Optimality or Sustainability?[1]

Geoffrey Heal

June 2001

[1]Prepared for presentation at the EAERE 2001 Conference, Southampton June 2001. This paper is a condensation of parts of my book *Valuing the Future: Economic Theory and Sustainability*, Columbia University Press, 1998, with a few new insights added. Proofs of all propositions can be found in that book.

Abstract

Does the present concern about sustainability raise fundamentally new issues for economics, or is it dealing with problems already on our agenda? There are two points that are central to sustainability: a concern for what happens in the long run, and a respect for the constraints that the natural world places on the dynamics of human societies and the well-being of their members. Concern for the long-run has a long and distinguished history in economics, going back to Sidgwick, Ramsey, Koopmans and others. We have not resolved these issues fully, but they are not new. Concern for the ecological limitations on society is a matter of specifying properly the constraints under which society operates. This does not raise fundamentally novel issues, although the precise specifications of these constraints, which could involve non-convexities and hysteresis effects, might be challenging. Here I explore optimal growth paths for economies with various specifications of the intertemporal objectives and constraints, and ask whether optimal paths are sustainable in a loose and intuitive sense. The answer is frequently affirmative. I argue that in fact most optimal paths are sustainable, using the terms "optimal" and "sustainable" in ways that command general assent.

Contents

1 What is Sustainability? 2
2 The Hotelling Model 4
2.1 Conclusions from the Hotelling model 6
3 Valuing a Depletable Stock 7
3.1 Utilitarian Optimal Paths 7
3.2 The Green Golden Rule 11
3.3 The Rawlsian Optimum 12
3.4 Overtaking 12
3.5 Summary 13
4 Renewable Resources 13
4.1 Stationary Solutions 15
4.2 Dynamic Behavior 17
4.3 The Green Golden Rule 18
4.4 The Rawlsian Solution 19
4.5 Overtaking 19
5 Equal Treatment Over Finite Horizons 20
6 Conclusion 21

1 What is Sustainability?

Two concerns lie at the heart of discussions of sustainability: a concern for the interests of those who will live in the distant future, and a concern for the constraints imposed on human activity by the ecological and biogeochemical foundations of our societies. These capture what is common to many definitions of sustainability, the best known of which is probably the Brundtland report's comment that "Sustainable development is development that meets the needs of the present without compromising the ability of future generations to meet their own needs." I have argued elsewhere that "best known" and "best" are not the same, and that this definition raises more questions than it answers,[1] though it does emphasize clearly the intertemporal welfare issue. Other definitions are more precise, requiring for example that utility levels be non-decreasing over time,[2] or that resource stocks be non-decreasing, or that capital stocks in total (including natural capital) be non-decreasing.[3] All of these formulations are attempting to ensure adequate welfare for future generations either directly by bounding this below or indirectly by bounding below the stocks instrumental in providing future welfare.

Do we need new concepts and models to talk intelligently about these issues, or are they already captured by existing economics? Although the potential for this has not been fully realized, it is possible to model these issues within the existing corpus of economics. Certainly a concern for the very long run raises old questions, dating back to Sidgwick and Ramsey [17]. Even if we have not answered these questions fully, we have discussed them at length. Likewise we may not yet have modeled well the constraints imposed by our society's biogeochemical infrastructure, but doing so probably raises no totally new theoretical issues. Focussing on these raises questions about how we specify the technological constraints under which society operates, whether these involve nonlinearities, non-convexities, irreversibilities, etc. There is room for plenty of new work here, especially on non-convexities and irreversibilities, but these are not new questions. In fact both of the issues defining sustainability - long-run welfare and constraints on growth - are also at the heart of another area of economics, optimal growth theory. Ramsey's classic work initiated this field and placed the issue of balancing the welfares of present and future squarely on the agenda: indeed Ramsey's comment that "Discounting future utilities is ethically indefensible and arises purely from a weakness of the imagination" states a position that most environmentalists would agree with, were they aware of it.

Here I explore what we know about the trade-off between present and future

[1] [8]

[2] Pezzey [14] and for a review see Asheim et al. [1]

[3] For a review see Smulders [19].

and about the specification of the resource constraints under which the economy operates, and relate the ideas emerging from sustainability to those from optimal growth. Specifically I ask "Are sustainable paths optimal?" and "Are optimal paths sustainable?"

I explore different approaches to the present-future trade-off, those due to Ramsey, von Weizäcker [22], Koopmans [11], Rawls [16], and others. I do this in the context of various specifications of constraints on the economy, constraints posed by exhaustible resources, renewable resources, resources that are a source of utility to consumers, resources used in production, etc. To see the overall picture, think of a table, with the rows representing different specifications of the present-future trade-off, and the columns representing different ways of specifying the constraints on the economy. The entries in this table are all the different ways of pairing objectives and constraints and defining optimal resource use.

	Pure depletion	Renewable resources	Resources and capital	Etc.
Discounted utilitarian	Hotelling 31	...	Dasgupta and Heal 74	...
Rawlsian	...	...	Solow 74	...
Overtaking	...	...	...	...
Stock-dependent utility	Krautkraemer 85, Heal 98	Krautkraemer 85, Heal 98	...	...

In the rest of the paper I explore selected entries from this table, as a way of enquiring into the sustainability-optimality relationship. Beginning with the simplest case, Hotelling's 1931 model, I progress through different specifications of preferences and constraints. A clear picture emerges, indicating that optimal paths are sustainable, provided that preferences and constraints reflect fully what we know about human society's dependence on environmental systems.[4] This proviso is crucial: if earlier generations of optimal growth models did not produce sustainable paths, it was largely because they did not reflect this dependence.[5] Different long-run welfare functions give different degrees of sustainability, but no reasonable definition of optimal choice would lead to the destruction of society's natural resource base. So it seems safe to assert that optimal paths are usually sustainable, using the terms "optimal" and "sustainable" in ways that command general assent. Sustainable paths, however, may not be optimal.

What are the implications of this conclusion? Sustainability, it seems, is not a separate goal from optimality: rather optimality is a refined form of sustainability. Instead of proselytizing about sustainability as a social goal, perhaps environmental economists should work to refine the concept of optimality generally used and ensure that it incorporates an understanding of human dependence on environmental systems.

[4] As summarized for example in Daily [3] or Heal [9].

[5] That sustainable paths are not optimal is sufficiently obvious taht it doe snot require formal proof. Sustainable paths are not required by their definitions to meet first order conditions for intertemporal efficiency.

2 The Hotelling Model

Consider first the simplest and most classical formulation of the problem of the optimal management of a natural resource. This formulation assumes the resource to be exhaustible and simplifies the analytical framework by neglecting two issues important in understanding sustainability: placing adequate value on the long-run and recognizing all sources of value from environmental assets. However it does recognize, albeit in a rather basic way, the constraints imposed by an exhaustible resource on consumption patterns. It provides analytical building blocks which become foundations for some of the subsequent structures.

One could think of this as a model of the depletion of an oil or gas reserve. These are finite in amount, non-renewable and of no value other than as inputs to production. In spite of its limitations, this formulation, introduced by Hotelling [10] in 1931, is instructive. It shows clearly the forces at work in the widely-used discounted utilitarian approach, provides a simple framework for the presentation of mathematical techniques which are central to the later work, and is a building block in the construction of more general and satisfactory frameworks. It tells a story which recurs as a subplot in the more complex dramas which capture more of our real concerns.

The problem in this framework is that of choosing the time-pattern of use of the exhaustible resource so as to maximize the integral of the discounted utilities obtained from consumption of the resource, subject of course to a constraint that the total amount of the resource used over time should not exceed the initial stock: symbolically, the problem is

$$\max \int_0^\infty u(c_t)\, e^{-\delta t} dt \text{ s.t. } \int_0^\infty c_t dt \leq s_0 \tag{1}$$

Here $u(c_t)$ is a utility function which is assumed throughout to be increasing, strictly concave and twice continuously differentiable, so that the first derivative is positive and the second negative. u' and u'' denote the first and second derivatives of u respectively. Problem (1) is a classical problem in physics, called an isoperimetric problem: it arises when we seek to minimize the energy in a hanging string of fixed length. We make the mathematics of this problem slightly easier if we replace the integral constraint $\int_0^\infty c_t dt \leq s_0$ by a differential equation equating the rate of change of the remaining stock to the consumption rate, and add an inequality requiring the remaining stock to be non-negative:

$$\max \int_0^\infty u(c_t)\, e^{-\delta t} dt \text{ s.t. } s_t \geq 0 \text{ and } \dot{s}_t = -c_t \tag{2}$$

where of course $s_t = s_0 - \int_0^t c_f df$. A dot over a variable is always used to denote its time derivative. Problems (1) and (2) are fully equivalent.

We solve this by introducing the Hamiltonian

$$H = u(c_t)\, e^{-\delta t} - \lambda_t e^{-\delta t} c_t \tag{3}$$

where λ_t is an adjoint variable or shadow price, and then maximizing the Hamiltonian H with respect to c_t, giving

$$u'(c_t) = \lambda_t \ \forall t : c_t > 0 \tag{4}$$

In addition, the first order conditions for a solution to (1) or (2) require that the present value of the adjoint variable or shadow price changes over time at a rate given by the negative of the derivative of the Hamiltonian with respect to the stock:[6]

$$\frac{d}{dt}\left(\lambda_t e^{-\delta t}\right) = -\frac{\partial H}{\partial s_t}$$

so that

$$\dot{\lambda}_t - \delta \lambda_t = 0, \text{ i.e. } \lambda_t = \lambda_0 e^{\delta t} \tag{5}$$

Equation (5) is know as the *"Hotelling Rule"*. It tells us that the present value of the shadow price of the resource has to be the same at all dates at which a positive amount is consumed. Of course, (4) tells us that the derivative of utility with respect to consumption, or marginal utility, has to equal the shadow price, and so marginal utility also has to grow at the discount rate and also be constant in present value terms. This aspect of the result is very intuitive: consumption has to be spread over the possible dates so that its incremental contribution to utility, in present value terms, (i.e., its contribution to the maximand) is the same at all dates. This is the usual result that we spread a fixed factor between uses (dates, in this case) so that the incremental contribution that it makes is the same in all. When this condition is satisfied, no small variation in the time pattern of consumption will lead to an increase in the maximand.

What are the implications of this for consumption paths? To start with, consider a simple case. Let $u(c_t) = \log c_t$. Then (4) and (5) imply that

$$c_t = c_0 e^{-\delta t}$$

i.e., consumption falls exponentially at the discount rate. Nothing is conserved or sustained for ever, and the present and future are treated very unequally. The ratio $\frac{c_t}{c_0}$ of initial consumption c_0 to consumption at date t, c_t, decreases exponentially with time: $\frac{c_t}{c_0} = e^{-\delta t}$. The inequality between generations increases exponentially over time.

[6] These are standard techniques from control theory or from calculus of variations. An intuitive exposition of these techniques can be found in Heal [6] and a more comprehensive treatment in Seirstad and Sydaeter [18].

In the general case, we have from (4) and (5) that

$$u''(c_t)\overset{\bullet}{c}_t = \delta u'(c_t)$$

so that

$$\frac{\overset{\bullet}{c}}{c} = -\frac{\delta}{\eta} \tag{6}$$

where $\eta = -c_t u''(c_t)/u'(c_t) > 0$ and is the elasticity of marginal utility of consumption: it is also a measure of risk aversion and of the curvature of the function $u(c)$. So we have the following general result:

Proposition 1 *If the utility function u has a constant elasticity of marginal utility, then consumption on an optimal path which solves (1) falls over time at a rate that is linear in the discount rate, with the constant of proportionality being the inverse of the elasticity of marginal utility:* $c_t = c_0 e^{-\frac{\delta}{\eta}t}$.

Again, we have inequality in the treatment of present and future: this can be reduced by reducing the discount rate, to give a flatter consumption profile, but the ratio of present to future consumption still grows exponentially. And setting the discount rate equal to zero is not a solution to the problem, for in that case (6) tells us that consumption should be constant over time, and the only feasible constant consumption level is zero. For $\delta = 0$, the problem (1) has no solution.[7] This is an example of the unsettling paradoxes to which a zero discount rate gives rise - see Heal [8].

2.1 Conclusions from the Hotelling model

To summarize, the Hotelling model of optimal depletion of an exhaustible resource leaves little room for any sensible discussions of sustainability. The set of possible paths is very limited: consumption must go to zero, and as consumption is the only source of welfare, the economy must ultimately collapse. In the next section we see that a relatively small change, acknowledging an explicit value for the resource stock, alters everything. It makes the problem qualitatively different. We still work with an exhaustible resource, so that the set of feasible paths is unaltered, but the valuation of the remaining stock alters optimal use patterns radically and introduces real substance into the discussion of sustainability. Making the resource renewable, which is the theme of section 4, takes this process even further. Another strategy for making the model richer is to allow for the accumulation of capital, which can to some degree substitute for the resource. This is the approach that was taken initially by Dasgupta and Heal [4][5], who showed that positive consumption levels may be sustained for ever even with an exhaustible resource, provided that there is considerable scope for substitution of produced capital for the resource.

[7] For a detailed discussion of this case, see Heal [6] and Dasgupta and Heal [5].

3 Valuing a Depletable Stock

Next we recognize more explicitly the mechanisms through which environmental assets contribute to economic well-being. First we change the pure depletion problem by adding the remaining stock of the resource as an argument of the utility function, so that we now recognize explicitly that the stock of the resource may be a source of value.

Examples of environmental resources for which this would be appropriate include biodiversity, which in the sense of the range of species or some measure of their variation, is a depletable asset: once it is reduced through extinction, it can never be restored to its original value. And clearly the stock of biodiversity is a source of many services.

Another example is a forest, which yields a flow of wood for consumption as well as recreational facilities and carbon sequestration services, removing CO_2 from the atmosphere. Forest are of course usually renewable rather than depletable, but a tropical hardwood forest may to a first approximation be thought of as depletable. Other examples are a landscape, which can be farmed to yield a flow of output or enjoyed as a stock, or the atmosphere which can be used to yield a flow of services as a sink for pollution or enjoyed as a stock of clean air.[8]

The basic problem is now

$$\max \int_0^\infty u\left(c_t, s_t\right) e^{-\delta t} dt \text{ s.t. } s_t \geq 0 \text{ and } \dot{s}_t = -c_t \tag{7}$$

where the only alteration from the previous section is in the inclusion of the stock as an argument of the utility function, leading to qualitatively different conclusions. Now, in contrast to the previous case, it may be optimal to preserve some of the resource stock indefinitely. How much is sensitive to the precise specification of the objective, and we investigate several alternatives.

3.1 Utilitarian Optimal Paths

In the case of problem (7) the Hamiltonian is

$$H = u\left(c_t, s_t\right) e^{-\delta t} - \lambda_t c_t e^{-\delta t}$$

and maximization with respect to consumption c_t gives the same result as before, namely that the derivative of utility with respect to consumption must be greater than or equal to the shadow price of the resource:

$$u_c\left(c_t, s_t\right) \leq \lambda_t, = \lambda_t \text{ if } c_t > 0$$

[8]This framework was introduced by Krautkraemer [12], and developed further by him in [13].

where $u_c \equiv \frac{\partial u(c,s)}{\partial c}$ etc. However, the condition describing the movement of the shadow price over time is different, and now a solution has to satisfy

$$\dot{\lambda}_t - \delta\lambda_t = -u_s\left(c_t, s_t\right)$$

For simplicity now consider the case when the utility function is additively separable:

$$u\left(c_t, s_t\right) = u_1\left(c_t\right) + u_2\left(s_t\right)$$

where as always the u functions are increasing, strictly concave and twice continuously differentiable. Separability implies that the marginal utility of consumption is independent of the level of the remaining stock, and vice versa: in other words, the valuation of each source of utility is independent of the level of the other. Then, letting a prime denote the derivative of a function of one variable with respect to its argument, the conditions for optimality become:

$$\left.\begin{array}{c} u_1'\left(c_t\right) \leq \lambda_t, = \lambda_t \text{ if } c_t > 0. \\ \dot{\lambda}_t - \delta\lambda_t = -u_2'\left(s_t\right) \end{array}\right\} \tag{8}$$

In the previous case, the shadow price of the resource grew indefinitely: now in contrast there may be a solution at which c_t, s_t and λ_t are constant. Note that if consumption is constant, it must of course be zero: this is the only feasible constant consumption. And note that if the shadow price is constant, then $\delta\lambda_t = u_2'\left(s_t\right)$. So at a stationary solution of the first order conditions (8)

$$\delta \leq \frac{u_2'\left(s^*\right)}{u_1'\left(0\right)}, \text{ with equality if } c_t > 0. \tag{9}$$

where s^* is the constant value of the remaining resource stock.

This equation has a simple interpretation: it requires that the slope of an indifference curve in the $s - c$ plane, the ratio of the marginal utilities of the stock and flow, equal (or exceed) the discount rate. If we rewrite it as $u_1'\left(0\right) = u_2'\left(s^*\right)/\delta$, we can see another interpretation. Consider postponing an increment of consumption Δc indefinitely. The loss of utility is the derivative of utility with respect to consumption times the drop in consumption, $u_1'\left(0\right)\Delta c$. The gain from an increased stock, which continues indefinitely, is the present value of the stream of incremental utilities accruing from an increased stock.

$$\int_0^\infty u_2'\left(s^*\right)\Delta c \exp[-\delta t]dt = \Delta c u_2'\left(s^*\right)/\delta$$

Equality of the incremental gains and losses implies (9), which can thus be interpreted as saying that the extra utility of an increment of consumption must equal the present value of the stream of incremental utility resulting from an increase in the stock. This is all very natural and straightforward.

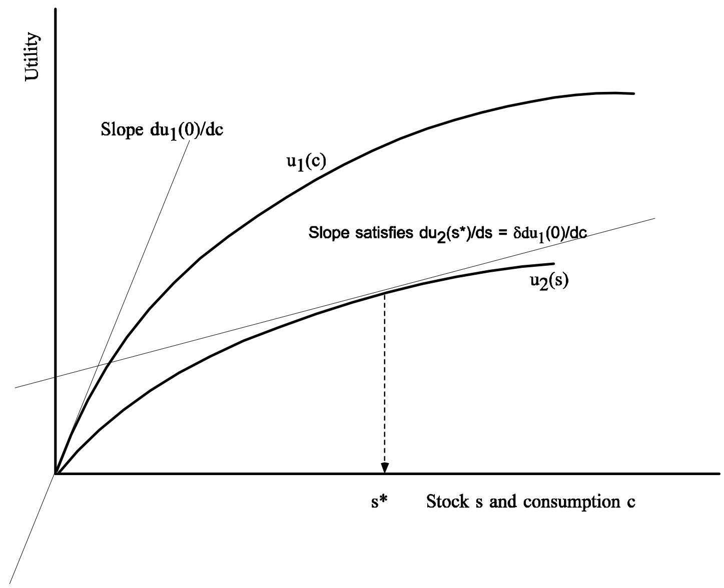

Figure 1:

The stationary configuration for this model is shown in figure 1: the constant level of the stock is one at which the derivative of utility with respect to the stock (the slope of the curve u_2) equals the slope of u_1, the utility-of-consumption function, at the origin, times the discount rate δ.

Figure 2 shows the dynamics of an optimal policy, which involve depleting the resource stock by consuming it until it is run down to s^*, and then stopping consumption and preserving the stock for ever. The optimal policy is described by the two differential equations

$$\begin{aligned} u_1''(c_t)\,\dot{c}_t - \delta u_1'(c_t) &= -u_2'(s_t) \\ \dot{s}_t &= -c_t \end{aligned} \tag{10}$$

whose phase portrait is shown in figure 2. To understand this portrait, note that $\dot{s}_t$ is always non-positive, so that the system moves to the left or is stationary. $\dot{c}_t$ is zero on the curve $\delta u_1'(c_t) = u_2'(s_t)$, and is negative to the right of this and positive to the left. There is a stationary solution to the equations (10) at the point $c = 0$, $s = s^*$. It is straightforward to verify that this stationary solution is approached from initial points on a one dimensional stable manifold, as shown in figure 2: this is shown by linearizing the systems (10) in a neighborhood of the stationary solution, and observing that the matrix of the linearized system has real eigenvalues, one positive

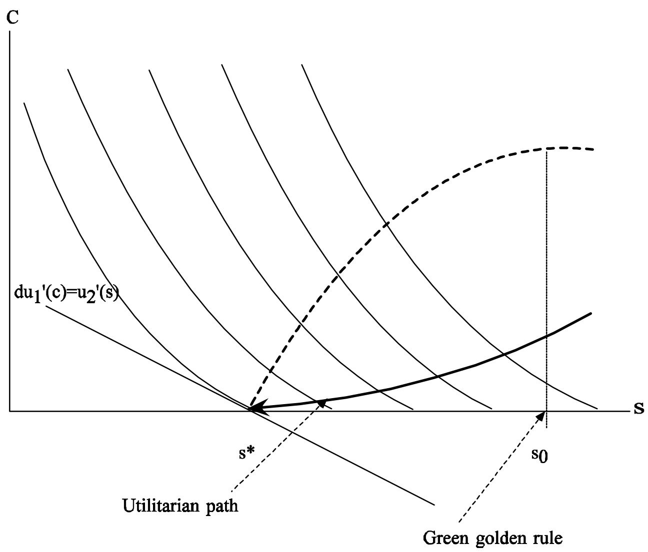

Figure 3.2: the dynamics of alternative solutions to the depletion problem.

Figure 2:

and one negative. The linearization is

$$\begin{pmatrix} \frac{u_2' u_1'''}{(u_1'')^2} + \frac{\delta\left([u_1'']^2 - u_1' u_1'''\right)}{\left(u_1''\right)^2} & \frac{-u_2''}{u_1''} \\ -1 & 0 \end{pmatrix}$$

and it follows that the signs of the eigenvalues are opposite as the determinant is negative.

There is an important difference between the solution to the present problem, and that to the classic Hotelling problem. In the present framework a positive stock may be preserved for ever on an optimal path: exactly how much depends on the discount rate and on the utility functions, but this is a qualitative difference between the two problems. So the concept of "sustainability" seems to have some relevance in the context of this solution.

When will it be optimal in this context to preserve a positive stock level for ever? This depends on the behavior of the utility function as the consumption level goes to zero. If the marginal utility of consumption goes to infinity as consumption goes to zero, as is often assumed, then equation (9) has no solution and there is no stationary state. This is the case with a Cobb-Douglas function, or with a separable utility function in which $u_1(c)$ has a constant elasticity of marginal utility. In any of these cases, the indifference curves in the $c - s$ plane do not cross the horizontal axis, but asymptote towards it. These are cases in which the flow from the resources is in some sense "essential." There is no substitute for it and it cannot be allowed to fall to zero. It is natural that in these cases no stock will be preserved. It is however counterintuitive that we preserve the stock when there is a substitute for the flow that it produces. Note that whether there is a substitute for the stock - i.e. the behavior of $u_2'(0)$ - is not important in determining whether a positive stock remains. We can summarize this as follows:

Proposition 2 *Consider an optimal solution to problem (7) when the utility function is additively separable,* $u(c,s) = u_1(c) + u_2(s)$. *A sufficient condition for this to involve the preservation of a positive stock for ever is that the marginal utility of consumption at zero is finite,* $u_1'(0) < \infty$*, and that there exists a finite stock level* s^**, the optimal stationary stock, such that* $u_1'(0)\delta = u_2'(s^*)$*. In this case, if the initial stock* $s_0 > s^*$*, then total consumption over time will equal* $s_0 - s^*$*: if* $s_0 \leq s*$*, then consumption will always be zero and the entire stock will be conserved on an optimal path. If on the other hand the marginal utility of consumption at* $c = 0$ *is infinite, then it will not be optimal to conserve any positive stock level indefinitely.*

3.2 The Green Golden Rule

A second difference arising from the inclusion of the stock as a source of utility comes when we ask the question: "Which configuration of the economy gives the maximum sustainable utility level?", a question motivated by the "golden rule of economic growth" introduced in the 1960s by Phelps [15], and by our present interest in sustainability. In the Hotelling formulation, there is no interesting answer to this question: the only utility level maintainable for ever is that associated with zero consumption. In the present model, however, the question is quite interesting, as there are many utility levels that can be maintained for ever. Clearly in the very long run no positive consumption level can be maintained and utility must be derived from the stock only. So the answer to the question "Which configuration of the economy gives the maximum sustainable utility level?" must be "the utility level associated with the initial stock (the biggest stock ever) and zero consumption." Formally, in finding the maximum utility that can be sustained indefinitely we are maximizing $u(0,s)$ where $s \leq s_0$, and the solution is clearly to preserve the entire stock and never consume anything. This is the solution that has been called the green golden rule [2]: it is the path which of all feasible paths gives the highest value of the long run level of utility[9]. It can be formalized as the solution to

$$\max_{feasible\ paths} \lim_{t\to\infty} u(c_t, s_t)$$

Formally,

[9] This is formalized as the maximum limiting utility value. However, other formalizations are in principle possible: for example, as the maximum of the lim sup of the utility values. The differences between alternatives become of significance only when positive limit sets of feasible trajectories may be limit cycles or other more complex attractors.

Proposition 3 *The maximum sustainable utility level is attained by conserving the entire initial stock.*

This point is illustrated in figure 2.

3.3 The Rawlsian Optimum

An alternative approach to sustainability is using the Rawlsian definition of justice between generations. In the case of the current model, the green golden rule happens to be the path which is optimal in the Rawlsian sense, i.e., which maximizes the welfare of the generation which is least well off. It is easy to see this point. On any path which involves positive consumption, the utility level is eventually non-increasing over time. So the least-well-off generation is the "last" generation: in fact there is no "last" generation, so more accurately the lowest welfare level is the limiting welfare level. But this is maximized by the green golden rule, which maximizes the sustainable, and so the limiting, welfare level over all feasible paths.

This coincidence of welfare criteria does not occur in all models, and in particular does not occur in the model of a renewable resource considered in the next section. In that model, the least well off generation may be the first, not the last. Nevertheless non-renewable environmental resources valued both as stocks as flows are probably sufficiently important that the present model has real relevance as an ideal type, so that the coincidence of criteria is a significant result.

3.4 Overtaking

In an attempt to avoid the problems of zero discount rates, and yet give equal weight to present and future, von Weizäcker [22] introduced the overtaking criterion:

Definition 4 *A path c^1 is said to weakly overtake a path c^2 if there exists a time T^* such that for all $T > T^*$, we have*

$$\int_0^T u\left(c_t^1\right) dt \geq \int_0^T u\left(c_t^2\right) dt$$

c^1 is said to strictly overtake c^2 if the inequality is strict.

This is an ingenious approach: it replaces infinite integrals by finite ones, and says that one path is better than another if from some date on cumulative utility on that path is greater. This is a relationship that can be checked even if both cumulative utility totals go to infinity as $T \to \infty$, so this approach does to some degree extend the applicability of an approach based on a zero discount rate. The overtaking criterion ranks paths with different limiting utility values according to those limiting values [8]. Consequently the overtaking optimal path is the green golden rule, the path along which nothing is consumed and the entire initial stock is maintained intact.

The argument is as follows. First recall that the limiting utility value on any path is $u\left(0, s'\right)$ where s' is the limiting resource stock. Provided that zero consumption does not incur an infinite utility penalty, we maximize this value by having as the limiting stock, the initial resource stock. The program which consumes nothing has the largest limiting utility value and so overtakes any other program. The overtaking optimum is also the Rawlsian optimum and the green golden rule. It is furthermore the only solution with non-decreasing utility levels. This is a remarkable coincidence of views. Note that with these three criteria complete conservation of the initial stock is optimal, whatever the size of the initial stock. Nothing in these arguments depends on the size of the initial stock. In some ways this is surprising: intuitively, one might feel that whether to conserve or not should depend on the size of the initial stock. This is true of the discounted utilitarian solution.

3.5 Summary

Explicit recognition of the resource stock as a source of utility gives substance to the concept of sustainability. Now there is a purpose to conservation, and indeed full conservation emerges optimal for several objectives. Optimality and sustainability overlap and perhaps even coincide: even discounted utilitarianism may recommend the conservation of a substantial stock *ad infinitum*.

4 Renewable Resources

Now we add further to the structure of the model, this time in the specification of the constraints and the dynamics of the resource. We assume the resource to be renewable, i.e., to have self-regenerating properties. The resource has a dynamic, a life, of its own. We model the interaction between this dynamic and the time path of its use by humans. Animals, fish and forests fall into this category. In fact, any ecosystem is of this type, and many of our most important natural resources are best seen as entire ecosystems rather than as individual species or subsystems. For example, soil is a renewable resource with a dynamic of its own, which interacts with the patterns of use by humans. Even for individual species such as whales or owls, one should ideally think of the validity and the dynamics of the entire ecosystem of which they are a part.

We shall see that the renewable nature of the resource makes a dramatic difference to the nature of optimal solutions. Now the future may actually be better treated than the present along an optimal path: if the initial resource stock is low, the optimal policy requires that consumption, stock and utility all rise monotonically over time. The point is that because the resource is renewable, both stocks and flows can be built up over time provided that consumption is less than the rate of regeneration.

In this reformulation, the maximand remains exactly as before: primarily the discounted integral of utilities from consumption and from the existence of a stock,

$\int_0^\infty u(c,s)\,e^{-\delta t}dt$, although as before some alternatives will be reviewed. However, the constraints are changed. We assume that the dynamics of the renewable resource are described by

$$\overset{\bullet}{s}_t = r(s_t) - c_t \tag{11}$$

Here r is the natural growth rate of the resource, assumed to depend only on its current stock. This describes its growth without human intervention. More complex models are of course possible, in which several such systems interact: a well-known example is the predator-prey system. In general, r is a concave function which attains a maximum at a finite value of s. This formulation has a long and classical history, which is reviewed in Dasgupta and Heal [5]. In the field of population biology, $r(s_t)$ is often taken to be quadratic, in which case an unexploited population (i.e., $c_t = 0 \forall t$) grows logistically. Here we assume that $r(0) = 0$, that there exists a positive stock level $\overline{s}$ at which $r(\overline{s}) = 0$, and that $r(s)$ is strictly concave and twice continuously differentiable.

Probably the weakest part of this specification, is the ecological dynamic. As noted above, most ecosystems are considerably more complex than suggested by the adjustment equation (11). In most cases they consist of many linked elements each with its own interacting dynamics. It is possible that under some conditions the simple representation used here can be thought of as an aggregate representation of the ecological system as a whole, with the variable s_t not the stock of an individual type but an aggregate measure such as biomass: this is a topic for further research. It is also true, fortunately, that the general qualitative conclusions which we reach, do not depend very sensitively on the precise specification of the ecological system.

The overall problem can now be specified as

$$\max \int_0^\infty u(c,s)\,e^{-\delta t}dt \text{ s.t. } \overset{\bullet}{s}_t = r(s_t) - c_t,\ s_0 \text{ given.} \tag{12}$$

The Hamiltonian in this case is

$$H = u(c_t, s_t)\,e^{-\delta t} + \lambda_t e^{-\delta t}\left[r(s_t) - c_t\right]$$

Maximization with respect to consumption gives as usual the equality of the marginal utility of consumption to the shadow price:

$$u_c(c_t, s_t) = \lambda_t$$

and the rate of change of the shadow price is determined by

$$\frac{d}{dt}\left(\lambda_t e^{-\delta t}\right) = -\left[u_s(c_t, s_t)\,e^{-\delta t} + \lambda_t e^{-\delta t} r'(s_t)\right]$$

A solution to the problem is characterized by

$$\left.\begin{array}{r} u_1'(c_t) = \lambda_t \\ \overset{\bullet}{s}_t = r(s_t) - c_t \\ \overset{\bullet}{\lambda}_t - \delta\lambda = -u_2'(s_t) - \lambda_t r'(s_t) \end{array}\right\} \tag{13}$$

which in fact reduce to the equations of the solution to the previous problem if $r(s)$ is identically zero. In studying these equations, we first analyze their stationary solution, and then examine the dynamics of this system away from the stationary solution.

4.1 Stationary Solutions

At a stationary solution, s is constant so that $r(s_t) = c_t$: in addition the shadow price is constant so that

$$\delta u_1'(c_t) = u_2'(s_t) + u_1'(c_t) r'(s_t)$$

Hence:

Proposition 5 *A stationary solution to (13) satisfies*

$$\left.\begin{array}{r} r(s_t) = c_t \\ \frac{u_2'(s_t)}{u_1'(c_t)} = \delta - r'(s_t) \end{array}\right\} \quad (14)$$

The first equation in (14) just tells us that a stationary solution must lie on the curve on which consumption of the resource equals its renewal rate: this is obviously a prerequisite for a stationary stock. The second gives us a relationship between the slope of an indifference curve in the $c - s$ plane and the slope of the renewal function at a stationary solution: the indifference curve cuts the renewal function from above. Such a configuration is shown as the utilitarian stationary solution in figure 7. This reduces to the earlier result that the slope of an indifference curve should equal the discount rate if $r'(s) = 0 \forall s$, i.e., if the resource is non-renewable.

There is a straightforward intuitive interpretation to the second equation in (14). This interpretation is exactly analogous to that given for the corresponding equation (9) of the previous section. Consider reducing consumption by an amount Δc and increasing the stock by the same amount. The welfare loss is $\Delta c u_1'$: there is a gain from increasing the stock of $\Delta c u_2'$, which continues for ever. But in addition the changed stock leads to a different stationary consumption level, in the amount $\Delta c r'$. Both of these effects—the changed stock and the changed stationary consumption—continue for ever and so we have to take their present values by dividing by the discount rate. Hence at an optimum

$$\Delta c u_1' = \Delta c[u_2' + r' u_1']/\delta$$

which reduces to the second equation in (14).[10] So (14) is a very natural and intuitive characterization of optimality.

[10] I am grateful to Jim Wilen for this interpretation.

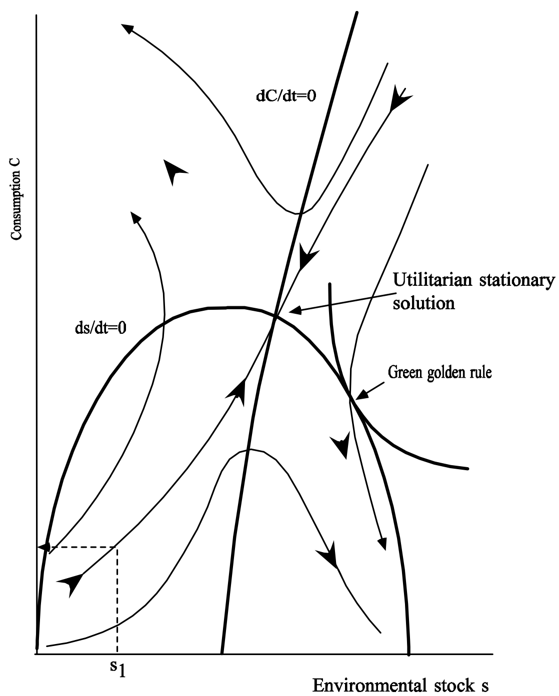
dC/dt=0
Consumption C
Utilitarian stationary solution
ds/dt=0
Green golden rule
s1
Environmental stock s

Figure 3:

4.2 Dynamic Behavior

What are the dynamics of this system outside of a stationary solution? These are also shown in figure 7. They are derived by noting the following facts:

1. beneath the curve $r(s) = c$, s is rising as consumption is less than the growth of the resource.

2. above the curve $r(s) = c$, s is falling as consumption is greater than the growth of the resource.

3. on the curve $r(s) = c$, s is constant.

4. from (13), the rate of change of c is given by

$$u_1''(c)\,\dot{c} = u_1'(c)\left[\delta - r'(s)\right] - u_2'(s)$$

The first term here is negative for small s and vice versa: the second is negative and large for small s and negative and small for large s. Hence c is rising for small s and vice versa: its rate of change is zero precisely when the rate of change of the shadow price is zero, which is on a line containing the stationary solution. The slope of this line is given by

$$\frac{\partial c}{\partial s} = \frac{u_1' r' + u_2''}{u_1''(\delta - r')}$$

The numerator is negative: the denominator is likewise if $\delta > r'$, in which case the slope of the $\dot{c} = 0$ line is positive at least in a neighborhood of the stationary solution.

5. by linearizing the system

$$\left.\begin{aligned} u_1''(c)\,\dot{c} &= u_1'(c)\left[\delta - r'(s)\right] - u_2'(s) \\ \dot{s}_t &= r(s_t) - c_t \end{aligned}\right\}$$

around the stationary solution, one can show that this solution is a saddle point.[11]

This shows that the utilitarian stationary solution (14) is a saddlepoint locally if it involves a stationary stock in excess of that giving the maximum sustainable yield. This is certainly the case for δ small enough. Hence the dynamics of paths satisfying the necessary conditions for optimality are as shown in figure 7, and we can establish:

[11] This is also true for small stocks for which $r' > \delta > 0$, and in other cases. To simplify the linearization I have taken the third derivative of u_1 to be zero, or at least small relative to the square of the second derivative.

Proposition 6 *For small enough discount rates, optimal paths for the problem (12) tend to the stationary solution (14). They do so along a path satisfying the first order conditions (13), and follow one of the two branches of the stable path in figure 7 leading to the stationary solution. Given any initial value of the stock s_0, there is a corresponding value of c_0 which will place the system on one of the stable branches leading to the stationary solution. The position of the stationary solution depends on the discount rate, and moves to higher values of the stationary stock as this decreases. As $\delta \to 0$, the stationary solution tends to a point satisfying $u_2'/u_1' = r'$, which means in geometric terms that an indifference curve of $u(c,s)$ is tangent to the curve $c = r(s)$ given by the graph of the renewal function.*

The renewable nature of the resource has clearly made a dramatic difference to the nature of optimal solutions. Now the future may actually be better treated than the present: if the initial resource stock is low, *the optimal policy requires that consumption, stock and utility all rise monotonically over time.* The point is that because the resource is renewable, both stocks and flows can be built up over time provided that consumption is less than the rate of regeneration, i.e., the system is inside the curve given by the graph of the renewal function $r(s)$.

In practice, unfortunately, many renewable resources are being consumed at a rate greatly in excess of their rates of regeneration: in terms of figure 7, the current consumption rate c_t is much greater than $r(s_t)$. So taking advantage of the regeneration possibilities of these resources would in many cases require sharp limitation of current consumption. Fisheries are a widely-publicized example: another is tropical hardwoods and tropical forests in general. Soil is a more subtle example: there are processes which renew soil, so that even if it suffers a certain amount of erosion or of depletion of its valuable components, it can be replaced. But in many cases human use of soils is depleting them at rates far in excess of their replenishment rates.

4.3 The Green Golden Rule

We can use the renewable framework to ask a question that we asked before: what is the maximum sustainable utility level? There is a simple answer.

First, note that a sustainable utility level must be associated with a sustainable configuration of the economy, i.e., with sustainable values of consumption and of the stock. But these are precisely the values that satisfy the equation

$$c_t = r(s_t)$$

for these are the values which are feasible and at which the stock and the consumption levels are constant. Hence in figure 7, we are looking for values which lie on the curve $c_t = r(s_t)$. Of these values, we need the one which lies on the highest indifference curve of the utility function $u(c,s)$: this point of tangency is shown in the figure. At this point, the slope of an indifference curve equals that of the renewal function, so that the marginal rate of substitution between stock and flow equals the marginal rate of transformation along the curve $r(s)$. Hence:

Proposition 7 *The maximum sustainable utility level (the green golden rule) satisfies*

$$\frac{u_2'(s_t)}{u_1'(c_t)} = -r'(s_t)$$

Recall from (14) that as the discount rate goes to zero, the stationary solution to the utilitarian case tends to such a point.

Note also that any path which approaches the tangency of an indifference curve with the reproduction function, is optimal according to the criterion of achieving the maximum sustainable utility. In other words, this criterion of optimality only determines the limiting behavior of the economy: it does not determine how the limit is approached. This clearly is a weakness: of the many paths which approach the green golden rule, some will accumulate far more utility than others. One would like to know which of these is the best, or indeed whether there is such a best. We return to this later.

4.4 The Rawlsian Solution

In the non-renewable context, we noted the coincidence of the Rawlsian optimum with the Green Golden Rule. In the present case things do not always fit together so neatly. Consider the initial stock level s_1 in figure 7: the utilitarian optimum from this is to follow the path that leads to the saddle point. In this case, as noted, consumption, stock and utility are all increasing. So the generation which is least well of, is the first generation, not the last, as it was in the non-renewable case. What is the Rawlsian solution in the present model, with initial stock s_1? It is easy to verify that this involves setting $c = r(s_1)$ for ever: this gives a constant utility level, and gives the highest utility level for the first generation compatible with subsequent levels being no lower. This remains true for any initial stock no greater than that associated with the Green Golden Rule: for larger initial stocks, the Green Golden Rule is a Rawlsian optimum and in this case we do still have the coincidence noted in the previous chapter. Formally,

Proposition 8 *For an initial resource stock s_1 less than or equal to that associated with the Green Golden Rule, the Rawlsian optimum involves setting $c = r(s_1)$ for ever. For s_1 greater than the green golden rule stock, the Green Golden Rule is a Rawlsian optimum.*

4.5 Overtaking

What does the overtaking criterion imply in the case of renewable resources?

First note that by quite standard arguments, an optimal path must satisfy the utilitarian first order conditions for optimality with the discount rate equal to zero:

$$\begin{aligned} u_1' &= \lambda \\ \dot{\lambda} &= -u_2' - u_1' r' \\ \dot{s} &= r(s) - c \end{aligned}$$

These conditions have as their saddle-point stable stationary solution the green golden rule.

Second, note that all paths satisfying the above first order conditions have well-defined limiting utility values.

Finally, note that the green golden rule is the highest possible limiting utility value, so that the path satisfying the above conditions and approaching the green golden rule overtakes any other path satisfying the same first order conditions. Overall, then, it is clear an overtaking optimal path follows the utilitarian first order conditions with a zero discount rate to the green golden rule.

5 Equal Treatment Over Finite Horizons

Much of our ethical and moral intuition is grounded in the consideration of finite horizons. Life on earth will certainly be of finite duration, although it is difficult to determine its final date. It is therefore important to determine whether the phenomena we have been discussing are an artifact of infinite horizons, or have a clear relationship to approaches which seem reasonable in the context of finite horizons.

This section will show that these phenomena can be seen as an extension to infinite horizons of the properties of optimal solutions for an intuitively appealing criterion for finite horizons, the criterion which values all generations equally. This we call the "finite equal treatment" criterion. Indeed, for a general class of dynamic optimization problems, we will see that as the finite horizon increases, the optimal solutions of equal treatment finite horizon problems spend an increasing amount of time progressively closer to the green golden rule. We refer to this property as a "turnpike" property.

Definition 9 *The equal treatment problem for horizon T is:*

$$\left.\begin{array}{c} \max \int_0^T u(c_t, s_t)\, dt \\ s.t.\ \dot{s}_t = r(s_t) - c_t,\ s_0 \text{ given.} \end{array}\right\} \qquad (15)$$

Its solution is called the equal treatment optimum over T generations.

Theorem 10 *The green golden rule is the "turnpike" of finite horizon problems (15) in which each generation is treated equally. This means that as the number of generations T increases, the equal treatment optima for T generations spend some of their*

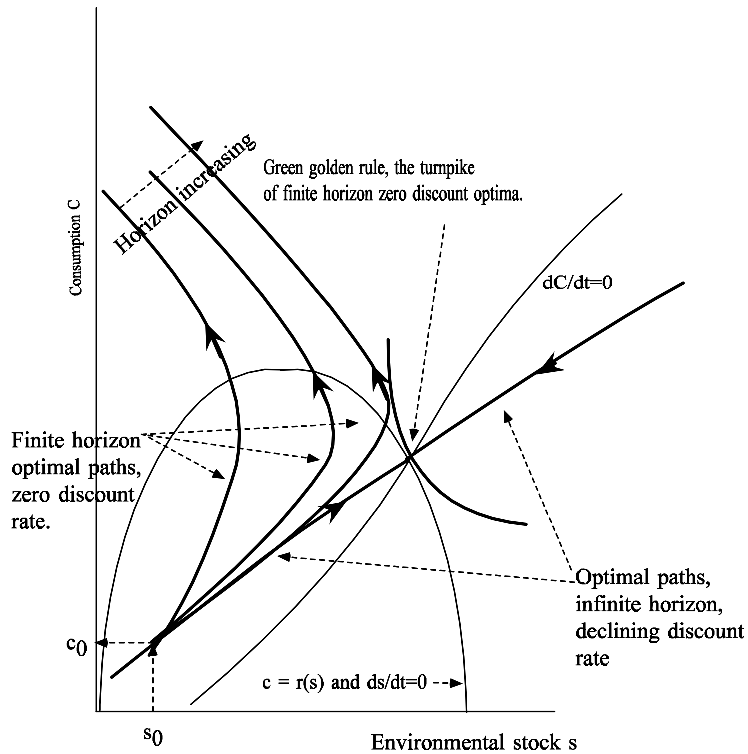

Figure 4:

time increasingly close to the green golden rule, and by the optima according to the overtaking criterion. Formally: the distance[12] *between the equal treatment optimal path for horizon T and the green golden rule g^* goes to zero as T goes to infinity.*

6 Conclusion

From a review of paths of resource use that are optimal for a variety of models, we see that many optimal paths are sustainable. That is, they involve maintaining at least a part of the initial resource stock intact for ever. In fact it is only the simple Hotelling model that does not produce sustainable paths, but this reflect the technology of

[12] As a measure of the distance between the path and the point g^* I use the Hausdorf distance, the shortest distance between a point on the path and the point g^*. Denoting the Hausdorf distance between points x and y by $HD(x, y)$, and the optimal path for horizon T by $\{Opt.T\}$, we have $HD(\{Opt.T\}, g^*) = \min_{x \in \{Opt.T\}} \|x, g^*\|$ where $\|x, y\|$ is the Euclidean distance between x and y.

the problem rather than a conflict between optimality and sustainability. With more environmentally appropriate preferences, even this specification of the technology can give optimal paths that are sustainable, and indeed maintain the entire initial stock intact. In the case of renewable resources, most possible optimal paths are sustainable in the sense of maintaining the resource base and indeed growing it. Some asymptote to the maximum possible utility level, the green golden rule, and are sustainable in a very strong sense: others settle at a lower utility level, but are still sustainable. The green golden rule occupies a strategic position in the analysis, in that most paths will move towards it or remain near it for long periods. My initial assertion that optimal paths are sustainable, provided that the preferences and constraints reflect fully what we know human dependence on environmental systems, seems well documented.

References

[1] Asheim Geir, Wolfgang Buchholz and Bertil Tungodden. "Justifying sustainability." *Journal of Environmental Economics and Management*, Vol 41, No. 3, May 2001.

[2] Beltratti, Andrea, Graciela Chichilnisky and Geoffrey M. Heal. "The green golden rule." *Economics Letters,* (1995), 49: 175-179.

[3] Daily, Gretchen (ed). *Nature's Services: Societal Dependence on Natural Ecosystems.* Island Press, Washington D.C., 1997.

[4] Dasgupta, Partha S. and Geoffrey M. Heal. "The optimal depletion of exhaustible resources." *Review of Economic Studies*, symposium (1974), pp. 3-28.

[5] Dasgupta, Partha S. and Geoffrey M. Heal. *Economic Theory and Exhaustible Resources*, Cambridge, England: Cambridge University Press, 1979.

[6] Heal, Geoffrey M. *The Theory of Economic Planning*, Amsterdam: North Holland Publishing Company, 1973.

[7] Heal, Geoffrey M. "Depletion and discounting: a classical issue in resource economics." In R. McElvey ed. *Environmental and Natural Resource Mathematics* (1985), 32: 33-43, Proceedings of Symposia in Applied Mathematics, American Mathematical Society, Providence, RI.

[8] Heal, Geoffrey M. *Valuing the Future: Economic Theory and Sustainability.* Columbia University Pess, 1998, New York.

[9] Heal, Geoffrey M. *Nature and the Marketplace.* Island Press, Washington D.C., 2000.

[10] Hotelling, Harold. "The economics of exhaustible resources." *Journal of Political Economy* (1931), 39, 137-75.

[11] Koopmans, Tjalling. "Stationary ordinal utility and impatience." *Econometrica* (1960), 28: 287-309.

[12] Krautkraemer, Jeffrey A. "Optimal growth, resource amenities and the preservation of natural environments." *Review of Economic Studies* (1985) 52: 153-170.

[13] Krautkraemer, Jeffrey A. "Optimal depletion with resource amenities and a backstop technology." *Resources and Energy* (1986), 8: 133-149.

[14] Pezzey John. "Economic analysis of sustainable growth and development." Environmental Department Working Paper No. 15, World Bank, Washington D.C., 1989.

[15] Phelps, Edmund S. "The golden rule of accumulation: a fable for growthmen." *American Economic Review* (1961), 638-643.

[16] Rawls, John. *A Theory of Justice*. Oxford, England: Clarendon Press, 1972.

[17] Ramsey, Frank. "A mathematical theory of saving." *Economic Journal* (1928), 38: 543-559.

[18] Seirstad, Atle and Knut Sydsæter. *Optimal Control Theory with Economic Applications.* Amsterdam: North Holland, 1987.

[19] Smulders, Sjak. "Economic growth and environmental quality." Intended as Chapter 21 of *Principles of Environmental Economics*, Henk Folmer and Landis Gabel (eds), Edward Elgar, 2001.

[20] Solow, Robert M. "A contribution to the theory of economic growth." *Quarterly Journal of Economics* (1956), 70 (1): 65-94.

[21] Solow Robert M. "Intergenerational equity and exhaustible resources." *Review of Economic Studies*, Symposium, (1974), pp. 29-45.

[22] von Weizäcker, Carl Christian. "Lemmas for a theory of approximately optimal growth." *Review of Economic Studies* (1967), pp. 143-151.