\tau})\right)\right] \cdot \mathbb{E}^{\frac{1}{2}}\left[\exp\left(-2\lambda(1 + \delta_i)v_i(1 - f(\lambda, v_i))n_i(\ell)\right)\right], \end{aligned} \tag{32}$$

where the above inequality follows from Cauchy-Schwartz inequality. Let \mathcal{F}_τ be the filtration corresponding to the history until epoch τ . Note that for any τ , $\mathbb{1}(i \in S_\tau)$ conditioned on F_τ is a constant and $\{\tilde{v}_{i,\tau} | \mathcal{F}_\tau\}$ is a geometric random variable. From lemma 1 ??, for all $\tau \geq 1$ and for any $0 < \lambda < \frac{1}{2} \log \frac{1+v_i}{v_i}$, we have,

$$\mathbb{E} \left(e^{2\lambda \tilde{v}_{i,\tau} \mathbb{1}(i \in S_\tau)} \middle| \mathcal{F}_\tau \right) = \left(\frac{1}{1 - v_i(e^{2\lambda} - 1)} \right)^{\mathbb{1}(i \in S_\tau)}.$$

Therefore, it follows that

$$\mathbb{E} \left(e^{(2\lambda \tilde{v}_{i,\tau} - 2f(\lambda, v_i)) \cdot \mathbb{1}(i \in S_\tau)} \middle| \mathcal{F}_\tau \right) \leq 1, \quad (33)$$

and

$$\begin{aligned} \mathbb{E} \left[\exp \left(\sum_{\tau=1}^{\ell} (2\lambda \tilde{v}_{i,\tau} - 2f(\lambda, v_i)) \cdot \mathbb{1}(i \in S_\tau) \right) \right] &= \mathbb{E} \left[\mathbb{E} \left\{ \exp \left(\sum_{\tau=1}^{\ell} (2\lambda \tilde{v}_{i,\tau} - 2f(\lambda, v_i)) \cdot \mathbb{1}(i \in S_\tau) \right) \middle| \mathcal{F}_\ell \right\} \right] \\ &= \mathbb{E} \left[\prod_{\tau=1}^{\ell-1} \exp \left(\sum_{\tau=1}^{\ell} (2\lambda \tilde{v}_{i,\tau} - 2f(\lambda, v_i)) \cdot \mathbb{1}(i \in S_\tau) \right) \cdot \mathbb{E} \left(e^{(2\lambda \tilde{v}_{i,\ell} - 2f(\lambda, v_i)) \cdot \mathbb{1}(i \in S_\ell)} \middle| \mathcal{F}_\ell \right) \right] \\ &\leq \mathbb{E} \left[\prod_{\tau=1}^{\ell-1} \exp \left(\sum_{\tau=1}^{\ell} (2\lambda \tilde{v}_{i,\tau} - 2f(\lambda, v_i)) \cdot \mathbb{1}(i \in S_\tau) \right) \right], \end{aligned}$$

where the inequality follows from (33). Similarly by conditioning with $\mathcal{F}_{\ell-1}, \dots, \mathcal{F}_1$, we obtain,

$$\mathbb{E} \left[\exp \left(\sum_{\tau=1}^{\ell} (2\lambda \tilde{v}_{i,\tau} - 2f(\lambda, v_i)) \cdot \mathbb{1}(i \in S_\tau) \right) \right] \leq 1.$$

From (31) and (32), we have

$$Pr \left(\sum_{\tau=1}^{\ell} \tilde{v}_{i,\tau} \mathbb{1}(i \in S_\tau) > (1 + \delta_i) v_i n_i(\ell) \right) \leq \mathbb{E}^{\frac{1}{2}} \left[\exp \left(-2\lambda(1 + \delta_i) v_i (1 - f(\lambda, v_i)) n_i(\ell) \right) \right].$$

Therefore, we have

$$Pr \left(\sum_{\tau=1}^{\ell} \tilde{v}_{i,\tau} \mathbb{1}(i \in S_\tau) > (1 + \delta_i) v_i n_i(\ell) \right) \leq \mathbb{E}^{\frac{1}{2}} \left[\min_{0 < \lambda < \frac{1}{2} \log \frac{1+v_i}{v_i}} \exp \left(-2\lambda(1 + \delta_i) v_i (1 - f(\lambda, v_i)) n_i(\ell) \right) \right]. \quad (34)$$

We have,

$$\operatorname{argmin}_{0 < \lambda < \frac{1}{2} \log \frac{1+v_i}{v_i}} \exp \left(-2\lambda(1 + \delta_i) v_i (1 - f(\lambda, v_i)) \cdot n_i(\ell) \right) = \operatorname{argmin}_{0 < \lambda < \frac{1}{2} \log \frac{1+v_i}{v_i}} -2(1 + \delta_i) \lambda n_i(\ell) v_i - n_i(\ell) \log \left(1 - v_i(e^{2\lambda} - 1) \right) \quad (35)$$

Noting that the right hand side in the above equation is a convex function in λ , we obtain the optimal λ by solving for the zero of the derivative. Specifically, at optimal t , we have

$$e^{2\lambda} = \frac{(1 + \delta_i)(1 + v_i)}{1 + v_i(1 + \delta_i)}.$$

Substituting the above expression in (34), we obtain the following bound.

$$Pr(\hat{v}_i(\ell) > (1 + \delta_i)v_i) \leq \mathbb{E}^{\frac{1}{2}} \left[\left(1 - \frac{\delta_i}{(1 + \delta_i)(1 + v_i)} \right)^{n_i(\ell)v_i(1 + \delta_i)} \left(1 + \frac{\delta_i v_i}{1 + v_i} \right)^{n_i(\ell)} \right]. \quad (36)$$

For notational brevity, we will use n to denote the random variable $n_i(\ell)$ and focus on bounding the right hand term in the above equation.

From Taylor series of $\log(1 - x)$, we have that

$$nv_i(1 + \delta_i) \log \left(1 - \frac{\delta_i}{(1 + \delta_i)(1 + v_i)} \right) \leq -\frac{n\delta_i v_i}{1 + v_i} - \frac{n\delta_i^2 v_i}{2(1 + \delta_i)(1 + v_i)^2},$$

From Taylor series for $\log(1 + x)$, we have

$$n \log \left(1 + \frac{\delta_i v_i}{1 + v_i} \right) \leq \frac{n\delta_i v_i}{(1 + v_i)}.$$

Note that if $\delta_i > 1$, we can use the fact that $\log(1 + \delta_i x) \leq \delta_i \log(1 + x)$ to arrive at the preceding result. Substituting the preceding two equations in (36), we have

$$Pr(\hat{v}_i(\ell) > (1 + \delta_i)v_i) \leq \mathbb{E}^{\frac{1}{2}} \left[\exp \left(-\frac{n\delta_i^2 v_i}{2(1 + \delta_i)(1 + v_i)^2} \right) \right]. \quad (37)$$

Bounding $Pr(\hat{v}_i(\ell) < (1 - \delta_i)v_i)$:

Now to bound the other one sided inequality, we use the fact that for any $\lambda > 0$,

$$\mathbb{E} \left(e^{-\lambda \tilde{v}_{i,\tau} \mathbb{1}(i \in S_\tau)} \middle| \mathcal{F}_\tau \right) = \left(\frac{1}{1 - v_i(e^{-\lambda} - 1)} \right)^{\mathbb{1}(i \in S_\tau)}.$$

and follow a similar approach. More specifically, from Markov Inequality, for any $\lambda > 0$ and $0 < \delta_i < 1$, we have

$$\begin{aligned} Pr \left(\sum_{\tau=1}^{\ell} \tilde{v}_{i,\tau} \mathbb{1}(i \in S_\tau) < (1 - \delta_i)v_i n_i(\ell) \right) &= Pr \left\{ \exp \left(-\lambda \sum_{\tau=1}^{\ell} \tilde{v}_{i,\tau} \mathbb{1}(i \in S_\tau) \right) > \exp(-\lambda(1 - \delta_i)v_i n_i(\ell)) \right\}, \\ &= Pr \left\{ \exp \left(-\lambda \sum_{\tau=1}^{\ell} \tilde{v}_{i,\tau} \mathbb{1}(i \in S_\tau) + \lambda(1 - \delta_i)v_i n_i(\ell) \right) > 1 \right\}, \\ &\leq \mathbb{E} \left[\exp \left(-\lambda \sum_{\tau=1}^{\ell} \tilde{v}_{i,\tau} \mathbb{1}(i \in S_\tau) + \lambda(1 - \delta_i)v_i n_i(\ell) \right) \right]. \end{aligned} \quad (38)$$

For notational brevity, denote $f(\lambda, v_i)$ by the function,

$$f(\lambda, v_i) = -\frac{\log(1 - v_i(e^{-2\lambda} - 1))}{2}.$$

We have,

$$\begin{aligned}
 & \mathbb{E} \left[\exp \left(-\lambda \sum_{\tau=1}^{\ell} \tilde{v}_{i,\tau} \mathbb{1}(i \in S_{\tau}) + \lambda(1 - \delta_i) v_i n_i(\ell) \right) \right] \\
 &= \mathbb{E} \left[\exp \left(\sum_{\tau=1}^{\ell} (-\lambda \tilde{v}_{i,\tau} - f(\lambda, v_i)) \cdot \mathbb{1}(i \in S_{\tau}) \right) \cdot \exp \left(\lambda(1 - \delta_i) v_i (1 + f(\lambda, v_i)) n_i(\ell) \right) \right] \\
 &\leq \mathbb{E}^{\frac{1}{2}} \left[\exp \left(\sum_{\tau=1}^{\ell} (-2\lambda \tilde{v}_{i,\tau} - 2f(\lambda, v_i)) \cdot \mathbb{1}(i \in S_{\tau}) \right) \right] \cdot \mathbb{E}^{\frac{1}{2}} \left[\exp \left(2\lambda(1 - \delta_i) v_i (1 + f(\lambda, v_i)) n_i(\ell) \right) \right],
 \end{aligned} \tag{39}$$

where the above inequality follows from Cauchy-Schwartz inequality. Let \mathcal{F}_{τ} be the filtration corresponding to the history until epoch τ . Note that for any τ , $\mathbb{1}(i \in S_{\tau})$ conditioned on \mathcal{F}_{τ} is a constant and $\{\tilde{v}_{i,\tau} | \mathcal{F}_{\tau}\}$ is a geometric random variable. Therefore, for all $\tau \geq 1$ and for any $\lambda > 0$, we have,

$$\mathbb{E} \left(e^{-2\lambda \tilde{v}_{i,\tau} \mathbb{1}(i \in S_{\tau})} \middle| \mathcal{F}_{\tau} \right) = \left(\frac{1}{1 - v_i(e^{-2\lambda} - 1)} \right)^{\mathbb{1}(i \in S_{\tau})}.$$

Therefore, it follows that

$$\mathbb{E} \left(e^{(-2\lambda \tilde{v}_{i,\tau} - 2f(\lambda, v_i)) \cdot \mathbb{1}(i \in S_{\tau})} \middle| \mathcal{F}_{\tau} \right) \leq 1, \tag{40}$$

and

$$\begin{aligned}
 \mathbb{E} \left[\exp \left(\sum_{\tau=1}^{\ell} (-2\lambda \tilde{v}_{i,\tau} - 2f(\lambda, v_i)) \cdot \mathbb{1}(i \in S_{\tau}) \right) \right] &= \mathbb{E} \left[\mathbb{E} \left\{ \exp \left(\sum_{\tau=1}^{\ell} (-2\lambda \tilde{v}_{i,\tau} - 2f(\lambda, v_i)) \cdot \mathbb{1}(i \in S_{\tau}) \right) \middle| \mathcal{F}_{\ell} \right\} \right], \\
 &= \mathbb{E} \left[\prod_{\tau=1}^{\ell-1} \exp \left(\sum_{\tau=1}^{\ell} (-2\lambda \tilde{v}_{i,\tau} - 2f(\lambda, v_i)) \cdot \mathbb{1}(i \in S_{\tau}) \right) \cdot \mathbb{E} \left(e^{(-2\lambda \tilde{v}_{i,\ell} - 2f(\lambda, v_i)) \cdot \mathbb{1}(i \in S_{\ell})} \middle| \mathcal{F}_{\ell} \right) \right] \\
 &= \mathbb{E} \left[\prod_{\tau=1}^{\ell-1} \exp \left(\sum_{\tau=1}^{\ell} (-2\lambda \tilde{v}_{i,\tau} - 2f(\lambda, v_i)) \cdot \mathbb{1}(i \in S_{\tau}) \right) \right],
 \end{aligned}$$

where the inequality follows from (40). Similarly by conditioning with $\mathcal{F}_{\ell-1}, \dots, \mathcal{F}_1$, we obtain,

$$\mathbb{E} \left[\exp \left(\sum_{\tau=1}^{\ell} (-2\lambda \tilde{v}_{i,\tau} - 2f(\lambda, v_i)) \cdot \mathbb{1}(i \in S_{\tau}) \right) \right] \leq 1.$$

From (38) and (39), we have

$$Pr \left(\sum_{\tau=1}^{\ell} \tilde{v}_{i,\tau} \mathbb{1}(i \in S_{\tau}) < (1 - \delta_i) v_i n_i(\ell) \right) \leq \mathbb{E}^{\frac{1}{2}} \left[\exp \left(2\lambda(1 - \delta_i) v_i (1 + f(\lambda, v_i)) n_i(\ell) \right) \right].$$

Therefore, we have

$$Pr \left(\hat{v}_i(\ell) < (1 - \delta_i) v_i \right) \leq \mathbb{E}^{\frac{1}{2}} \left[\min_{\lambda > 0} \exp \left(2\lambda(1 - \delta_i) v_i (1 + f(\lambda, v_i)) n_i(\ell) \right) \right].$$

Following similar approach as in optimizing the previous bound (see (34)) to establish the following result. For notational brevity, we will use n to denote the random variable $n_i(\ell)$.

$$Pr(\hat{v}_i(\ell) < (1 - \delta_i)v_i) \leq \mathbb{E}^{\frac{1}{2}} \left[\left(1 + \frac{\delta_i}{(1 - \delta_i)(1 + v_i)} \right)^{nv_i(1 - \delta_i)} \left(1 - \frac{\delta_i v_i}{1 + v_i} \right)^n \right].$$

Now we will use Taylor series for $\log(1 + x)$ and $\log(1 - x)$ in a similar manner as described for the other bound to obtain the required result. In particular, since $1 - \delta_i \leq 1$, we have for any $x > 0$ it follows that $(1 + \frac{x}{1 - \delta_i})^{(1 - \delta_i)} \leq (1 + x)$. Therefore, we have

$$Pr(\hat{v}_i(\ell) < (1 - \delta_i)v_i) \leq \mathbb{E}^{\frac{1}{2}} \left[\left(1 + \frac{\delta_i}{1 + v_i} \right)^{nv_i} \left(1 - \frac{\delta_i v_i}{1 + v_i} \right)^n \right]. \quad (41)$$

Note that since $\tilde{v}_{i,\tau} \geq 0$ for all i, τ , we have a zero probability event if $\delta_i > 1$. Therefore, without loss of generality, we assume $\delta_i < 1$ and from Taylor series for $\log(1 - x)$, we have

$$n \log \left(1 - \frac{\delta_i v_i}{1 + v_i} \right) \leq -\frac{n\delta_i v_i}{1 + v_i},$$

and from Taylor series for $\log(1 + x)$, we have

$$n \log \left(1 + \frac{\delta_i v_i}{1 + v_i} \right) \leq \frac{n\delta_i}{(1 + v_i)} - \frac{n\delta_i^2 v_i}{6(1 + v_i)^2} \left(3 - \frac{2\delta_i v_i}{1 + v_i} \right).$$

Therefore, substituting the preceding equations in (41), we have,

$$Pr(\hat{v}_i < (1 - \delta_i)v_i) \leq \exp \left(-\frac{n\delta_i^2 v_i}{6(1 + v_i)^2} \left(3 - \frac{2\delta_i \mu}{1 + v_i} \right) \right). \quad (42)$$

The result follows from (37) and (42).

Lemma 10 *If $v_i \leq 1$, for all $i = 1, \dots, N$, then for all $i = 1, \dots, N$, and any ℓ, m , we have,*

1. $\mathcal{P} \left(|\hat{v}_i(\ell) - v_i| > 4\sqrt{\frac{\hat{v}_i(\ell)(\hat{v}_i(\ell) + 1)m \log(\ell + 1)}{n_i(\ell)} + \frac{24m \log(\ell + 1)}{n_i(\ell)}} \right) \leq \frac{5}{\ell^m}.$
2. $\mathcal{P} \left(|\hat{v}_i(\ell) - v_i| \geq \sqrt{\frac{12v_i m \log(\ell + 1)}{n_i(\ell)} + \frac{24m \log(\ell + 1)}{n_i(\ell)}} \right) \leq \frac{4}{\ell^m}.$

Proof Proof. Let $\delta_i = \sqrt{\frac{4v_i(v_i+2)m \log(\ell+1)}{v_i n_i(\ell)}}$. We analyze the cases $\delta_i \leq \frac{1}{2}$ and $\delta_i \geq \frac{1}{2}$ separately.

Case 1: $\delta_i \leq \frac{1}{2}$: For any $v_i \leq 1$ and $\delta_i \leq 1/2$, we have,

$$\frac{v_i \delta_i^2 n_i(\ell)}{2(1 + \delta_i)(1 + v_i)^2} \geq \frac{v_i \delta_i^2 n_i(\ell)}{6(1 + v_i)} \geq m \log(\ell + 1),$$

and

$$\frac{v_i \delta_i^2 n_i(\ell)}{6(1 + v_i)^2} \left(3 - \frac{2\delta_i v_i}{1 + v_i} \right) \geq \frac{v_i \delta_i^2 n_i(\ell)}{6(1 + v_i)} \geq m \log(\ell + 1).$$

Therefore, substituting $\delta_i = \sqrt{\frac{4(v_i+2)m \log(\ell+1)}{v_i n_i(\ell)}}$ in Theorem 9 with δ_i , we have,

$$\begin{aligned} \mathcal{P}(2\hat{v}_i(\ell) \geq v_i) &\geq 1 - \frac{1}{\ell^m}, \\ \mathcal{P}\left(|\hat{v}_i(\ell) - v_i| < \sqrt{\frac{4v_i(v_i+2)m \log(\ell+1)}{n_i(\ell)}}\right) &\geq 1 - \frac{2}{\ell^m}. \end{aligned} \quad (43)$$

From the above three results, we have,

$$\mathcal{P}\left(|\hat{v}_i(\ell) - v_i| < \sqrt{\frac{16\hat{v}_i(\ell)(\hat{v}_i(\ell)+1) \log(\ell+1)}{n_i(\ell)}}\right) \geq \mathcal{P}\left(|\hat{v}_i(\ell) - v_i| < \sqrt{\frac{4v_i(v_i+2) \log(\ell+1)}{n_i(\ell)}}\right) \geq 1 - \frac{3}{\ell^m}. \quad (44)$$

By assumption, $v_i \leq 1$. Therefore, we have $v_i(v_i+2) \leq 3v_i$ and,

$$\mathcal{P}\left(|\hat{v}_i(\ell) - v_i| < \sqrt{\frac{12v_i \log(\ell+1)}{n_i(\ell)}}\right) \geq 1 - \frac{3}{\ell^m}.$$

Case 2: $\delta_i > \frac{1}{2}$: Now consider the scenario, when $\sqrt{\frac{4v_i(v_i+2)m \log(\ell+1)}{v_i n_i(\ell)}} > \frac{1}{2}$. Then, we have,

$$\bar{\delta}_i \triangleq \frac{8v_i(v_i+2)m \log(\ell+1)}{v_i n_i(\ell)} \geq \frac{1}{2},$$

which implies for any $v_i \leq 1$,

$$\begin{aligned} \frac{nv_i \bar{\delta}_i^2}{2(1+\bar{\delta}_i)(1+v_i)^2} &\geq \frac{nv_i \bar{\delta}_i}{12(1+v_i)}, \\ \frac{n\bar{\delta}_i^2 v_i}{6(1+v_i)^2} \left(3 - \frac{2\bar{\delta}_i v_i}{1+v_i}\right) &\geq \frac{nv_i \bar{\delta}_i}{12(1+v_i)}. \end{aligned}$$

Therefore, substituting the value of $\bar{\delta}_i$ in Theorem 9, we have

$$\mathcal{P}\left(|\hat{v}_i(\ell) - v_i| > \frac{24m \log(\ell+1)}{n}\right) \leq \frac{2}{\ell^m}.$$