Growth to Value: 
Option Exercise and the Cross-Section of Equity Returns

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Abstract

We put forward a model that links the cross-sectional variation in expected equity returns to firms' life cycle dynamics. We show that value premium arises naturally in equilibrium under two simple conditions: inelastic supply of investment in physical capital and mean reverting aggregate risk. Growth assets in the model are options on assets in place (i.e., value assets). The cost of option exercise, endogenously determined in equilibrium, is highly sensitive to aggregate consumption risks. This provides a hedge against risks in assets in place, making growth options less risky than value assets. Our model features long-run consumption risks (as in Bansal and Yaron (2004)) and replicates the empirical failure of the traditional CAPM and CCAPM. We calibrate the model and show that it is able to account for the observed pattern in mean returns on book-to-market sorted portfolios.

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Introduction

We put forward an equilibrium model that links the cross-section of expected equity returns to firms’ life cycle dynamics. The model specifies exogenously preferences of a representative agent and aggregate consumption process and focuses on investment decisions of firms. We use the framework of Bansal and Yaron (2004) to introduce two sources of risks in the economy: long-run and short-run fluctuations in consumption growth. Our model generates an endogenous mechanism, through which growth options have lower exposure to long-run consumption risks than assets in place (i.e., value assets) and, consequently, carry a low risk premium. In the model, the null hypothesis of the conditional CAPM fails – an asset has a high $\alpha$ in the one-factor conditional CAPM regression if it has high exposure to long-run consumption risks. Thus, we provide a rational explanation of the value premium and, simultaneously, are able to account for the failure of the standard CAPM as well as the consumption-based CAPM to price book-to-market sorted portfolios.

Assets in the model differ by the amount of installed capital they carry: a value asset has one unit of installed capital and produces consumption goods; a growth asset does not carry any installed capital, nor does it participate in production. Growth assets are options on assets in place. Exercise of a growth option requires one unit of installed capital. The cost of installed capital, which is endogenously determined in equilibrium, is highly sensitive to persistent fluctuations in aggregate consumption. Since growth options are long positions in assets in place and short positions in installed capital, the cost of installed capital acts as a hedge against long-run risks and makes growth options less risky than value assets.

We prove that the necessary and sufficient condition for the value premium to exist in the model equilibrium is that firms’ dividend payments are more exposed to long-run risks than aggregate consumption. Indeed, recent empirical evidence suggests that dividends of many assets are highly sensitive to low frequency risks in consumption (Bansal, Dittmar, and Lundblad (2005), Hansen, Heaton, and Li (2006)). The intuition of our result can be understood as follows. The model assumes a fixed supply of both growth options and installed capital. In good times, value assets produce a lot of consumption goods, and become more valuable. This drives up the demand for installed capital. Because capital is inelastically supplied, its price has to increase to clear the market. In bad times, the price of value assets is low, and so are the demand for installed capital and its price. Thus, an ownership of installed capital is risky. Further, if installed capital is more sensitive to persistent consumption risks than value assets, growth options will be less risky than assets in place. We prove that as long as dividend exposure to long-run risks exceeds that of aggregate consumption, this will always be the case. Our model, thus, provides a simple explanation of the observed value premium.

Our model also predicts that value assets with small market capitalization have higher exposure to long-run risks than those with large market capitalization. Operating a value asset requires a
cost. High market capitalization assets are more productive and generate enough cash flow to cover the operating cost and, thus, are less likely to exit the industry when the economy is hit by a negative persistent shock. Small and less productive assets, however, have to respond to negative long-run risks by shutting down because of the cost of operation and, thus, are highly exposed to long-run risks in consumption. Value assets with small market capitalization, therefore, have to carry a high expected return as a compensation for their exposure to long-run risks.

We show that in the model, assets with high exposure to long-run risks will always have high $\alpha$'s in the conditional CAPM and wealth-CAPM regressions. This is consistent with the well-known failure of the CAPM and consumption-based CAPM in the data (see, for example, Mankiw and Shapiro (1986), Fama and French (1992), Lewellen and Nagel (2005), and Zhang and Petkova (2005)). Cross-sectional implications of our model are also consistent with the empirical evidence in the long-run risks literature that highlights the importance of long-run consumption risks in explaining the dispersion in risk premia along book-to-market dimension (Bansal, Dittmar, and Lundblad (2005), Hansen, Heaton, and Li (2006), Kiku (2006), Malloy, Moskowitz, and Vissing-Jorgensen (2006), Bansal, Dittmar, and Kiku (2007), and Bansal, Kiku, and Yaron (2007)). These papers, in particular, show that firms’ exposures to low-frequency consumption fluctuations account for a significant portion of the value premium in the data. Our model rationalizes this observation endogenously as an equilibrium phenomenon. We calibrate the model and show that, quantitatively, the model-implied value premium matches the observed difference in mean returns on book-to-market sorted portfolios.

Real option based models typically imply that growth options are riskier than assets in place and, therefore, entail high risk compensations (see, for example, Berk, Green, and Naik (1999)\textsuperscript{1}, Gomes, Kogan, and Zhang (2003), Carlson, Fisher, and Giammarino (2004), Panageas and Yu (2006)). These papers generate the value premium through a different mechanism: high book-to-market ratio stocks are option intensive assets, and therefore are riskier. This implication, however, does not conform well with the data - empirically, high book-to-market stocks are lower duration assets while low book-to-market stocks tend to pay off far in the future (for example, Dechow, Sloan, and Soliman (2004) and Da (2006)). The argument that growth options are riskier is based on the intuition that options are long positions in the underlying asset and short positions in the strike asset. If the strike asset is risk free, then options are effectively leveraged positions in the underlying asset and must carry a high risk premium. For traded options, the strike asset is a cash amount and will, indeed, be risk free. For real options, strike assets are installed physical capital, the price of which is highly pro-cyclical, which is supported by empirical evidence on aggregate $q$. Berk, Green, and Naik (1999) and Carlson, Fisher, and Giammarino (2004) are partial equilibrium models and the strike assets in their settings are assumed to be risk-free exogenously. In the general equilibrium model of Gomes, Kogan, and Zhang (2003), the relative price of the strike asset with

\textsuperscript{1}In Berk, Green, and Naik (1999), growth options could be riskier or less risky than assets in place depending on parameter values.
respect to consumption is fixed to be one by technology and, therefore, is always risk-free. We provide a simple equilibrium mechanism that makes the installed capital risky, consistent with both the time-series evidence on marginal-\( q \) and the cross-sectional evidence on the value premium.

Many other papers address the cross-sectional variation of expected returns inside partial or general equilibrium frameworks (for example, Cooper (2006), Gala (2005), Gourio (2006), Panageas and Yu (2006), Zhang (2005), among others). A key difference between our model and the above is that these models are typically built on only one source of risks. Therefore, although they are able to generate a high expected return of value firms and/or small firms, in all these models, the conditional CAPM still holds. In contrast, we allow for two independent sources of risks, long-run and short-run fluctuations in aggregate consumption growth, and are able to account for the failure of the conditional CAPM in the data.

On a technical level, we obtain closed form solutions for asset prices in the economy and the cross-sectional distribution of assets’ productivity. General equilibrium models with aggregate risk and heterogenous firms are notoriously hard to solve, even numerically. Thus, for tractability, we have to sacrifice on a general equilibrium framework. Our model is not a fully-specified general equilibrium in the sense that aggregate dividend payments of firms do not add up to aggregate consumption and the determination of the discrepancy between the two is outside of the model.\(^2\) We focus on a sector of the economy, whose assets are publicly traded. This allows us to obtain closed form solutions and provide sharp theoretical characterization of the value premium. Similar techniques of solving for the cross-sectional distribution of firms have been used by Miao (2005) in an industry equilibrium model and Luttmer (2007) in a general equilibrium economy without aggregate uncertainty. Optimal stopping problems in economies with recursive preference and long-run risks are studied by Bhamra, Kuehn, and Strebulaev (2007) and Chen (2008).

The paper is organized as follows. Section I provides a simple example to illustrate the main intuition of the model. Section II sets up the model and defines an appropriate notion of equilibrium. Section III discusses the solution to firms’ maximization problems and characterizes the cross-sectional distribution of assets. Section IV provides an analysis of the failure of the conditional CAPM and conditions for the value premium to exist in equilibrium. Section V calibrates the model and discusses its quantitative implications. Section VI concludes.

I A Simple Example

Here we use a simple example to demonstrate the basic intuition behind the economic mechanism in our model that generates the value premium. The key premise of our result is that the cost of installed capital is riskier than assets in place. A growth option is a long position in assets in place,

\(^2\)Jovanovic (2007) introduces investment options in a general equilibrium with aggregate uncertainty and representative firms.
and a short position in one unit of installed capital. If the cost of the installed capital is riskier, it offsets the risk in assets in place, and makes the growth option less risky. The following example illustrates the differential risk exposure of growth options and assets in place as an equilibrium phenomenon when installed capital is inelastically supplied.

Consider an economy with two dates, \( t = 1, 2 \). At \( t = 1 \), the economy is endowed with measure 1 of ideas and measure \( \frac{1}{2} \) of installed capital. The quality of ideas, denoted \( Z \), is uniformly distributed in the interval \([0, 1]\).

At date \( t = 1 \), an idea can be combined with one and only one unit of installed capital to make a production unit, which produces consumption goods. The productivity of production units is subject to a common shock \( \theta_t \), which follows a two-state Markov chain, with state space \( \{ \theta_H, \theta_L \} \), where \( \theta_H > \theta_L \). The transition probabilities of the Markov chain are given by:

\[
P(\theta_2 = \theta_H | \theta_1 = \theta_H) = P(\theta_2 = \theta_L | \theta_1 = \theta_L) = 1 - p
\]

where \( p \in [0, 1] \). An idea with quality \( Z \in [0, 1] \) will produce \( Z\theta_1 \) units of consumption goods at date \( t = 1 \) once it becomes a production unit, nothing at \( t = 2 \) (i.e., the production unit evaporates completely in the second period). If an idea is not matched with installed capital at date \( t = 1 \), it can be used to produce \( \delta \theta_2 \) units of consumption good at date \( t = 2 \), where \( \delta < \frac{1}{2} \).

For simplicity, assume that the representative agent is risk-neutral and interest rate is 0. Although risks in \( \theta_t \) are not priced in the economy, we can still use this example to discuss the basic mechanism in our model that makes value of assets in place more sensitive to productivity shocks \( (\theta_t) \) than that of growth options. One can easily introduce a correlation of \( \theta_t \) with aggregate risk, in which case the difference in sensitivities with respect to shocks in \( \theta_t \) will translate directly into difference in expected returns.

In this economy an idea matched with a unit of installed capital is an asset in place. The value of assets in place is determined by the amount of consumption goods it produces. Therefore the value of the asset in place is given by \( Z\theta_1 \) if it involves an idea of quality \( Z \). Note that assets in place are risky, since their values depend on \( \theta_1 \). An unmatched idea is an option: it can be matched with a unit of installed capital at \( t = 1 \), or it can wait for one period and produce consumption good at date \( t = 2 \). If we use \( f(\theta) \) to denote the equilibrium price of installed capital, then the value of an idea with quality \( Z \) is given by:

\[
\max\{Z\theta_1 - f(\theta_1), \quad E[\delta\theta_2 | \theta_1]\}.
\]

It is clear that ”in-the-money” ideas are long positions in assets in place, and a short positions in installed capital. We are interested in the determination of \( f(\theta) \), and hence the risk exposure of growth options and assets in place.
In equilibrium, only half of all ideas will be implemented at \( t = 1 \), since the total supply of installed capital is \( \frac{1}{2} \). \( f(\theta) \) will be determined by the profit maximization problem of a marginal idea that acquires a unit of installed capital. In equilibrium, high quality ideas will be implemented first. Therefore, any \( Z \geq \frac{1}{2} \) will be matched with a unit of installed capital at \( t = 1 \), and any \( Z < \frac{1}{2} \) will be used to produce consumption goods at \( t = 2 \). The equilibrium price of installed capital, \( f(\theta) \) must be such that the marginal idea \( Z = \frac{1}{2} \) is indifferent between acquiring a unit of installed capital today, and wait to produce consumption good at time 2. The total payoff of acquiring a unit of installed capital for an idea of quality \( Z \) is \( Z\theta_1 - f(\theta_1) \). Therefore, profit maximization of the marginal idea \( Z = \frac{1}{2} \) requires:

\[
\frac{1}{2}\theta_H - f(\theta_H) = (1 - p)\delta\theta_H + p\delta\theta_L \quad (1)
\]
\[
\frac{1}{2}\theta_L - f(\theta_L) = (1 - p)\delta\theta_L + p\delta\theta_H \quad (2)
\]

The above equations imply that the price of installed capital is given by:

\[
f(\theta_H) = \frac{1}{2}\theta_H - \delta[(1 - p)\theta_H + p\theta_L] \quad (3)
\]
\[
f(\theta_L) = \frac{1}{2}\theta_L - \delta[(1 - p)\theta_L + p\theta_H] \quad (4)
\]

It follows that for all \( Z \in [0,1] \),

\[
\frac{f(\theta_H)}{f(\theta_L)} > \frac{Z\theta_H}{Z\theta_L} \quad (5)
\]

That is, when the economy is hit by a negative shock in \( \theta \), the cost of installed capital drops by a higher percentage than the value of assets in place. Thus, the cost of installed capital is riskier than assets in place.

To understand the intuition behind equation (5), first consider the case when \( p \to 0 \). In this case, the probability of regime switching in \( \theta \) is very small, and

\[
f(\theta_H) = \left( \frac{1}{2} - \delta \right)\theta_H \quad (6)
\]
\[
f(\theta_L) = \left( \frac{1}{2} - \delta \right)\theta_L. \quad (7)
\]

Therefore,

\[
\frac{f(\theta_H)}{f(\theta_L)} = \frac{\theta_H}{\theta_L} = \frac{Z\theta_H}{Z\theta_L}.
\]

That is, installed capital is as risky as assets in place. This is not surprising. Given that the productivity of the idea in both days is proportional to \( \theta \), the market clearing price \( f(\theta) \) defined in equations (1) and (2) must be proportional \( \theta \). When \( \theta \) changes, prices of all assets in the economy move proportionally and growth assets are as risky as value assets.
The possibility of a regime switch in $\theta$ introduces an additional incentive for ideas to exercise options in the good state ($\theta_1 = \theta_H$), and a disincentive for ideas to exercise options in the bad state ($\theta_1 = \theta_L$). For the market clearing condition to hold, the price of installed capital must be higher than that in equation (6) in the good state, and lower than that in equation (7) in the bad state. Note that the left-hand side of equation (1) is the benefit of exercising the option at $t = 1$, and the right-hand side of equation (1) is the benefit of waiting until $t = 2$ to start production. Because of the possibility of a regime switch in $\theta$, the benefit of waiting in the good state is smaller: in the case of a regime switch in $\theta$, the idea will produce less consumption good at date $t = 2$. This induces more incentives to exercise the option at $t = 1$. To deter entrance, the equilibrium price of installed capital, therefore, must be higher than in equation (6). Similarly, in the bad state, the possibility of a regime switch in $\theta$ implies a higher benefit of waiting: in the case of a regime switch, the idea will produce more consumption good at $t = 2$, compared to case of no regime switch. This induces an additional incentive for ideas to wait in the bad state. Consequently, the equilibrium price of installed capital must drop to clear the market. Therefore, the market clearing price of installed capital in equation (4) is lower than that in the case of $p = 0$. To summarize, the combination of inelastic supply of installed capital and the mean reverting nature of $\theta$ creates a mechanism that makes the price of installed capital vary more than assets in place, generating the value premium.

In the rest of the paper, we imbed the above mechanism in a fully dynamic equilibrium model with long-run risks. The model, as we show, is able to generate the value premium, which is not captured by the conditional CAPM, and is consistent with empirical dynamics of asset dividends and returns documented in the recent long-run risks literature.

II Set-up of the Model

A Preference

Consider an infinite-horizon economy with a representative consumer. The flow rate of aggregate consumption is assumed to follow the process:

$$dC_t = C_t [\theta_t dt + \sigma_C (\theta_t) dB_C t]$$

(8)

where $\{B_C t\}_{t \geq 0}$ is a one-dimensional standard Brownian motion. $\{\theta_t\}_{t \geq 0}$ is a two-state Markov process with state space $\Theta = \{\theta_H, \theta_L\}$, where $\theta_H > \theta_L$. The transition probability of $\theta_t$ over an infinitesimal time interval $\Delta$ is given by:

$$
\begin{bmatrix}
    e^{-\lambda_H \Delta} & 1 - e^{-\lambda_H \Delta} \\
    1 - e^{-\lambda_L \Delta} & e^{-\lambda_L \Delta}
\end{bmatrix}
$$
The Markov chain $\{\theta_t\}_{t \geq 0}$ is assumed to be persistent to capture the idea of long-run risks (Bansal and Yaron (2004)). We also allow the diffusion coefficient of consumption growth to depend on the state variable $\theta$. This provides a parsimonious way of modeling stochastic volatility of consumption. When $\theta = \theta_H$, the expected growth rate of consumption is high, the economy is in a boom. When $\theta = \theta_L$, the expected consumption growth is low, consequently, the economy is in a recession.

The representative consumer’s intertemporal preferences are represented by the Kreps and Porteus (1978) utility with constant relative risk aversion parameter $\gamma > 0$ and constant intertemporal elasticity of substitution parameter (IES) $\psi > 0$. Following Duffie and Epstein (1992a) and Duffie and Epstein (1992b), we represent preferences as stochastic differential utility in continuous time.\(^3\) Since the aggregate consumption growth contains a persistent component $\{\theta_t\}_{t \geq 0}$, the economy is a continuous time version of the Bansal and Yaron (2004) model. We use $\{U_t\}_{t \geq 0}$ to denote the utility process of the representative agent. Given a consumption process, $\{C_s : s \geq 0\}$, for every $t \geq 0$, the date-$t$ utility of the agent, denoted $U_t$ is defined recursively by:\(^4\):

$$U_t = E_t \left[ \int_t^\infty f(C_s, U_s) \, ds \right]. \quad (9)$$

In the above equation, $f(C, U)$ is the aggregator of the recursive preference and is given by:\(^5\)

$$f(C, U) = \frac{\beta}{1 - 1/\psi} \frac{C^{1-1/\psi} - ((1 - \gamma)U)^{1-1/\psi}}{((1 - \gamma)U)^{1-1/\psi} - 1}, \quad (10)$$

where $\gamma$ and $\psi > 0$. We assume that $\gamma \neq 1$. We allow $\psi = 1$ with the understanding that in this case,

$$f(C, U) = \beta (1 - \gamma) \left[ \ln C - \frac{1}{1 - \gamma} \ln [(1 - \gamma)U] \right] \quad (11)$$

The representative consumer’s preferences and consumption determine the pricing kernel of the economy, denoted by $\{\pi_t\}_{t \geq 0}$ and characterized in Proposition 1 in Section III.A.

**B Life Cycle Dynamics of Firms**

We focus directly on the equilibrium in the public equity sector and do not attempt to endogenize aggregate consumption. We specify the endowment and production technology of the public equity

\(^3\)A discrete time version of preferences is discussed in Kreps and Porteus (1978), Epstein and Zin (1989), and Weil (1989).

\(^4\)Representation of SDU in the infinite horizon case is discussed in Duffie and Epstein (1992b). Existence and uniqueness of SDU of the Kreps and Porteus (1978) type are discussed in Duffie and Lions (1992) and Schroder and Skiadas (1999).

\(^5\)In general, recursive preferences are characterized by a pair of aggregators $(f, A)$. Duffie and Epstein (1992b) show that one can always normalize $A = 0$. The aggregator $f$ used here is the normalized aggregator.
sector traded on the stock market and aim to understand the economic mechanism that generates differences in expected asset returns.

There are two types of endowment in this economy: endowment of ideas and endowment of installed capital. Ideas arrive exogenously at rate $m_X$ per unit of time, and endowment of installed capital arrives exogenously at rate $m_Z$ per unit of time, where $m_X > m_Z$.

Ideas are storable. An idea by itself does not produce any consumption good. An idea starts producing consumption good once it is matched with one unit of installed capital. In other words, an idea is an option on a stream of future consumption goods, which could be obtained once a unit of installed capital is acquired. Ideas capture the essence of growth assets. They do not carry any installed capital and are long-duration assets in the sense that they do not generate any cash flow immediately but are expected to generate cash-flow stream in the future. For an unmatched idea $i$, we use the notation $t^i$ to denote the calendar time at which it acquires a unit of installed capital.

Unmatched ideas differ by their quality. The initial quality of an idea $i$ is drawn from a time-invariant continuous density $u(\cdot)$ with support $[0, \bar{X}]$. After birth, an unmatched idea die at Poisson rate $\kappa_X > 0$. Conditioning on survival, the quality of the idea, denoted $X^i_t$ evolves randomly according to the following stochastic differential equation:

$$dX^i_t = X^i_t \left[ \mu_X dt + \sigma_X dB^i_t \right], \quad t \leq t^i,$$

until the idea is matched with a unit of installed capital, or hit by the Poisson death shock. In equation (12), $\{B^i_t\}_{t \geq 0}$ are standard Brownian motions and are independent among ideas. That is, the quality of ideas are geometric Brownian motions, and the Brownian motion risks are idiosyncratic.

Installed capital is not storable unless it is matched with ideas. Each idea can be matched with at most one unit of installed capital. An idea matched with one unit of installed capital becomes a production unit and produces consumption good. Production units are value assets. They carry one unit of installed capital each and are short duration assets (when compared to unmatched ideas) – they produce consumption goods, which are paid right away as dividend to owners of production units.

Production units die at Poisson rate $\kappa_Z > 0$. Conditioning on survival, a production unit produces consumption goods at rate $Z^i_tC_t$, where $Z^i_t$ is the productivity of the production unit $i$ at calendar time $t$. In other words, at any time $t$, a production unit produces a fraction of aggregate consumption, where the fraction depends on its productivity at time $t$. The initial productivity of a production unit depends on the quality of the idea that is used to set up the production unit. For idea $i$, which acquires one unit of installed capital and becomes a production unit at time $t^i$,
the initial productivity of the production unit at time $t^i$ is:

$$Z^i_{t^i} = X^i_{t^i}.$$  

In order to operate the production unit, a per period flow cost has to be paid in the amount of:

$$o(\theta_t) C_t, \quad \text{for } t \geq t^i.$$  

The above specification allows the operating cost of production units to depend on the aggregate state of the economy through function $o(\cdot)$. This allows the equilibrium failure rate of firms to differ in booms and recessions. To summarize, the rate of net cash flow generated by a production unit $i$ is given by:

$$(Z^i_t - o(\theta_t)) C_t \quad \text{for } t \geq t^i$$  

Claims to production units pay cash flow in the amount given in (13) as dividends.

Conditioning on survival, productivity of a production unit evolves according to the following stochastic differential equation:

$$dZ^i_t = Z^i_t \left[ \mu_Z(\theta_t) dt + \sigma_Z(\theta_t) dB^i_t \right], \quad t \geq t^i$$  

(14)

The drift and diffusion coefficient of the productivity process are identical among all firms and are assumed to depend on the long-run risk state variable $\theta$. The Brownian motion shocks $B^i_t$ are independent among production units. This specification captures the idea that dividend payments of firms in the public equity sector have different exposure to long-run risks than aggregate consumption. In particular, $\mu_Z(\theta_H) > \mu_Z(\theta_L)$ implies that dividend payment has higher exposure to long-run risks than aggregate consumption, which is consistent with the cross-sectional evidence in the long-run risks literature (eg., Bansal, Dittmar, and Lundblad (2005) and Hansen, Heaton, and Li (2006)).

A profit maximization firm who owns an unmatched idea can choose to purchase a unit of installed capital and turn the idea into a production unit at any time. The decision to match installed capital with an idea is irreversible and once matched, installed capital cannot be matched productively with a different idea later on. The price of installed capital will be determined by equilibrium conditions. A firm can choose to abandon an idea or a production unit at any time. Once abandoned, both the ideas and installed capital evaporate. The decision to abandon an idea or a production unit is irreversible.
C Notion of Competitive Equilibrium

An unmatched idea has an option to acquire a unit of installed capital and begin to produce consumption good, and an option to exit. Since no cost is needed to keep the idea alive, it is never optimal for an idea to exit voluntarily. We denote the equilibrium price of installed capital, measured in units of current period consumption numeraire as \( q_t \). The optimization problem of idea \( i \) at calendar time \( t \leq t_i \) is written as:

\[
V_{GR}(i,t) \equiv \max_{\tau} E_t \left[ \frac{t^\tau \pi_t}{\pi_t} \left\{ V_{VA}(i,\tau) - q_\tau \right\} \right], \ t \leq t_i
\]

(15)

where \( V_{GA}(i,t) \) is the value of the idea, and \( V_{VA}(i,\tau) \) denotes the value of a production unit set up by idea \( i \) at the time of exercise. In the class of equilibria we consider in this paper, the optimal decision rule for an idea can be characterized by an optimal exit threshold \( X^* \) in terms of its quality. It is optimal for idea \( i \) to exercise growth option if \( X^i_t \geq X^* \). That is, if we denote the optimal stopping time for the maximization problem in (15) as \( \tau_{GR}(i,t) \), then,

\[
\tau_{GR}(i,t) = \inf \left\{ t : X^i_t > X^* \right\}
\]

(16)

Figure 1 depicts the dynamics of a cohort of new ideas born at calendar time \( t \). At time \( t \), a cohort of ideas are drawn from the distribution \( u(\cdot) \) with support \( [0,X] \). The rate at which these ideas are drawn is \( m_X \) per unit of time. Some of the ideas (group [1] in Figures 1 and 2) are born with quality higher than the option exercise threshold, \( X^* \). They purchase one unit of installed capital immediately and become production units right after birth. Others (group [2] in Figure 1) are born with quality lower then the option exercise threshold \( X^* \). These ideas will wait and exercise growth options later once their quality becomes high enough. The quality of these idea will evolve according to the law of motion in equation (12). Some of them will eventually exercise growth options and become production units. These ideas are represented by group [4] in Figure 1 and Figure 2. Others, born with quality lower than \( X^* \), die prematurely at the exogenous rate \( \kappa_X \). These ideas are represented by group [3] in Figure 1. If the quality of an idea reaches the threshold level \( X^* \), it will purchase one unit of capital and starts generating cash flow. Therefore, \( X^* \) is the absorbing barrier of the cross-section distribution of the quality of unmatched ideas.

A production unit faces the following optimal stopping problem: it has an option to stop operation and exit the public equity sector. It will do so if productivity is too low to justify the operating cost. Mathematically, a production unit’s optimization problem is written as:

\[
V_{VA}(i,t) = \max_{\tau} E_t \left[ \int_0^\tau e^{-\kappa_X(t+s)} \frac{\pi_{t+s}}{\pi_t} \left( Z_{t+s}^i - o(\theta_{t+s}) \right) C_{t+s}ds \right], \ t \geq t_i.
\]

(17)

In the above expression, \( \{\pi_t\}_{t \geq 0} \) is the state price density of the economy determined by aggregate consumption and the representative consumer’s preferences, and \( V_{VA}(i,t) \) denotes the value of the
production unit $i$ at calendar time $t \geq t^i$. The maximization is taken over all stopping times $\tau$ (that are adapted to an appropriately defined filtration). The optimal stopping time for production units can be characterized by a pair of state-contingent optimal exit thresholds $\{Z^*(\theta_H), Z^*(\theta_L)\}$ in terms of their productivity. If the current state of the economy is $\theta_H$, then it is optimal for the production unit $i$ to exit if $Z^i_t \leq Z^*(\theta_H)$. Similarly, if the current state of the economy is $\theta_L$, then it is optimal for the production unit to exit if $Z^i_t \leq Z^*(\theta_L)$. Mathematically, if we denote the optimal stopping time for the maximization problem in (17) as $\tau_{VA}(i, t)$, then,

$$
\tau_{VA}(i, t) = \inf \left\{ t : \theta_t = \theta_H, \text{ and } Z^i_t < Z^*(\theta_H) \right\} \cup \left\{ t : \theta_t = \theta_L, \text{ and } Z^i_t < Z^*(\theta_L) \right\}
$$

(18)

Figure 2 shows the dynamics of a cohort of production units set up at time $t$. A production unit may be created by a new born idea with quality higher than $X^*$, or by an old idea that just hit the absorbing barrier $X^*$. Some production units die exogenously because of the Poisson shock which arrives at rate $\kappa_Z$. These are denoted by group [5]. Others exit the public equity sector voluntarily if their productivity is lower than the exit threshold $Z^*(\theta)$. These ideas are represented by group [6].

The equilibrium price of installed capital is determined by the market clearing condition. During any time interval, the total measure of ideas that exercise growth option must be equal to the total supply of installed capital. The total measure of ideas that exercise options will depend on the cross-sectional distribution of the quality of ideas. The cross-sectional distribution of the quality of ideas is, in general, history dependent, which makes the equilibrium hard to characterize. We focus on equilibria, in which the cross-sectional distribution of the quality of ideas has a stationary distribution. Section III of the paper shows that under very general conditions the cross-section distribution of the quality of ideas will converge to a unique stationary distribution asymptotically. We call such an equilibrium Competitive Equilibrium with Stationary Distribution of Quality of Ideas, or CE with Stationary Distribution for short. The precise definition of CE with stationary distribution is given below.

**Definition 1:** CE with Stationary Distribution

A CE with Stationary Distribution consists of:

1) Price of the firms: $\{V_{GR}(i, t), V_{VA}(i, t)\}_{i, t}$, and price of installed capital $\{q_t\}_t$.

2) Optimal stopping times for ideas and production units: $\{\tau_{GR}(i, t), \tau_{VA}(i, t)\}_{i, t}$

3) Optimal option exercise threshold for ideas $X^*$, and optimal exit threshold for production units $\{Z^*(\theta)\}_{\theta = \theta_H, \theta_L}$.

4) Density of the stationary measure of the cross-section distribution of the quality of ideas: $\Phi(\cdot)$. 

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such that the above quantities satisfy:

   a) Share-holder value maximization for production units.

\[ \forall i, t, V_{VA}(i, t) \text{ and } \tau_{VA}(i, t) \text{ are the value function and solution to the optimal stopping problem described in (17). Furthermore, } \tau_{VA}(i, t) \text{ takes the form given in (18).} \]

   b) Share-holder value maximization for ideas.

\[ \forall i, t, V_{GR}(i, t) \text{ and } \tau_{GR}(i, t) \text{ are the value function and solution to the optimal stopping problem described in (15). Furthermore, } \tau_{GR}(i, t) \text{ takes the form given in (16).} \]

c) Market clearing for installed capital:

\[ \forall t, \ m_X \int_{X^*}^{\mathbb{Z}} u(z) \, dz + m_{EXIT} [\Phi, X^*] = m_Z \]  

where \( m_{EXIT} [\Phi, X^*] \) denotes the absorbing rate of measure \( \Phi \) at the absorbing barrier \( X^* \). The above condition says that, in equilibrium, the total rate of exercise of growth options is equal to \( m_Z \).

d) Consistency of Macro- and Micro- variables. The density of the cross-sectional distribution of the quality of ideas \( \Phi \) with absorbing barrier \( X^* \) is consistent with the law of motion of the quality of individual ideas as described in (12).

The equilibrium requirements a) and b) are straightforward. The market clearing condition in (19) states that the demand and supply of installed capital must equal in equilibrium. The left hand-side of equation (19) has two terms. The first term is the demand for installed capital from the newly born ideas whose quality is high enough to exercise growth options immediately after birth. The second term in (19) is the demand for installed capital from old ideas crossing the absorbing barrier \( X^* \). The flow rate of these ideas is \( m_{EXIT} [\Phi, X^*] \) per unit of time. Since we have a model of heterogenous ideas, we need some consistency condition to guarantee that the variables that describe macroeconomic quantities are consistent with the individual behavior of the ideas. Technically, this implies that \( \Phi \) has to satisfy a version of the Komogorov forward equation. Details of the technical conditions are discussed in Appendix III.

The next section is devoted to the construction of the equilibrium defined above.

### III Characterization of Equilibrium

Below, we construct the Competitive Equilibrium with Stationary Distribution defined in Section II.C. Readers who are anxious to move on to the asset pricing implications of the model can skip the current section and go to Section IV directly.
A Preliminaries

In this subsection, we outline some assumptions that are needed to ensure the existence of the Competitive Equilibrium with Stationary Distribution. We also solve for the pricing kernel of the economy implied by preferences and aggregate consumption of the representative consumer.

We first make some simplifying assumptions on the parameter values that will allow us to avoid cumbersome mathematical expressions and derive sharp and intuitive characterization of the equilibrium. Our model can be solved for arbitrary preference parameter values of $\gamma$ and $\psi$. Assuming $\psi = 1$ greatly simplifies the math while still allows the model to capture the mechanism of long-run risks as long as $\gamma > 1^{6}$. We will maintain these assumption in Sections III and IV. We do allow for $\psi \neq 1$ in Section V when we calibrate the model. Assumptions on preference parameters are summarized below.

**Assumption 1:** The risk aversion and intertemporal elasticity of substitution parameters satisfy:

$$\gamma > 1, \text{ and } \psi = 1$$

In our framework, the necessary and sufficient condition for long-run risks to require a positive premium in equilibrium is that the agent has higher utility in booms (i.e., when $\theta_t = \theta_H$) than in recessions (when $\theta_t = \theta_L$). Technically, this requires that the volatility of consumption growth in booms is not too high relative to the volatility of consumption growth in recessions. This assumption is supported by the counter-cyclical variation of consumption growth volatility in the data. The technical condition on the drift and volatility of consumption growth is summarized in Assumption 2.

**Assumption 2:** The drift and diffusion coefficient of aggregate consumption growth satisfy:

$$(\theta_H - \theta_L) - \frac{1}{2} \gamma \left[ \sigma_C^2 (\theta_H) - \sigma_C^2 (\theta_L) \right] > 0 \quad (20)$$

Given the above conditions, the pricing kernel, or the state price density in the language of Duffie (2001), is characterized in the following proposition.

**Proposition 1 State Price Density of the Economy**

The state price density of the economy, denoted $\{\pi_t\}_{t \geq 0}$, is a Levy process of the form:

$$d\pi_t = \pi_t \left[ -r (\theta_t^-) \, dt - \gamma \sigma (\theta_t^-) \, dB_t - \eta_x (\theta_t^-)^T d\bar{N}_t \right] \quad (21)$$

---

6If $\gamma < \psi$, then the utility function features preferences for late resolution of uncertainty and the model will generate negative premium for long-run risks.
where $\tilde{N}_t$ is a compensated Poisson measure defined in Appendix I, and

$$\eta_n(\theta) = \left[ (1 - \omega^{-1}) I_{(\theta_H)}(\theta), (1 - \omega) I_{(\theta_L)}(\theta) \right],$$

where $\omega < 1$ is a constant given in Appendix I. Furthermore,

$$r(\theta) = \beta + \theta - \gamma \sigma C^2(\theta)$$

is the risk-free rate of the economy.

The quality of ideas and productivity of production units could potentially grow without bound. Together with Assumptions 1 and 2, the following assumption guarantees that the value of ideas and production units stay finite.

Assumption 3:

$$\beta + \kappa_Z - \mu(\theta) > 0 \text{ for } \theta = \theta_H, \theta_L$$
$$\beta + \kappa_X - \mu_X > 0$$

As long as the hazard rate of death is positive, that is $\kappa_X, \kappa_Z > 0$, the total measure of ideas and production units in the economy will remain finite. To guarantee that the share of total production of the public equity sector in aggregate consumption is always finite, the following assumption has to be satisfied.\(^7\)

Assumption 4:

$$\kappa_Z > \mu_Z(\theta) \text{ for } \theta = \theta_H, \theta_L$$

Clearly, Assumption 4 implies (24). Note that in our model, individual production units are not cointegrated with aggregate consumption; however, under the above assumption, the dividend stream of the aggregate market portfolio and consumption share a common stochastic trend. Assumptions 1-4 will be maintained throughout Sections III and IV.

**B Firms’ Optimization Problems**

We construct the equilibrium of the economy via the following procedure. We use the market clearing condition (19) to determine the unique $X^*$ in a competitive equilibrium with stationary distribution. We then use the optimality condition of option exercise together with the market clearing $X^*$ to solve for the equilibrium price of installed capital.

\(^7\)As long as the total production in the public equity sector is finite, we can always make sure that it is less than aggregate consumption by normalizing $\bar{X}$. Since changing $\bar{X}$ does not affect the asset pricing implications of the model, we do not impose any conditions on $\bar{X}$ except in Section V when we calibrate the model.
The optimal stopping problem of production units does not depend on the price of installed capital and can be characterized without solving for the equilibrium. The solution to the optimal stopping problem for production units is summarized in the following proposition.

**Proposition 2 Optimal Stopping for Production Units**

The value function of the optimization problem for production units is of the following form:

\[ V_{VA}(i, t) = G_{VA}(Z_i, \theta_t) C_t, \quad (26) \]

where

\[ G_{VA}(Z, \theta) = a(\theta) Z - b(\theta) o(\theta) + \sum_{i=1}^{2} e_{VA}(i, \theta) K_i Z^{\alpha_{VA}(i)}, \quad (27) \]

where

\[ G_{VA}(Z) = [G_{VA}(Z, \theta_H), G_{VA}(Z, \theta_L)]^T \]

and \( \{\alpha_{VA}(i)\}_{i=1,2} \) are constants, and \( a(\theta), b(\theta), [e_{VA}(i, \theta)]_{i=1,2,3,4} \) are functions of \( \theta \) given in Appendix II.

Furthermore, \( Z^*(\theta_H) \) and \( Z^*(\theta_L) \) along with the two constants \( K_1 \) and \( K_2 \) are jointly determined by the two value matching conditions:

\[ \begin{bmatrix} G_{VA}(Z^*(\theta_H), \theta_H) \\ G_{VA}(Z^*(\theta_L), \theta_L) \end{bmatrix} = 0 \quad (28) \]

and the smooth pasting conditions:

\[ \begin{bmatrix} G'_{VA}(Z^*(\theta_H), \theta_H) \\ G'_{VA}(Z^*(\theta_L), \theta_L) \end{bmatrix} = 0 \quad (29) \]

**Proof.** See Appendix III. ■

Recall that production units exit the public equity sector optimally if their productivity hits \( Z^*(\theta_H) \) from above in state \( \theta_H \), or hits \( Z^*(\theta_L) \) from above in state \( \theta_L \), whichever happens earlier. According to the above proposition, the value function is homogenous in aggregate consumption \( C_t \). We will refer to function \( G_{VA}(Z, \theta) \) as the normalized value function of production units. Note that term \( a(\theta) Z \) is the present value of cash flow \( \{Z_i C_t\}_{i=0}^{\infty} \), and \( b(\theta) o(\theta) \) is the present value of the operating cost \( \{o(\theta_t) C_t\}_{i=0}^{\infty} \). The term \( \sum_{i=1}^{2} e_{VA}(i, \theta) K_i Z^{\alpha_{VA}(i)} \) captures the value of the exit option.

The solution to the optimization problem of ideas will depend on the equilibrium price of installed capital. In the class of equilibria we construct, the equilibrium price of installed capital
will take a simple form:
\[ \forall t \geq 0, \quad q_t = f(\theta_t) C_t \]  
(30)
for some function \( f(\theta) \), which will be determined endogenously by equilibrium conditions. Condition (30) greatly simplifies the problem, and allows us to derive closed-form solutions to the optimization problem of ideas, which are summarized in the following proposition.

**Proposition 3 Optimal Stopping for Ideas**

Suppose the equilibrium price of installed capital is given in (30), then the value function of the optimization problem for ideas is of the form:
\[
V_{GR}(i,t) = G_{GR}(X_i^t, \theta_t) C_t,
\]
where
\[
G_{GR}(X, \theta) = \sum_{i=3}^{4} e_{GR}(i, \theta) L_i X^{\alpha_{GR}(i)}
\]  
(31)
where \( \{\alpha_{GR}(i)\}_{i=3,4} \) are constants, and \( \{e_{GR}(i, \theta)\}_{i=1,2,3,4} \) are functions of \( \theta \) given in Appendix II.

The optimal stopping time, denoted \( \tau_{GR} \) is given by:
\[
\tau_{GR} = \inf \{ t : \theta_t = \theta_H, \text{ and } X_i^t > X^*(\theta_H) \} \cup \{ t = \theta_L, \text{ and } X_i^t > X^*(\theta_L) \}
\]

where \( X^*(\theta_H) \) and \( X^*(\theta_L) \) along with the two constants \( \{L_i\}_{i=3,4} \) are jointly determined by the value matching and smooth pasting conditions:
\[
\begin{bmatrix}
G_{GR}(X^*(\theta_H), \theta_H) \\
G_{GR}(X^*(\theta_L), \theta_L)
\end{bmatrix} =
\begin{bmatrix}
G_{VA}(X^*(\theta_H), \theta_H) - f(\theta_H) \\
G_{VA}(X^*(\theta_L), \theta_L) - f(\theta_L)
\end{bmatrix}
\]
(32)
\[
\begin{bmatrix}
G_{GR}(X^*(\theta_H), \theta_H) \\
G_{GR}(X^*(\theta_L), \theta_L)
\end{bmatrix} =
\begin{bmatrix}
G_{VA}(X^*(\theta_H), \theta_H) \\
G_{VA}(X^*(\theta_L), \theta_L)
\end{bmatrix}
\]
(33)

**Proof.** See Appendix II. ■

The above proposition implies that the solution to the optimization problem of unmatched ideas can also be characterized by a pair of exit barriers, one for each state. An idea will exercise growth option once its quality is high enough, that is, once \( X_i^t \) hits the option exercise boundary \( X^*(\theta_H) \) from below in state \( \theta_H \), or \( X_i^t \) hits \( X^*(\theta_L) \) from below in state \( \theta_L \). Since ideas do not produce any consumption good, the value of ideas comes completely from their growth option feature.

The next section closes the model by imposing market clearing for installed capital and stationarity of the quality of ideas.
C  Stationary Distribution of the Quality of Ideas

In this section, we explicitly solve for the stationary distribution of the quality of ideas, \( \Phi \) for a given absorbing barrier \( X^* \). This will allow us to impose the market clearing condition (19) and solve for the unique \( X^* \) that equates supply and demand of installed capital. The equilibrium price and quantities will be completely characterized once the market clearing \( X^* \) is determined.

To facilitate closed form solution for the stationary distribution of the quality of ideas, we assume that the initial quality of ideas is of uniform distribution. That is:

**Assumption 5:**

\[
u(X) = \frac{1}{X}, \quad X \in [0, X]\]

In the equilibrium we construct, the distribution of the quality of ideas is time-invariant, so is the option exercise threshold for ideas, \( X^* \). At any point in time, the total measure of ideas that enter into the economy is \( m_X \), and their initial quality is uniformly distributed on the interval \([0, X]\). Measure \( \frac{X^*-X}{X}m_X \) of them has quality higher than the option exercising barrier \( X^* \), and acquires installed capital to become production units immediately after birth. This is group \([1]\) in Figure 1. A measure \( \frac{X^*}{X}m_X \) of ideas has initial quality below the option exercise boundary \( X^* \). This is group \([2]\) firms depicted in Figure 1. They exit if either they are hit by the exogenous death shock at Poisson rate \( \kappa X \), or their quality hits the option exercise threshold \( X^* \). Given the absorbing barrier \( X^* \), we can solve for the stationary distribution of the quality of ideas in closed form, which is done in Appendix III. Once the distribution of the quality of ideas is known, we can find the exit rate of ideas at the given absorbing barrier \( X^* \). Together with the market clearing condition for installed capital, this determines the equilibrium option exercise threshold \( X^* \). The exit rate at a given absorbing barrier \( X^* \) is characterized by the following proposition.

**Proposition 4  Option Exercise Rate of Unmatched Ideas**

Suppose the option exercise barrier of ideas is a constant \( X^* \), then the stationary distribution of the quality of ideas exits. If \( X^* \leq X \), then the exit rate of ideas at the absorbing barrier of the stationary distribution, \( X^* \) is:

\[
m_{EXIT}[\Phi, X^*] = \frac{1}{1 - \eta_2 \frac{X^*}{X}} m_X, \quad (34)
\]

where \( \eta_2 \) is a function of the parameters \((\kappa_X, \mu_X, \sigma_X)\) given in Appendix III.

If \( X^* > X \), then the exit rate at the absorbing barrier, \( X^* \), is given by:

\[
m_{EXIT}[\Phi, X^*] = \frac{1}{1 - \eta_2 \left(\frac{X^*}{X}\right)^{\eta_2}} m_X \quad (35)
\]
Proof. See Appendix III. □

Using equations (34) and (35), the market clearing condition (19) can be written in closed form. We can then solve for the market clearing option exercise threshold \( X^* \) as a function of the primitive parameters of the model. This is stated below as a Corollary of the above proposition.

**Corollary 1 Market Clearing Option Exercise Threshold**

The unique option exercise threshold \( X^* \) determined by the market clearing condition (19) is given by:

\[
X^* = \delta \bar{X}
\]

If \( \frac{m_Z}{m_X} \in \left[ \frac{1}{1-\eta_2}, 1 \right) \), then \( \delta \) is given by:

\[
\delta = \frac{1 - \frac{m_Z}{m_X}}{\frac{1}{1-\eta_2}} \in (0, 1]
\]

If \( \frac{m_Z}{m_X} \in \left( 0, \frac{1}{1-\eta_2} \right) \), then \( \delta > 1 \) is determined by the unique solution to the following equation on \((1, \infty)\):

\[
\frac{1}{1-\eta_2} \delta^{\eta_2} = \delta - 1 + \frac{m_Z}{m_X}
\]

Since \( m_Z < m_X \), and installed capital is not storable unless matched immediately with ideas, in any Competitive Equilibrium with Stationary Distribution, there will always be a surplus of ideas, but no idle installed capital. The above corollary implies that if the supply of installed capital is relatively abundant, that is if \( \frac{m_Z}{m_X} \geq \frac{1}{1-\eta_2} \), then the optimal threshold level is below \( \bar{X} \). In this case, there is enough supply of installed capital to absorb some new ideas of high quality, so that \( X^* < \bar{X} \). In this scenario, some new-born ideas always become production units immediately after birth. If the supply of installed capital is small relative to the supply of ideas, that is, when \( \frac{m_Z}{m_X} < \frac{1}{1-\eta_2} \), then all newly born ideas have to wait before they could become production units.

The equilibrium price and quantity can be obtained through the following procedure. We first solve the optimization problem of production units and obtain the normalized value function \( G_{VA}(Z, \theta) \) using Proposition 2. We use the above corollary to obtain the equilibrium level of the optimal option exercise threshold \( X^* \). Given \( X^* \), we use Proposition 3 to solve for the equilibrium price \( q_t \) and the optimal stopping problem for ideas. In fact, once we impose \( X^* (\theta_H) = X^* (\theta_L) = X^* \) the two value matching conditions (32) and the smooth pasting conditions (33) can be used to jointly to determine \( f (\theta_H) \) and \( f (\theta_L) \), and the constants \( L_3 \) and \( L_4 \).
IV Asset Pricing Implications

There are two sources of aggregate risks in the economy: short-run consumption risks (represented by the Brownian motion $B_{Ct}$) and long-run consumption risks (represented by $\theta_t$) that in equilibrium carry different risk premia. Every asset $i$ in our economy has different exposure to the two consumption risks, i.e., different long-run and short-run consumption betas: $\beta^i_{LR} \neq \beta^i_{SR}$. The expected return of any asset will depend on its two betas and the two market prices of risk. Therefore, a one-factor conditional CAPM is bound to fail in this economy. Section IV.A discusses the determination of the risk premium in the economy. We characterize asset $\alpha$'s in the conditional CAPM regressions in Section IV.B and provide conditions under which assets with high exposure to long-run risks obtain high $\alpha$'s. In Section IV.C, we prove the key result of the paper: value premium exists in equilibrium if and only if firm’s dividend payments have higher exposure to long-run risks than aggregate consumption.

A Risk Premium of Firms’ Assets

In the economy constructed above, the value of a generic asset $i$ is of the form:

$$V^i_t = G(i, \theta_t) C_t$$  \hspace{1cm} (36)

with the understanding that for ideas, $G(i, \theta_t) = G_{GR} (X^i_t, \theta_t)$, and for production units, $G(i, \theta_t) = G_{VA} (Z^i_t, \theta_t)$. Let $D_{i,t}$ denote the rate of the dividend payment of asset $i$ at time $t$. We can define the cumulative return process of asset $i$, $\{R^i_t\}_{t \geq 0}$ via the following equation:

$$dR^i_t = \frac{1}{V^i_t} [dV^i_t + D^i_t dt]$$ \hspace{1cm} (37)

The expected return of holding the asset during a small interval $[t, t + \Delta]$ is therefore given by:

$$E_t \left[ \frac{R^i_{t+\Delta}}{R^i_t} - 1 \right]$$

The risk premium of asset $i$, denoted $RP(i,t)$, is defined by:

$$RP(i,t) = \lim_{\Delta \to 0} \frac{1}{\Delta} E_t \left[ \frac{R^i_{t+\Delta}}{R^i_t} - 1 \right] - r(\theta_t)$$

where $r(\theta_t)$ is the instantaneous risk-free rate of this economy as given in equation (23). The risk premium of asset $i$ with value of the form in (36) is given by the following proposition.

Proposition 5 Risk Premium
This risk premium of firm $i$ whose value given by (36) is:

$$RP(i,t) = \gamma \sigma^2_C \left( \theta_L \right) + \lambda_L (1 - \omega) \left[ \frac{G(i, \theta_H)}{G(i, \theta_L)} - 1 \right] \quad \text{if } \theta_t = \theta_L \quad (38)$$

$$RP(i,t) = \gamma \sigma^2_C \left( \theta_H \right) + \lambda_H (1 - \omega^{-1}) \left[ \frac{G(i, \theta_L)}{G(i, \theta_H)} - 1 \right] \quad \text{if } \theta_t = \theta_H \quad (39)$$

where $\omega < 1$ governs the volatility of the stochastic discount factor and is formally defined in Appendix I.

**Proof.** Appendix IV. □

The above proposition has a simple intuitive interpretation. The risk premium of a firm’s asset has two components: compensation for short-run risks and compensation for long-run risks. The first term in (38) and (39) can be written as:

$$\gamma \sigma^2_C \left( \theta_t \right) = \gamma \lim_{\Delta \rightarrow 0} Cov_t \left( \frac{V_i^{t+\Delta} - V_i^t}{V_i^t}, \frac{C_{t+\Delta} - C_t}{C_t} \right).$$

Clearly, this term is the compensation for the covariance of return of the asset with contemporaneous innovations in consumption growth, that is, the compensation for the asset’s exposure to short-run consumption risks. The second component of the risk premium is the compensation for the covariance of the asset return with innovations in expected consumption growth: $\theta_t$. Note that $\lambda_L (1 - \omega) > 0$. Therefore, the higher the size of the jump in the firm’s asset value associated with a regime switch in $\theta$, the higher is the second component of risk premium. In fact, the second component is proportional to:

$$\lim_{\Delta \rightarrow 0} Cov_t \left( \frac{V_i^{t+\Delta} - V_i^t}{V_i^t}, \frac{\theta_{t+\Delta} - \theta_t}{\theta_t} \right)$$

In the model economy, all firms have the same exposure to short-run consumption risks, but differ in their exposure to long-run consumption risks. The exposure of an asset to long-run risks depends on the asset’s characteristics, which is captured by the $G(i, \theta_t)$ function. Ideas are growth assets and have different $G(i, \theta_t)$ functions from production units, which are already assets in place. Production units with different productivity levels also differ in their exposure to long-run risks, which is captured by the $G(i, \theta_t)$ function. Below, we show that this feature of the model makes the conditional CAPM fail, and generates the value premium as an equilibrium phenomenon.
B Failure of Conditional CAPM

As shown in section IV.A, the risk premium of an asset is determined by both the asset’s exposures to short-run and long-run consumption risks. The null hypothesis of the conditional CAPM fails because it is a one-factor model. The key result in this section is Proposition 6 that provides conditions under which assets with high exposure to long-run risks will obtain a high $\alpha$ in the conditional CAPM regressions. These conditions are in fact empirically plausible and are satisfied in most calibrated long-run risks models.

For any asset $i$, let $R^i_t$ denote the cumulative return process of the asset as defined in (37). We consider the following conditional CAPM regressions in the model economy:

1. Conditional CAPM with a reference asset:

   $$R^i_{t,t+\Delta} - r_{t,t+\Delta} = \alpha^i_{t,\Delta} + \left(R^{ref}\frac{R^i_{t,t+\Delta}}{R^i_t} - r_{t,t+\Delta}\right) \beta^i_{t,\Delta} + \varepsilon_{t,t+\Delta}.$$  

   (40)

   In the above specification,

   $$R^i_{t,t+\Delta} \equiv \frac{R^i_{t,t+\Delta} - R^i_t}{R^i_t},$$

   where $R^i_t$ is the cumulative return process of asset $i$ defined in equation (37). Therefore, $R^i_{t,t+\Delta}$ is the return of asset $i$ during the time interval $[t, t + \Delta]$. $r_{t,t+\Delta}$ is the return of the locally risk-free asset during the same time interval. Similarly,

   $$R^{ref}_{t,t+\Delta} = \frac{R^{ref}_{t,t+\Delta} - R^{ref}_t}{R^{ref}_t},$$

   is the return to a reference asset with cumulative return process $\{R^{ref}_t\}_{t \geq 0}$. The reference asset could be the aggregate wealth of the economy, in which case we will refer to the regression in (40) as to Wealth CAPM. Alternatively, the reference asset could be the aggregate equity portfolio that we will refer to as to Market CAPM.

2. Conditional CCAPM:

   $$R^i_{t,t+\Delta} - r_{t,t+\Delta} = \alpha^i_{t,\Delta} + \left(C_{t+\Delta} - C_t \frac{C_t}{C^i_t}\right) \beta^i_{t,\Delta} + \varepsilon_{t,t+\Delta}.$$  

   (41)

   Under the null hypothesis of the above CAPM regressions, the $\alpha$’s do not depend on asset’s individual characteristics, that is, $\alpha$’s do not depend on $i$. This would be true if the right-hand side variable were perfectly correlated with the true state price density of the economy. However, as shown in equation (21), innovations in the stochastic discount factor are driven by two risk factors: long-run and short-run risks in consumption. The CAPM regressions are one factor models and, in general, are not able to capture differential exposures of assets to the two sources of consumption.
risks.

In particular, consider the theoretical values of $\alpha$ in the CAPM with a reference asset:

$$\alpha_{i,t} = (E_t [R_{i,t+\Delta}] - r_{t+\Delta}) - \frac{Cov_t \left(R_{i,t+\Delta}, R_{ref,t+\Delta}^{ref}\right)}{Var_t \left(R_{ref,t+\Delta}^{ref}\right)} \left(E_t \left[R_{ref,t+\Delta}^{ref}\right] - r_{t+\Delta}\right)$$

(42)

and in the CCAPM:

$$\alpha_{i,t} = (E_t [R_{i,t+\Delta}] - r_{t+\Delta}) - \frac{Cov_t \left(R_{i,t+\Delta}, C_{t+\Delta} - C_t\right)}{Var_t \left(C_{t+\Delta} - C_t\right)} \left(E_t \left[C_{t+\Delta} - C_t\right]/C_t\right)$$

(43)

To characterize the alpha in the CAPM regression, assume that the value of the reference asset takes the following form:

$$V_{ref,t} = s(\theta_t) C_t$$

(44)

This specification includes the Wealth CAPM as a special case. In fact, the total value of aggregate wealth in the economy is equal to $\frac{1}{\beta}C_t$.\(^8\) Therefore, the CCAPM and Wealth CAPM are equivalent in the model. The Market CAPM is also a special case of (44), if the market index is calculated using a fixed set of equity.\(^9\) It is more convenient to deal with the continuous time limit in our framework:

$$\alpha_{i,t} = \lim_{\Delta \to 0} \frac{1}{\Delta} \alpha_{i,t,\Delta}.$$  

**Proposition 6** In the CAPM with a reference asset, whose value is of the form in (44),

$$\alpha_{i,t} = (\gamma - \chi_t) \sigma_C^2(\theta_H) + \lambda_H \left[(1 - \omega^{-1}) - \chi_t \left(s(\theta_L)/s(\theta_H) - 1\right)\right] \left[G(i, \theta_L) - G(i, \theta_H) - 1\right] \quad \text{if } \theta_t = \theta_H$$

(45)

and

$$\alpha_{i,t} = (\gamma - \chi_t) \sigma_C^2(\theta_L) + \lambda_L \left[(1 - \omega) - \chi_t \left(s(\theta_H)/s(\theta_L) - 1\right)\right] \left[G(i, \theta_H) - G(i, \theta_L) - 1\right] \quad \text{if } \theta_t = \theta_L$$

(46)

where

$$\chi_t = \lim_{\Delta \to 0} \frac{E_t \left[R_{ref,t+\Delta}^{ref}\right] - r_{t+\Delta}}{Var_t \left(R_{ref,t+\Delta}^{ref}\right)}$$

(47)

\(^8\)This result follows from the fact that the wealth-consumption ratio is constant under the assumption of unit IES.

\(^9\)The total value of the equity traded in the public equity sector in the model, however, depends on the distribution of production units, which is a function of the history of $\theta_t$. Therefore, the total market value in the model is not of the form in (44).
In particular, in the CCAPM, and Wealth CAPM,

\[ \alpha_t^i = (\gamma - \chi_t) \sigma_C^2(\theta_H) + \lambda_H (\omega^{-1} - 1) \left[ 1 - \frac{G(i, \theta_L)}{G(i, \theta_H)} \right] \text{ if } \theta_L = \theta_H \]  

(48)

and

\[ \alpha_t^i = (\gamma - \chi_t) \sigma_C^2(\theta_L) + \lambda_L (1 - \omega) \left[ \frac{G(i, \theta_H)}{G(i, \theta_L)} - 1 \right] \text{ if } \theta_L = \theta_H \]  

(49)

**Proof.** Use equation (42) and (43) along with Proposition 5. ■

It follows from the above proposition that in the CCAPM and the Wealth CAPM, assets with high exposure to long-run risks, that is, a high \( \frac{G(i, \theta_H)}{G(i, \theta_L)} \) ratio, will always have high \( \alpha \)'s. In general, assets with high exposure to long-run risks will obtain a high \( \alpha \) if

\[ (\omega^{-1} - 1) > \chi_t \left( 1 - \frac{s(\theta_L)}{s(\theta_H)} \right) \]  

(50)

and

\[ (1 - \omega) > \chi_t \left( \frac{s(\theta_H)}{s(\theta_L)} - 1 \right) \]  

(51)

Note that \( \omega^{-1} - 1 \) is the size of the jump of the state price density when \( \theta_L \) changes from \( \theta_H \) to \( \theta_L \), and \( 1 - \omega \) is the size of the jump of the state price density when \( \theta_L \) changes from \( \theta_L \) to \( \theta_H \). Therefore \( \omega^{-1} - 1 \) and \( 1 - \omega \) measure the sensitivity of the state price density with respect to shifts in \( \theta_t \). Similarly, \( 1 - \frac{s(\theta_L)}{s(\theta_H)} \) and \( \frac{s(\theta_H)}{s(\theta_L)} - 1 \) measure the sensitivity of the return to the reference asset with respect to the jumps in \( \theta_t \). Intuitively, conditions (50) and (51) imply that the state price density is \( \chi_t \) times more sensitive to long-run risks than the return to the reference asset.

Note that \( \chi_t \) is the Sharpe ratio of the reference asset return divided by its volatility. Consider the standard CAPM with the aggregate equity market portfolio as a reference asset. Assume a market risk-premium of \( 6-8\% \), and a volatility of the market return of \( 15-20\% \), then the plausible value of \( \chi_t \) falls in-between 1.5 and 3.5. If the sensitivity of the pricing kernel exceeds the exposure of the market return to long-run risks by more than this amount, the Market CAPM will generate high alphas for assets with high exposure to low-frequency risks. This, in fact, will hold in most calibrated long-run risks models. For example, in Bansal and Yaron (2004), the stochastic discount factor is 4.5 times more sensitive to long-run risks than the market return.

### C Value Premium

In this section, we provide conditions under which ideas have less exposure to long-run risks than production units. Proposition 7 below proves that value premium exists if and only if dividends are more sensitive to long-run risks then aggregate consumption. We also show that the model implies
that value assets with small capitalization have higher exposure to long-run risks than large value assets.

To compare the riskiness of value and growth assets, first note that by Proposition 5, if a production unit with productivity $Z_i$ is riskier with respect to long-run risks than an idea with quality $X_j$, then

$$\frac{G_{VA}(Z_i, \theta_H)}{G_{VA}(Z_i, \theta_L)} > \frac{G_{GR}(X_j, \theta_H)}{G_{GR}(X_j, \theta_L)}. \quad (52)$$

For simplicity, consider a special case in which $X_j = Z_i = X^*$. That is, we compare the riskiness of an idea just before the option exercise and immediately after the option exercise. In this case, the value of the idea after the option exercise is the value of the production unit it makes. The value of this production unit is equal to the value of the same idea before the option exercise plus the value of installed capital, that is:

$$G_{VA}(X^*, \theta) = G_{GR}(X^*, \theta) + f(\theta), \quad \text{for} \quad \theta = \theta_H, \theta_L. \quad (53)$$

Using (53), we can write equation (52) as:

$$\frac{G_{VA}(X^*, \theta_H)}{G_{VA}(X^*, \theta_L)} > \frac{G_{VA}(X^*, \theta_H) - f(\theta_H)}{G_{VA}(X^*, \theta_L) - f(\theta_L)}. \quad (54)$$

It is clear that equation (52) is equivalent to:

$$\frac{f(\theta_H)}{f(\theta_L)} > \frac{G_{VA}(X^*, \theta_H)}{G_{VA}(X^*, \theta_L)}. \quad (55)$$

That is, the growth option is less risky than the asset in place precisely when the cost of installed capital is riskier than the asset in place.

The above argument is illustrated in Figure 3, which plots the normalized value functions for ideas and production units and illustrates the mechanism of the model.\textsuperscript{10} The dashed lines are the normalized value functions for production units in the high state (thick line) and in the low state (thin line). The solid lines are the normalized value functions of ideas. At the option exercise threshold $X^*$, the distance between the two dashed lines is higher than that between the solid lines. That is, when there is a regime switch in the long-run risk state variable $\theta$, in relative terms, the value of production units changes by more than the value of ideas due to the simultaneous adjustment of the price of installed capital. The cost of option exercise is captured by the double-arrowed distances in Figure 3. Notice that the price of installed capital is much higher in good states than in recessions and, in fact, is riskier that assets in place (as shown in (55)). The risk in installed capital, therefore, partially offsets the risk in assets in place, making growth options less risky relative the value assets.

\textsuperscript{10}Figures 3 and 4 are based on the parameter values that we subsequently use in our calibration exercise in Section V.
Implication (55) arises due to the fixed supply of installed capital in our model and the mean reverting nature of long-run risks. The economic intuition behind this has been demonstrated in the simple example in Section I of the paper. Essentially, if the propagability of regime switching in \( \theta \) is close to 0, installed capital must be as risky as assets in place to clear the market. When the economy switches from \( \theta_L \) to \( \theta_H \), the cost of installed capital must increase by the same proportion as the value of assets in place to equate supply and demand of installed capital. The possibility of a regime shift introduces more incentives to exercise the option when \( \theta_t = \theta_H \) and more disincentives to exercise the option when \( \theta_t = \theta_L \). Intuitively, in the good state, holders of ideas want to exercise the option early, because they are worried that the long-run risk may hit the economy in the next period, in which case waiting for one more period will lower the value of the option. To deter entrance and restore the equilibrium, the price of installed capital must be higher in the good state relative to case of no regime switch in \( \theta \). Similarly, the possibility of a regime switch introduces more incentive for holders of ideas to delay the exercise of the option: they want to wait for a regime switch to happen, in which case the idea will have a higher value.

To facilitate analytically simple results and make the intuition as transparent as possible, we make a simplifying assumption that \( o(\theta_H) = o(\theta_L) = 0 \), that is, the operating cost for value assets is assumed to be 0. In this case, growth assets are less risky with respect to long-run risks if and only if dividends paid by production units have more exposure to long-run risks than aggregate consumption. The formal statement is given by the following proposition.

**Proposition 7 Value Premium**

Assume that \( o(\theta_H) = o(\theta_L) = 0 \), then

\[
\frac{G_{VA}(Z, \theta_H)}{G_{VA}(Z, \theta_L)} > \frac{G_{GR}(X^*, \theta_H)}{G_{GR}(X^*, \theta_L)}
\]

for all \( Z > 0 \) if and only if \( \mu_Z(\theta_H) > \mu_Z(\theta_L) \).

**Proof. Appendix**

Figure 4 plots the risk premia of ideas and production units against the quality of ideas and productivity of production units.\(^{11}\) The dashed lines are the risk premia of production units, and the solid lines are the risk premia of ideas. The think lines are risk premia of assets in the high state \( (\theta_t = \theta_H) \), and the two thin lines are risk premia of assets when \( \theta_t = \theta_L \). It is clear that, uniformly, production units (value assets) are riskier than ideas (growth assets).

Notice also that as the productivity (or market capitalization) of a value asset gets smaller, the risk premium of the production unit increases. Thus, our model is able to generate the size premium. Recall that production units have an option to exit the industry. They will do so when

\(^{11}\)Note that Figure 4, as well as Figure 3 discussed above, assume non-zero operating cost parameters.
the productivity is too low and the option value of waiting is not high enough to cover the operating cost. Low productivity production units in our model are more likely to fail when $\theta_t = \theta_L$ and, consequently, will have higher exposure to long-run risks than high productivity units, giving rise to the size premium.

V Calibration

A Parameter Configuration

We evaluate asset pricing implications of the model by calibrating it to the US economy. As we show, the model is able to quantitatively account for the value premium, as well as replicate the failure of the CAPM observed in the data. We simulate the model on the monthly frequency but aim to match time-series dynamics of annual consumption and aggregate dividends. We focus on annual moments in order to avoid any seasonal and measurement biases in the data. To be specific, we simulate monthly series over 80 years, aggregate the simulated variables to the annual frequency and report various moments of the resulting annual data. To remove the effect of initial conditions, we effectively simulate 160 years of data and discard the first half of the sample. We find that increasing the size of the initial simulated sample does not alter the results.

Preference and time-series parameters used in simulations are presented in Table I. We choose preference parameters following the long-run risks literature (see Bansal-Yaron (2004)). In particular, we set the time-discount factor at 0.01, we choose risk aversion of 10 and set the elasticity of intertemporal substitution at 1.5. This choice of preferences (together with technology parameters, discussed below) allows the model to match the dynamics and the level of the risk-free rate, as well as the magnitude of the risk premium in the economy.

Consumption growth parameters are chosen to match time-series properties of observed consumption data. We target moments of real annual per-capita consumption of non-durables and services from the NIPA tables available from the Bureau of Economic Analysis. Our consumption as well as all asset data span the period from 1929 to 2007. The dynamics of simulated consumption series and the corresponding moments of the actual consumption growth are presented in Table II. Our calibration implies the volatility of consumption growth of about 1.9%, comparable to 2.1% in the data. The model-implied first-order autocorrelation of consumption growth of 0.42 matches well with the observed persistence of growth rates. Consistent with the NBER-dated business cycle fluctuations, our calibration assumes a longer duration of expansions than that of recessions. Expansions in the model are defined by high mean and low volatility of consumption growth. Similarly, recessions are associated with both low growth and high uncertainty about future consumption. Thus, our calibration accounts for a negative covariation between growth and
uncertainty observed in the data.\textsuperscript{12}

We choose firm’s cash flow parameters to match the dynamics of dividend growth rates of the aggregate stock market. All asset data come from the Center for Research in Securities Prices (CRSP) and the Compustat database. Table III illustrates the fit of the calibrated model. Data estimates along with the Newey-West (1987) standard errors are presented in the left column; model-implied statistics are reported on the right. Overall, our calibration is consistent with the dynamics of the observed dividend data. In particular, we are able to match the mean growth rate of aggregate dividends, their volatility, as well as their correlation with consumption growth. The only dimension, which the model has some difficulty to address is the persistence of dividend growth. In the data, the first-order autocorrelation of dividend growth rates is 0.22, whereas the corresponding statistics in the model is about 0.56. Although undesirable, this implication of the model does not seem to be critical. In fact, a relatively high serial correlation of dividend growth rates induced by the channel of long-run risks can easily be reduced by introducing another common (orthogonal to consumption risks) component into firm’s cash flows. We do not entertain such an extension as it will have no effect on the pricing implications of the model.

The bottom panel of Table III illustrates that the model can successfully account for the historically high equity premium, as highlighted in Bansal-Yaron (2004). The model-implied average return of the market portfolio is about 7.5%. As in the data, the correlation between equity returns and consumption growth is low. The model also correctly predicts higher volatilities of asset prices relative to dividends – the standard deviation of the market return is about 19% in the data as well as in the model.

In addition to preference and time-series parameters, we need to decide on the entry configuration and the dynamics of the new-born ideas. We set $\bar{X}$ at 1, which is purely a normalization that has no qualitative or quantitative effect.\textsuperscript{13} We set $\frac{m_Z}{m_X}$ at 0.7, which implies that 70% of new ideas eventually obtain capital and become production units and the rest dies prematurely. Finally, we choose $\mu_X = 0.02$ and $\sigma_X = 0.40$. We find the model implications to be robust to the choice of parameters that govern the entry and the evolution of unexercised ideas.

\textbf{B Cross-Sectional Distribution of Firm Dividends and Returns}

Relying on the assumed time-series dynamics and model solutions, we simulate a pool of ideas and production units. To take the model to the data, we randomly sample assets from the simulated pool and staple them into firms. Guided by the cross-sectional dispersion of sales and market capitalization in the data, we assume that the initial distribution of ideas and production units

\textsuperscript{12}In our data set, the correlation between consumption growth and consumption volatility (measured by absolute residuals from an AR(1) fitted to consumption growth) is about -31%. This evidence is robust to various measures of economic growth and uncertainty.

\textsuperscript{13}Without loss of generality, we set the initial distribution of the quality of ideas to a point mass at $\bar{X} = 1$.  

27
across firms is Pareto. We track each firm over time, replacing extinct units with brand-new ideas. Our cross-section consists of 2,000 firms. We sort the simulated sample of firms into five book-to-market portfolios following the standard sorting procedure as in Fama and French (1992, 1993) and others.

Table IV illustrates the dynamics of per-share dividend growth rates of book-to-market sorted portfolios. Note that, in the data, value portfolio features high unconditional growth, whereas firms in growth portfolio, on average, exhibit low per-share growth. Our calibration captures this feature of the data. In particular, as the book-to-market ratio increases, the unconditional growth of the per-share dividends raises from 0.3% to about 5.4% in the data, and from -1% to about 3.5% in the model. Although the model-implied volatilities of growth rates are somewhat lower than their data counterparts, they are all within the two standard error band except for the highest book-to-market portfolio. This, however, speaks in favor rather than against our calibration. The high volatility of observed growth rates of value portfolio is driven by few virtually-zero dividend observations at the beginning of the sample, which might well be spurious.

Average returns of the simulated portfolios along with their empirical counterparts are presented in Table V. Consistent with the data, the model features a significant value premium – the mean return of high book-to-market firms is much higher than the average compensation for holding growth firms. Quantitatively, the model-implied value premium is about 3.6%. For comparison, the difference in mean returns on high and low book-to-market portfolios in the data amounts to about 5.9%. Thus, the model is able to account for a sizable portion of the cross-sectional variation in mean returns on book-to-market sorted portfolios.

Importantly, in the model, as in the data, the standard CAPM fails. Tables VI and VII quantify the deviations of the model from the unconditional and conditional CAPM predictions. Table VI reports the unconditional-CAPM alphas and the corresponding t-statistics for each of the book-to-market portfolios. The unconditional CAPM is strongly rejected – four out of five model-implied alphas are sizable and significant. As in the data, the CAPM tends to underprice growth stocks (by about 1.6% per annum) and overprice value stocks (by about 2.5%). Overall, both the pattern and the magnitude of simulated alphas are consistent with the CAPM mis-pricing in the data. Similarly, the conditional CAPM falls short in explaining the cross-sectional dispersion in risk premia. To evaluate the performance of the conditional market betas, we run the three-year rolling window regressions of monthly excess returns of book-to-market portfolios on monthly excess returns of the aggregate stock market. Average alphas and their t-statistics are presented in Table VII. On average, the model-implied conditional alphas are monotonically increasing, from about -2% for growth firms to almost 3% for value portfolio replicating an empirically strong positive relationship between alphas and book-to-market characteristics. Quantitatively, average conditional alphas in simulations conform to the failure of the conditional CAPM in the observed data.

To summarize, we show that the model calibrated to match the observed time-series dynamics
of aggregate consumption and market dividends, is able to generate a cross-section of firm returns, which is consistent with key stylized features of the observed asset prices. In particular, the model is able to simultaneously reconcile the high value premium and empirical failure of the standard CAPM regressions.

VI Conclusion

We put forward a general equilibrium model that links the cross-sectional variation of expected returns to firm’s life cycle dynamics. In the model, assets have different exposures to short-run and long-run consumption risks (Bansal and Yaron (2004)). An econometrician who uses conditional CAPM regression to predict asset returns will obtain high $\alpha$ for assets with high exposure to long-run risks. In the model, growth options have low exposure to long-run risks than assets in place because the cost of exercising growth options is highly sensitive to low-frequency fluctuations in aggregate consumption and, therefore, provides a hedge against risks in assets in place.

Appendix

Appendix I: State Price Density

Proof of Proposition 1

It is notationally convenient to represent the Markov chain as a compound Poisson process. In particular, let $\{N_{H,t}\}_{t \geq 0}$ be a Poisson process with intensity $\lambda_H$, and $\{N_{L,t}\}_{t \geq 0}$ be a Poisson process with intensity $\lambda_L$. let $I_{\{x\}}$ be the indicator function, that is,

$$ I_{\{x\}}(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases} $$

Then $\{\theta_t\}$ can be represented as the following compound Poisson process:

$$ d\theta_t = \Delta \eta(\theta^-)_t^T dN_t $$

where

$$ \Delta = \theta_H - \theta_L $$

and $\eta(\theta)$, and $N(t)$ are vector notations:

$$ \eta(\theta) = [-I_{\{\theta_H\}}(\theta), I_{\{\theta_L\}}(\theta)]^T $$

(58)
\[ N (t) = [N_{Ht}, \ N_{Lt}]^T \]

Here we use the convention that \( \{\theta_t\} \) is right-continuous with left limits, and use the notation

\[ \theta_t^- = \lim_{s \to t, s < t} \theta_s \]

We first guess the utility function of the representative agent is of the following form:

\[ v_t = V (\theta_t, C_t) = \frac{1}{1 - \gamma} H (\theta_t) C_t^{1 - \gamma} \]  

where

\[ H (\theta_H) = \exp \left\{ \frac{1}{\beta} [(1 - \gamma) \theta_H - \frac{1}{2} \gamma (1 - \gamma) \sigma^2_C (\theta_H) + \lambda_H (\omega^{-1} - 1)] \right\} \]  

(61)

\[ H (\theta_L) = \exp \left\{ \frac{1}{\beta} [(1 - \gamma) \theta_L - \frac{1}{2} \gamma (1 - \gamma) \sigma^2_C (\theta_L) + \lambda_L (\omega - 1)] \right\} \]  

(62)

where \( \omega \) is the unique solution on \((0, \infty)\) to the following equation:

\[ \beta \ln \omega = (1 - \gamma) \left[ (\theta_H - \theta_L) - \frac{1}{2} \gamma \left( \sigma^2_C (\theta_H) - \sigma^2_C (\theta_L) \right) \right] + \lambda_H (\omega^{-1} - 1) + \lambda_L (1 - \omega) \]  

(63)

One can show that equation (63) has a unique solution on \((0, \infty)\), and the solution satisfies \( \omega \in (0, 1) \). Define

\[ LHS (\omega) = \beta \ln \omega \]

and

\[ RHS (\omega) = (1 - \gamma) \left[ (\theta_H - \theta_L) - \frac{1}{2} \gamma \left( \sigma^2_C (\theta_H) - \sigma^2_C (\theta_L) \right) \right] + \lambda_H (\omega^{-1} - 1) + \lambda_L (1 - \omega) \]

Then \( LHS (\omega) \) is strictly increasing on \((0, \infty)\), and \( LHS (\omega) \to -\infty \) as \( \omega \to 0 \); \( LHS (1) = 0 \). Also, \( RHS (\omega) \) is strictly decreasing on \((0, \infty)\). \( RHS (\omega) \to +\infty \) as \( \omega \to 0 \), and

\[ RHS (1) = (1 - \gamma) \left[ (\theta_H - \theta_L) - \frac{1}{2} \gamma \left( \sigma^2_C (\theta_H) - \sigma^2_C (\theta_L) \right) \right] < 0 \]

under assumptions 1 and 2. It then follows from (61), (62) and (63)

\[ \frac{H (\theta_H)}{H (\theta_L)} = \omega < 1 \]
The state price process of this economy is given by (Duffie and Epstein (1992b)):

\[
\frac{d\pi_t}{\pi_t} = \frac{df_C (C_t, V_t)}{f_C (C_t, V_t)} + f_V (C_t, V_t) \, dt
\]

Using generalized Ito’s formula (Øksendal and Sulem (2004)), one can show \( \{\pi_t\} \) is a Levy process of the form:

\[
d\pi_t = \pi_t \left[ -\bar{r} (\theta_t) \, dt - \gamma \sigma_C (\theta_t) dB_t - \eta \pi (\theta_t^+) T \, dN_t \right]
\]

where

\[
\bar{r} (\theta) = \beta + \theta - \gamma \sigma_C^2 (\theta) - \left[ \lambda_H I_H (\theta) (1 - \omega^{-1}) + \lambda_L I_L (\theta) (1 - \omega) \right]
\]

and

\[
\eta (\theta) = \left[ (1 - \omega^{-1}) I_{\{H\}} (\theta), (1 - \omega) I_{\{L\}} (\theta) \right]
\]

Furthermore, the risk-free rate of this economy is given by:

\[
r (\theta) = \beta + \theta - \gamma \sigma_C^2 (\theta)
\]

Appendix II: Optimal Stopping Problems

**Lemma 1** General Solutions to the Optimal Stopping Problem

Consider the following general formulation of the optimal stopping problem:

\[
V (\theta_t, \xi_t, C_t) = E \left[ \int_t^\tau e^{-\kappa (s-t)} \frac{\pi_s}{\pi_t} D (\xi_s, \theta_s) C_s ds + e^{-\kappa (\tau-t)} \frac{\pi_t}{\pi_t} B (\xi_\tau, \theta_\tau) C_\tau \, t \right]
\]

\[
d\xi = \xi \left[ \mu (\theta_t) \, dt + \sigma (\theta_t) \, dB_t \right]
\]

\[
dC_t = C_t \left[ \theta_t dt + \sigma_C (\theta_t) dB_t \right]
\]

\[
d\pi_t = \pi_t \left[ -\bar{r} (\theta_t) \, dt - \gamma \sigma_C (\theta_t) dB_{Ct} - \eta \pi (\theta_t^+) T \, dN_t \right]
\]

\[
d\theta_t = (\theta_H - \theta_L) \times \eta (\theta_t^+) T \, dN_t,
\]

where \( \tau \) is a \( \{\mathcal{F}_t\}_{t \geq 0} \) adapted stopping time. Then

\[
V (\theta, \xi, C) = G (\xi, \theta) C
\]

where \( G (\xi, \theta) \) is the solution to the following coupled ODE:

\[
D (\xi, \theta_H) - G (\xi, \theta_H) \beta_H + G' (\xi, \theta_H) \xi \mu (\theta_H) + \frac{1}{2} G'' (\xi, \theta_H) \xi^2 \sigma^2 (\theta_H)
\]

\[
+ \lambda_H \left[ \omega^{-1} G (\xi, \theta_L) - G (\xi, \theta_H) \right] = 0
\]
and
\[
D(\xi, \theta_L) - G(\xi, \theta_L)\beta_L + G'(\xi, \theta_L)\xi \mu(\theta_L) + \frac{1}{2} G''(\xi, \theta_L)\xi^2 \sigma^2(\theta_L) + \lambda_L [\omega G(\xi, \theta_H) - G(\xi, \theta_L)] = 0
\]  
(68)

where we denote
\[
\beta_H = \kappa + \bar{r}(\theta_H) - \theta_H + \gamma \sigma^2(\theta_H) = \beta + \kappa - \lambda_H (1 - \omega^{-1}) 
\]  
(69)
\[
\beta_L = \kappa + \bar{r}(\theta_L) - \theta_L + \gamma \sigma^2(\theta_L) = \beta + \kappa - \lambda_L (1 - \omega) 
\]  
(70)

\textbf{Proof.} By equation 8, we have:
\[
e^{-\kappa t} \pi_t V(\xi_t, \theta_t, C_t) + \int_0^t e^{-\kappa s} \pi_s D(\xi_s, \theta_s) C_s ds
\]
\[
= \max_{\tau} E_t \left[ \int_0^\tau e^{-\kappa(s-t)} \frac{\pi_s}{\pi_t} D(\xi_s, \theta_s) C_s ds + e^{-\kappa(\tau-t)} \frac{\pi_\tau}{\pi_t} B(\xi_\tau, \theta_\tau) C_\tau \right]_{\theta_t, \xi_t, C_t}
\]
is a martingale. Therefore,
\[
e^{-\kappa t} \pi_t D(\xi_t, \theta_t) C_t + \mathcal{L} \{ e^{-\kappa t} \pi_t V(\xi_t, \theta_t, C_t) \} = 0
\]  
(71)

where $\mathcal{L}$ is the generator associated with the vector Markov process $\{\xi_t, C_t, \pi_t, \theta_t\}$. Using homogeneity, $V(\theta_t, \xi_t, C_t) = G(\xi_t, \theta_t) C_t$, we have
\[
\mathcal{L} \{ e^{-\kappa t} \pi_t V(\xi_t, \theta_t, C_t) \} = -\kappa e^{-\kappa t} \pi_t G(\xi_t, \theta_t) + e^{-\kappa t} \mathcal{L} \{ \pi_t C_t G(\xi_t, \theta_t) \}
\]  
(72)

Using generalized Ito’s formula (Øksendal and Sulem (2004)), we have:
\[
\frac{1}{\pi_t C_t} \mathcal{L} \{ \pi_t C_t G(\xi_t, \theta_t) \} = G(\xi_t, \theta_t) \left[ -\bar{r}(\theta_t) + \theta_t - \gamma \sigma^2(\theta_t) \right]
\]
\[
+ G'(\xi_t, \theta_t) \xi \mu(\theta_t) + \frac{1}{2} G''(\xi_t, \theta_t) \xi^2 \sigma^2(\theta_t)
\]
\[
+ I_{\{\theta_H\}}(\theta_t) \lambda_H \left[ \omega^{-1} G(\xi_t, \theta_L) - G(\xi_t, \theta_H) \right]
\]
\[
+ I_{\{\theta_L\}}(\theta_t) \lambda_L \left[ \omega G(\xi_t, \theta_H) - G(\xi_t, \theta_L) \right]
\]
Therefore equation (71) is written as:

$$D(\xi_t, \theta_t) + G(\xi_t, \theta_t) \left[ -\kappa - \tau(\theta_t) + \theta_t - \gamma \sigma_C^2(\theta_t) \right]$$

$$+ G'(\xi_t, \theta_t) \xi_t \mu(\theta_t) + \frac{1}{2} G''(\xi_t, \theta_t) \xi_t^2 \sigma^2(\theta_t)$$

$$+ I(\theta_H)(\theta_t) \lambda_H \left[ \omega^{-1} G(\xi_t, \theta_L) - G(\xi_t, \theta_H) \right]$$

$$+ I(\theta_L)(\theta_t) \lambda_L \left[ \omega G(\xi_t, \theta_H) - G(\xi_t, \theta_L) \right] = 0$$

This is the same as (67) and (68).

**Lemma 2** Consider the homogenous part of the coupled ODE (67) and (68):

$$-G(\xi, \theta_H) \beta_H + G'(\xi, \theta_H) \xi \mu(\theta_H) + \frac{1}{2} G''(\xi, \theta_H) \xi^2 \sigma^2(\theta_H)$$

$$+ \lambda_H \left[ \omega^{-1} G(\xi, \theta_L) - G(\xi, \theta_H) \right] = 0$$

$$-G(\xi, \theta_L) \beta_L + G'(\xi, \theta_L) \xi \mu(\theta_L) + \frac{1}{2} G''(\xi, \theta_L) \xi^2 \sigma^2(\theta_L)$$

$$+ \lambda_L \left[ \omega G(\xi, \theta_H) - G(\xi, \theta_L) \right] = 0$$

is of the following form:

$$G(\xi, \theta_H) = \sum_{i=1}^{4} K_i e_{i,H} \xi^{\alpha_i}, \quad G(\xi, \theta_L) = \sum_{i=1}^{4} K_i e_{i,L} \xi^{\alpha_i}$$

(75)

where \( \{K_i\}_{i=1,2,3,4} \) are constants. \{\( \alpha_i \)\}_{i=1,2,3,4} are the eigenvalues of the following quadratic eigenvalue problem:

$$\frac{1}{2} M e \alpha^2 + N e \alpha + L e = 0$$

(76)

where

$$M = \begin{bmatrix} \sigma^2(\theta_H) & 0 \\ 0 & \sigma^2(\theta_L) \end{bmatrix}, \quad N = \begin{bmatrix} \mu(\theta_H) - \frac{1}{2} \sigma^2(\theta_H) & 0 \\ 0 & \mu(\theta_L) - \frac{1}{2} \sigma^2(\theta_L) \end{bmatrix}$$

$$L = \begin{bmatrix} - (\lambda_H + \beta_H) & \lambda_H \omega^{-1} \\ \lambda_L \omega & -(\lambda_L + \beta_L) \end{bmatrix}$$

and

$$e_i = [e_{i,H}, e_{i,L}]^T, \quad for \quad i = 1, 2, 3, 4$$

are the eigenvectors associated with \( \{\alpha_i\}_{i=1,2,3,4} \).

Without loss of generality, we will always normalize the eigenvectors \([e(i,H), e(i,L)]^T\)_{i=1,2,3,4}
so that
\[ e(i, H) = 1, \quad i = 1, 2, 3, 4 \]

**Proof.** We seek a solution to the homogenous part of (67) and (68) of the following form:
\[ G(\xi, \theta_H) = e_H^\alpha; \quad G(\xi, \theta_L) = e_L^\alpha \]

Then the homogenous part of equation (67) and (68) are written as:
\[
-\beta e_H^\alpha + \mu(\theta_H)e_H^\alpha + \frac{1}{2}\sigma^2(\theta_H)e_H^\alpha (\alpha - 1) + \lambda_H [\omega^{-1}e_L^\alpha - e_H^\alpha] = 0
\]
\[
-\beta_L e_L^\alpha + \mu_L e_L^\alpha + \frac{1}{2}\sigma^2(\theta_L)e_L^\alpha (\alpha - 1) + \lambda_L [\omega e_H^\alpha - e_L^\alpha] = 0
\]

Divide by \( \xi^\alpha \) on both sides and re-arrange, we have:
\[
\frac{1}{2}\sigma^2(\theta_H)e_H^\alpha + \left[ \mu(\theta_H) - \frac{1}{2}\sigma^2(\theta_H) \right] e_H^\alpha (\beta + \lambda_H) e_H + \lambda_H \omega^{-1} e_L = 0 \quad (77)
\]
\[
\frac{1}{2}\sigma^2(\theta_L)e_L^\alpha + \left[ \mu(\theta_L) - \frac{1}{2}\sigma^2(\theta_L) \right] e_L^\alpha (\beta + \lambda_L) e_L + \lambda_L \omega e_H = 0 \quad (78)
\]

Clearly, equation (77) and (78) can be written in matrix notation as (76).

**Corollary 2** Consider a production unit. Suppose \( o(\theta_H) = o(\theta_L) = 0 \), then the value of the production unit is of the form in (26), where
\[ G_{VA}(Z, \theta_H) = a(\theta_H) Z, \quad G_{VA}(Z, \theta_L) = a(\theta_L) Z \]

and
\[
a(\theta_H) = \frac{\beta + \kappa_Z + \lambda_L \omega + \lambda_H \omega^{-1} - \mu_Z(\theta_L)}{[\beta + \kappa_Z + \lambda_H \omega^{-1} - \mu_Z(\theta_H)] [\beta + \kappa_Z + \lambda_L \omega - \mu_Z(\theta_L)] - \lambda_H \lambda_L} \quad (79)
\]
\[
a(\theta_L) = \frac{\beta + \kappa_Z + \lambda_L \omega + \lambda_H \omega^{-1} - \mu_Z(\theta_H)}{[\beta + \kappa_Z + \lambda_H \omega^{-1} - \mu_Z(\theta_H)] [\beta + \kappa_Z + \lambda_L \omega - \mu_Z(\theta_L)] - \lambda_H \lambda_L} \quad (80)
\]

**Proof.** Let
\[ \xi = Z_i^1, \quad (81) \]
\[ \kappa = \kappa_Z, \quad \mu(\theta) = \mu_Z(\theta), \quad \sigma(\theta) = \sigma_Z(\theta) \quad \text{for} \quad \theta = \theta_H, \theta_L, \quad (82) \]
\[ D(Z_i^1, \theta) = Z_i^1, \quad B(Z_i^1, \theta) = 0, \quad (83) \]
\[ \tau = \infty, \quad (84) \]

and apply Lemma 1 and Lemma 2. ■
Corollary 3  Consider a production unit with dividend given in equation (13). Suppose it does not have the option to exit, then the value of the production unit is of the form in (26), where

\[ G_{VA}(Z, \theta_H) = a(\theta_H)Z - b(\theta_H), \quad G_{VA}(Z, \theta_L) = a(\theta_L)Z - b(\theta_L) \]

and

\[ b(\theta_H) = \frac{o(\theta_H) [\beta + \kappa + \lambda L \omega] + o(\theta_L) \lambda H \omega^{-1}}{[\beta + \kappa + \lambda H \omega^{-1}] [\beta + \kappa + \lambda L \omega] - \lambda H \lambda L} \] (85)

\[ b(\theta_L) = \frac{o(\theta_L) [\beta + \kappa + \lambda H \omega^{-1}] + o(\theta_H) \lambda L \omega}{[\beta + \kappa + \lambda H \omega^{-1}] [\beta + \kappa + \lambda L \omega] - \lambda H \lambda L} \] (86)

**Proof.** Apply Lemma 1 and Lemma 2 to the special case in which (81), (82), (84) and

\[ D(Z_i^t, \theta) = Z_i^t - o(\theta), \quad B(Z_i^t, \theta) = 0. \] (87)

Proof of Proposition 2

Note the optimization problem for production units stated in equation (17) is a special case of (66) with (81), (82), (84), (87), and

\[ \tau = \tau_{VA}(i, t) \] (88)

where \( \tau_{VA}(i, t) \) is the solution to the optimal stopping problem (17). By Lemma 1 and Lemma 2, the general solution to the optimal stopping problem in (17) must be of the form as given in (26) and (27), where \( a(\theta_H), a(\theta_L) \) are given in (79) and (80), \( b(\theta_H), b(\theta_L) \) are given in (85) and (86), and \( \left\{ \alpha_{VA}(i), [e_{VA}(i, \theta_H), e_{VA}(i, \theta_L)]^T \right\}_{i=1,2,3,4} \) are eigenvalues and eigenvectors of the quadratic eigenvalue problem in (76) with parameter restrictions (82). Assumption 3 implies we can arrange the eigenvalues so that \( \alpha_{VA}(1) < \alpha_{VA}(2) < 0 < \alpha_{VA}(3) < \alpha_{VA}(4) \) without loss of generality (For proof, see Guo (2001)). Furthermore, boundary conditions at \( Z_i^t \to \infty \) implies \( K_3 = K_4 = 0 \) in (75). Optimality requires the value matching and smooth pasting conditions (28) and (29) to be satisfied, which determine the coefficients \( K_1, K_2 \) and the optimal option exercise barrier \( Z^*(\theta_H), Z^*(\theta_L) \).

Proof of Proposition 3

Note the optimization problem for ideas stated in equation (15) is a special case of (66) with

\[ \xi = X_i^t, \]
\[ \kappa = \kappa_X, \; \mu(\theta) = \mu(\theta_L) = \mu_X, \; \sigma(\theta) = \sigma(\theta_L) = \sigma_X, \] (89)

and

\[ D(X^i_t, \theta) = 0, \; B(X^i_t, \theta) = G_{VA}(X^i_t, \theta) - f(\theta) \quad \text{for} \; \theta = \theta_H, \theta_L. \] (90)

Lemma 2 implies \( G_{GR}(X, \theta) \) is of the form:

\[
G_{GR}(X, \theta_H) = \sum_{i=1}^{4} L_i e_{GR}(i, H) X^{\alpha_{GR}(i)}, \quad G_{GR}(X, \theta_L) = \sum_{i=1}^{4} L_i e_{GR}(i, L) X^{\alpha_{GR}(i)}
\]

where \([\alpha_{GR}(i)]_{i=1,2,3,4}\) and \([e_{GR}(i, H), e_{GR}(i, L)]_{i=1,2,3,4}\) are the eigenvalues and their corresponding eigenvectors of the quadratic eigenvalue problem in (76) with parameter restrictions (89). Again, \([\alpha_{GR}(i)]_{i=1,2,3,4}\) can be ordered so that \(\alpha_{GR}(1) < \alpha_{GR}(2) < 0 < 1 < \alpha_{GR}(3) < \alpha_{GR}(4)\). Since \(\alpha_{GR}(1), \alpha_{GR}(2) < 0\), the boundary condition

\[
\lim_{z \to 0} G_{GR}(X, \theta) \geq 0 \quad \text{for} \; \theta = \theta_H, \theta_L
\]

implies \(L_1 = L_2 = 0\). \(L_3, L_4\), together with the option exercise threshold \(X^*(\theta_H), X^*(\theta_L)\) are determined by the value matching and smooth pasting conditions, (32) and (33).

Appendix III: The Cross-Section Distribution of Ideas

The density \(\phi\) of the cross-sectional distribution of log of \(X\) satisfies the following condition:

\[
0 = \kappa_X \phi(y) + e^y u(e^y) - \left(\mu_X - \frac{1}{2} \sigma_X^2\right) \frac{d}{dy} \phi(y) + \frac{1}{2} \sigma_X^2 \frac{d^2}{dy^2} \phi(y)
\] (91)

We first observe that the characteristic quadratic associated with the homogeneous part of the above equation,

\[
\kappa_X + \left(\mu_X - \frac{1}{2} \sigma_X^2\right) \eta - \frac{1}{2} \sigma_X^2 \eta^2 = 0,
\] (92)

has two roots:

\[
\eta_1 = \left(\frac{\mu_X}{\sigma_X^2} - \frac{1}{2}\right) + \sqrt{\left(\frac{\mu_X}{\sigma_X^2} - \frac{1}{2}\right)^2 + \frac{2\kappa}{\sigma_X^2}} > 0
\] (93)

\[
\eta_2 = \left(\frac{\mu_X}{\sigma_X^2} - \frac{1}{2}\right) - \sqrt{\left(\frac{\mu_X}{\sigma_X^2} - \frac{1}{2}\right)^2 + \frac{2\kappa}{\sigma_X^2}} < 0.
\] (94)

It is more convenient to consider the distribution of

\[ x^i_t = \ln X^i_t. \] (95)
Let $a$ denote the age of the idea, and let $\nu(a, \cdot)$ denote the density of the logarithm of the quality of ideas of age $a$. $\nu(0, \cdot)$ is therefore the density of the initial distribution of $\ln X$. For $a > 0$, $\nu(a, \cdot)$ has to satisfy the following forward equation:

$$D_a \nu(a,x) = -\kappa_X \nu(a,x) - \left(\mu_X - \frac{1}{2}\sigma_X^2\right) D_y \nu(a,x) + \frac{1}{2} \sigma_X^2 D_{yy} \nu(a,x)$$  \hspace{1cm} (96)

along with the boundary condition at the absorbing barrier:

$$\forall a > 0, \quad \nu(a, \ln X^*) = 0$$  \hspace{1cm} (97)

**Lemma 3** The solution to (96) with the boundary condition (97) is:

$$\nu(a,y) = \int_{-\infty}^{+\infty} \left[ e^{-\kappa_X a} \int_{-\infty}^{+\infty} h(t \sigma_X^2, y-x-(\mu_X - \frac{1}{2}\sigma_X^2) a) \right] \nu(0,x) dx$$

and

$$h(t,y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}}$$

The solution can be found in Luttmer (2007). The density of the logarithm of quality of existing ideas can be found by integrating $\nu(a,y)$ over all $a$:

$$\phi(y) = \int_{0}^{\infty} \nu(a,y) da$$  \hspace{1cm} (98)

Recall that the initial quality of ideas are drawn from the uniform distribution on $[0, X]$. This implies that the density of $\ln X$ is given by:

$$\frac{1}{X} m_X e^x, \quad x \in (-\infty, \ln X].$$  \hspace{1cm} (99)

There are two possibilities: $X^* < X$ and $X^* \geq X$. If $X^* < X$, that is the absorbing barrier $X^*$ is below $X$. This implies some newly born ideas become production units immediately right after birth. The density of the quality of new ideas that will have to wait is given by:

$$\nu(0,x) = \frac{1}{X} m_X e^x, \quad x \in (-\infty, \ln X^*].$$

If $X^* \geq X$, then none of ideas exercise options, and the density of their quality is given by:

$$\nu(0,x) = \frac{1}{X} m_X e^x, \quad x \in (-\infty, \ln X]$$

The density of the stationary distribution of the quality of ideas are given by the following lemma.
Lemma 4 \( \kappa_X > 0 \) implies the integral in (98) exists and density of the stationary distribution of \( \ln X \) is given by:

**Case 1:** \( X^* < \bar{X} \):

\[
\phi (y) = \frac{m_X}{\bar{X}} \frac{1}{\kappa_X + \mu_X - \sigma_X^2} \left[ e^{y} - e^{(1 - \eta_1) \ln X^* + \eta_1 y} \right], \quad y \in (-\infty, \ln X^*)
\]

**Case 2:** \( X^* \geq \bar{X} \): For \( y \in (\ln \bar{X}, \ln X^*) \)

\[
\phi (y) = \frac{m_X}{\bar{X}^{\eta_2}} \frac{1}{(1 - \eta_2) \sqrt{\left( \mu_X - \frac{1}{2} \sigma_X^2 \right)^2 + 2 \kappa_X \sigma_X^2}} \left[ e^{\eta_2 y} - e^{(\eta_2 - \eta_1) \ln X^* + \eta_1 y} \right],
\]

and for \( y \in (-\infty, \ln \bar{X}] \),

\[
\phi (y) = \frac{m_X}{\bar{X}} \left\{ \frac{\bar{X}^{1-\eta_1}}{\sqrt{\left( \mu_X - \frac{1}{2} \sigma_X^2 \right)^2 + 2 \kappa_X \sigma_X^2}} \left[ \frac{1}{1-\eta_1} - \frac{1}{1-\eta_2} \left( \frac{X^*}{\bar{X}} \right)^{\eta_2 - \eta_1} \right] e^{\eta_1 y} \right\}
\]

where \( \eta_1 \) and \( \eta_2 \) are defined in (93) and (94).

Using Lemma 4, the density of the stationary distribution of \( X \) can be written as:

**Case 1:** \( X^* < \bar{X} \):

\[
\Phi (X) = \frac{m_X}{(\kappa + \mu_X - \sigma_X^2) \bar{X}} \left[ 1 - \left( \frac{X}{X^*} \right)^{\eta_1 - 1} \right], \quad X \in [0, X^*]
\]

(100)

**Case 2:** \( X^* > \bar{X} \). For \( X \in (\bar{X}, X^*) \):

\[
\Phi (X) = \frac{m_X}{\bar{X}} \frac{1}{(1 - \eta_2) \sqrt{\left( \mu_X - \frac{1}{2} \sigma_X^2 \right)^2 + 2 \kappa_X \sigma_X^2}} \left[ \left( \frac{X}{\bar{X}} \right)^{\eta_2 - 1} - \left( \frac{X^*}{\bar{X}} \right)^{\eta_2 - \eta_1} \left( \frac{X}{\bar{X}} \right)^{\eta_1 - 1} \right],
\]

(101)

and for \( X \in (0, \bar{X}) \):

\[
\Phi (X) = \frac{m_X}{\bar{X}} \left\{ \frac{\bar{X}^{1-\eta_1}}{\sqrt{\left( \mu_X - \frac{1}{2} \sigma_X^2 \right)^2 + 2 \kappa_X \sigma_X^2}} \left[ \frac{1}{1-\eta_1} + \frac{1}{1-\eta_2} \left( \frac{X^*}{\bar{X}} \right)^{\eta_2 - \eta_1} \right] X^{\eta_1 - 1} \right\}.
\]

(102)

Furthermore, 4 can be proved by noting that the absorbing rate at the absorbing barrier must be given by:

\[
\frac{1}{2} \sigma_X^2 |\phi' (\ln X^*)|
\]
Appendix IV: Asset Pricing Implications

Proof of Proposition 5

Since \( \{ \pi_t \}_{t \geq 0} \) is the state price density, for any asset \( i \), \( \pi_t R_{i,t}^i \) is a martingale, where \( R_{i,t}^i \) is the cumulative return process of the asset defined in equation (37). Using generalized Ito’s formula (Øksendal and Sulem (2004)),

\[
\lim_{\Delta \to 0} \frac{1}{\Delta} E_t \left[ \frac{R_{i,t+\Delta}}{R_{i,t}} - 1 \right] - r(\theta_t) = Cov_t \left[ \frac{d\pi_t}{\pi_t}, \frac{dR_{i,t}^i}{R_{i,t}^i} \right]
\]

One can therefore prove Proposition 5 accordingly by using the functional form of \( \pi_t \) in (21) and \( V_t^i \) in (36).

Proof of Proposition 6

Take limit as \( \Delta \to 0 \) on both sides of equation (42), we have:

\[
\alpha_t^i = RP(i,t) - \chi_t Cov_t \left( \frac{dR_t^i}{R_t^i}, \frac{dR_{ref,t}^i}{R_{ref,t}^i} \right)
\]

where \( \chi_t \) is as defined in (47) and \( RP(ref,t) \) is the risk premium of the reference asset at time \( t \). Using Proposition 5,

\[
RP(i,t) = \gamma \sigma_C^2(\theta_t) + \sum_{j=H,L} \lambda_j \eta_{i,j}
\]

where

\[
\begin{bmatrix}
\eta_{\pi,H} \\
\eta_{\pi,L}
\end{bmatrix} = 
\begin{bmatrix}
1 - \omega^{-1} \\
1 - \omega
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
\eta_{i,H} \\
\eta_{i,L}
\end{bmatrix} = 
\begin{bmatrix}
\frac{G(i,\theta_L)}{G(i,\theta_H)} - 1 \\
\frac{G(i,\theta_H)}{G(i,\theta_L)} - 1
\end{bmatrix}
\]

Also, since the value of the reference asset is of the form in (44), the cumulative return process of the reference asset can be written as:

\[
\frac{dR_t^i}{R_t^i} = [EP(ref,t) + r(\theta_t)] dt + \sigma_C(\theta_t) dB_t + \eta_{ref}(\theta^-_t) dN_t
\]

where

\[
\eta_{ref}(\theta) = 
\begin{bmatrix}
\eta_{ref,H} I_{\theta_H}(\theta) \\
\eta_{ref,L} I_{\theta_L}(\theta)
\end{bmatrix} = 
\begin{bmatrix}
\frac{s(\theta_L)}{s(\theta_H)} - 1 \\
\frac{s(\theta_H)}{s(\theta_L)} - 1
\end{bmatrix}
\begin{bmatrix}
I_{\theta_H}(\theta) \\
I_{\theta_L}(\theta)
\end{bmatrix}
\]
Therefore,
\[ \text{Cov}_t \left( \frac{dR^*_t}{R_t^t}, \frac{dR^*_{t\text{ref}}}{R_t^*_{t\text{ref}}} \right) = \sigma_C^2 (\theta_t) + \sum_{j=H,L} \lambda_j \eta_{\text{ref},j} \eta_{i,j} \]

Therefore (103) can be written as:
\[
\alpha_t = \gamma \sigma_C^2 (\theta_t) + \sum_{j=H,L} \lambda_j \eta_{\pi,j} \eta_{i,j} - \chi_t \left[ \sigma_C^2 (\theta_t) + \sum_{j=H,L} \lambda_j \eta_{\text{ref},j} \eta_{i,j} \right] 
\]
\[
= (\gamma - \chi_t) \sigma_C^2 (\theta_t) + \sum_{j=H,L} \lambda_j \eta_{i,j} \left[ \eta_{\pi,j} - \chi_t \eta_{\text{ref},j} \right] 
\]

This proves equations (45) and (46).

To prove (48) and (49), note that in this economy, consumption-to-wealth ratio is constant and equal to $\beta$. Therefore the Consumption CAPM and CAPM with aggregate wealth as the reference asset should produce the same $\alpha$. With aggregate wealth as the reference asset, we have
\[
s (\theta_H) = s (\theta_L) = \frac{1}{\beta} \tag{104}
\]

Equations (48) and (49) can therefore be proved by imposing condition (104) in equations (45) and (46).

**Appendix V: Value Premium**

In this section, we prove of Proposition 7 through a series of lemmas. First note that the optimization problem of ideas in (15) is a special case of (66) with the parameter restriction (89) and (90). We first state and prove a lemma that gives a complete characterization of the eigenvalue and eigenvectors of the quadratic eigenvalue problem of ideas.

**Lemma 5** The eigenvalues of the quadratic eigenvalue problem in (66) with the parameter restriction (89) and (90) satisfies
\[
a_{GR} (1) < a_{GR} (2) < 0 < 1 < a_{GR} (3) < a_{GR} (4).
\]
The corresponding normalized eigenvectors are given by:
\[
\begin{bmatrix}
 e_{GR} (1, H) \\
 e_{GR} (1, L)
\end{bmatrix} = \begin{bmatrix}
 e_{GR} (4, H) \\
 e_{GR} (4, L)
\end{bmatrix} = \begin{bmatrix}
 1 \\
 \frac{\lambda_l \omega^2}{\lambda_H} 
\end{bmatrix}
\]

and
\[
\begin{bmatrix}
 e_{GR} (2, H) \\
 e_{GR} (2, L)
\end{bmatrix} = \begin{bmatrix}
 e_{GR} (3, H) \\
 e_{GR} (3, L)
\end{bmatrix} = \begin{bmatrix}
 1 \\
 1 
\end{bmatrix}
\]

**Proof.** Note that the optimization problem of ideas in (15) is a special case of (66) with the parameter restriction (89) and (90). Therefore the quadratic eigenvalue problem in (76) can be
written as:

\[ \frac{1}{2} \sigma x e_H \alpha^2 + \left[ \mu_X - \frac{1}{2} \sigma_X \right] e_H \alpha - \left( \beta_H + \lambda_H \right) e_H + \lambda_H \omega^{-1} e_L = 0 \]  
(105)

\[ \frac{1}{2} \sigma X e_L \alpha^2 + \left[ \mu_X - \frac{1}{2} \sigma_X \right] e_L \alpha - \left( \beta_L + \lambda_L \right) e_L + \lambda_L \omega e_L = 0 \]  
(106)

Divide both sides of equation (105) by \( e_H \), and both sides of equation (106) by \( e_L \), we have:

\[ \frac{1}{2} \sigma X \alpha^2 + \left[ \mu_X - \frac{1}{2} \sigma_X \right] \alpha - \left( \beta_H + \lambda_H \right) \lambda_H \omega^{-1} e_L = 0 \]  
(107)

\[ \frac{1}{2} \sigma X \alpha^2 + \left[ \mu_X - \frac{1}{2} \sigma_X \right] \alpha - \left( \beta_L + \lambda_L \right) \lambda_L e_L e_L = 0 \]  
(108)

We can rearrange (107) and (108) and get:

\[ \frac{e_H}{e_L} = -\frac{1}{2} \sigma X \alpha^2 - \left[ \mu_X - \frac{1}{2} \sigma_X \right] \alpha + (\beta_L + \lambda_L) \frac{\lambda_H \omega^{-1}}{\lambda_L \omega} = \frac{\lambda_H \omega^{-1}}{-\frac{1}{2} \sigma X \alpha^2 - \left[ \mu_X - \frac{1}{2} \sigma_X \right] \alpha + (\beta_H + \lambda_H)} \]

It is therefore easy to see that the eigenvectors and eigenvalues can be constructed in the following way:

1. \( \alpha \) is an eigenvalue if \( \exists x \) such that

\[ \frac{\beta_L + \lambda_L - x}{\lambda_L \omega} = \frac{\lambda_H \omega^{-1}}{\beta_H + \lambda_H - x} \]  
(109)

and

\[ \frac{1}{2} \sigma X \alpha^2 + \left[ \mu_X - \frac{1}{2} \sigma_X \right] \alpha = x \]  
(110)

2. The corresponding normalized eigenvector can be constructed by:

\[ \frac{e_H}{e_L} = \frac{\beta_L + \lambda_L - x}{\lambda_L \omega}, \quad e_H = 1 \]  
(111)

Use equation (69) and (70), equation (109) can be written as:

\[ \frac{\beta + \kappa + \lambda_L \omega - x}{\lambda_L \omega} = \frac{\lambda_H \omega^{-1}}{\beta + \kappa + \lambda_H \omega^{-1} - x} \]  
(112)

and equation (111) can be written as:

\[ \frac{e_H}{e_L} = \frac{\beta + \kappa + \lambda_L \omega - x}{\lambda_L \omega}, \quad e_H = 1 \]
Equation (112) has two solutions,

\[ x_1 = \beta + \kappa, \quad x_2 = \beta + \kappa + \lambda_H \omega^{-1} + \lambda_L \omega \]

Therefore there are two independent eigenvectors:

\[ \frac{e_{A,H}}{e_{A,L}} = 1 \quad \text{and} \quad \frac{e_{B,H}}{e_{B,L}} = \frac{\lambda_H}{\lambda_L} \omega^{-2} \]

There are two eigenvalues corresponding to the eigenvector \([e_{A,H}, e_{A,L}]\), denoted \(\alpha_{A,1}, \alpha_{A,2}\), which are solutions to the following quadratic:

\[
\frac{1}{2} \sigma_X \alpha^2 + \left[ \mu_X - \frac{1}{2} \sigma_X \right] \alpha = \beta + \kappa \tag{113}
\]

Without loss of generality, we require \(\alpha_{A,1} < \alpha_{A,2}\). There are two eigenvalues corresponding to the eigenvector \([e_{B,H}, e_{B,L}]\), denoted \(\alpha_{B,1}, \alpha_{B,2}\), which are solutions to the following quadratic:

\[
\frac{1}{2} \sigma_X \alpha^2 + \left[ \mu_X - \frac{1}{2} \sigma_X \right] \alpha = \beta + \kappa + \lambda_H \omega^{-1} + \lambda_L \omega \tag{114}
\]

Without loss of generality, we require \(\alpha_{B,1} < \alpha_{B,2}\). It is straightforward to show \(\alpha_{B,1} < \alpha_{A,1} < 0 < \alpha_{A,2} < \alpha_{B,2}\). Furthermore, assumption 3 implies \(\alpha_{A,2} < \alpha_{B,2}\). This proves the lemma. \(\blacksquare\)

The following lemma links \(a(\theta)\) to the sign of the coefficient of the value function for ideas, \(L_4\), in equation (31).

**Lemma 6** \(L_4 > 0\) if and only if

\[ a(\theta_H) > a(\theta_L) \tag{115} \]

where \(a(\theta_H)\) and \(a(\theta_L)\) are given by equation (79) and (80), and \(L_4\) is the coefficient in equation (31).

**Proof.** Given the functional form of \(G_{GR}(Z, \theta)\) in (31), the value matching and smooth pasting conditions, (32) and (33), can therefore be written as:

\[
\begin{align*}
    e_{GR}(3, H)(X^*)^{a_{GR}(3)} L_3 + e_{GR}(4, H)(X^*)^{a_{GR}(4)} L_4 &= a(\theta_H) X^* - f(\theta_H) \\
    e_{GR}(3, L)(X^*)^{a_{GR}(3)} L_3 + e_{GR}(4, L)(X^*)^{a_{GR}(4)} L_4 &= a(\theta_L) X^* - f(\theta_L)
\end{align*}
\]
Given $X^*$, equation (118) and (119) determine $L_3$ and $L_4$, and equation (116) and (117) determine $f(\theta_H)$ and $f(\theta_L)$. Using equation (118) and (119), we have:

$$L_4 = \frac{a(\theta_L) e_{GR}(3, H) - a(\theta_H) e_{GR}(3, L)}{e_{GR}(4, L) e_{GR}(3, H) - e_{GR}(4, H) e_{GR}(3, L) \alpha_{GR}(4) (X^*)^{\alpha_{GR}(4) - 1}}$$

$$= \frac{\lambda_L \omega^2 + 1}{\lambda_H \omega^2 + 1} \alpha_{GR}(4) (X^*)^{\alpha_{GR}(4) - 1}$$

where the second line above uses Lemma 5. This proves the lemma since $\alpha_{GR}(4) > 1$ by Lemma 5.

The following lemma gives a necessary and sufficient condition for the inequality (56) in terms of the eigenvectors of (76).

**Lemma 7** (56) is true if and only if (115) is true.

**Proof.** $o(\theta_H) = o(\theta_L) = 0$ implies

$$G_{VA}(Z, \theta_H) = a(\theta_H) Z; \quad G_{VA}(Z, \theta_L) = a(\theta_L) Z$$

Equation (120), along with the value matching condition (32) implies that condition (56) is equivalent to

$$\frac{a(\theta_H) X^* - f(\theta_H)}{a(\theta_L) X^* - f(\theta_L)} < \frac{a(\theta_H)}{a(\theta_L)}.$$  

Therefore we only need to prove the equivalence between (115) and (121).

Combine equation (116) and (118), we have:

$$\alpha_{GR}(3) \left[ a(\theta_H) X^* - f(\theta_H) \right] = a(\theta_H) X^* - L_4 \left[ \alpha_{GR}(4) - \alpha_{GR}(3) \right] e_{GR}(4, H) (X^*)^{\alpha_{GR}(4)}$$

Similarly, we combine equation (117) and (119) to get:

$$\alpha_{GR}(3) \left[ a(\theta_L) X^* - f(\theta_L) \right] = a(\theta_L) X^* - L_4 \left[ \alpha_{GR}(4) - \alpha_{GR}(3) \right] e_{GR}(4, L) (X^*)^{\alpha_{GR}(4)}$$

Note both sides of equation (122) and (123) must be positive by the value matching conditions.
Combine equation (122) and (123), we have:

\[
\frac{a(\theta_H) X^* - f(\theta_H)}{a(\theta_L) X^* - f(\theta_L)} = \frac{a(\theta_H) X^* - L_4 [\alpha_{GR}(4) - \alpha_{GR}(3)] e_{GR}(4, H) (X^*)^{\alpha_{GR}(4)}}{a(\theta_L) X^* - L_4 [\alpha_{GR}(4) - \alpha_{GR}(3)] e_{GR}(4, L) (X^*)^{\alpha_{GR}(4)}}
\]

\[
= \frac{a(\theta_H) X^* - L_4 [\alpha_{GR}(4) - \alpha_{GR}(3)] (X^*)^{\alpha_{GR}(4)}}{a(\theta_L) X^* + \frac{L_4}{X^*} \omega^2 L_4 [\alpha_{GR}(4) - \alpha_{GR}(3)] (X^*)^{\alpha_{GR}(4)}}
\]

\text{(124)}

where the second equality makes use of Lemma 5. Equation (124) implies

\[
\frac{a(\theta_H) X^* - f(\theta_H)}{a(\theta_L) X^* - f(\theta_L)} < \frac{a(\theta_H)}{a(\theta_L)}
\]

if and only if \( L_4 > 0 \). By Lemma 6, this is equivalent to \( a(\theta_H) > a(\theta_L) \), as needed.

To complete the proof of Proposition 7, note that Lemma 7 implies that (56) is true if and only if

\[
\frac{a(\theta_H)}{a(\theta_L)} > 1.
\]

\text{(125)}

Using the definition of \( a_H \) and \( a_L \) are given by equation (79) and (80), it is easy to see that (125) is equivalent to \( \mu_Z(\theta_H) > \mu_Z(\theta_L) \).

References


Table I
Model Parameters

<table>
<thead>
<tr>
<th>Preferences</th>
<th>Consumption</th>
<th>Cash Flows</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>$0.01$</td>
<td>$\theta_H$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$10$</td>
<td>$\theta_L$</td>
</tr>
<tr>
<td>$\psi$</td>
<td>$1.5$</td>
<td>$\lambda_H$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_L$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\sigma_H$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\sigma_L$</td>
</tr>
</tbody>
</table>

Table I presents the chosen configuration of preferences and time-series parameters of the model. All the parameters of consumption and cash-flow growth dynamics are stated in annual terms.
**Table II**
Consumption Growth Dynamics

<table>
<thead>
<tr>
<th>Statistic</th>
<th>— Data —</th>
<th>— Model —</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[\Delta c]$</td>
<td>1.94 (0.34)</td>
<td>1.92</td>
</tr>
<tr>
<td>$\sigma(\Delta c)$</td>
<td>2.13 (0.38)</td>
<td>1.89</td>
</tr>
<tr>
<td>$AC(1)$</td>
<td>0.44 (0.13)</td>
<td>0.42</td>
</tr>
</tbody>
</table>

Table II presents moments of consumption growth in the data and in the model. $E[\Delta c]$ denotes the unconditional mean of consumption growth, $\sigma(\Delta c)$ is the standard deviation, and $AC(1)$ is the first-order autocorrelation of growth rates. Means and volatilities are expressed in percentage terms. Consumption data, taken from the BEA, are real, annual, per capita series of non-durable expenditure and services from 1929 to 2007. Standard errors of the data statistics, reported in parentheses, are calculated using the Newey-West (1987) variance-covariance estimator with 4 lags. The model-implied moments are based on 80 years of simulated data.
Table III
Dynamics of Aggregate Market Portfolio

<table>
<thead>
<tr>
<th>Statistic</th>
<th>— Data —</th>
<th>— Model —</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Dividend Growth</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E[\Delta d]$</td>
<td>0.81 (1.46)</td>
<td>0.60</td>
</tr>
<tr>
<td>$\sigma(\Delta d)$</td>
<td>11.92 (1.97)</td>
<td>11.27</td>
</tr>
<tr>
<td>$AC(1)$</td>
<td>0.22 (0.14)</td>
<td>0.56</td>
</tr>
<tr>
<td>$Corr(\Delta d, \Delta c)$</td>
<td>0.66 (0.18)</td>
<td>0.72</td>
</tr>
<tr>
<td><strong>Return</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E[R]$</td>
<td>8.46 (1.85)</td>
<td>7.53</td>
</tr>
<tr>
<td>$\sigma(R)$</td>
<td>19.40 (1.74)</td>
<td>18.97</td>
</tr>
<tr>
<td>$AC(1)$</td>
<td>-0.03 (0.09)</td>
<td>-0.02</td>
</tr>
<tr>
<td>$Corr(R, \Delta c)$</td>
<td>0.07 (0.10)</td>
<td>0.26</td>
</tr>
</tbody>
</table>

Table III presents data- and model-based moments of dividend growth rates and returns of the aggregate stock market portfolio. $E[\cdot]$ and $\sigma(\cdot)$ denote the unconditional mean and standard deviation, respectively. $AC(1)$ is the first-order autocorrelation coefficient, and $Corr(\cdot, \Delta c)$ denotes the unconditional correlation between the corresponding variable and consumption growth. Means and volatilities are expressed in percentage terms. The aggregate stock market index is the value-weighted portfolio of NYSE, AMEX and NASDAQ traded firms recorded by CRSP. Nominal growth rates and returns are converted to real using the personal consumption expenditure deflator. Consumption data, taken from the BEA, are real, annual, per capita expenditure on non-durables and services. Data series cover the period from 1929 to 2007. Standard errors of the data statistics, reported in parentheses, are calculated using the Newey-West (1987) variance-covariance estimator with 4 lags. The model-implied moments are based on a simulated pool of 2,000 firms over 80 years.
Table IV

Dividend Growth Rate Dynamics

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$E[\Delta d]$</td>
<td></td>
</tr>
<tr>
<td>B1</td>
<td>0.30 (1.49)</td>
<td>-0.97</td>
</tr>
<tr>
<td>B2</td>
<td>0.51 (1.52)</td>
<td>0.80</td>
</tr>
<tr>
<td>B3</td>
<td>2.34 (1.85)</td>
<td>1.73</td>
</tr>
<tr>
<td>B4</td>
<td>2.24 (2.29)</td>
<td>2.28</td>
</tr>
<tr>
<td>B5</td>
<td>5.42 (2.92)</td>
<td>3.50</td>
</tr>
<tr>
<td></td>
<td>$\sigma(\Delta d)$</td>
<td></td>
</tr>
<tr>
<td>B1</td>
<td>15.77 (2.01)</td>
<td>14.53</td>
</tr>
<tr>
<td>B2</td>
<td>13.13 (1.74)</td>
<td>10.97</td>
</tr>
<tr>
<td>B3</td>
<td>14.13 (2.37)</td>
<td>10.76</td>
</tr>
<tr>
<td>B4</td>
<td>18.76 (3.26)</td>
<td>10.77</td>
</tr>
<tr>
<td>B5</td>
<td>23.94 (3.40)</td>
<td>12.55</td>
</tr>
</tbody>
</table>

Table IV presents means ($E[\Delta d]$) and standard deviations ($\sigma(\Delta d)$) of the per-share dividend growth rates for book-to-market sorted portfolios in the data and in the model. Means and volatilities are expressed in percentage terms. Book-to-market portfolios (B1-B5) are constructed by sorting NYSE, AMEX and NASDAQ firms into five bins at the end of June of each year and following their returns over the next twelve months. All return data are taken from the CRSP dataset and cover the period from 1929 to 2007. Dividend growth rates are expressed in real annual terms. Personal consumption expenditure deflator has been used to convert nominal quantities to real values. Standard errors of the data statistics, reported in parentheses, are calculated using the Newey-West (1987) variance-covariance estimator with 4 lags. The model-implied moments are based on a simulated pool of 2,000 firms over 80 years.
Table V
Average Returns

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>— Data —</th>
<th>— Model —</th>
</tr>
</thead>
<tbody>
<tr>
<td>B1</td>
<td>7.64</td>
<td>6.26</td>
</tr>
<tr>
<td>B2</td>
<td>8.19</td>
<td>7.79</td>
</tr>
<tr>
<td>B3</td>
<td>10.01</td>
<td>8.69</td>
</tr>
<tr>
<td>B4</td>
<td>11.44</td>
<td>8.95</td>
</tr>
<tr>
<td>B5</td>
<td>13.56</td>
<td>9.83</td>
</tr>
</tbody>
</table>

Table V presents average returns on book-to-market sorted portfolios in the data and in the model. Book-to-market portfolios (B1-B5) are constructed by sorting NYSE, AMEX and NASDAQ firms into five bins at the end of June of each year and following their returns over the next twelve months. All return data are taken from the CRSP dataset and cover the period from 1929 to 2007. Returns are expressed in real annual terms. Personal consumption expenditure deflator has been used to convert nominal quantities to real values. Standard errors of the data statistics, reported in parentheses, are calculated using the Newey-West (1987) variance-covariance estimator with 4 lags. The model-implied moments are based on a simulated pool of 2,000 firms over 80 years.
Table VI

Unconditional-CAPM Alpha's

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>— Data —</th>
<th>— Model —</th>
</tr>
</thead>
<tbody>
<tr>
<td>B1</td>
<td>-0.49 (-0.74)</td>
<td>-1.69 (-1.96)</td>
</tr>
<tr>
<td>B2</td>
<td>0.19 (0.35)</td>
<td>0.59 (0.95)</td>
</tr>
<tr>
<td>B3</td>
<td>1.24 (1.18)</td>
<td>1.43 (2.29)</td>
</tr>
<tr>
<td>B4</td>
<td>1.88 (1.53)</td>
<td>1.66 (2.78)</td>
</tr>
<tr>
<td>B5</td>
<td>2.75 (1.90)</td>
<td>2.54 (4.11)</td>
</tr>
</tbody>
</table>

Table VI illustrates the performance of the unconditional CAPM in the data as well as the model. The entries represent intercepts and the corresponding t-statistics from an OLS regression of excess returns of book-to-market portfolios on the excess return of the aggregate stock market. Book-to-market portfolios (B1-B5) are constructed by sorting NYSE, AMEX and NASDAQ firms into five bins at the end of June of each year and following their returns over the next twelve months. The risk-free rate in the data is measured using the 3-month Treasury bill. All return data are taken from the CRSP dataset and cover the period from 1929 to 2007. The model-implied moments are based on a simulated pool of 2,000 firms over 80 years.
Table VII

Average Conditional-CAPM Alpha’s

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>— Data —</th>
<th>— Model —</th>
</tr>
</thead>
<tbody>
<tr>
<td>B1</td>
<td>-0.93 (-2.00)</td>
<td>-2.17 (-2.46)</td>
</tr>
<tr>
<td>B2</td>
<td>-0.00 (-0.01)</td>
<td>0.92 (1.73)</td>
</tr>
<tr>
<td>B3</td>
<td>1.76 (3.50)</td>
<td>1.70 (2.01)</td>
</tr>
<tr>
<td>B4</td>
<td>2.31 (3.15)</td>
<td>2.06 (2.51)</td>
</tr>
<tr>
<td>B5</td>
<td>3.44 (3.43)</td>
<td>2.98 (3.54)</td>
</tr>
</tbody>
</table>

Table VII illustrates the performance of the conditional CAPM in the data and in the model. The entries are average (annualized) alphas and their t-statistics from the 36-month rolling regressions of excess returns of book-to-market portfolios on the excess return of the aggregate stock market. Book-to-market portfolios (B1-B5) are constructed by sorting NYSE, AMEX and NASDAQ firms into five bins at the end of June of each year and following their returns over the next twelve months. The risk-free rate in the data is measured using the 3-month Treasury bill. All return data are taken from the CRSP dataset, sampled on the monthly frequency, and cover the period from 1929 to 2007. The model-implied moments are based on a simulated pool of 2,000 firms over 80 years.
Figure 1. Dynamics of Ideas
Figure 2. Dynamics of Production Units
Figure 3. Normalized Value Functions of Ideas and Production Units
Figure 4. Equity Premia of Ideas and Production Units