Optimal Stopping of a Random Sequence with Unknown Distribution*

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March 11, 2019

Abstract

The subject of this paper is the problem of optimal stopping of a sequence of i.i.d. random variables with unknown distribution. We propose a stopping rule that is based on relative ranks and study its performance as measured by the maximal relative regret over suitable nonparametric classes of distributions. It is shown that the proposed rule is first order asymptotically optimal and nearly rate–optimal in terms of the rate at which the relative regret converges to zero. We also develop a general method for numerical solution of sequential stopping problems with no distributional information, and use it in order to implement the proposed stopping rule. Some numerical experiments are presented as well, illustrating performance of the proposed stopping rule.

Keywords: optimal stopping, secretary problems, extreme–value distribution, relative ranks, full information, no information, minimax regret.

2000 AMS Subject Classification:

1 Introduction

Background and problem formulation. Let $X_1, \ldots, X_n$ be integrable, independent identically distributed random variables with common continuous distribution G. Denote $\mathscr{X}_t := \sigma(X_1, \ldots, X_t)$ the σ–field generated by $X_1, \ldots, X_t$, and $\mathscr{X} = (\mathscr{X}_t, 1 \le t \le n)$ the corresponding filtration. Let $\mathscr{T}(\mathscr{X})$ be the class of all stopping times τ with respect to the filtration $\mathscr{X}$, i.e., the class of all integer-valued random variables τ such that event $\{\tau = t\}$ belongs to $\mathscr{X}_t$ for every $1 \le t \le n$. The optimal stopping problem, hereafter referenced as (SP1) for short, is:

(SP1) find the stopping time $\tau^* \in \mathscr{T}(\mathscr{X})$ such that $\mathrm{E}_G(X_\tau)$ is maximized, i.e.,

$$v_n(G) := \mathrm{E}_G(X_{\tau^*}) = \sup_{\tau \in \mathscr{T}(\mathscr{X})} \mathrm{E}_G(X_\tau). \tag{1}$$

Here and in all what follows E_G is the expectation with respect to the probability measure P_G of the observations $X_1, X_2, \ldots, X_n$ when the underlying distribution is G.

*The research was supported by the grants BSF 2010466, ISF 361/15 and NSF CNS-0964170.

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The optimal reward $v_n = v_n(G) = \mathrm{E}_G(X_{\tau^*})$ is given by the well–known recursive representation

$$v_1 = \mathrm{E}_G X, \;\; v_{t+1} = \mathrm{E}_G\{X \vee v_t\}, \;\; t = 1, 2, \ldots, \tag{2}$$

and the optimal stopping time is

$$\tau_* = \min\{1 \le t \le n-1 : X_t > v_{n-t}\}, \tag{3}$$

provided the set in the parentheses is non–empty, and $\tau_* = n$ otherwise; see Chow et al. (1971).

When the distribution G is known, a case that will be referred to as *full information*, the optimal stopping problem (SP1) is solved by backward induction (2)–(3), and asymptotic behavior of the reward sequence $v_n = v_n(G)$ for different distributions was studied in the literature. Moser (1956) showed that if G is the uniform distribution on $[0, 1]$ then $\lim_{n\to\infty} n(1-v_n) = 2$ [see also Gilbert and Mosteller (1966, Section 5a)]. Guttman (1960) considered the problem for the normal distribution. The limiting behavior of $n[1 - G(v_n)]$ under general assumptions on the upper tail of G was studied in Kennedy & Kertz (1991). In particular, it is shown there that $n[1 - G(v_n)]$ converges to a constant whose value is determined by the (extreme value) *domain of attraction* of the distribution G.

The problem of optimal stopping with *partial information* arises when the distribution G is unknown. Here the optimal policy (2)–(3) is not directly applicable, and learning of G should be incorporated in the policy construction. Stopping problems with partial information have been studied to a much lesser extent than their full information counterparts. Stewart (1978) considered problem (SP1) for random variables distributed uniformly on an unknown interval, and proposed a Bayesian stopping procedure. Building on Stewart's work, Samuels (1981) constructed a minimax stopping rule and computed the asymptotic minimax risk in the class of all uniform distributions. A minimax stopping rule for normally distributed random variables with unknown mean and variance was derived in Petruccelli (1985). It was shown there that the proposed rule is asymptotically as good as the optimal stopping rule based on full information in the following sense: the ratio of the expectation of the selected observation to the expectation of the largest order statistic converges to one. Boshuizen & Hill (1992) derived minimax stopping rules for a sequence of independent uniformly bounded random variables when only the means and/or variances are known. There are also works dealing with best–choice problems under partial information; we refer the reader to Petruccelli (1980), Gnedin and Krengel (1996) and references therein.

In this paper we study the optimal stopping problem with partial information within a minimax framework. We adopt *the relative regret* as a measure of the quality of any stopping rule $\hat\tau \in \mathscr{T}(\mathscr{X})$

$$\mathcal{R}_n[\hat\tau; G] := \frac{\mathrm{E}_G[X_{(n)} - X_{\hat\tau}]}{\mathrm{E}_G[X_{(n)}]} \;.$$

If Σ is a class of distribution functions G, then the maximal relative regret of $\hat\tau$ on the class Σ is

$$\mathcal{R}_n[\hat\tau; \Sigma] := \sup_{G\in\Sigma} \mathcal{R}_n[\hat\tau; G],$$

and the minimax regret is defined as

$$\mathcal{R}^*_n[\Sigma] := \inf_{\hat\tau\in\mathscr{T}(\mathscr{X})} \mathcal{R}_n[\hat\tau; \Sigma].$$

We say that the stopping time $\hat\tau_* \in \mathscr{T}(\mathscr{X})$ is *rate–optimal* on Σ if

$$\mathcal{R}_n[\hat\tau_*; \Sigma] \le C_n \mathcal{R}^*_n[\Sigma], \quad \sup_n C_n < \infty.$$

Main contributions. The main contribution of this paper is two–fold.

1. We develop a nearly rate–optimal stopping rule for solution of problem (SP1) when the underlying distribution G is unknown and belongs to sufficiently large nonparametric classes Σ of distribution functions. The proposed stopping rule is based solely on the relative ranks of the observations; in particular, it maximizes the probability of selecting one of the k largest observations with k being a tuning parameter. The latter problem is known in the literature as the Gusein–Zade stopping problem [see Gusein–Zade (1966)]. We characterize classes of distributions for which there exist stopping rules with relative regret tending to zero. In other words, this rule sequentially selects an observation that approaches, in a suitable sense, the maximal observation in the sequence. Concurrently we show that there is a complementary class of distributions for which there is *no stopping rule* that achieves this type of performance. In particular, this dichotomy in first order asymptotic optimality is determined by the domain of attraction of extreme-value distributions. When first order asymptotic optimality is achievable, the more refined asymptotic behavior of the relative regret of the proposed stopping rule is studied over suitably restricted nonparametric classes of distribution functions. It is shown that for a proper choice of the tuning parameter k and depending on the domain of attraction, the relative regret of the proposed rule converges to zero at the rate which is slower by only a $\ln\ln\ln n$ or $\ln n$–factor than the rate achievable in the full information case. In that sense, the proposed stopping rule is nearly rate-optimal.

2. To address the computational challenges and implementation of the proposed stopping rule, we develop a general method for exact numerical solution of stopping problems with no distributional information. The structure of the original Gusein–Zade stopping rule, that seeks to localize to the top k absolute ranks, is relatively straightforward to characterize, yet its implementation for general k and n is hardy tractable. The existing literature suggests exact computations for some specific cases and various approximate solutions [the reader is referred to Section 4 for detailed discussion]. Our proposed numerical solution is based on the fact that every stopping problem with no information can be represented as a stopping problem for a suitably defined sequence of independent (though not identically distributed) random variables. Then for these random variables the optimal rule is always of the single threshold type, and once the distributions of the elements in this constructed sequence are determined, the exact recursive algorithm for calculating the stopping rule is straightforward. Using this numerical procedure we implement our rule and present some results illustrating its performance.

Organization of the paper. The paper is structured as follows. In Section 2 we introduce the proposed stopping rule based on relative ranks; we motivate its construction and present a result on its properties. Section 3 contains the main results of this paper. In Section 4 we develop a general method for numerical solution of stopping problems with no information; this method provides a basis for implementation of the proposed stopping rule. Some numerical results on the stopping rule performance are presented in Section 5. Proofs of all statements are given in the concluding Section 6.

Notation and conventions. In what follows we use the following notation and conventions. As mentioned above, P_G denotes the probability measure of observations $X_1, \ldots, X_n$ when the underlying distribution is G, and E_G is the expectation with respect to P_G. The probability and expectation symbols will not be indexed by G when the distribution of involved random variables does not depend on G.

Recall that the order statistics of the sample $X_1, \ldots, X_n$ are denoted $X_{(1)} \leq X_{(2)} \leq \cdots \leq$

$X_{(n)}$. The relative and absolute ranks of X_t are given by

$$R_t := \sum_{j=1}^{t} \mathbf{1}(X_t \le X_j), \quad A_t := \sum_{j=1}^{n} \mathbf{1}(X_t \le X_j), \quad t = 1, \dots, n \tag{4}$$

respectively. With these definitions the largest observation has the smallest rank. We denote $\mathscr{R}_t = \sigma(R_1, \dots, R_t)$ the σ–field generated by $R_1, \dots, R_t$, and $\mathscr{R} = (\mathscr{R}_t, 1 \le t \le n)$ stands for the corresponding filtration. The class of all stopping rules with respect to a filtration $\mathscr{Y} = (\mathscr{Y}_t, 1 \le t \le n)$ is denoted $\mathscr{T}(\mathscr{Y})$.

2 The Proposed Stopping Rule

In the absence of information on the distribution G that governs observations $X_1, \dots, X_n$, we propose to use stopping procedures based on relative ranks of said observations.

Consider the following auxiliary stopping problem, which will be referred to hereafter as (SP2) for short.

(SP2) Let $k \in \{1, \dots, n\}$ be a fixed integer and define $\tau_k^* \in \mathscr{T}(\mathscr{R})$ as the solution to:

$$w_n(k) := \mathrm{P}\{A_{\tau_k^*} \le k\} = \sup_{\tau \in \mathscr{T}(\mathscr{R})} \mathrm{P}\{A_\tau \le k\}. \tag{5}$$

The problem (SP2) was first considered by Gusein–Zade (1966) who established the existence of the stopping time τ_k^* and characterized the solution to (5). In what follows we propose to use τ_k^* that solves (SP2), with k being a design parameter, to solve the original optimal stopping problem (SP1) under incomplete information. In particular, we will study its performance in said problem as viewed through the lens of the relative regret. Later, in Section 4, we present a recursive procedure that solves (SP2) and computes τ_k^*.

The next result provides a lower bound on the optimal value of the above auxiliary problem.

Proposition 1 *For any $\alpha \in (0,1)$ such that $(1-\alpha)\ln(\frac{1}{1-\alpha}) > \frac{1}{n}$ and any $k \in \{1, \dots, n\}$ one has*

$$\mathrm{P}\big\{A_{\tau_k^*} > k\big\} \le (1-\alpha)^{k/2} + (1-\alpha)^{(k-1)\alpha/4} + \frac{1-\alpha}{1 - e^{-3\alpha^2/32}} \exp\Big\{-\frac{3\alpha^2 k}{32}\Big\}. \tag{6}$$

In particular, if $\alpha = 2/3$ and $n > 7$ then

$$\mathrm{P}\big\{A_{\tau_k^*} > k\big\} \le 11 e^{-k/24}. \tag{7}$$

Remark 1

(i) *The bound (6) shows that the risk of the optimal stopping rule τ_k^* decreases exponentially in k for all values of n and k. Frank & Samuels (1980) derived asymptotic approximations for the value $w_k^* = \lim_{n\to\infty} w_n(k)$ as k tends to infinity. In particuluar, their results imply that for large values of n and k*

$$w_n(k) = \mathrm{P}\big\{A_{\tau_k^*} \le k\big\} \approx 1 - (1-t^*)^k,$$

where $t^* \approx 0.2834$. Although our bound in (7) is conservative, it holds for all n and k in contrast to the above asymptotic result. This fact is of particular importance for our purposes because we choose the tuning parameter k depending on n.

(ii) *The proof of Proposition 1 is based on the construction of a suboptimal (yet analytically more tractable) stopping rule for which the bound (6) is achieved.*

The following statement is key to bounding the relative regret of any stopping rule $\hat{\tau} \in \mathscr{T}(\mathscr{R})$. It provides a decomposition of $\mathrm{E}_G[X_{(n)} - X_\tau]$ which will be used in conjunction with Proposition 1 in the regret analysis.

Proposition 2 *Let $\hat{\tau} \in \mathscr{T}(\mathscr{R})$; then for any distribution G and any $k \in \{1, \dots, n\}$*

$$\mathrm{E}_G\big[X_{(n)} - X_{\hat{\tau}}\big] \;\le\; \mathrm{E}_G[X_{(n)} - X_{(n-k)}] \;+\; \mathrm{E}_G\big[X_{(n-k)} - X_{(1)}\big]\mathrm{P}\{A_{\hat{\tau}} > k\}. \tag{8}$$

This decomposition puts forward two terms that are present on the RHS of the above inequality. The first, which captures the *gap in order statistics* is controlled by properties of the functional class Σ, in particular tail behavior of distributions in the class Σ relative to which the regret is measured. The second reflects the error of the stopping rule $\hat{\tau}$ in *localizing* to the top k order statistics; this is controlled by Proposition 1. The design parameter k of the auxiliary problem will be chosen so as to balance these two error terms.

3 Main Results

Without loss of generality, in all of what follows we assume that $X_1, \dots, X_n$ are non–negative random variables. For the purpose of our main results we will recall some basic facts about extreme-value distributions and introduce necessary notation; cf. Leadbetter, Lingren & Rootzen (1986), Resnick (1987) and de Haan and Ferreira (2006).

Extreme value distributions. A distribution G is said to belong to the *domain of attraction* of G_γ, $\mathcal{D}(G_\gamma)$, if there exist real-valued sequences $a_n > 0$ and b_n such that

$$\lim_{n\to\infty} G^n(a_n x + b_n) = G_\gamma(x) := \exp\big\{-(1+\gamma x)^{-1/\gamma}\big\} \tag{9}$$

for all x satisfying $1 + \gamma x > 0$ with $\gamma \in \mathbb{R}$. Depending on the sign of γ we have three different types of extreme–value distributions. If $\gamma > 0$ we get the Fréchet class of distributions, $\Phi_\alpha(x) = \exp\{-x^{-\alpha}\}$, $x \ge 0$ (here $\alpha = 1/\gamma$); for $\gamma = 0$ we have Gumbel's distribution $\Lambda(x) = \exp\{-e^{-x}\}$, $x \in \mathbb{R}$; and for $\gamma < 0$ we have the reverse–Weibull class of distributions,

$$\Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^\alpha\}, & x < 0 \\ 1, & x \ge 0. \end{cases}$$

Let the function U be the left-continuous inverse of $1/(1-G)$:

$$U(t) := \big(\tfrac{1}{1-G}\big)^{\leftarrow}(t) = G^{\leftarrow}\big(1 - \tfrac{1}{t}\big) = \inf\{x : 1 - G(x) \le \tfrac{1}{t}\}. \tag{10}$$

Denote also $x_G^* := U(\infty) = \sup\{x : G(x) < 1\}$.

We recall also the following facts about norming constants (a_n) and (b_n) for distributions from domains of attraction $\mathcal{D}(\Lambda)$ and $\mathcal{D}(\Psi_\alpha)$.

(i) The distribution function G belongs to the domain of attraction $\mathcal{D}(\Lambda)$ if and only if there exists a positive function $\psi(t)$ such that

$$\lim_{t\uparrow x_G^*} \frac{1 - G(t + x\psi(t))}{1 - G(t)} = e^{-x}$$

for all $x \in \mathbb{R}$. Here one can take

$$\psi(t) := \frac{\int_t^{x_G^*}[1 - G(x)]\mathrm{d}x}{1 - G(t)}, \quad t < x_G^*, \tag{11}$$

and (9) holds with $a_n = a(n) := \psi(U(n))$ and $b_n = U(n)$.

(ii) The distribution function G belongs to the domain of attraction $\mathcal{D}(\Psi_\alpha)$ if and only if $x_G^* < \infty$ and

$$\lim_{h\downarrow 0} \frac{1-G(x_G^*-xh)}{1-G(x_G^*-h)} = x^\alpha$$

for $\alpha > 0$ and any $x > 0$. Here (9) holds with $a_n = x_G^* - U(n)$, $b_n = U(n)$.

From now on $\{a_n\}$ and $\{b_n\}$ stand for the sequences defined (i) and (ii) above.

First order asymptotics. With the above definitions in hand we have the following characterization of the first order behavior of our proposed stopping rule for the classes of distributions G that belong to $\mathcal{D}(\Lambda)$ and $\mathcal{D}(\Psi_\alpha)$, and a negative result for *any* stopping rule for distributions that belong to the domain of attraction of the Fréchet class $\mathcal{D}(\Phi_\alpha)$.

Theorem 1 ***(first order asymptotics)*** *Let $G \in \mathcal{D}(\Lambda)$ or $G \in \mathcal{D}(\Psi_\alpha)$. Then there exists $k := k_n \to \infty$ as $n \to \infty$ such that the stopping rule $\tau^*_{k_n}$ given in (5) is first order asymptotically optimal:*

$$\lim_{n\to\infty} \mathcal{R}_n[\tau^*_{k_n}; G] = 0. \tag{12}$$

In contrast, for every $G \in \mathcal{D}(\Phi_\alpha)$ and any stopping rule $\tau \in \mathscr{T}(\mathscr{X})$

$$\liminf_{n\to\infty} \mathcal{R}_n[\tau; G] \geq c > 0.$$

Relative regret bounds for our proposed stopping rule. With this result as a departure point, we next study regret asymptotics, i.e., second order behavior of the regret. Since the above theorem rules out first order asymptotic optimality for distributions belonging to the domain of attraction of the Fréchet class, our study will restrict attention to distributions in the other two classes of domain of attraction. We further refine these classes below by considering subsets of distributions with specific behavior of the expected gap in order statistics. To do so, we require the following definitions.

Definition 1 *Let $A > 0$, $\beta > 0$, and $\kappa_i > 0$, $i = 1, 2$ be fixed constants. We say that a distribution function $G \in \mathcal{D}(\Lambda)$ belongs to the class $\mathcal{G}_\beta(A)$ if the following conditions holds: there exist $t_0 > 1$ such that for all $t \geq t_0$*

$$\frac{U(t)-U(t/2)}{a(t/2)} \geq \kappa_1, \quad \frac{a(t/2)}{a(t)} \geq \kappa_2, \tag{13}$$

where $a(t) = \psi(U(t))$ and ψ is defined in (11), and for $x_0 > 1$

$$\frac{U(tx)}{U(t)} - 1 \leq \frac{A \ln x}{(\ln t)^\beta}, \quad \forall t \geq t_0, \;\; x \geq x_0. \tag{14}$$

Several remarks on the above definition are in order. The conditions in (13) are rather weak. Indeed, it is well known that for $G \in \mathcal{D}(\Lambda)$ and $x > 0$ one has

$$\frac{U(tx)-U(t)}{a(t)} \to \ln x \quad \text{as} \quad t \to \infty, \tag{15}$$

and $a_n = a(n)$ is the normalization in (9). In addition, for $G \in \mathcal{D}(\Lambda)$ the function $a(t) = \psi(U(t))$ is slowly varying, i.e., for any $x > 0$

$$\frac{a(tx)}{a(t)} \to 1, \quad t \to \infty, \tag{16}$$

and, in particular, $a(t/2)/a(t) \to 1$ as $t \to \infty$. Therefore (13) can be viewed as *uniformity* conditions on standard asymptotic results that hold for all distributions from the domain of attraction $\mathcal{D}(\Lambda)$.

If $G \in \mathcal{D}(\Lambda)$ then U is slowly varying, i.e., $\lim_{t\to\infty} U(tx)/U(t) = 1$ for any $x > 0$. The condition (14) specifies the rate of convergence of $U(tx)/U(t)$ to 1. The specific form of the expression on the right hand side of (14) is rather natural. Indeed, if $U(tx)/U(t) - 1 \sim h(x)g(t)$ as $t \to \infty$ for some slowly varying function g, then necessarily $h(x) = c \ln x$; see, e.g., Goldie and Smith (1987) where different asymptotic relations for convergence of $U(tx)/U(t)$ to 1 are discussed. Thus, our definition of the class $\mathcal{G}_\beta(A)$ corresponds to the above asymptotic relation with a particular choice of $g(t) = [\ln t]^{-\beta}$. Moreover, (14) is fulfilled for a wide family of distributions, as demonstrated in the following examples.

Example 1 *Let $G(x) = 1 - e^{-\lambda x}$; then $U(t) = \frac{1}{\lambda}\ln t$ and*

$$\frac{U(tx)}{U(t)} = \frac{\ln(tx)}{\ln t} = 1 + \frac{\ln x}{\ln t}.$$

Thus (14) holds with $t_0 = 1$, $x_0 = 1$, $A = 1$ and $\beta = 1$.

Example 2 *Let G be the distribution function of the standard normal random variable; then [see, e.g., Boucheron & Thomas (2012)]*

$$[2\ln t - \ln\ln t - \ln(4\pi)]^{1/2} \le U(t) \le [2\ln t - \ln\ln t - \ln\pi]^{1/2}, \quad t \ge 6.$$

If $t \ge 4\pi$ then

$$\begin{aligned}\frac{U(tx)}{U(t)} &\le \left[\frac{2\ln(tx) - \ln\ln(tx) - \ln\pi}{2\ln t - \ln\ln t - \ln(4\pi)}\right]^{1/2} \le \left[1 + \frac{2\ln x + \ln 4}{2\ln t - \ln\ln t - \ln(4\pi)}\right]^{1/2} \\ &\le \left[1 + \frac{4\ln x + 2\ln 4}{\ln t}\right]^{1/2} \le 1 + \frac{2\ln x + \ln 4}{\ln t} \le 1 + \frac{3\ln x}{\ln t},\end{aligned}$$

provided that $x \ge 4$. Hence (14) holds with $t_0 = 4\pi$, $x_0 = 4$, $A = 3$ and $\beta = 1$.

Examples 1 and 2 provide distributions from $\mathcal{G}_\beta(A)$ with $\beta = 1$. The next example presents a distribution from $\mathcal{G}_\beta(A)$ with $\beta > 1$.

Example 3 *Let $x_* > \delta > 0$, $\beta > 1$, $\theta > 0$ and $G(x) = 1 - \exp\{-\theta/(x_* - x)^{1/(\beta-1)}\}$ on the interval $[x_* - \delta, x_*]$; then $U(t) = x_* - (\ln t/\theta)^{-\beta+1}$. We have for large enough $t \ge t_0(x_*, \beta, \delta)$ that*

$$\frac{U(tx)}{U(t)} = \frac{x_* - [\ln(tx)/\theta]^{-\beta+1}}{x_* - (\ln t/\theta)^{-\beta+1}} = 1 + \frac{[\ln(tx)]^{\beta-1} - [\ln t]^{\beta-1}}{[\ln(tx)]^{\beta-1}[x_*(\ln t/\theta)^{\beta-1} - 1]} \le 1 + \frac{c(\beta-1)\theta^{\beta-1}\ln x}{x_*(\ln t)^\beta},$$

where c is an absolute constant. Thus $G \in \mathcal{G}_\beta(A)$ provided that $A \ge c(\beta-1)\theta^{\beta-1}/x_*$.

The following result establishes bounds on the relative regret with respect to our proposed stopping rule relative to the above subclass of the Gumbel domain of attraction.

Theorem 2 ***(regret bounds for $\mathcal{G}_\beta(A)$)*** *Let $\tau^*_{k_n}$ be the stopping rule specified in (5) with $k_n = \lfloor 24\beta \ln\ln n\rfloor$; then*

$$\limsup_{n\to\infty}\left\{\phi_n^{-1}\mathcal{R}_n[\tau^*_{k_n}; \mathcal{G}_\beta(A)]\right\} \le C_1, \quad \phi_n := \frac{A\ln\ln\ln n}{(\ln n)^\beta}, \tag{17}$$

where C_1 is a constant depending on κ_1 and κ_2 only.

We now proceed to study the behavior of our proposed stopping procedure for distributions in a suitable subclass of the reverse Weibull domain of attraction $\mathcal{D}(\Psi_\alpha)$.

Definition 2 *We say that a distribution function G belongs to the class $\mathcal{W}_\alpha(A)$ if $G \in \mathcal{D}(\Psi_\alpha)$, $x_G^* = U(\infty) < \infty$, and there exist $A > 0$, $\alpha > 0$ and $\delta > 0$ such that*

$$1 - \frac{U(t)}{U(\infty)} \le At^{-1/\alpha}, \qquad \forall\, t\ :\ U(t) \ge U(\infty) - \delta. \tag{18}$$

Note that for all distributions $G \in \mathcal{D}(\Psi_\alpha)$ it holds that [cf. de Haan and Ferreira (2006, p.23)]

$$\frac{U(\infty) - U(tx)}{U(\infty) - U(t)} \to x^{-1/\alpha} \quad \text{as } t \to \infty.$$

The above definition can be viewed as a non-asymptotic variant of this condition. In fact, (18) characterizes the behavior of the distribution G in the vicinity of its right end point x_G^*.

Example 4 *Let*

$$G(x) = 1 - L(x_G^* - x)^\alpha, \quad x_G^* - L^{-1/\alpha} \le x \le x_G^*,$$

where $L > 0$ and $x_G^* < \infty$. Then $U(t) = x_G^* - (Lt)^{-1/\alpha} = U(\infty) - (Lt)^{-1/\alpha}$. Hence the condition of Definition 2 is satisfied with $A = [L^{1/\alpha}U(\infty)]^{-1}$. This example includes the uniform distribution among others.

The following result present an upper bound on the relative regret for our proposed stopping rule with respect to the above defined class of distributions.

Theorem 3 ***(regret bounds for $\mathcal{W}_\alpha(A)$)*** *Let $\tau_{k_n}^*$ be the stopping rule specified in (5) with $k_n = \lfloor (24/\alpha)\ln n \rfloor$; then*

$$\limsup_{n\to\infty} \Big\{ \varphi_n^{-1} \mathcal{R}_n[\tau_{k_n}^*; \mathcal{W}_\alpha(A)] \Big\} \ \le\ C_2, \quad \varphi_n := A\Big(\frac{\ln n}{n}\Big)^{1/\alpha},$$

where the constant $C_2 = C_2(\alpha)$ depends on α only.

Lower bounds on the minimax relative regret. The next theorem establishes limits of achievable performance that hold for any stopping rule belonging to $\mathscr{X}$. In particular, it identifies a lower bound on the rate of convergence of the relative regret to zero given by scaling sequences $\{\phi_{n,G}\}$ and $\{\varphi_{n,G}\}$ that depend on whether the distribution G is in the domain of attraction of $\mathcal{D}(\Lambda)$ or $\mathcal{D}(\Psi_\alpha)$, respectively. The subsequent corollary will identify these rates explicitly for the subclasses of distributions which are the subject of Theorems 2 and 3.

Theorem 4 ***(lower bounds on the relative regret)*** *Let Σ be a family of distributions such that $\Sigma \subseteq \mathcal{D}(\Lambda)$; then for any $\tau \in \mathscr{T}(\mathscr{X})$*

$$\liminf_{n\to\infty} \bigg[\sup_{G\in\Sigma} \{\phi_{n,G}^{-1} \mathcal{R}_n[\tau; G]\} \bigg] \ge c_1 > 0, \qquad \phi_{n,G} := \frac{\psi(U(n))}{U(n)},$$

where c_1 is an absolute constant. Furthermore, if Σ is a family of distributions such that $\Sigma \subseteq \mathcal{D}(\Psi_\alpha)$ then for any $\tau \in \mathscr{T}(\mathscr{X})$

$$\liminf_{n\to\infty} \bigg[\sup_{G\in\Sigma} \{\varphi_{n,G}^{-1} \mathcal{R}_n[\tau; G]\} \bigg] \ge c_2 > 0, \qquad \varphi_{n,G} := \frac{U(\infty) - U(n)}{U(\infty)},$$

where c_2 is an absolute constant.

A direct interpretation of Theorem 4 is that the ratio of the norming constants a_n/b_n, defined in (9), provides a lower bound on the relative regret for any admissible stopping rule. The next result, an immediate consequence of Theorem 4, identifies these norming constants explicitly and establishes lower bounds on the minimax relative regret with respect to the classes of distributions $\mathscr{G}_\beta(A)$ and $\mathcal{W}_\alpha(A)$.

Corollary 1 ***(lower bounds on the minimax regret)*** *One has*

$$\liminf_{n\to\infty} \Big\{ A^{-1}(\ln n)^{\beta}\, \mathcal{R}_n^*[\mathscr{G}_\beta(A)] \Big\} \geq c_3 > 0, \tag{19}$$

$$\liminf_{n\to\infty} \Big\{ A^{-1} n^{1/\alpha}\, \mathcal{R}_n^*[\mathcal{W}_\alpha(A)] \Big\} \geq c_4 > 0, \tag{20}$$

where $c_3 = c_3(\kappa_1)$ *and* $c_4 = c_4(\alpha)$.

Corollary 1 together with Theorems 2 and 3 shows that the proposed stopping rule $\tau^*_{k_n}$ is nearly rate-optimal in the class of all stopping rules of $\mathscr{X}$ in the following sense: its relative regret is within a $\ln\ln\ln n$–factor of the best relative regret for the class $\mathscr{G}_\beta(A)$, and within of a $\ln n$–factor of the best relative regret for $\mathcal{W}_\alpha(A)$. It is important to stress that the proposed rule $\tau^*_{k_n}$ does not require any information on the underlying distribution G outside of the subclass $\mathscr{G}_\beta(A)$ or $\mathcal{W}_\alpha(A)$ to which it belongs.

4 Numerical Solution for Stopping Problems with No Information

In this section we present a general method for numerical solution of stopping problems with no information, and then specialize it to the Gusein–Zade stopping problem (SP2).

It is shown in Gusein–Zade (1966) that the optimal stopping rule for solution of problem (SP2) with fixed k has the following structure. It is determined by k natural numbers $1 \leq \pi_1 \leq \cdots \leq \pi_k$ and proceeds as follows: pass the first $\pi_1 - 1$ observations, $\{X_t, t = 1, \ldots, \pi_1 - 1\}$, and among observations $\{X_t, t = \pi_1, \pi_1 + 1, \ldots, \pi_2 - 1\}$ choose the first observation with relative rank one; if it does not exist then among observations $\{X_t, t = \pi_2, \pi_2 + 1, \ldots, \pi_3 - 1\}$ choose the first observation with relative rank at most two, etc. Gusein–Zade (1966) studied limiting behavior of the numbers $\pi_1, \ldots, \pi_k$, and showed that $\lim_{n\to\infty} w_n(2) \approx 0.574$ [cf. (5)].

In its original form the optimal stopping rule of Gusein–Zade (1966) requires determination of thresholds $\pi_1, \ldots, \pi_k$. Although the structure of the optimal policy is relatively straightforward, explicit determination of $\pi_1, \ldots, \pi_k$ for general n and k is hardly tractable, and does not lend itself to implementation. Based on general results of Mucci (1973), Frank & Samuels (1980) suggested asymptotic approximations and computed numerically $\lim_{n\to\infty} w_n(k)$ for a range of different values of k. Exact results for the case $k = 3$ are given in Quine & Law (1996), and the recent paper Dietz et al. (2011) studies some approximate policies.

We propose a general method for exact calculation of optimal policies in stopping problems with no information. It is based on the idea that any stopping problem with no information can be represented as a problem of stopping a sequence of independent random variables whose distributions (while not identical) can be easily calculated. Then the resulting optimal stopping rule is always of the single threshold type and, as will be seen, can be easily implemented. To the best of our knowledge this approach, while reasonably straightforward, was not studied in antecedent literature. Moreover, the scope of application of this method is clearly broader than just the optimal stopping problem discussed in this paper.

4.1 General idea

Assume that we observe sequentially relative ranks $R_1, R_2, \dots, R_n$ and we would like to find a stopping rule $\tau_* \in \mathscr{T}(\mathscr{R})$ so that

$$\mathrm{E}q(A_{\tau_*}) = \sup_{\tau \in \mathscr{T}(\mathscr{R})} \mathrm{E}q(A_\tau), \tag{21}$$

where $\{A_t\}$ are the absolute ranks [cf. (4)], and q is a fixed payoff function. The Gusein–Zade stopping problem corresponds to $q(a) = \mathbf{1}(a \le k)$, while the payoff $q(a) = \mathbf{1}(a = 1)$ leads to the classical secretary problem. It is well known that $R_1, \dots, R_n$ are independent random variables, $\mathrm{P}(A_t = a | R_1 = r_1, \dots, R_t = r_t) = \mathrm{P}(A_t = a | R_t = r_t)$, and

$$\mathrm{P}(A_t = a | R_t = r) \;=\; \frac{\binom{a-1}{r-1}\binom{n-a}{t-r}}{\binom{n}{t}}, \quad r \le a \le n - t + r.$$

Therefore setting

$$u_t(r) := \mathrm{E}\{q(A_t) \,|\, R_t = r\} = \sum_{a=r}^{n-t+r} q(a) \frac{\binom{a-1}{r-1}\binom{n-a}{t-r}}{\binom{n}{t}}, \quad r = 1, \dots, t, \tag{22}$$

$$Y_t := u_t(R_t), \quad 1 \le t \le n, \tag{23}$$

we note that (21) can be represented as a stopping problem for a sequence of independent (though not identically distributed) random variables $Y_1, \dots, Y_n$:

$$\mathrm{E}q(A_{\tau_*}) = \mathrm{E}Y_{\tau_*} = \sup_{\tau \in \mathscr{T}(\mathscr{Y})} \mathrm{E}Y_\tau.$$

Solution to the latter problem is given by the following procedure [see, e.g., Derman, Lieberman & Ross (1972)]. If F_t is the distribution function of Y_t then the optimal stopping rule is of the threshold type, $\tau_* = \min\{1 \le t \le n : Y_t \ge v_{n-t+1}\}$, and

$$v_{t+1} = \int_{v_t}^{\infty} z \mathrm{d}F_{n-t+1}(z) + v_t F_{n-t+1}(v_t), \quad v_2 = \mathrm{E}Y_{n-1}. \tag{24}$$

$$\mathrm{E}Y_{\tau_*} = \sup_{\tau \in \mathscr{T}(\mathscr{Y})} \mathrm{E}Y_\tau = v_{n+1}. \tag{25}$$

In order to apply the recursive procedure (24)-(25) to the sequence of random variables $\{Y_t\}$ defined in (22)–(23) we need to determine distribution functions F_t of Y_t, $1 \le t \le n$. To this end, let $y_t(1), \dots, y_t(\ell_t)$ denote distinct points of the set $\{u_t(1), \dots, u_t(t)\}$. Then the distribution of Y_t is supported on $\{y_t(1), \dots, y_t(\ell_t)\}$ and

$$f_t(j) := \mathrm{P}\{Y_t = y_t(j)\} = \frac{1}{t} \sum_{r=1}^{t} \mathbf{1}\{u_t(r) = y_t(j)\},$$

$$F_t(z) = \sum_{j=1}^{\ell_t} f_t(j) \mathbf{1}\{y_t(j) \le z\},$$

where we have used the fact that $\mathrm{P}\{R_t = r\} = 1/t$, $\forall r = 1, \dots, t$. Armed with these formulas, we can rewrite (24) in the following form

$$v_{t+1} = \sum_{j=1}^{\ell_{n-t+1}} \big[v_t \vee y_{n-t+1}(j)\big] f_{n-t+1}(j), \quad v_2 = \sum_{j=1}^{\ell_{n-1}} y_{n-1}(j) f_{n-1}(j). \tag{26}$$

It is convenient to define vectors $f_t = \{f_t(j), j = 1, \dots, \ell_t\}$, $y_t = \{y_t(j), j = 1, \dots, \ell_t\}$; then (26) reduces to

$$v_{t+1} = (v_t \vee y_{n-t+1})^T f_{n-t+1}, \quad v_2 = y_{n-1}^T f_{n-1}, \tag{27}$$

where $\vee$ stands for the coordinate–wise maximum of the vector y_{n-t+1} and the real number v_t. We are going to implement the relationship (27) to obtain the solution of problem (SP2).

4.2 A method for solving the auxiliary problem (SP2)

Application of (27) to the Gusein–Zade problem, (SP2), requires determination of quantities $\{\ell_t, t = 1, \dots, n\}$, and corresponding vectors $y_t \in \mathbb{R}^{\ell_t}$, $f_t \in \mathbb{R}^{\ell_t}$ in the case when

$$u_t(r) = \begin{cases} 0, & k+1 \le r \le t, \\ \sum_{a=r}^{(n-t+r)\wedge k} \frac{\binom{a-1}{r-1}\binom{n-a}{t-r}}{\binom{n}{t}}, & 1 \le r \le k, \end{cases} \qquad t = 1, \dots, n.$$

As shown in Mucci (1973, Proposition 2.1), for any payoff function q and for $u_t(r)$ defined in (22) the following recursive formula holds

$$u_t(r) = \frac{r}{t+1} u_{t+1}(r+1) + \Big(1 - \frac{r}{t+1}\Big) u_{t+1}(r), \quad r = 1, \dots, t. \tag{28}$$

For the Gusein–Zade problem, where $q(a) = \mathbf{1}\{a \le k\}$, we have

$$u_n(r) = \begin{cases} 1, & r = 1, \dots, k \\ 0, & r = k+1, \dots, n. \end{cases} \tag{29}$$

This relationship in conjunction with (28) implies the following facts.

1. If $1 \le t \le k$ then all the values $u_t(1), \dots, u_t(t)$ are positive and distinct. Thus

$$\ell_t = t; \quad y_t(j) = u_t(j),\ j = 1, \dots, t; \quad f_t(j) = \frac{1}{t},\ \ j = 1, \dots, t. \tag{30}$$

2. If $k+1 \le t \le n-k+1$ then the set $\{u_t(1), \dots, u_t(t)\}$ contains $k+1$ distinct values: $u_t(1), \dots, u_t(k)$ are positive distinct, and $u_t(k+1) = \cdots = u_t(t) = 0$. Therefore

$$\ell_t = k+1; \quad y_t(j) = \begin{cases} u_t(j), & j = 1, \dots, k \\ 0, & j = k+1; \end{cases} \qquad f_t(j) = \begin{cases} 1/t, & j = 1, \dots, k, \\ 1 - k/t, & j = k+1. \end{cases} \tag{31}$$

3. If $n-k+2 \le t \le n$ then the set $\{u_t(1), \dots, u_t(t)\}$ contains $n-t+2$ distinct values. Here $u_t(j) = 1$ for $j = 1, \dots, k-n+t$, $u_t(j) = 0$ for $j = k+1, \dots, t$ and takes distinct values in $(0,1)$ for $j = k-n+t+1, \dots, k$. Therefore

$$\ell_t = n-t+2; \quad y_t(j) = \begin{cases} 1, & j = 1, \\ u_t(k-n+t-1+j), & j = 2, \dots, n-t+1, \\ 0, & j = n-t+2, \end{cases} \tag{32}$$

and

$$f_t(j) = \begin{cases} 1 - (n-k)/t, & j = 1, \\ 1/t, & j = 2, \dots, n-t+1, \\ 1 - k/t, & j = n-t+2. \end{cases} \tag{33}$$

It remains to compute v_2. In view of (28) and (29), $u_{n-1}(r) = 1$ for $r = 1, \ldots, k$, $u_{n-1}(r) = 0$ for $r = k+1, \ldots, n$, and $u_{n-1}(r) = 1 - k/n$. Thus, $\ell_{n-1} = 3$, and we have that

$$y_{n-1}(1) = 1, \;\; y_{n-1}(2) = 1 - \frac{k}{n}, \;\; y_{n-1}(3) = 0,$$
$$f_{n-1}(1) = \frac{k-1}{n-1}, \;\; f_{n-1}(2) = \frac{1}{n-1}, \;\; f_{n-1}(3) = \frac{n-k}{n-1}.$$

Substitution of these formulas in (26) yields $v_2 = k/n$.

Formulas (30)–(33) provide complete description of the quantities $\{\ell_t, t = 1, \ldots, n\}$ and vectors (y_t) and (f_t); this leads to a simple implementation of (27). We note that computation of vectors (u_t), $t = n, n-1, \ldots, 1$ using (28) requires at most $O(n^2)$ flops while computation of the sequence (v_t) with vectors (y_t) and (f_t) defined in (30)–(33) requires $O(nk + k^2)$ flops. Thus, the computational cost of the algorithm does not exceed $O(n^2 + nk + k^2)$ flops.

5 Numerical Experiments and Discussion

In this section we provide some numerical examples that illustrate performance of the proposed stopping rule. The focus is on three aspects. First we look at dependence of the performance upon the choice of the tuning parameter k. Then we consider the effect of the sample size n on the regret behavior (essentially providing a "picture proof" of the results in our main theorems). Finally, we consider an example that explores the "value of information." Here we compare the performance of our rule to a benchmark that is designed (via dynamic programming) under full information on the underlying distribution. The example elucidates the loss in performance due to having only "partial information" on the underlying distributions, and simultaneously illustrates the sensitivity (lack of robustness) of the full information rule to model misspecification.

Choice of the tuning parameter k. The choice of the tuning parameter k prescribed by Theorems 2 and 3 is based on the conservative bound in Proposition 1 and leads to values of k that tend to be conservatively large. To illustrate dependence of the stopping rule performance on the parameter k, we consider two reference distributions that are representative of the nonparametric classes studied in the paper, uniform ($[0, 1]$) and exponential (with mean 1). In all experiments the sample size is set to $n = 100$, and each experiment is simulated $N = 1000$ runs to generate the histograms in Figures 1 and 2.

As evident in Figures 1 and 2, the performance of our stopping rule exhibits some sensitivity to the choice of k. This manifests most prominently in the spread of the distribution of outcomes, both for the choice of selected rank as well as the relative regret performance. In particular, smaller values of k result in higher probability of selecting observations with small rank, yet it also tends to select more often observations with high rank. This is can be seen in the top row in the figures, where it is also evident that the relative regret has a more significant left tail. On the other hand, if k is large then practically all runs result in the selected rank not exceeding k, but since k is large this increases the range of selected observations which results in less runs having small relative regret (compared to the case of small k).

The presented numerical results indicate that there is a range of reasonably good values of k that is typically much smaller than the prescriptions of Theorems 2 and 3. A practical, less conservative, choice of k can follow from the asymptotic result of Frank & Samuels (1980) alluded to in Remark 1(i). In particular, if we loosely assume that the probability in (7) is bounded from above by $(1 - t_*)^k \approx e^{-0.3332k}$ then the resulting choice of k would be $k \approx \lfloor 3\beta \ln \ln n \rfloor$ for the class $\mathcal{G}_\beta(A)$ and $k \approx \lfloor (3/\alpha) \ln n \rfloor$ for the class $\mathcal{W}_\alpha(A)$. The stopping rule associated with

RegK1.pdf

(a) $k = 1$

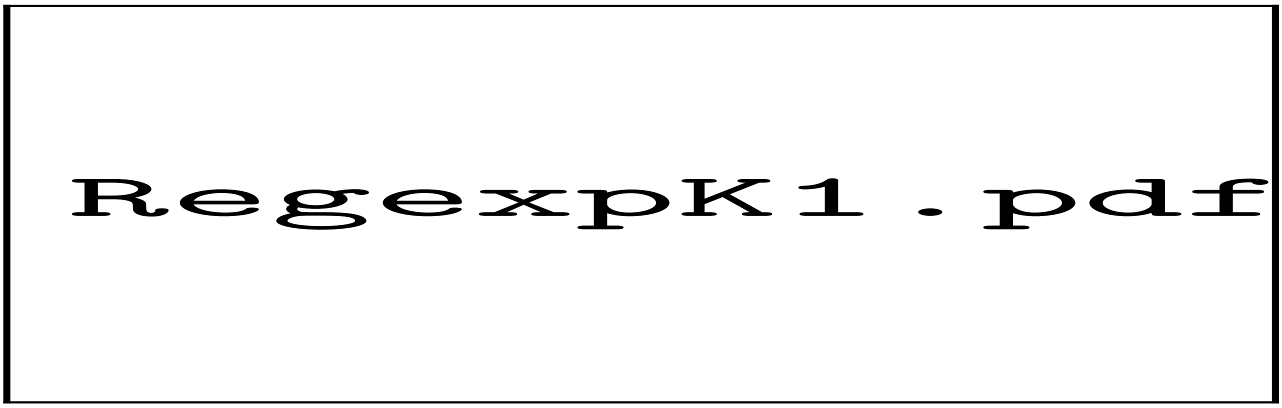
RegK5.pdf

(b) $k = 5$

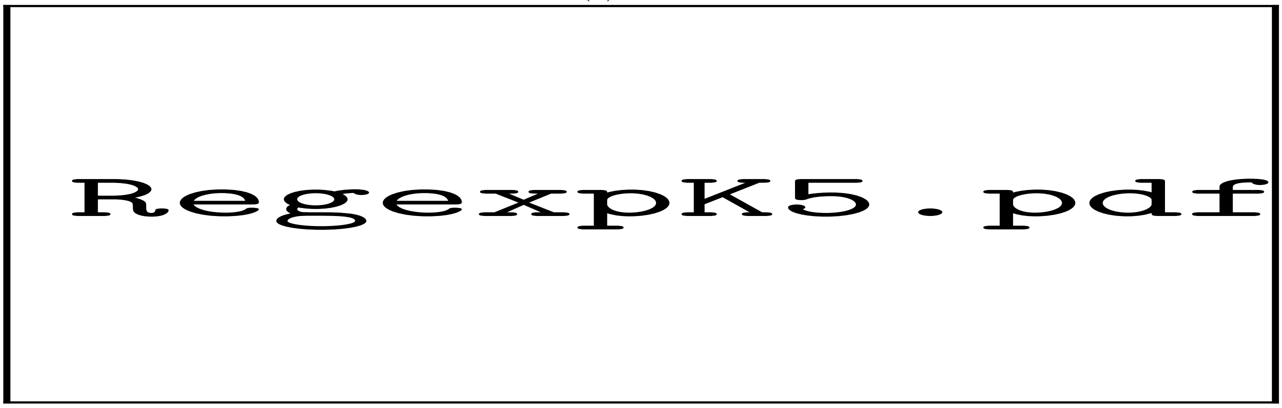
RegK20.pdf

(c) $k = 20$

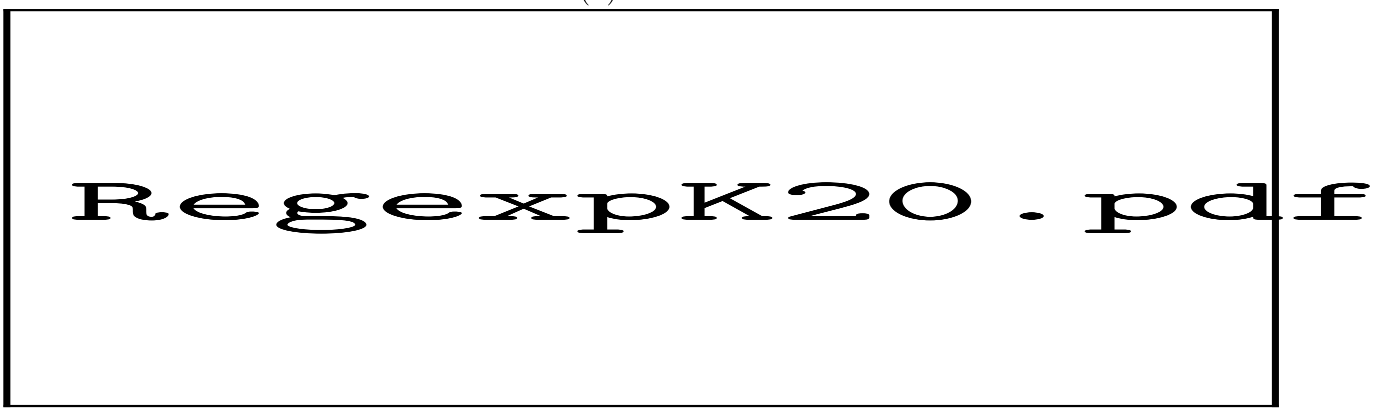
Figure 1: The histograms of the selected rank (left column) and the relative regret (right column) over $N = 1000$ simulation runs with samples from the uniform distribution on $[0, 1]$ for different values of the tuning parameter k.



(a) $k = 1$

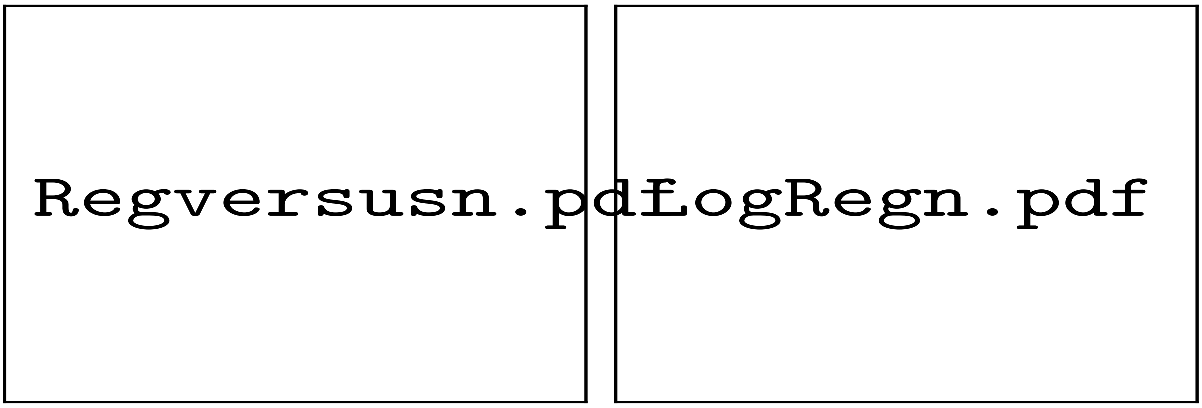


(b) $k = 5$



(c) $k = 20$

Figure 2: The histograms of the selected rank (left column) and the relative regret (right column) over $N = 1000$ simulation runs with samples from the exponential distribution with the parameter 1 for different values of the tuning parameter k.



(a) (b)

Figure 3: The graphs of: (a) the relative regret versus the sample size; and (b) the logarithm of the relative regret versus the logarithm of the sample size computed over $N = 1000$ simulation runs for the proposed stopping rule associated with $k = \lfloor 3 \ln n \rfloor$. The observations were generated according to the uniform distribution on $[0, 1]$.

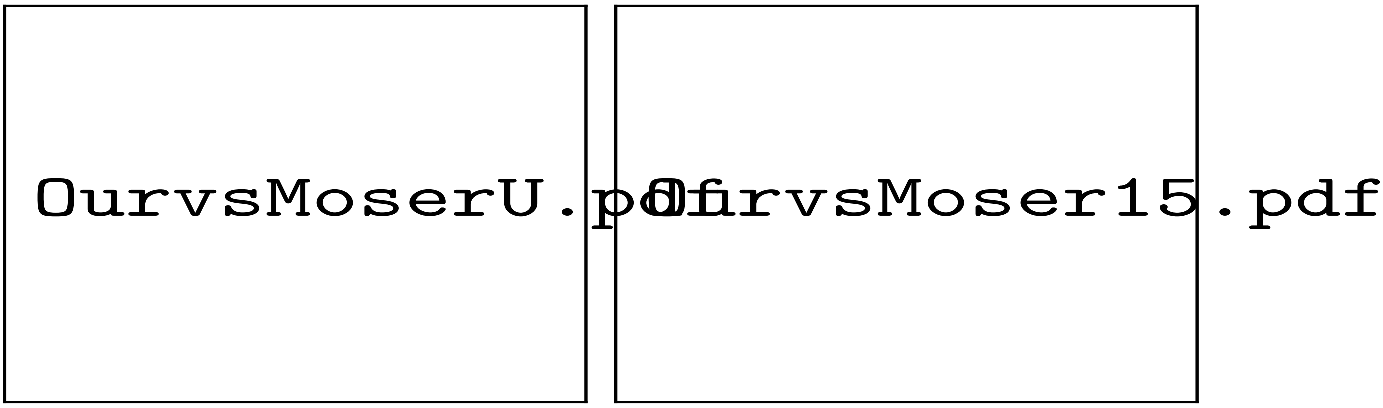
these values of k demonstrate good performance in numerical experiments, and we recommend this choice of k for practical use.

Effect of the sample size on relative regret. We consider the uniform distribution on $[0, 1]$ and report on the (estimated) expected regret as a function of sample size $n \in \{100, 250, 500, 1000, 2000\}$. The tuning parameter k is set to $k = \lfloor 3 \ln n \rfloor$ and the experiment is performed for $N = 1000$ runs. Figure 3(b) is consistent with the the result of Theorem 3 where it is stated that the relative regret decays roughly as $O(1/n)$ (up to logarithmic terms). Hence the straight line in the log-log plot of regret versus sample size.

Full information stopping rule and model misspecification. In this experiment we consider again the uniform distribution on $[0, 1]$ and contrast the performance of our proposed stopping rule (with tuning parameter $k = \lfloor 3 \ln n \rfloor$) with that of the full information optimal rule due to Moser (1956). In particular, the latter solves the dynamic program to obtain a simple recursion for the thresholds of the optimal policy. Specifically, the thresholds $\{v_t : 1 \le t \le n\}$ are given by:

$$v_{t+1} = \tfrac{1}{2}(1 + v_t^2) \quad v_1 = \tfrac{1}{2}.$$

As expected, in Figure 4(a), the well-specified case, the full information rule outperforms the proposed rank-based rule. The gap in performance indicates the value of full information when designing the stopping rule. In contrast consider the results in Figure 4(b) that show the performance of the full information rule under misspecification. Here the full information rule is constructed for the case of $U[0, 1]$ distribution, but the observations are generated from the distribution $U[0, 1.5]$. As evident, the rank-based rule, which is agnostic to this model misspecification, outperforms the full information rule.



(a) (b)

Figure 4: Comparison of the performance of the full information optimal rule that is tuned for the uniform distribution on $[0, 1]$ and the proposed rank-based rule for sample size $n = 100$ computed over $N = 1000$ simulation runs; the proposed rule uses $k = \lfloor 3 \ln n \rfloor$. The boxplot (a) reports on the well-specified case where observations were generated according to the uniform distribution on $[0, 1]$, and boxplot (b) reports the misspecified case where observations were generated according to the uniform distribution on $[0, 1.5]$.

6 Auxiliary Results and Proofs

For ease of reference we start with some well known auxiliary results and facts that are extensively used in the proofs.

6.1 Auxiliary results and facts

Absolute and relative ranks.

(i) The relative ranks $R_1, \ldots, R_n$ are independent random variables, and

$$\mathrm{P}(R_t = r) = \frac{1}{t}, \quad 1 \le r \le t, \quad 1 \le t \le n.$$

(ii) One has $\mathrm{P}(A_t = a | R_1 = r_1, \ldots, R_t = r_t) = \mathrm{P}(A_t = a | R_t = r_t)$, and

$$\begin{aligned} \mathrm{P}(A_t = a | R_t = r) &= \frac{\binom{a-1}{r-1}\binom{n-a}{t-r}}{\binom{n}{t}}, \quad r \le a \le n - t + r, \\ \mathrm{E}(A_t | R_t) &= \frac{n+1}{t+1} R_t, \quad \forall 1 \le t \le n. \end{aligned}$$

(iii) The random vectors $(X_{(1)}, \ldots, X_{(n)})$ and $(A_1, \ldots, A_n)$ are independent, and for any function h

$$\mathrm{E}\big[h(X_1, \ldots, X_n) | A_1 = a_1, \ldots, A_n = a_n\big] = \mathrm{E}\big[h(X_{(a_1)}, \ldots, X_{(a_n)})\big].$$

Hypergeometric distribution. Recall some properties of the hypergeometric distribution that will be used in the proof of Proposition 1. A discrete random variable ξ has the hypergeometric distribution with parameters M, N and n if

$$\mathrm{P}(\xi = k) = \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}}, \quad k = 0, \dots, \min\{n, M\}.$$

Here $N \in \{1, 2, \dots\}$, $M \in \{0, 1, \dots, N\}$, $n \in \{1, 2, \dots, N\}$, and, by convention, $\binom{a}{b} = 0$ for $a < b$. The expectation and the variance of ξ are given by the formulas

$$\mathrm{E}\xi = \frac{M}{N} n, \quad \mathrm{var}(\xi) = \frac{M(N-M)(N-n)}{N^2(N-1)}\, n.$$

We also state a version of the Bernstein inequality for the hypergeometric random variable which is an immediate consequence of the results by Hoeffding (1963).

Lemma 1 *Let ξ be a random variable having hypergeometric distribution with parameters M, N and n. Then for any $\lambda > 0$*

$$\mathrm{P}\big\{|\xi - \mathrm{E}\xi| \ge \lambda\big\} \le 2 \exp\Big\{ -\frac{\lambda^2}{2n\sigma^2 + \frac{2}{3}\lambda} \Big\},$$

where $\sigma^2 = M(N-M)/N^2$.

6.2 Proofs

Proof of Proposition 1. The proof proceeds in steps.

0^0. Consider the following stopping rule

$$\hat{\tau}_k = \min\big\{t : R_t \le r_t\big\},$$

where (r_t) is a sequence of integer numbers such that

$$r_1 = \cdots = r_s = 0, \quad 1 \le r_t \le k, \;\; \forall t = s+1, \dots, n-1, \;\; r_n = n.$$

Note that choice $r_n = n$ guarantees that $\hat{\tau}_k \le n$. The stopping rule $\hat{\tau}_k$ is completely determined by number s and sequence r_t, $t = s+1, \dots, n-1$; they will be specified in the sequel. Because $\mathrm{P}\{A_{\tau_k^*} > k\} \le \mathrm{P}\{A_{\hat{\tau}_k} > k\}$, it will be sufficient to establish the upper bound on $\mathrm{P}\{A_{\hat{\tau}_k} > k\}$.

1^0. Let B be the event that the k largest order statistics $X_{(n)}, X_{(n-1)}, \dots, X_{(n-k+1)}$ appear among the first s observations $X_1, \dots, X_s$. We can write

$$\mathrm{P}\big\{A_{\hat{\tau}_k} > k\big\} = \mathrm{P}\big\{A_{\hat{\tau}_k} > k, B\big\} + \mathrm{P}\big\{A_{\hat{\tau}_k} > k, B^c\big\}, \tag{34}$$

and the first term is evidently bounded from above by

$$\mathrm{P}\{B\} = \frac{\binom{n-k}{s}}{\binom{n}{s}} = \prod_{j=0}^{s-1} \Big(\frac{n-k-j}{n-j}\Big) \le \exp\Big\{ -k \sum_{j=0}^{s-1} \frac{1}{n-j} \Big\} = \exp\Big\{ -k \sum_{j=n-s+1}^{n} \frac{1}{j} \Big\}.$$

Using the well known fact that $C + \ln n \le \sum_{j=1}^{n} (1/j) \le C + \ln n + \frac{1}{2n}$, $\forall n$, where C is the Euler constant, [see, e.g., Gradshtein and Ryzhik (1965, formula 0.131)] we conclude

$$\mathrm{P}\{B\} \le \exp\Big\{ -k \ln\Big(\frac{n}{n-s}\Big) + \frac{k}{2(n-s)} \Big\}. \tag{35}$$

Turning to the second term on the right hand side of (34) we have

$$\mathrm{P}\{A_{\hat\tau_k}>k,B^c\}=\mathrm{P}\{A_{\hat\tau_k}>k,\hat\tau_k=n,B^c\}+\sum_{t=s+1}^{n-1}\mathrm{P}\{A_{\hat\tau_k}>k,\hat\tau_k=t,B^c\}=:I_1+I_2.$$

Using statement (i) of Section 6.1 we have

$$I_1\le \mathrm{P}\{R_{s+1}>r_{s+1},\dots,R_{n-1}>r_{n-1}\}=\prod_{t=s+1}^{n-1}\mathrm{P}\{R_t>r_t\}=\prod_{t=s+1}^{n-1}\Big(1-\frac{r_t}{t}\Big)\le\exp\Big\{-\sum_{t=s+1}^{n-1}\frac{r_t}{t}\Big\}.$$

Furthermore, for $t=s+1,\dots,n-1$

$$\begin{aligned}
\mathrm{P}\{A_{\hat\tau_k}>k,\hat\tau_k=t,B^c\} &\le \mathrm{P}\{A_t>k,R_{s+1}>r_{s+1},\dots,R_{t-1}>r_{t-1},R_t\le r_t\}\\
&= \sum_{j=1}^{r_t}\mathrm{P}\{A_t>k,R_{s+1}>r_{s+1},\dots,R_{t-1}>r_{t-1},R_t=j\}\\
&= \sum_{j=1}^{r_t}\mathrm{P}\{A_t>k|R_t=j\}\,\mathrm{P}\{R_t=j\}\prod_{m=s+1}^{t-1}\mathrm{P}\{R_m>r_m\}\\
&\le \frac{1}{t}\sum_{j=1}^{r_t}\mathrm{P}(A_t>k|R_t=j\}\exp\Big\{-\sum_{m=s+1}^{t-1}\frac{r_m}{m}\Big\}\\
&= \frac{1}{t}\sum_{j=1}^{r_t}\sum_{a=k+1}^{n}\frac{\binom{a-1}{j-1}\binom{n-a}{t-j}}{\binom{n-1}{t-1}}\frac{t}{n}\exp\Big\{-\sum_{m=s+1}^{t-1}\frac{r_m}{m}\Big\},
\end{aligned}$$

where in the third and last lines we used (ii) of Section 6.1. Thus,

$$\begin{aligned}
I_2 &\le \frac{1}{n}\sum_{t=s+1}^{n-1}\exp\Big\{-\sum_{m=s+1}^{t-1}\frac{r_m}{m}\Big\}\sum_{j=1}^{r_t}\sum_{a=k+1}^{n}\frac{\binom{a-1}{j-1}\binom{n-a}{t-j}}{\binom{n-1}{t-1}}\\
&= \frac{1}{n}\sum_{t=s+1}^{n-1}\exp\Big\{-\sum_{m=s+1}^{t-1}\frac{r_m}{m}\Big\}\sum_{a=k+1}^{n}\mathrm{P}\{\xi_{t,a}\le r_t-1\},
\end{aligned}$$

where $\xi_{t,a}$ stands for the hypergeometric random variable with parameters $a-1$, $n-1$ and $t-1$. Combining the obtained inequalities for I_1, I_2 with (35) and (34) we obtain

$$\begin{aligned}
\mathrm{P}\big\{A_{\hat\tau}>k\big\} &\le \exp\Big\{-k\ln\Big(\frac{n}{n-s}\Big)+\frac{k}{2(n-s)}\Big\}+\exp\Big\{-\sum_{t=s+1}^{n-1}\frac{r_t}{t}\Big\}\\
&\quad+\frac{1}{n}\sum_{t=s+1}^{n-1}\exp\Big\{-\sum_{m=s+1}^{t-1}\frac{r_m}{m}\Big\}\sum_{a=k+1}^{n}\mathrm{P}\{\xi_{t,a}\le r_t-1\}\\
&=: J_1+J_2+J_3.
\end{aligned}$$

Now we bound from above J_1, J_2 and J_3 using specific choice of s and (r_t).

2^0. Let $s=\alpha n$ for some $\alpha\in(0,1)$, and without loss of generality assume that s is integer. With this choice we obtain that

$$J_1\le\exp\Big\{-k\ln\Big(\frac{1}{1-\alpha}\Big)+\frac{k}{2n(1-\alpha)}\Big\}\le(1-\alpha)^{k/2}, \tag{36}$$

provided that α satisfies

$$(1-\alpha)\ln\Big(\frac{1}{1-\alpha}\Big) > \frac{1}{n}.$$

Now we proceed with bounding J_3. Note that $\mathrm{E}\xi_{t,a} = (t-1)(a-1)/(n-1)$; then

$$\mathrm{P}\{\xi_{t,a} \le r_t - 1\} = \mathrm{P}\Big\{\xi_{t,a} \le \mathrm{E}\xi_{t,a} - (t-1)\Big(\frac{a-1}{n-1} - \frac{r_t-1}{t-1}\Big)\Big\}.$$

Choose

$$r_t = 1 + \frac{k-1}{2}\Big(\frac{\alpha n}{n-1}\Big),\quad \forall t = \alpha n+1,\ldots,n-1. \tag{37}$$

Note that with this choice for $a = k+1,\ldots,n-1$, and $t = \alpha n+1,\ldots,n-1$

$$\frac{r_t-1}{t-1} = \frac{(k-1)(\alpha n/(n-1))}{2(t-1)} \le \frac{k-1}{2(n-1)} \le \frac{a-1}{2(n-1)}. \tag{38}$$

Therefore, applying Lemma 1 we obtain

$$\begin{aligned}
\mathrm{P}\{\xi_{t,a} \le r_t - 1\} &\le \exp\Bigg\{-\frac{(t-1)^2\big[\frac{a-1}{n-1} - \frac{r_t-1}{t-1}\big]^2}{2n\frac{(a-1)(n-a)}{(n-1)^2} + \frac{2}{3}(t-1)\big[\frac{a-1}{n-1} - \frac{r_t-1}{t-1}\big]}\Bigg\}\\
&\le \exp\Bigg\{-\frac{\frac{1}{4}(t-1)^2\big[\frac{a-1}{n-1}\big]^2}{2n\frac{(a-1)(n-a)}{(n-1)^2} + \frac{2}{3}(t-1)\big[\frac{a-1}{n-1}\big]}\Bigg\}\\
&\le \exp\Big\{-\frac{(\alpha n)^2(a-1)}{8n(n-a) + \frac{8}{3}(n-1)^2}\Big\} \;\le\; \exp\Big\{-\frac{3\alpha^2(a-1)}{32}\Big\},
\end{aligned}$$

where in the second line we have used (38), and the third line follows from $n \ge t \ge \alpha n + 1$. This bounds yield

$$J_3 \;\le\; \frac{1}{n}\sum_{t=\alpha n+1}^{n-1}\sum_{a=k+1}^{n} \exp\Big\{-\frac{3\alpha^2(a-1)}{32}\Big\} \;\le\; \frac{1-\alpha}{1-e^{-3\alpha^2/32}}\exp\Big\{-\frac{3\alpha^2 k}{32}\Big\}. \tag{39}$$

Furthermore, with r_t defined in (37)

$$\begin{aligned}
J_2 \le \exp\Big\{-\frac{1}{2}(k-1)\alpha\sum_{t=\alpha n+1}^{n-1}\frac{1}{t}\Big\} &\le \exp\Big\{-\frac{1}{2}(k-1)\alpha\Big[\ln\Big(\frac{1}{1-\alpha}\Big) - \frac{1}{2n(1-\alpha)}\Big]\Big\}\\
&\le \exp\Big\{-\frac{\alpha}{4}(k-1)\ln\Big(\frac{1}{1-\alpha}\Big)\Big\} \;=\; (1-\alpha)^{\alpha(k-1)/4}.
\end{aligned} \tag{40}$$

Combining (36), (40) and (39) we complete the proof. ∎

Proof of Proposition 2. For any $\hat{\tau} \in \mathscr{T}(\mathscr{R})$ and $k \in \{1,\ldots,n\}$ we have

$$\begin{aligned}
\mathrm{E}_G(X_{\hat{\tau}}) &= \mathrm{E}_G\big[X_{\hat{\tau}}\mathbf{1}(A_{\hat{\tau}} \le k)\big] + \mathrm{E}_G\big[X_{\hat{\tau}}\mathbf{1}(A_{\hat{\tau}} > k)\big]\\
&\ge \mathrm{E}_G X_{(n-k)} + \mathrm{E}_G\big[(X_{(1)} - X_{(n-k)})\mathbf{1}\{A_{\hat{\tau}} > k\}\big]\\
&= \mathrm{E}_G X_{(n)} - \mathrm{E}_G[X_{(n)} - X_{(n-k)}] - \mathrm{E}_G\big[(X_{(n-k)} - X_{(1)})\mathbf{1}\{A_{\hat{\tau}} > k\}\big].
\end{aligned}$$

Now, let $\mathscr{A}_t^n = \sigma(A_t, A_{t+1}, \ldots, A_n)$ be the σ–field generated by the absolute ranks $A_t, \ldots, A_n$. It is evident that R_t is $\mathscr{A}_t^n$–measurable, and $\{\hat{\tau} = t\} \in \mathscr{A}_1^n$. This implies

$$\begin{aligned}
&\mathrm{E}_G\big[(X_{(n-k)} - X_{(1)})\mathbf{1}\{A_{\hat{\tau}} > k\}\mathbf{1}\{\hat{\tau} = t\}\big] \\
&= \mathrm{E}_G\Big\{\mathbf{1}\{A_{\hat{\tau}} > k\}\mathbf{1}\{\hat{\tau} = t\}\mathrm{E}_G\big[X_{(n-k)} - X_{(1)}|\mathscr{A}_1^n\big]\Big\} \\
&= \mathrm{E}_G\big[X_{(n-k)} - X_{(1)}\big]\mathrm{P}\big\{A_{\hat{\tau}} > k, \hat{\tau} = t\big\},
\end{aligned}$$

where the third line follows from independence of the order statistics and $\mathscr{A}_1^n$ [see statement (iii) of Section 6.1]. Then the result follows by summation over t. ∎

Proof of Theorem 1. Here we will prove the second statement only; the proof of the first statement is contained in the proofs of Theorems 2 and 3.

For arbitrary stopping time $\tau \in \mathscr{T}(\mathscr{X})$ and real number $\delta > 0$

$$\mathrm{E}_G\big[X_{(n)} - X_\tau\big] \geq \delta \mathrm{P}_G\{X_{(n)} - X_\tau \geq \delta\} = \delta\big[1 - \mathrm{P}_G\{X_{(n)} - X_\tau \leq \delta\}\big]$$

Moreover,

$$\begin{aligned}
\mathrm{P}_G\{X_{(n)} - X_\tau \leq \delta\} &= \mathrm{P}_G\{X_{(n)} - X_\tau \leq \delta, A_\tau = 1\} + \mathrm{P}_G\{X_{(n)} - X_\tau \leq \delta, A_\tau > 1\} \\
&\leq \mathrm{P}_G\{A_\tau = 1\} + \mathrm{P}_G\{X_{(n)} - X_{(n-1)} \leq \delta\}.
\end{aligned}$$

Since for any rule $\tau \in \mathscr{T}(\mathscr{X})$ and any distribution G, $\mathrm{P}_G(A_\tau = 1) < \frac{3}{5}$ for large enough n [see Gilbert and Mosteller (1966)], we obtain that

$$\mathrm{E}_G\big[X_{(n)} - X_\tau\big] \geq \sup_{\delta > 0} \delta\big[\tfrac{2}{5} - \mathrm{P}_G\{X_{(n)} - X_{(n-1)} \leq \delta\}\big]. \tag{41}$$

First, for any $G \in \mathcal{D}(\Phi_\alpha)$ and large enough n we have $\mathrm{E}_G[X_{(n)}] \leq c_1 U(n)$ with $c_1 = c_1(\alpha)$. Second, by the joint convergence of the largest maxima [cf. Theorem 2.3.2 in Leadbetter, Lingren & Rootzen (1986)] the vector $(U^{-1}(n)X_{(n)}, U^{-1}(n)X_{(n-1)})$ converges in distribution to a non–degenerate random variable. Therefore, if we put $\delta = c_2 U(n)$ then for a small enough constant c_2 we can ensure that $\lim_{n\to\infty} \mathrm{P}_G\{X_{(n)} - X_{(n-1)} \leq c_2 U(n)\} < 1/5$. With this choice of δ, using (41) we have $\lim_{n\to\infty} \mathcal{R}_n[\tau; G] \geq \frac{c_2}{5c_1}$, as claimed. ∎

Proof of Theorem 2. The proof is divided in three steps. First, we derive a non-asymptotic bound on the relative regret $\mathcal{R}_n[\tau_k^*; G]$ of the stopping rule τ_k^* for any $G \in \mathcal{D}(\Lambda)$. Second, this bound is used in order to show that the tuning parameter $k = k_n$ can be chosen in such a way that $\mathcal{R}_n[\tau_{k_n}^*; G] \to 0$ as $n \to \infty$. This proves the first statement of Theorem 1 for $G \in \mathcal{D}(\Lambda)$. At the third step we complete the proof of Theorem 2.

1^0. We have

$$\begin{aligned}
\mathrm{E}_G\big[X_{(n)} - X_{(n-k)}\big] &= \int_0^\infty \Big[\mathrm{P}_G\{X_{(n)} > t\} - \mathrm{P}_G\{X_{(n-k)} > t\}\Big]\mathrm{d}t \\
&= \int_0^\infty \mathrm{P}_G\{1 \leq Z_n(t) \leq k\}\mathrm{d}t,
\end{aligned}$$

where $Z_n(t) := \sum_{i=1}^n \mathbf{1}\{X_i > t\}$ is the binomial random variable with parameters n and $1 - G(t)$.

Let U be the left–continuous inverse of $1/(1 - G)$; see (10). In general, $U(t)$ is defined for $t > 1$. With slight abuse of the definition of U we denote the left end point of G as

$U(1) := \inf\{x : G(x) > 0\}$. Note that $U(1)$ is non-negative since $X_1, \ldots, X_n$ are assumed to be non–negative random variables.

By definition, if $t < U(n/2k)$ then $1 - G(t) > 2k/n$, and $\mathrm{E}_G Z_n(t) = n[1 - G(t)] > 2k$. Therefore for $t < U(n/2k)$ by the Bernstein inequality

$$\begin{aligned}
\mathrm{P}_G\{1 \le Z_n(t) \le k\} &\le \mathrm{P}_G\Big\{Z_n(t) - \mathrm{E}_G Z_n(t) \le -\big[n(1-G(t)) - k\big]\Big\} \\
&\le \exp\bigg\{-\frac{[n(1-G(t)) - k]^2}{2n(1-G(t))G(t) + \frac{2}{3}[n(1-G(t)) - k]}\bigg\} \\
&\le \exp\big\{-\tfrac{3}{32} n[1-G(t)]\big\} \;\le\; e^{-3k/16}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{E}_G\big[X_{(n)} - X_{(n-k)}\big] \;&\le\; U\big(n/2k\big)e^{-3k/16} + \int_{U(\frac{n}{2k})}^{\infty} \mathrm{P}\{X_{(n)} > t\}\mathrm{d}t \\
&\le\; U(n/2k)e^{-3k/16} + \big[U(n) - U(n/2k)\big] + \int_{U(n)}^{\infty} [1 - G^n(t)]\mathrm{d}t. \quad (42)
\end{aligned}$$

Since $G \in \mathcal{D}(\Lambda)$, we have

$$\int_{U(n)}^{\infty} [1-G^n(t)]\mathrm{d}t \;\le\; n\int_{U(n)}^{x_G^*} [1-G(t)]\mathrm{d}t \;=\; n\psi(U(n))[1-G(U(n))] \;\le\; \psi(U(n)) = a_n, \quad (43)$$

where function ψ is given in (11), and we have used that $n[1-G(U(n))] \le 1$. We also note that $\mathrm{E}_G[X_{(n)}] \le U(n) + \psi(U(n)) = U(n) + a_n$ for any $G \in \mathcal{D}(\Lambda)$.

Furthermore, letting $\Delta_j := \big(U(n/j), U(n/(j-1)\big]$, $j = 1, 2, \ldots, n$, where, by convention, $U(1) = \inf\{x : G(x) > 0\}$ and $U(\infty) = x_G^*$ we have

$$\begin{aligned}
\mathrm{E}_G[X_{(n)}] \;&=\; \int_0^{\infty} \mathrm{P}_G\{X_{(n)} > t\}\mathrm{d}t = U(1) + \int_{U(1)}^{\infty} \mathrm{P}_G\{Z_n(t) \ge 1\}\mathrm{d}t \\
&\ge\; U(1) + \sum_{j=2}^{n-1} \int_{\Delta_j} \mathrm{P}_G\{Z_{(n)}(t) \ge 1\}\mathrm{d}t + \int_{U(1)}^{U(n/(n-1))} \mathrm{P}_G\{Z_{(n)}(t) \ge 1\}\mathrm{d}t \\
&=:\; U(1) + \sum_{j=2}^{n-1} I_j + I_n, \quad (44)
\end{aligned}$$

and our current goal is to bound the terms on the right hand side. We have

$$\begin{aligned}
\sum_{j=2}^{n-1} I_j \;&=\; \sum_{j=2}^{n-1} \int_{\Delta_j} \mathrm{P}_G\{Z_n(t) \ge 1\}\mathrm{d}t = \sum_{j=2}^{n-1} \int_{\Delta_j} \sum_{m=1}^{n} \tbinom{n}{m} [1-G(t)]^m [G(t)]^{n-m}\mathrm{d}t \\
&\ge\; \sum_{j=2}^{n-1} \sum_{m=1}^{n} \tbinom{n}{m} \int_{\Delta_j} \big(\tfrac{j-1}{n}\big)^m \big(1 - \tfrac{j}{n}\big)^{n-m}\mathrm{d}t \\
&=\; \sum_{j=2}^{n-1} \big[U\big(n/(j-1)\big) - U\big(n/j\big)\big] \big(1 - \tfrac{j}{n}\big)^n \sum_{m=1}^{n} \tbinom{n}{m} \big(\tfrac{j-1}{n-j}\big)^m \\
&=\; \sum_{j=2}^{n-1} \big[U\big(n/(j-1)\big) - U\big(n/j\big)\big] \big[\big(1 - \tfrac{1}{n}\big)^n - \big(1 - \tfrac{j}{n}\big)^n\big] \\
&\ge\; \big[\big(1 - \tfrac{1}{n}\big)^n - \big(1 - \tfrac{2}{n}\big)^n\big] \big[U(n) - U(n/(n-1))\big] \ge c_1[U(n) - U(n/(n-1))], \quad (45)
\end{aligned}$$

where $c_1 = (2e)^{-1} - e^{-2} \approx 0.0486$. The third line follows from the binomial formula and the penultimate inequality is a consequence of $\big(1 - \frac{1}{n}\big)^n \geq e^{-1}(1 - \frac{1}{2(n-1)}) \geq (2e)^{-1}$ for $n \geq 2$ [see e.g., Leadbetter, Lingren & Rootzen (1986, Lemma 2.1.1)], and $(1 - \frac{2}{n})^n \leq e^{-2}$. Moreover,

$$I_n = \int_{U(1)}^{U(n/(n-1))} [1 - G^n(t)]\mathrm{d}t \geq \int_{U(1)}^{U(n/(n-1))} [1 - G(t)]\mathrm{d}t \geq \tfrac{n-1}{n}[U(n/(n-1)) - U(1)].$$

Combining this inequality with (45) and (44) we obtain

$$\mathrm{E}_G[X_{(n)}] \geq c_1 U(n). \tag{46}$$

We note in passing that the derivation of (46) does not use any assumption on the domain of attraction to which G belongs. Thus, (46) is valid for distributions from $\mathcal{D}(\Lambda)$, $\mathcal{D}(\Psi_\alpha)$, and $\mathcal{D}(\Phi_\alpha)$.

Taking into account (46), (43) and (42) we obtain

$$\frac{\mathrm{E}_G[X_{(n)} - X_{(n-k)}]}{\mathrm{E}_G[X_{(n)}]} \quad \leq \quad \frac{1}{c_1}\bigg[e^{-3k/16} + \frac{U(n) - U(n/2k)}{U(n)} + \frac{a_n}{U(n)}\bigg]. \tag{47}$$

In view of (8), (7) and (47)

$$\begin{aligned} \frac{\mathrm{E}_G[X_{(n)} - X_{\tau_k^*}]}{\mathrm{E}_G[X_{(n)}]} \quad &\leq \quad \frac{\mathrm{E}_G[X_{(n)} - X_{(n-k)}]}{\mathrm{E}_G[X_{(n)}]} + 11e^{-k/24} \\ &= \quad \frac{1}{c_1}\bigg\{e^{-3k/16} + \frac{U(n/2k)}{U(n)}\bigg[\frac{U(n)}{U(n/2k)} - 1\bigg] + \frac{a_n}{U(n)}\bigg\} + 11e^{-k/24}. \end{aligned} \tag{48}$$

2^0. Now we show that for any $G \in \mathcal{D}(\Lambda)$ there exists $k = k_n \to \infty$, $n \to \infty$ such that the right hand side of (48) converges to zero as $n \to \infty$. This will imply that the corresponding stopping rule $\tau^*_{k_n}$ is first order asymptotically optimal. The fact that $G \in \mathcal{D}(\Lambda)$ implies that $\lim_{n\to\infty} a_n/U(n) = 0$; see, e.g., de Haan and Ferreira (2006, Lemma 1.2.9). Therefore it is sufficient to show existence of $k = k_n \to \infty$ such that $U(n)/U(n/2k_n) \to 1$ as $n \to \infty$.

For $G \in \mathcal{D}(\Lambda)$ the function U is slowly varying; by definition, it is also non-decreasing. Therefore U is a *normalized slowly varying function* [cf. Bingham et al. (1987, Theorem 1.5.5)], i.e., it admits representation

$$U(t) = c_2 \exp\bigg\{\int_{c_3}^{t} \frac{\varepsilon(x)}{x}\mathrm{d}x\bigg\},$$

for positive constants c_2, c_3 and a function $\varepsilon(x) \to 0$, $x \to \infty$. The function $\varepsilon(x)$ can be chosen non-negative; it is almost everywhere equal to $xU'(x)/U(x)$. Let

$$\eta(t) := \int_t^{2t} \frac{\varepsilon(x)}{x}\mathrm{d}x = \ln\bigg[\frac{U(2t)}{U(t)}\bigg].$$

The function $\eta(t)$ is positive, slowly varying, and $\eta(t) \to 0$ as $t \to \infty$. Therefore we have $[U(2t)/U(t)] - 1 = e^{\eta(t)} - 1 = O(\eta(t))$, $t \to \infty$. This fact implies that if $\rho(t) := \exp\{[\eta(t)]^{-\alpha}\}$ with any $\alpha \in (0,1)$ then

$$\bigg[\frac{U(2t)}{U(t)} - 1\bigg]\ln\rho(t) = O([\eta(t)]^{1-\alpha}) \to 0, \quad t \to \infty.$$

Note that $\rho(t)$ varies slowly, and $\rho(t) \to \infty$ as $t \to \infty$; therefore the function $t^\gamma \rho(t)$ is eventually increasing for any $\gamma \in (0,1)$. Then it follows from Theorem 2 of Bojanic & Seneta (1971) that

$$\lim_{t\to\infty} \frac{U(t\rho^\delta(t))}{U(t)} = 1, \quad \forall \delta > 0.$$

Therefore if we choose, say, $k = k_n = \frac{1}{2}\rho(n)$ then

$$\lim_{n\to\infty} \frac{U(n)}{U(n/2k_n)} = \lim_{m\to\infty} \frac{U(m\rho(m))}{U(m)} = 1.$$

This proves the first statement of Theorem 1 for domain of attraction $\mathcal{D}(\Lambda)$.

3^0. Now assume that $G \in \mathcal{G}_\beta(A)$. Then by (13) and by monotonicity of $U(\cdot)$ we have for $n \geq t_0$ and $n/2k \geq t_0$

$$\frac{U(n) - U(n/2k)}{U(n)} \geq \frac{U(n) - U(n/2)}{a_{[n/2]}} \cdot \frac{a_{[n/2]}}{a_n} \cdot \frac{a_n}{U(n)} \geq \kappa_1\kappa_2 \frac{a_n}{U(n)}.$$

Therefore

$$\frac{\mathrm{E}_G[X_{(n)} - X_{(n-k)}]}{\mathrm{E}_G[X_{(n)}]} \leq c_4\left[e^{-3k/16} + \frac{U(n) - U(n/2k)}{U(n)}\right],$$

where c_2 depends on κ_i, $i = 1,2$ only. Applying (14) we obtain

$$\frac{U(n)}{U(n/2k_n)} - 1 \leq \frac{A\ln(2k_n)}{[\ln(n/2k_n)]^\beta} \leq c_5 A \frac{\ln\ln\ln n}{(\ln n)^\beta}$$

which together with (48) yields (17). ∎

Proof of Theorem 3. The proof goes along the same lines as the proof of Theorem 2. In particular, instead of (42) we have

$$\mathrm{E}_G\big[X_{(n)} - X_{(n-k)}\big] \leq U(n/2k)e^{-3k/16} + \big[U(\infty) - U(n/2k)\big]$$

which together with (46) leads to

$$\begin{aligned}\frac{\mathrm{E}_G[X_{(n)} - X_{\tau_k^*}]}{\mathrm{E}_G[X_{(n)}]} &\leq \frac{\mathrm{E}_G[X_{(n)} - X_{(n-k)}]}{\mathrm{E}_G[X_{(n)}]} + 11e^{-k/24} \\ &= c_2\left\{\frac{U(n/2k)}{U(n)}e^{-3k/16} + \frac{U(\infty) - U(n/2k)}{U(n)}\right\} + 11e^{-k/24}. \qquad (49)\end{aligned}$$

If $k = k_n \to \infty$ and $n/k_n \to \infty$ as $n \to \infty$ then $U(n/2k)/U(n) \to 1$ and

$$\frac{U(\infty) - U(n/2k)}{U(n)} \to 0.$$

This yields the first statement of Theorem 1 for $G \in \mathcal{D}(\Psi_\alpha)$.

If $G \in \mathcal{W}_\alpha(A)$ then for large enough n

$$\frac{U(\infty) - U(n/2k)}{U(n)} \leq \frac{U(\infty) - U(n/2k)}{U(\infty)(1 - An^{-1/\alpha})} \leq 2A\left(\frac{2k}{n}\right)^{1/\alpha}.$$

Then substitution of $k = k_n = (24/\alpha)\ln n$ in (49) leads to the announced result. ∎

Proofs of Theorem 4 and Corollary 1. The proof of Theorem 4 proceeds along the same lines as the proof of the second statement of Theorem 1. In particular, with (41) established, we note that if $G \in \mathcal{D}(\Lambda)$ then putting $\delta = c_1 a_n = c_1\psi(U(n))$, by choice c_1 we can ensure that for all n large enough $\mathrm{P}_G\{X_{(n)} - X_{(n-1)} \le \delta\} \le 1/5$. Note that c_1 is completely determined by the limiting distribution of $X_{(n)} - X_{(n-1)}$ (which is dictated by the domain of attraction of G). Moreover, as mentioned in the proof of Theorem 2, for $G \in \mathcal{D}(\Lambda)$ one has $\mathrm{E}_G[X_{(n)}] \le U(n) + \psi(U(n))$. Therefore for any $G \in \mathcal{D}(\Lambda)$, any $\tau \in \mathscr{T}(\mathscr{X})$ and all large enough n

$$\mathcal{R}_n[\tau; G] = \frac{\mathrm{E}_G[X_{(n)} - X_\tau]}{\mathrm{E}_G[X_{(n)}]} \ge \frac{c_1\psi(U(n))}{5[U(n) + \psi(U(n))]}.$$

This inequality, together with the fact that $\psi(U(n))/U(n) \to 0$ as $n \to \infty$ for $G \in \mathcal{D}(\Lambda)$, implies that the RHS above is lower bounded by, say, $(c_1/10)\phi_{n,G}$, and hence the lower bound in the theorem follows since c_1 is uniform over any $\Sigma \subseteq \mathcal{D}(\Lambda)$.

The case of $\Sigma \subseteq \mathcal{D}(\Psi_\alpha)$ is treated similarly. Here we choose $\delta = c_2[U(\infty) - U(n)]$ and since $\mathrm{E}_G[X_{(n)}] \le U(\infty)$ we obtain that for any $\tau \in \mathscr{T}(\mathscr{X})$ and for large enough n

$$\mathcal{R}_n[\tau; G] = \frac{\mathrm{E}_G[X_{(n)} - X_\tau]}{\mathrm{E}_G[X_{(n)}]} \ge \frac{c_2[U(\infty) - U(n)]}{5U(\infty)}.$$

This implies the lower bound in the theorem since the RHS above can be identified as $(c_2/5)\varphi_{n,G}$, and c_2 is uniform over $\Sigma \subseteq \mathcal{D}(\Psi_\alpha)$.

The proof of Corollary 1 follows immediately from Theorem 4 and the definitions of the classes $\mathcal{G}_\beta(A)$ and $\mathcal{W}_\alpha(A)$. Indeed, if $G \in \mathcal{G}_\beta(A)$ then in view of (13) and (14)

$$\phi_{n,G} = \frac{\psi(U(n))}{U(n)} = \frac{a(n)}{U(n)} \le \frac{1}{\kappa_1} \cdot \frac{U(2n) - U(n)}{U(n)} \le \frac{A \ln 2}{\kappa_1 (\ln n)^\beta}.$$

Similarly, if $G \in \mathcal{W}_\alpha(A)$ then by (18)

$$\varphi_{n,G} = 1 - \frac{U(n)}{U(\infty)} \le A n^{-1/\alpha}.$$

These inequalities together with the statement of Theorem 4 complete the proof of Corollary 1. ∎

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