A Note on the Relationship Among Capacity, Pricing and Inventory in a Make-to-Stock System

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Abstract

We address the simultaneous determination of pricing, production and capacity investment decisions by a monopolistic firm in a multi-period setting under demand uncertainty. We analyze the optimal decision with particular emphasis on the relationship between price and capacity. We consider models that allow for either bi-directional price changes or models with markdowns only, and in the latter case we prove that capacity and price are *strategic substitutes*.

Short Title: Relationship between pricing and capacity decisions

Keywords: capacity investment, pricing, inventory, stochastic demand.

1 Introduction

1.1 Background and Overview of Main Findings

Recent years have witnessed an increased interest in the use of pricing in operations management practices, with a particular focus on the integration of inventory control and dynamic (state-dependent) pricing strategies. Concomitantly, studies focusing on the interface between capacity investment and replenishment strategies have led to further understanding of capacitated inventory systems and supply chains. A very useful qualitative insight in this context has been the understanding that capacity and inventory are in essence *strategic substitutes*. Roughly speaking,

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decision variables are said to be strategic substitutes if increasing the value of one variable decreases the return from increasing the other; a more precise definition will be advanced in section 4. One of the main motivations for the present paper is to develop similar insights that pertain to pricing and capacity decisions. As the literature review at the end of this section indicates, we are only aware of a few papers to date that focus on the problem of joint capacity planning and pricing strategies, and even less that go on to explore the three-way relationship between capacity, inventory and pricing decisions.

In this paper we study a stylized problem in which a centralized monopolistic firm sells a product over a finite selling horizon; the number of periods constituting this time horizon measure the time elapsed from the first introduction of the product to the market, up until the point where the firm terminates its production and sale. The firm reviews the state of the system periodically and at the beginning of each period makes three decisions: i.) invest or disinvest in production capacity; ii.) replenish inventory (constrained by production capacity); and iii.) fix a price for the produced goods that will take effect in the following period. Subsequent to these decisions, demand is observed. We first allow the firm to carry inventory from one period to the next, and orders are allowed to be backlogged. Subsequently, we introduce a restriction that disallows carry-over of inventory from period to period; the firm must then either satisfy inventory shortage using emergency replenishment or by paying penalty fees. In the next stage, we also restrict the firm's pricing flexibility by only allowing markdowns, i.e., the price of the product can only decrease over its life cycle.

The main contribution of this paper is in studying the relationship between pricing and capacity decisions in the context of a dynamic optimization problem that has capacity, inventory and price as its variables. The analysis proceeds by first showing that the optimal capacity investment policy in the presence of pricing and inventory decisions is of a *target interval* form (see Theorem 1). Given a fixed capacity level, the optimal joint pricing-inventory decisions are seen to follow a modified base-stock list-price policy (see Theorem 2). These results serve as a basis for studying a model where no inventory carry-over is allowed, and pricing is restricted to markdowns. In this important scenario we show that price and capacity are strategic substitutes both as decision variables and as state variables (see Theorem 3); an important implication is that these levers can be used in a complementary manner (see discussion in Section 6). Several numerical examples illustrate some of these findings.

This section concludes with a review of related literature and known results. Section 2 describes the model and sets up the dynamic optimization problem. Section 3 provides the first set of results. Section 4 discusses the cases where no carry-overs are allowed and when pricing is restricted to markdowns. Section 5 summarizes some qualitative insights that are gleaned from the main results and also provides a simple example that illustrates the key findings. All proof are collected in the appendix.

1.2 Literature review and positioning of the present paper

Given the voluminous literature on the topic of interest in this paper, we restrict our review to work that is closely related in terms of thrust and problem formulation. For a recent survey and further references on pricing, inventory and capacity decisions the reader is referred to Chan et. al. (2004, Section 4.2).

Inventory decisions in capacitated systems. Federgruen and Zipkin (1987) consider a single-item, periodic-review inventory model with uncertain demands. Under the assumption of finite production capacity in each period, they show that a modified basic-stock policy is optimal. Some extensions include Aviv and Federgruen (1997) that deals with non-stationary demand, and Ozer and Wei (2004) that deals with information acquisition and replenishment costs.

Inventory and pricing decisions. Federgruen and Heching (1999) study the relationship between price and inventory in an uncapacitated system with stochastic demand. They characterize the structure of the optimal price-inventory policy, and show that *inventory and price* are strategic substitutes. Further references in this stream of literature are surveyed in Elmaghraby and Keskinocak (2003); a deterministic analysis of such problems dates back to Thomas (1974) and Kunreuther and Schrage (1973).

Inventory and capacity decisions. Angelus and Porteus (2002) study capacity decisions in cases where a firm can and cannot hold inventories. In the former case, they establish that *capacity and inventory* are strategic substitutes. (For an example of an analysis of a deterministic model see Rao (1976).) Eberley and Van Mieghem (1997) study a problem that can be viewed as a generalization of the "no-carry-over" version of Angelus and Porteus (2002). They characterize the optimal capacity policy when capacity is multi-dimensional and it is costly to reverse capacity investments. Duenyas and Ye (2007) generalize Angelus and Porteus' (2002) "no-carry-over" model by allowing fixed and variable costs in capacity adjustments.

Joint pricing, production and capacity decisions. Maccini (1984) studies the effects of inventory dynamics and capital on pricing and capacity decisions from a macroeconomic perspective. He finds that excess capacity tends to cause prices to decrease below their acceptable long run levels. Gaimon (1987) shows, by means of a numerical study, that upgrading capacity lowers the firm's per unit production cost and thus the prices it charges. Li (1988) introduces a point process model of a production firm with intensities parameterized by production, capacity and price, respectively. A distinction is made between static decision making (capacity levels are set at time zero), and dynamic operating decisions (pricing and production). Van Mieghem and Dada (1999) study different possible postponement strategies in a single period problem when firms make three decisions: capacity investment, production (inventory) quantity, and price. (See also Boyaci and Ozer (2007) for a study of information acquisition through advance selling.)

A notable entry absent from the above list is work focusing on joint pricing and capacity decisions in an inventory setting, and the present paper indeed strives to fill that gap. In terms of the model and analysis tools, our work is most closely related to that by Angelus and Porteus (2003) and Federgruen and Heching (1999): the former studies the relationship between inventory and capacity, and the latter discusses inventory and prices. Our research is intended to complement theirs.

2 Problem Formulation

We consider a monopolistic firm that produces a single product whose capacity, inventory, and price are reviewed periodically. At the beginning of each period the firm makes three decisions: (i) capacity investment (or disinvestment); (ii) production level; and (iii) the price it will charge for the product. We assume that capacity investments and produced goods become available instantaneously. The life cycle of the product, and therefore the time horizon, is set to be T periods. The sequence of events in each period, $t = 1, \dots, T$, is as follows:

1. Investment or disinvestment in capacity, setting it to a level equal to z_t.

2. Production (if needed) to set the inventory level to y_t.

3. A price p_t is set and held fixed up until period $t+1$.

4. Demand is realized and satisfied if it is less than available inventory, or backlogged otherwise. Backlog and holding costs are incurred.

Demand in period t, D_t, depends on the prevailing price which is given by a general additive stochastic demand function

$$D_t = d_t(p_t) + \epsilon_t, \tag{1}$$

where

$$\begin{aligned} p_t &= \text{price charged in period } t, \\ \epsilon_t &= \text{sequence of independent random variables.} \end{aligned}$$

Feasible price levels are confined to the interval $\left[\underline{p}, \overline{p}\right]$, where $\overline{p}$ and $\underline{p}$ are the highest and lowest prices, respectively. (In Section 4 and Section 5 we indicate how the main results extend to more general demand functions.) Let

$$\begin{aligned} x_t &= \text{inventory level at the beginning of period } t\text{, before ordering,} \\ y_t &= \text{inventory level at the beginning of period } t\text{, after ordering.} \end{aligned}$$

The firm incurs two types of production and inventory costs: the end-of-period inventory carrying (and backlogging) costs, and a variable production cost. Specifically,

$$\begin{aligned} h_t(x) &= \text{inventory (or backlogging) cost incurred in period } t \text{ with} \\ & \quad\ \text{terminal inventory level equals } x, \\ c_t &= \text{per unit purchasing or production cost in period } t. \end{aligned}$$

Let

$$G_t(y, p) = \mathbb{E}h_t(y - D_t) = \mathbb{E}h_t(y - d_t(p, \epsilon_t)), \tag{2}$$

denote the single-period expected inventory and backlogging costs for period t, for a given price p and an inventory level (after ordering) y, where the expectation here, as well as in the remainder of the paper, is taken with respect to the distribution of the random noise term. We assume that:

(A1) $\mathbb{E}|\epsilon_t| < \infty$, for all $t = 1, \ldots, T$,

(A2) $h_t(\cdot)$ is convex for all $t = 1, \ldots, T$.

(A3) $d_t(p_t, \epsilon_t) = a_t - b_t p_t + \epsilon_t$ where $a_t, b_t > 0$, $a_t/b_t \geq \overline{p}$, for all $t = 1, \ldots, T$.

These assumptions ensure that the cost functions $G_t(y, p)$ are well defined, finite, and jointly convex in y and p for all $t = 1, \ldots, T$.

Remark 1 The assumption of linear demand can be generalized to any demand function which is continuous and strictly decreasing in the price variable, and for which the revenue rate $d\mathbb{E}(d_t^{-1}(d, \epsilon_t))$ is concave in d, where d_t^{-1} is the inverse function of d_t for fixed ϵ_t. This assumption is rather benign and quite standard in the revenue management literature; Ferguson et al. (2006) provide example of a linear function and an exponential demand function that satisfy these conditions; see also Chen and Simchi-Levi (2002) for further discussion. In that case, one would need to impose directly that $G(\cdot, \cdot)$ is jointly convex; see Federgruen and Heching (1999) for conditions ensuring that this holds.

Let $\gamma_t(y, p)$ denote the expected contribution in profits in period t, if the firm has y units at the beginning of the period (i.e., post production) and it charges p per produced unit that is sold on the market. That is, in period t

$$\gamma_t(y, p) = p\mathbb{E}[d_t(p, \epsilon_t)] - c_t y - G_t(y, p). \tag{3}$$

Let

$z_t =$ the capacity level at the beginning of period t, after adjustment.

We define three capacity related costs:

$$\begin{aligned} K &= \text{the cost of adding a unit of capacity,} \\ k &= \text{the return from selling a unit of capacity,} \\ h_c &= \text{the capacity overhead cost per unit.} \end{aligned}$$

Hence h_c amalgamates all costs that are associated with maintaining production, but are independent of the production volume. We assume that $K \geq k$ which reflects the fact that capacity is usually sold for less than the purchase price. Revenues and costs are discounted with a discount factor $\alpha \in (0, 1]$. We note that all capacity-related costs are taken to be time-homogeneous for simplicity, and the analysis that follows can easily be adjusted to account for such temporal dependency. We assume that a firm begins the life-cycle of the product with capacity level z_0 and inventory level x_0 (allowing for the possibility of $x_0 = 0, z_0 = 0$). Note, that our work pertains primarily to industries in which there is short long lead time for capacity changes, as well as low fixed costs for capacity changes (apart for the friction of selling-buying). In industries, such as medical devices, in which soft-tooling is being used, such quick capacity changes are possible.

Let $f_t(z, x)$ be the maximum expected present value of the total net profits that can be earned in months t and on, given that the capacity level is z and inventory level is x at the beginning of period t. That is,

$$f_t(x,z) = \max\Big\{\gamma_t(y,p) + c_t x - C(z'-z) - h_c z' + \alpha \mathbb{E} f_{t+1}(y - d_t(p,\epsilon_t), z') :$$
$$z' \geq 0, x \leq y \leq x + z', \underline{p} \leq p \leq \overline{p}\Big\}, \quad (4)$$

for $t = 1, \ldots, T$, where

$$C(z) = \begin{cases} kz & \text{if} \quad z \leq 0 \\ Kz & \text{if} \quad z > 0. \end{cases} \quad (5)$$

At the terminal period we assume that demand is satisfied and the remaining capacity is sold immediately thereafter, that is, we set

$$f_{T+1}(x,z) = kz - h_{T+1}(x).$$

To recapitulate, at the beginning of each period $t = 1, \ldots, T$, the firm must determine a capacity investment level z', an inventory level y, and a price p based on the initial inventory and capacity, x and z. These decisions are held fixed throughout period t. The objective of the firm is to maximize the sum of discounted profits over the time horizon T with respect to the abovementioned decision variables; the maximum value of this dynamic optimization problem is given by $f_1(x, z)$.

For future purposes it will be convenient to rewrite $f_t(x, z)$ as follows (see Angelus and Porteus (2002)):

$$f_t(x,z) = \max_{z' \geq 0} \left[c_t x - C(z'-z) - h_c z' + \Gamma_t(x,z')\right], \quad (6)$$

where

$$\begin{aligned} \Gamma_t(x,z) &= \max\Big\{a_t(y,p,z) \ : \ y \in [x, x+z], p \in [\underline{p}, \overline{p}]\Big\}, \\ a_t(y,p,z) &= \gamma_t(y,p) + \alpha \mathbb{E} f_{t+1}(y - d_t(p,\epsilon_t), z), \end{aligned} \quad (7)$$

for all $t = 1, \ldots, T$.

3 The Optimal Policy and Key Relations

3.1 Main results

In this section we characterize the structure of a policy that maximizes the expected discounted profits. The results are very much in the spirit of those in Federgruen and Heching (1999) (albeit for a non-capacitated system) and Angelus and Porteus (2002) (for a model with exogenously given prices).

Recall, the maximum value of this objective is given by $f_1(\cdot,\cdot)$, where $f_t(\cdot,\cdot)$ is defined in (6). We will begin by analyzing the optimal capacity investment policy. Then, given the optimal capacity at the beginning of a period, we will derive the optimal joint inventory-pricing policy. It is important to note that the three decisions are made simultaneously; the optimal policy is described in a sequential manner to allow for a more transparent representation.

To characterize the optimal capacity investment policy, we first describe a family of ISD policies (Invest/Stay Put/Disinvest), often referred to as *target interval* policies.

Definition 1 *A sequence $\{z_t\}_{t=1}^T$ constitutes a target interval policy with respect to a sequence of non-negative real number $\{L_t, U_t\}_{t=1}^T$, if:*

(i) $L_t \leq U_t$

(ii) *L_t and U_t are independent of z_{t-1};*

(iii) $z_t = \begin{cases} L_t & \text{if} \quad z_{t-1} < L_t, \\ z_{t-1} & \text{if} \quad L_t \leq z_{t-1} \leq U_t \\ U_t & \text{if} \quad z_{t-1} > U_t, \quad \text{for all } t = 1, \ldots, T \end{cases}$

The upper and lower targets L_t and U_t can be functions of the state of the system (and past information observed up until time t,) and the notation $L_t(\cdot)$ and $U_t(\cdot)$ will be used to indicate this dependence; in the following theorem, both are functions of the initial inventory x.

Theorem 1 ***(Optimal capacity investment policy)*** *The optimal capacity investment decision follows a target interval policy in each period, with lower- and upper- capacity targets $L_t(x)$ and $U_t(x)$ for each $t = 1, \ldots, T$ and each initial inventory level $x \in \mathbb{R}$.*

Based on the optimal capacity investment, we will now show that the optimal joint production-pricing decision takes the form of a modified base-stock list-price policy (we use the term "modified" because of the capacity constraint on the production). This policy is characterized by a base-stock level and a list-price combination $(\widehat{y}_t(x,z), \widehat{p}_t(x,z))$ which are given as a function of the initial inventory and capacity (x, z); the former and latter functions are the optimal inventory position and price levels, respectively, given that period t begins with capacity z and inventory level x, and are derived as follows:

$$(\widehat{y}_t(x,z), \widehat{p}_t(x,z)) = \arg\max \left\{ a_t(y,p,z) \ : \ y \in [x, x+z], p \in [\underline{p}, \overline{p}] \right\}.$$

The existence and uniqueness of $\widehat{y}_t(x,z)$ and $\widehat{p}_t(x,z)$ for given initial capacity and inventory levels, x and z, are established in the proof of the Theorem 2. Note that the "hat" notation is used to distinguish the optimal policy. For the purpose of the following theorem, we introduce the following definitions.

Definition 2 *Variables $u, v \in \mathbb{R}$ are said to be strategic substitutes with respect to a function $f(u,v) : \mathbb{R}^2 \to \mathbb{R}$, if $f(u,v)$ is submodular in u and v.*

For a definition of submodularity see, e.g., Topkis (1978), and for further discussion of economic implications and interpretation see, e.g., Milgrom and Roberts (1990).

Theorem 2 ***(Optimal pricing-inventory policy)***

(a) *An optimal inventory-pricing policy is a base-stock list-price with base-stock $\widehat{y}_t(x,z)$ and list-price $\widehat{p}_t(x,z)$, for $t = 1, \dots, T$. At period $t \in \{1, \dots, T\}$ and given a capacity level z: if $x \le \widehat{y}_t(x,z) \le x+z$, it is optimal to order up to the base-stock level $\widehat{y}_t(x,z)$ and to charge the list-price $\widehat{p}_t(x,z)$; if $x > \widehat{y}_t(x,z)$, it is optimal not to order and to charge $p_t \le \widehat{p}_t(x,z)$; and if $\widehat{y}_t(x,z) > x+z$, it is optimal to order z units and charge $p_t \ge \widehat{p}_t(x,z)$.*

(b) *For each period $t \in \{1, \dots, T\}$ and fixed capacity and inventory state values $x, z \in \mathbb{R}$, the price and inventory decision variables (y_t, p_t) are strategic substitutes with respect to the function $a_t(\cdot, \cdot, z)$ given in (7).*

Remark 2 (discussion) The result of the theorem states that if the inventory level, x, is below the base-stock level, it is increased to that value and the list-price is charged. If the inventory level is above the base-stock level, then nothing is ordered, and a price discount is offered, i.e., the price charged is below the list price. (The higher the excess in the initial inventory level, the larger the optimal discount offered.) If the sum of inventory and capacity is below the base-stock level, the maximum possible amount is produced (i.e., the production level equals the capacity level), and the price charged is higher than the list price. No discounts are offered unless the product is overstocked, and no higher-than-list-prices are charged unless the product is in shortage, which happens when the current capacity is not sufficient to support the "desired" inventory level. Numerical illustrations of this result are given in Section 5.

Remark 3 (structural results) The proof of the theorem establishes certain monotonicity results for $y_t(x, z)$ and $p_t(x, z)$ with respect to the variables x and z. For example, Lemma 3 establishes that $y_t(\cdot, z)$ is non-decreasing. The results of Lemma 4 imply that $p_t(\cdot, z)$ is non increasing. Similar statements can be derived for the dependence on z for the post-adjustment capacity.

4 Joint Capacity Planning and Pricing

In this section we analyze a particular instance of the joint capacity planning and pricing problem when inventory cannot be carried over from period to period and prices can only be decreased throughout the time horizon. This situation arises when firms can not use inventory produced in "off-peak" periods to absorb "peak-demand." To this end, we assume that stockouts are satisfied at the end of the period in which they occur; Federgruen and Heching (1999) describe such a mechanism as *emergency purchases* or production runs.

4.1 Main results

Let $f_t^M(z, p)$ denote the maximum expected present value of the total profits that can be earned in periods t up until T, given that period t starts with capacity z and price p. The optimality equations for $t = 1, \ldots, T$ are given by

$$\begin{aligned} f_t^M(z, p) &= \max_{z' \geq 0, p'} \left\{ \gamma_t(y, p) - C(z' - z) - h_c z' + \alpha \mathbb{E} f_{t+1}^M(p', z') \;:\; 0 \leq y \leq z', \underline{p} \leq p' \leq p \right\}, \\ f_{T+1}^M(z) &= kz. \end{aligned}$$

The decision variables in the above equation are the price (p') and capacity (z') set in period t. We then have the following result.

Theorem 3 *Assume a firm cannot carry inventories and increase prices from period to period. Then, the following properties hold for all* $t = 1, \ldots, T$*:*

(a) $f_t^M(z, p)$ *is submodular and jointly concave in the state variables* (p, z).

(b) *The decision variables* p' *and* z' *are strategic substitutes with respect to* $f_t^M(\cdot, \cdot)$

(c) *The optimal capacity policy is a target interval policy in each period. The capacity targets* $L_t(p)$ *and* $U_t(p)$ *satisfy* $L_t(p) \leq U_t(p)$ *for each* $t = 1, \ldots, T$*, and each initial price* p.

(d) $L_t(p)$ *and* $U_t(p)$ *are non-increasing in* p *for each* $t = 1, \ldots, T$.

Note that the upper and lower barriers $L_t(\cdot), U_t(\cdot), t = 1 \ldots, T$ are now functions of the price in the beginning of the period, unlike the case where inventory carry-overs and bi-directional price changes are allowed, in which case these barriers were functions of the inventory level in the beginning of the period.

Remark 4 The model can be extended to treat non-linear demand functions by assuming that $p\mathbb{E}d_t(p, \epsilon_t)$ is concave in p and that $G_t(y, p)$ is jointly concave in (y, p), for all $t = 1, \ldots, T$. The first condition is easily satisfied for a broad family of demand functions. For a discussion of conditions that ensure the joint concavity of $G_t(\cdot, \cdot)$ see Federgruen and Heching (1999).

5 Discussion and Qualitative Insights

5.1 An illustrative example of the Joint pricing-inventory-capacity model

A two-period problem with quadratic holding costs. To illustrate the relationship highlighted in Theorem 2, we analyze a two-period problem (one period in which a decision is being made and a terminal period). The demand in period $t = 1, 2$ is given by $d_t(p_t, \epsilon_t) = a_t - b_t p_t + \epsilon_t$ and the inventory holding cost is given by $h_t(x) = hx^2$. Thus we get that $G_t(y, p) = h\left[\sigma_t^2 + (y - a_t + b_t p_t)^2\right]$ and $\gamma_t(y, p) = p(a_t - b_t p) - c_t y - h\sigma_t^2 - h(y - a_t + b_t p)^2$, where $\sigma_t^2 = Var(\epsilon_t)$. Note that $f_{T+1}(x, z) = kz - hx^2$. It is easy to show that

$$\widehat{p_T} = \frac{a_T}{2b_T} + \frac{c_T}{2}, \quad \text{and} \quad \widehat{y}_T = \frac{a_T}{2} - \frac{c_T}{2}\left[b_T + \frac{1}{(\alpha+1)h}\right]. \tag{8}$$

Put $\phi \equiv b_T + ((\alpha + 1)h)^{-1}$. Then the optimal pricing-inventory policy (given capacity z) can be described as follows:

- if $x < \widehat{y}_T < x + z$, order up to $\widehat{y}_T$ and set price to $\widehat{p}_T$;
- if $\widehat{y}_T > x + z$, order z units and set price to $\widehat{p}(x+z) = a_t\left(1/2 + \phi/2b_T\right)/\phi - (x+z)/\phi$;
- if $x > \widehat{y}_T$, order no more units and set price to $\widehat{p}(x) = a_t\left(1 + \phi/b_T\right)/\phi - x/\phi$.

One can observe that given the capacity level (after adjustment), z, the target inventory level at the beginning of the period is independent of of the inventory level at the end of the previous period, x if $x < a_T/2 - c_T\phi/2 < x + z$. Otherwise, keeping the the capacity level fixed, the target inventory level is increasing linearly in x. Note, that changing x, may affect the optimal capacity level, and thus, indirectly affect the optimal inventory level. We next study the impact of the optimal capacity on the optimal ordering policy. It is easy to see that unless $a_T/2 - c_T\phi/2 > x + z$, the optimal target inventory level is independent of z.

In order to find the optimal capacity policy we need to compute the boundary functions $L_T(x)$ and $U_T(x)$. It is straightforward to show that

$$\begin{aligned} L_T(x) &= \frac{K}{b_T/\phi^2 + 2\left(1 - b_T/\phi\right)^2(\alpha+1)h} - x + M \\ U_T(x) &= \frac{k}{b_T/\phi^2 + 2\left(1 - b_T/\phi\right)^2(\alpha+1)h} - x + M, \end{aligned}$$

where M is a constant that depends explicitly on the problem parameters.

Thus, the width of the inactivity band can be computed in closed form and we observe that, as anticipated, the inactivity region increases with the difference between the cost of increasing capacity, and the price for sold capacity. Moreover, we observe that as the initial inventory level increases, both the upper level and the lower level of the inactivity region decrease, and thus as the inventory level increases, the optimal capacity level (weakly) decreases.

Note that if the initial capacity level $z_0 \in (L_t(x), U_t(x))$, the capacity level is independent of the initial inventory. If this is indeed the case, the optimal inventory level depends on x only directly, as was discussed above. Thus, an increases in the initial inventory increases the target-inventory level unless $x < a_T/2 - c_T\phi/2 < x + z_0$. Note that in this region, the target inventory level does depend on the initial capacity level, and it increases with the initial capacity level. In all other regions, the target inventory level does not depend on the initial capacity level, but may depend on the target capacity level.

If, on the other hand, $z_0 > U_t(x)$, then $z = U_t(x) \equiv \widehat{U}_t - x$. Note that, then, an increase in x, will decrease z, yet the sum of the two will remain constant. Since the target inventory level depends only on the $x + z$, the target inventory level is independent of x, once x is above a level such that $z_0 > U_t(x)$.

We observe that for fixed b_T, as the holding cost h decreases to zero, the inactivity region shrinks. Thus, capacity adjustments are always made if holding costs are negligible. At the same time, for any given value of h, if the price sensitivity b_T decreases to zero the inactivity region will remain proportional to the holding cost. Thus, the higher the holding cost, the *less* likely that capacity adjustments will be made. To summarize: as the holding cost increases or price sensitivity decreases, the value of adjusting capacity decreases. As mentioned above, the inactivity region also depends on the difference between the cost of purchasing capacity and its selling value. In many firms, capacity changes are fairly costly. In these cases the difference between the costs will increase the size of the inactivity region, thus allowing the firms to only modify price and inventory levels, while keeping the capacity level unchanged.

Numerical illustration. Consider a firm that produces and sells a product during three periods, and a fourth period being the terminal one in which the firm sells off its capacity. The firm starts off with no capacity and zero inventory. Demand is anticipated to be low in the first period, increase during the middle period, and then return to its initial level in the final period. To encode this using our model parameters, we put $a_1 = a_3 = 5$ and $a_2 = 10$ in the demand function. We set $b_1 = b_3 = 1$ and $b_2 = 2$. For purposes of this example, we take the error term ϵ_t to follow a Poisson distribution with mean 1, independent for each period $t = 1, 2, 3$. The firm's variable cost of production is $c_1 = c_2 = c_3 = 1$. We set the holding cost to $h(x) = h^+ \max(x, 0) - h^- \min(x, 0)$ where $h_t^- = 1.9$, and $h_t^+ = 1.5$ for $t = 1, 2, 3$. The discount factor is $\alpha = 0.9$.

Figure 1 depicts the target inventory levels and optimal capacity levels for different levels of initial inventory. First, one can observe that in this case the optimal inventory level is a non-decreasing function of the initial inventory level, and that capacity level is a non-increasing function of the initial inventory. Note that the crosses line depicts the sum of the initial inventory and the optimal capacity level. This allows us to identify three production regions. As long as the initial inventory x is below 2, there is limited capacity and the firm sets the target inventory to $x + z$. When the inventory is the interval $2 \leq x \leq 5$, the optimal inventory level is set to $\hat{y}$, which is

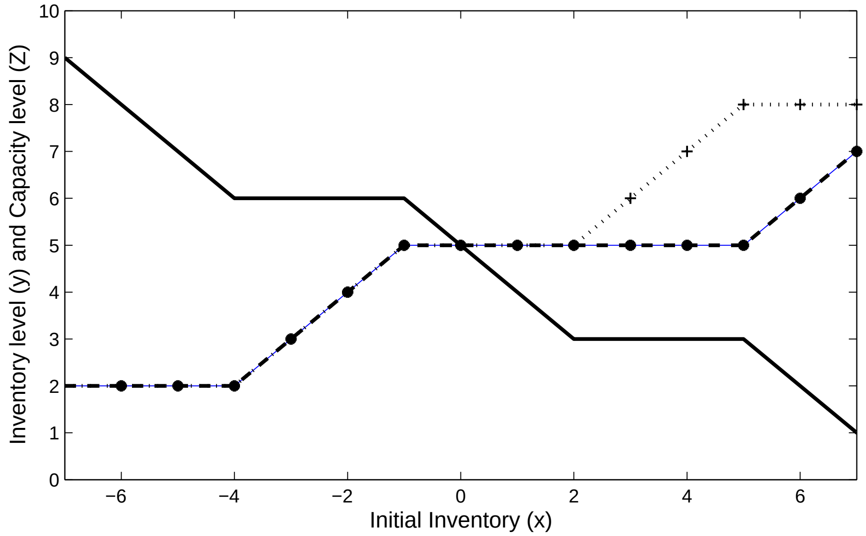

Figure 1: **Optimal capacity and inventory levels as a function of initial inventory.** The solid line depicts the optimal capacity level, the starred line depicts the optimal inventory level, and the crossed line depicts the sum of the initial inventory and the optimal capacity level.

independent of x and z. When the initial inventory level is greater than 5, the target inventory is identical to the initial inventory, i.e. $\hat{y} = x$. The capacity policy is as follows: as long as the initial inventory is below 6, the capacity is set to the upper limit, which decreases with the initial inventory. Once the initial inventory reaches -4, the capacity level reaches the inactivity region, and the capacity remains unchanged until the inventory level reaches -1. Note that due to the change in the production region, the slope of the capacity function changes again around inventory levels 2 and 5. One can also observe that when z is below the lower limit (for example, when $-7 \leq x \leq -4$), $x + z$ is kept constant, and thus the optimal inventory is independent of either, as predicted by the analysis of the two-period model.

5.2 Illustrative examples of the joint capacity-pricing problem

Two period problem with quadratic holding cost. We again, analyze the two period model with quadratic cost in order to gain insights into the structure of the solution of the joint capacitypricig model with no carry overs. One can show that, if $p < a_T/2b_T + c_T/2$, then

$$F(z,p) = \begin{cases} (p - c_T)(a_T - b_T p) + c_T^2/2h - h\sigma_T^2 + \alpha k z & \text{if} \quad z > A_T, \\ p(a_T - b_T p) - c_T z - h\sigma_T^2 - h(z - a_T + b_T p)^2 + \alpha k z & \textit{otherwise}, \end{cases}$$

where $A_T = a_T/2 - c_T/2\,(1/h + b_T)$. We observe that once the current price is below this threshold level, it will remain at its current level. Note that if $z > A_T$, the firm's optimal capacity level is A_T, while if it is below we have to examine the upper and lower boundary functions. We observe that $L_T(p) = a_T - b_T p - (c_T - \alpha k + K - hc)/2h$, and $U_T(p) = a_T - b_T p - (c_T - \alpha k + k - hc)/2h$. The inactivity region is given by $((K-k)(1-\alpha))/2h$. Note that here (i.e., when $p < a_T/2b_T + c_T/2$), the capacity inactivity region is independent of price. Since the firm cannot use a price lever anymore, the decision whether to use the other two decision variables depends only on the ratio between the capacity cost difference $(K - k)$ and the holding cost (h). If this ratio is "high" (i.e., cost of adjusting capacity is high relative to the holding cost), we expect the firm to restrict use to inventory in order to meet variability in demand.

In the region in which the initial price $p > a_T/2b_T + c_T/2$, we have that if z is below a certain level, and $K \gg k$, then both price and capacity are kept fixed. Since capacity is below the level A_T, the firm cannot further reduce price without incurring excessive shortage costs, and thus it will keep the price fixed. Capacity cannot change since related costs are too high. In this situation, inventory is essentially the only useful lever.

A three period example illustrating the relationship between price markdown and capacity investment decisions. We again, consider a firm that produces and sells a product during three periods; the fourth being the terminal period in which the firm sells off its capacity. The firm starts off with no capacity and zero inventory. Demand is anticipated to be low in the first period, increase during the middle period, and then return to its initial level in the final period. To encode this using our model parameters, we put $a_1 = a_3 = 8$ and $a_2 = 10$ in the demand function. We set $b_t \equiv 1$, for $t = 1, 2, 3$. For purposes of this example, we take the error term ϵ_t to follow a Poisson distribution with mean 1, independent for each period $t = 1, 2, 3$. The firm's variable cost of production is $c_1 = c_2 = c_3 = 1$. To reflect the fact that the firm cannot carry inventory and is thus inclined to resolve any excess demand within the period, we set $h_t^- = 3$, $h_t^+ = 1.5$ for $t = 1, 2, 3$. The discount factor is $\alpha = 0.9$.

Figure 2 is concerned with three capacity investment irreversibility values: $K/k = 2, 6, 8$ (dotted, dashed, and solid lines, respectively). For each of these ratios we computed the optimal policy that maximizes the average profit over the finite horizon using standard dynamic programming. The figure depicts the optimal policy under a "typical" path which is obtained by setting the noise variable ϵ_t to its mean value. We observe that as long as the ratio is lower than 6, the firm essentially uses the same pricing scheme, charging \$5, \$4 and \$3, and lowers the level of acquired capacity. However, once the ratio increases above 8, the firm utilizes a different pricing scheme, charging \$6, \$5 and \$4 while lowering the capacity level it purchases. Since the firm can foresee that it will not be able to absorb demand using a high level of production (and capacity), and since it cannot increase its price in the middle of the product life cycle, it elects to charge a relatively high price in the first period even though the demand in this period is not greater than other periods. The firm then decreases prices in each subsequent period. In terms of capacity planning: the firm always invests in capacity in the first period, may invest in the second period (to accommodate the peak-demand anticipated in period 2), and "stays-put" in the third period (even tough demand is expected to be lower than in the second period). The above may be viewed as an illustration of complementarity between price and capacity. To wit, the first period commences with a relatively high price, and a relatively low level of capacity, leading to a high utilization of this capacity. In the second period, the firm increases capacity level to its maximum, and lowers price to increase demand. In the third period, since the firm already has acquired a significant level of capacity, it will again lower its price to allow for full utilization of the capacity, even though the expected demand is lower than that in the middle period.

5.3 Additional discussion

Price and capacity as strategic substitutes. The fact that price and capacity are strategic substitutes is equivalent to a complementarity relation between the level of capacity investment and the level of price decrease (relative to the maximum price $\overline{p}$). The notion of complementarity that we are referring to is due to Edgeworth, according to which activities are considered *complements* if increasing the level of any one of them results in an increase in the return of engaging more in the other; see Milgrom and Roberts (1990, 1995) that summarize the principal results of the theory of supermodular optimization which underlies the notion of complementarity. They describe supermodularity as a way to formalize the intuitive idea of synergistic effects. In our example, a

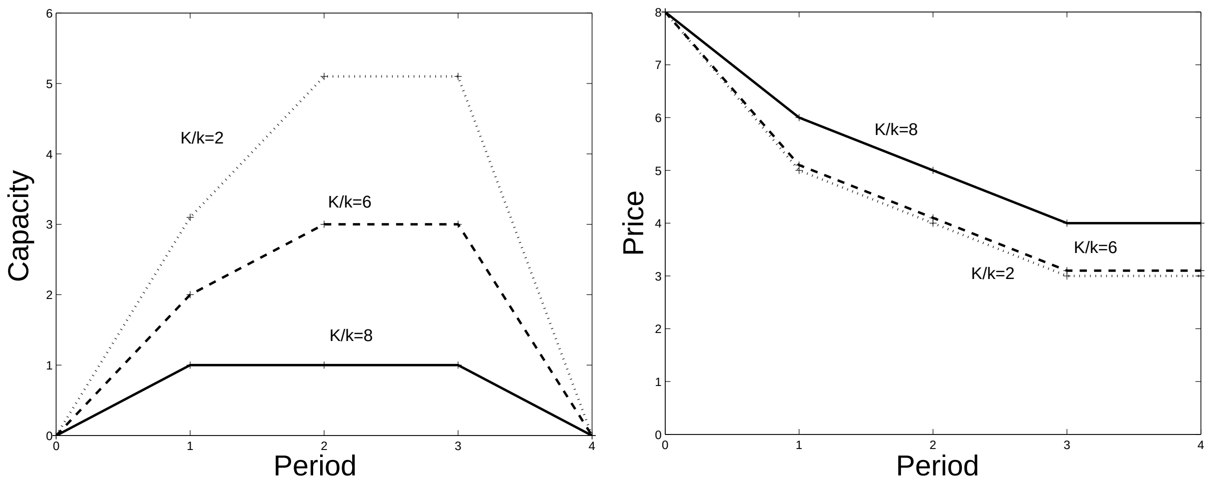

Figure 2: **Optimal capacity and price levels for different capacity investment irreversibility ratios K/k.** The left (right) panel depicts the optimal capacity (pricing) level over the planning horizon for a "typical" demand path.

firm that coordinates sales planning and capacity investment has the potential to increase its profits on the basis of the observed complementarity.

Benefits of capacity flexibility in the presence of restrictions on price changes (Table 1). To explore further the importance of capacity flexibility, we compare the expected profits of a firm in two configurations: (i) the firm sets its capacity level at the beginning of the life-cycle; and (ii) the firm is capable of adjusting its capacity periodically. For each of these settings we compute the optimal average profit function when beginning with zero inventory on-hand and zero capacity, using standard dynamic programming. In both cases, the firm is only allowed to markdown its prices, and cannot carry inventories from period to period.

We observe in Table 1 that when the cost of adjusting capacity (i.e., the ratio K/k) is low, the value added from capacity flexibility is negligible. In particular, the firm can sell the capacity at the end of the life-cycle without incurring any losses, and thus will probably invest in the maximum required capacity.

Future research. In this note we have made several simplifying assumptions for purposes of mathematical tractability and facilitating the analysis Among these, the most desirable extensions include relaxing the stipulation of zero lead time for both procurement and capacity adjustment, as well as verifying the applicability of the main results for more general demand models. Fixed

K/k	1	2	6	8
Fixed capacity	44.5365	40.5018	28.0212	26.0132
Flexible capacity	44.8183	40.7027	30.0208	29.70307
Percentage increase	0.63 %	0.50 %	7.14 %	14.18 %

Table 1: **Expected Profits: The value of capacity flexibility.** The first row depicts expected profits when capacity level is set at the beginning of the horizon. The second row depicts expected profits when capacity can be adjusted periodically. The third row depicts percentage improvement due to flexibility.

capacity investment costs will complicate the analysis in a significant manner, and lead to potentially much more complex policy structures; some indication of this can be seen in Duenyas and Ye (2007) in the more restricted setting of joint capacity and inventory control.

References

Angelus, A. and E. Porteus (2002) "Simultaneous capacity and production management of short-life-cycle, produce-to-stock goods under stochastic demand," *Management Science* 48, 339-413

Aviv Y. and A. Federgruen (1997) "Stochastic inventory models with limited production capacity and varying parameters". *Probab. Engrg. Inform. Sci.* 11 107-135.

Boyaci, T. and O. Ozer (2007) "Information acquisition via pricing and advance selling for capacity planning: When to stop and act?" *Working paper, Columbia University.*

Boyd, S. and L. Vandenberghe, (2004) *Convex Optimization*, Cambridge University Press, MA.

Chan, L. M. A., Z. J. Max Shen, D. Simchi Levi, J. L. Swann (2004) "Coordination of pricing and inventory decisions: a survey and classifications," in *Handbook of Quantitative Supply Chain Analysis: Modelling in E-Business Era*, Kluwer Academic Publishers, Boston, MA.

Chen, X. and D. Simchi-Levi, (2002) "Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: the finite horizon case," *Operations Research*, 52, 887-896.

Duenyas, I. and Q. Ye, (2007) "Optimal Capacity Investment Decisions with Two-sided Fixed

Capacity Adjustment Costs," *Operations Research* 55, 272-283

Eberly, J. C. and J. A. Van Mieghem (1997) "Multi-factor dynamic investment under uncertainty," *Journal Of Economic Theory* 75, 345-387.

Elmaghraby W. and P. Keskinocak (2003) "Dynamic pricing in the presence of inventory considerations: research overview, current practices and future directions," *Management Science* 49, 1287-1309.

Federguren, A. and A. Heching (1999) "Combined pricing and inventory control under uncertainty," *Operations Research* 47, 454-475.

Federgruen, A. and P. Zipkin (1986) "An inventory model with limited production capacity and uncertain demands II. The discounted-cost criterion", *Math. Oper. Res.* 11, 208-215

Ferguson, M., M. E. Ketzenberg and R. Kuik (2006) "Purchasing Speculative Inventory for Price Sensitive Demand", working paper.

Gaimon, C. (1988) "Simultaneous and dynamic price, production, inventory and capacity decisions," *European Journal of Operational Research* 35, 426-441.

Heyman, D. and M. Sobel (1984), *Stochastic Models in Operations Research, Volume II*, Mc-Graw-Hill, New-York.

Kunreuther, H. and L. Schrage (1973) "Joint pricing and inventory decisions for non-seasonal items," *Econometrica* 39, 173-175.

Li, L. D. (1988) "A stochastic theory of the firm," *Mathematics of Operations Research* 13, 447-466.

Maccini, L. J. (1984) "The interrelationship between price and output decisions and investments," *Journal of Monetary Economics* 13, 41-65.

Milgrom, P. and J. Roberts (1990) "The economics of modern manufacturing: technology, strategy, and organization," *American Economic Review* 80, 511-28.

Milgrom, P. and J. Roberts (1995). "Complementarities and fit strategy, structure, and organizational change in manufacturing," *Journal of Accounting and Economics* 19, 179-208.

Ozer O. and W. Wei (2004) "Inventory control with limited capacity and advance demand information. Operations Research" 52(6), 988-1000.

Rao, M.R. (1976) "Optimal capacity expansion with inventory," *Operations Research* 24, 291-300.

Royden, H. (1988) *Real Analysis*, Prentice Hall, NJ.

Thomas, L. J. (1974) "Price and production decisions with random demand," *Operations Research* 22, 513-518.

Topkis, D. (1978) "Minimizing a submodular function on a lattice." *Operations Research* 26, 305-321.

Van Mieghem, J. A. and M. Dada (1999) "Price versus production postponement: capacity and competition," *Management Science* 45, 1631-1649.

6 Proofs of the Main Results

All notations in this appendix follows that in the paper. The proofs of auxiliary results are deferred to Appendix B.

Proof of Theorem 1: Fix x. We first show that the solution can be expressed in terms of two function $L_t(x)$ and $U_t(x)$ that satisfy the three conditions in Definition 1, and then solve

$$\max_{z'\geq 0} f(x,z) = \max_{z'\geq 0} \left\{\Gamma_t(x,z') - C(z'-z) - h_c z'\right\}.$$

To this end, we need the following result whose proof is deferred to Appendix B.

Lemma 1 *$f_t(x,z)$ is jointly concave for all $t = 1,\ldots,T$.*

Define

$$\begin{aligned} L_t(x) &= \arg\max_{z'\geq 0}\{\Gamma_t(x,z') - K(z'-z) - h_c z'\} \\ U_t(x) &= \arg\max_{z'\geq 0}\{\Gamma_t(x,z') - k(z'-z) - h_c z'\} \end{aligned}$$

Let $\nabla_z f_t(x,z)$ denote the subgradient of $f_t(x,z)$ at the point (x,z), i.e., $f_t(x,v) \leq f_t(x,z) + \nabla_z f_t(x,z)(v-z), \forall v$

Lemma 2 *(Royden [38,p. 113]) For all $t = 1,\ldots,T$, $f_t(x,z)$ is continuous and has non-increasing left and right partial derivatives with respect to z which are equal almost everywhere.*

Thus, the subgradient $\nabla_z f_t(x,z)$ is unique and equal to the gradient of $f_t(x,z)$, except on a set of Lebesgue measure zero. Since $f_t(x,z)$ is concave, the first order (sub)differential conditions are sufficient. Thus, we can write

$$L_t(x) = \begin{cases} 0 & \text{if} \quad \nabla_z \Gamma_t(x, z')|_{z'=0} < K - h_c, \\ \sup\{z' : \nabla_z \Gamma(x, z') \geq K - h_c\} & \textit{otherwise}, \end{cases}$$

$$U_t(x) = \begin{cases} \infty & \text{if} \quad \nabla_z \Gamma_t(x, z') > K - h_c \forall z' > 0, \\ \inf\{z' : \nabla_z \Gamma(x, z') \leq K - h_c\} & \textit{otherwise}. \end{cases}$$

Since both $L_t(x)$ and $U_t(x)$ are independent of z, we can partition the space into the following three regions: (i) $z < L_t(x)$; (ii) $L_t(z) \leq z \leq U_t(x)$; and (iii) $z > U_t(x)$. In each of these regions we will compare the three possible decisions: investing, disinvesting and staying put.

Region (i): if the firm decides to invest, by the definition of $L_t(z)$, it is the optimal value. If the firm decides to disinvest, then, since $U_t(x) > L_t(x) > z$, it is better to stay put. However, staying put is inferior to investing in this region, since $L_t(x) > z$ (if staying put were better, then $L_t(x)$ would equal z).

Region (iii): if the firm decides to disinvest, by the definition of $U_t(z)$, it is be the optimal value. If the firm decides to invest, then, since $z > U_t(x) > L_t(x)$, it is is better to stay put. However, staying put is inferior to disinvesting in this region, since $U_t(x) < z$ (if staying put were better, then $U_t(x)$ would equal z.)

Region (ii): if the firm decides to invest, since $z \geq L_t(x)$, it is better to stay put. If the firm decides to disinvest, since $z \leq U_t(x)$, it is better off staying put as well. Therefore it is optimal in this region to stay put.

Thus, we have established the existence of two functions that satisfy the conditions of Definition 1, which completes the proof. ■

Proof of Theorem 2: Fix $t \in \{1, \ldots, T\}$. We will begin by analyzing the relationship between the optimal inventory level after ordering, y_t, and the starting inventory level x_t.

Lemma 3 *If $x \leq \widehat{y}_t(x, z) \leq x + z$, it is optimal to order up to the base-stock level $\widehat{y}_t(x, z)$ and to charge the list-price $\widehat{p}_t(x, z)$; if $x > \widehat{y}_t(x, z)$, it is optimal not to order, and if $\widehat{y}_t(x, z) > x + z$, it is optimal to order z units.*

To prove that a base-stock list-price policy is optimal, it suffices to show that the optimal price to be selected in any given period is non-increasing in the prevailing inventory level. In other words, under higher starting inventory levels, a price is selected that is no larger than the optimal price under a lower starting inventory. Monotonicity of the optimal price level, p_t, depends on the submodularity of the function $a_t(y,p,z)$.

We would like first to show that $a_t(y,p,z)$ is submodular in (y,p). Since the sum of submodular functions is submodular, it suffices to establish submodularity of each of the terms $\gamma_t(y,p)$ and $\mathbb{E}f_{t+1}(y-d_t(p,\epsilon_t),z)$. To show submodularity of $\gamma_t(y,p)$, it suffices, by definition, to show supermodularity of $G_t(y,p)$. Fix ϵ_t. Then, the function $h_t(y-d_t(p,\epsilon_t))$ is supermodular in y and p by the following lemma.

Lemma 4 *If $g(\cdot)$ is a convex function and $h(\cdot)$ in a non-decreasing function, then $g(u+h(v))$ is supermodular in u,v.*

The stated supermodularity therefore applies to the function $G_t(y,p)=\mathbb{E}h_t(y-d_t(p,\epsilon_t))$, and thus to the function $\gamma_t(y,p)$. Since $f_t(x,z)$ is concave in x, by Lemma 4, for fixed ϵ_t, $f_{t+1}(y-d_t(p,\epsilon_t),z)$ is submodular in y and p. Taking expectation preserves this property and hence completes the proof of part (b), i.e., the submodularity of $a_t(y,p,z)$ with respect to y and p.

The decision problem in period t, given capacity z_t (after adjustments) can be viewed as consisting of two stages. In the first stage, the inventory after ordering y_t is chosen, and in the second stage the corresponding price p_t is set. The second stage thus has $\mathcal{S}=\mathbb{R}$ as its state space, and $\mathcal{A}=[\underline{p},\overline{p}]$ as the set of feasible (price) actions. Since $a_t(y,p,z)$ is strictly concave in (y,p), and the feasible set is convex, the optimal price p_t is unique. Since $a_t(y,p,z)$ is submodular, it follows from Theorem 8-4 in Heyman and Sobel (1984) that the optimal price p_t is nonincreasing in the state y_t, and hence in x. The proof is complete. ■

Proof of Theorem 3: $f^M_{T+1}(z,p)=kz$ is clearly jointly concave and submodular in z,p. We assume that $f^M_{t+1}(z,p)$ is submodular and jointly concave in (z,p), and prove that this implies that for $t\in\{1,\ldots,T\}$, $f_t(z,p)$ is submodular and jointly concave in (z,p). Note that

$$\gamma_t(y,p)=p\mathbb{E}d_t(p,\epsilon_t)-c_ty-G_t(y,p)$$

was shown in the proof of Theorem 2 to be jointly concave and submodular in y,p, thus $\gamma_t(y,p)$ is supermodular and jointly concave in $(-y,p)$. Define

$$g_t(z,p)=\max\{\gamma_t(y,p)\ :\ y\leq z\}=\max\{\gamma_t(y,p)\ :\ -y\geq -z\}$$

We now use the following lemma.

Lemma 5 *if $g(y,v)$ is jointly concave and supermodular in (y,v), then* $G(y,u) = \max\Big\{g(y,v)\ :\ v \geq u, \underline{v} \leq v \leq \overline{v}\Big\}$ *is jointly concave and supermodular in (y,u).*

Consequently, $g_t(z,p)$ is jointly concave and supermodular in $(-z,p)$, and therefore submodular and jointly concave in (z,p). Let

$$\Gamma_t(z,p) := \alpha \mathbb{E} f^M_{t+1}(z,p) + g_t(z,p).$$

By the induction assumption, $\Gamma_t(z,p)$ is submodular and jointly concave in (z,p), and thus jointly concave and supermodular in $(z,-p)$. Define

$$F(z,p) = \max\left\{\Gamma_t(z,p')\ :\ p' < p, \underline{p} \leq p' \leq \overline{p}\right\} = \max\left\{\Gamma_t(z,p')\ :\ -p' > -p, -\underline{p} \geq -p' \geq -\overline{p}\right\}$$

By Lemma 5, $F(z,p)$ is supermodular and jointly concave in $(z,-p)$, and thus jointly concave and submodular in (z,p). Let

$$\widehat{F}_t(z^B, z^A, p) = F_t(z^A, p) - h_c z^A - C(z^A - z^B)$$

where z^A and z^B are the inventory levels *after* and *before* adjustment (investment / disinvestment), respectively. Now, $C(\cdot)$ is convex, therefore by Lemma 4, $C(z^A - z^B)$ is submodular in (z^A, z^B), and therefore $-C(z^A - z^B)$ is supermodular and jointly concave in (z^A, z^B). Since this function is independent of p, it is (trivially) jointly concave and supermodular in $(z^A, z^B, -p)$. $F(z^A, p)$ is supermodular and jointly concave in (z^A, p), and since it is independent of z^B, we can conclude using the same reasoning that $\widehat{F}(z^A, z^B, -p)$ is supermodular and jointly concave in $(z^A, z^B, -p)$. We then write

$$f^M_t(z,-p) = \max\Big\{\widehat{F}_t(z,z',-p)\ :\ z' \geq 0\Big\}.$$

Since $\{z' \geq 0\}$ is a lattice and a convex set, $f^M_t(z,-p)$ is supermodular in $(z,-p)$, and its joint concavity and supermodularity are immediate from the the preservation under maximization theorems (Theorem 4.3 of Topkis (1978), and Proposition B-4 of Heyman and Sobel (1984), respectively). Thus, f^M_t is jointly concave and submodular in (z,p), which completes the induction proof and the proofs of parts (a) and (b).

For the proof of part (c) define

$$L_t(p) = \begin{cases} 0 & \text{if} \quad \nabla_z F_t(p,z')|_{z'=0} < K - h_c, \\ \sup\{z' : \nabla_z F_t(p,z) \geq K - h_c\} & \textit{otherwise} \end{cases}$$

$$U_t(p) = \begin{cases} \infty & \text{if} \quad \nabla_z F_t(p, z') > K - h_c, \text{ for all } z' > 0, \\ \inf\{z' : \nabla_z F(p, z') \le K - h_c\} & \textit{otherwise.} \end{cases}$$

Now repeat the arguments given in the proof of Theorem 1 to conclude that the optimal capacity policy is a target interval policy with $L_t(p)$ and $U_t(p)$ as its barrier functions.

Since $L_t(p)$ and $U_t(p)$ are maximizers of submodular functions in (z, p), it follows (again, from Theorem 8-4 in Heyman and Sobel (1984)) that both $L_t(p)$ and $U_t(p)$ are non-decreasing in p. This completes the proof. ■

7 Proof of Auxiliary Results

Proof of Lemma 1: $f_{T+1}(x, z) = kz - h_{T+1}(x)$ is concave in (x, z) since h_{T+1} is convex in x. Fix $t \in \{1, \ldots, T\}$, and suppose that $f_{t+1}(x, z)$ is concave. We shall show that f_t is concave. We first prove that $a_t(y, p, z)$ is jointly concave in (y, p, z). We will prove the concavity in each of its two elements. Fix ϵ_t. Since $d_t(p_t, \epsilon_t)$ is linear in p, $y - d_t(p, \epsilon_t)$ is an affine function of (y, p). By the concavity assumption for $f_{t+1}(x, z)$, and since affine mappings preserve concavity (see Boyd and Vandenberghe (2004) section 3.2.2), $f_{t+1}(y - d_t(p, \epsilon_t))$ is jointly concave. (Note that concavity is preserved under expectation with respect to the random variable ϵ_t.) We now establish that $\gamma_t(y, p) = p\mathbb{E}d_t(p, \epsilon_t) - c_t y - G_t(y, p)$ is jointly concave. First, note that $G_t(y, p)$ is jointly convex. Thus, we are only required to show that $p\mathbb{E}d_t(p, \epsilon_t)$ is jointly concave in (y, p). Since $d_t(p, \epsilon_t)$ is linear and decreasing in p, it is straightforward that if we fix ϵ_t, $pd_t(p, \epsilon_t)$ is concave in p. Again, concavity is preserved under expectation with respect to ϵ_t. Now, note that the set

$$\{(y, p, z, x) \ : \ x \ge 0, z \ge 0, x \le y \le x + z, \underline{p} \le p \le \overline{p}\} \tag{9}$$

is convex. Thus, by the concavity preservation under maximization theorem (see Proposition B-4, Heyman and Sobel (1984)), $\Gamma(x, z)$ is jointly concave in (x, z). Since $C(\cdot)$ is convex using again, the concavity preservation under maximization theorem, $f_t(x, z)$ is jointly concave, which completes the induction proof. ■

Proof of Lemma 3: Fix $t \in \{1, \ldots, T\}$ and $z \in \mathbb{R}$. Since $a_t(y, p, z)$ is jointly concave in (y, p), $(\widehat{y}_t(x, z), \widehat{p}_t(x, z))$ is the optimal decision pair when $x \le \widehat{y}_t(x, z) \le x + z$, i.e., in this region it is optimal to order up to the base stock level $\widehat{y}_t(x, z)$ and to charge the list price $\widehat{p}_t(x, z)$ if $x \le \widehat{y}_t(x, z) \le x + z$. Similarly, it is optimal to choose $y_t = x$ if $x > \widehat{y}_t(x, z)$, i.e., not to produce.

Now, if $x > \widehat{y}_t(x,z)$, and a decision pair (y,p') is chosen with $y > x$, then for the pair (x,p'') on the line segment connecting $(\widehat{y}_t(x,z),\widehat{p}_t(x,z))$ with (y,p'), $a_t(x,p'',z) \geq a_t(y,p',z)$, using the joint concavity of $a_t(y,p,z)$. Using the same logic, we can show that if $\widehat{y}_t(x,z) > x+z$, it is optimal to set $y_t = x+z$, namely, to produce the maximum possible amount. In particular, if $\widehat{y}_t(x,z) > x_t$ and a decision pair (y,p') is chosen with $y < x+z$, then for the pair $(x+z,p'')$ on the line segment connecting $(\widehat{y}_t(x,z),\widehat{p}_t(x,z))$ with (y,p') we have that, $a_t(x+z,p'',z) \geq a_t(y,p',z)$, using the joint concavity of $a_t(y,p,z)$. We conclude that y_t is nondecreasing in x. This completes the proof. ■

Proof of Lemma 4: Assume without loss of generality that $u_1 > u_2$, and $v_1 > v_2$. Then,

$$\begin{aligned} g(u_1+h(v_1)) - g(u_2+h(v_1)) &= g(u_2+h(v1)+(u_1-u_2)) - g(u_2+h(v_1)) \\ &\geq g(u_2+h(v_2)+(u_1-u_2)) - g(u_2+h(v_2)) \\ &= g(u_1+h(v_2)) - g(u_2+h(v_2)), \end{aligned}$$

where the inequality follows from the convexity of g and the fact that h is increasing. ■

Proof of Lemma 5: Let $v^*(y)$ denote the smallest maximizer of $g(y,\cdot)$ on $[\underline{v},\overline{v}]$ (clearly the function has a maximizer on a bounded interval). Since $g(y,v)$ is concave in v, for a given y

$$G(y,u) = \begin{cases} g(y,v^*(y_1)) & \text{if} \quad u \leq v^*(y) \\ g(y,u) & \text{if} \quad v^*(y) \leq u. \end{cases} \tag{10}$$

Since $g(\cdot,\cdot)$ is supermodular, $v^*(y)$ is nondecreasing in y. Therefore, if $y_1 > y_2$, then $v^*(y_1) \geq v^*(y_2)$. Thus, we can write

$$G(y_1,u) - G(y_2,u) = \begin{cases} g(y_1,v^*(y_1)) - g(y_2,v^*(y_2)) & \text{if} \quad u < v^*(y_2) \leq v^*(y_1) \\ g(y_1,v^*(y_1)) - g(y_2,u) & \text{if} \quad v^*(y_2) \leq u \leq v^*(y_1) \\ g(y_1,u) - g(y_2,u) & \text{if} \quad v^*(y_1) \leq u. \end{cases} \tag{11}$$

If $u \leq v^*(y_2)$ then the function is constant. Since for all $u \geq v^*(y_2)$, $g(y_2,v^*(y_2)) \geq g(y_2,u)$ by concavity, thus the function $g(y_1,v^*(y_1)) - f(y_2,u)$ is non-decreasing. For $u > v^*(y_1)$, $G(y_1,u) - G(y_2,u) = g(y_1,u) - g(y_2,u)$ has increasing differences in view of g having increasing differences. Joint concavity of $G(\cdot,\cdot)$ is immediate from the concavity preservation under maximization theorem (see Proposition B-4, Heyman and Sobel (1984)). This completes the proof. ■