Intermediate Financial Theory
Danthine and Donaldson

Solutions to Additional Exercises

Chapter 1

1.6. Consider a two agent –two good economy. Assume well-behaved utility functions (in particular, indifference curves don't exhibit flat spots). At a competitive equilibrium, both agents maximize their utility given their budget constraints. This leads each of them to select a bundle of goods corresponding to a point of tangency between one of his or her indifference curves and the price line. Tangency signifies that the slope of the IC and the slope of the budget line (the price ratio) are the same. But both agents face the same market prices. The slope of their indifference curves are thus identical at their respective optimal point.

Now consider the second requirement of a competitive equilibrium: that market clear. This means that the respective optimal choices of each of the two agents correspond to the same point of the Edgeworth-Bowley box.

Putting the two elements of this discussion together, we have that a competitive equilibrium is a point in the box corresponding to a feasible allocation where both agents' indifference curves are tangent to the same price line, have the same slope, and, consequently, are tangent to one another. Since the contract curve is the locus of all such points in the box at which the two agents' indifference curves are tangent, the competitive equilibrium is on the contract curve.

Of course, we could have obtained this result simply by invoking the First Welfare Theorem.

1.7. Indifference curves of agent 2 are non-convex.
Point A is a PO : the indifference curves of the two agents are tangent. This PO cannot be obtained as a Competitive Equilibrium, however. Let a price line tangent to I_1 at point A. It is also tangent to I_2, but "in the wrong direction": it corresponds to a local minimum for agent 2 who, at those prices, can reach higher utility levels. The difficulty is generic when indifference curves have such a shape. The geometry is inescapable, underlining the importance of the assumption that preferences should be convex.

Chapter 4

4.9 Certainty equivalent.
The problem to be solved is: find Y such that

$$\frac{1}{2}U(Y+1000)+\frac{1}{2}U(Y-1000)\equiv U(Y-500)$$
$$\frac{1}{(Y+1000)}+\frac{1}{(Y-1000)}=\frac{2}{(Y-500)}$$
$$\frac{(Y+1000)+(Y-1000)}{(Y+1000)(Y-1000)}=\frac{2}{(Y-500)}$$
$$\frac{2Y}{(Y^2-1000^2)}=\frac{2}{(Y-500)}$$
$$Y^2-1000^2=Y^2-500Y$$
$$Y=2000$$

The logarithmic utility function is solved in the same way; the answer is Y = 1250.

4.10. Risk premium.

The problem to be solved is: find P such that
$$\frac{1}{2}(\ln(Y+1000)+\ln(Y-1000))\equiv\ln(Y-P)$$
$$Y-P=\exp\left(\frac{1}{2}(\ln(Y+1000)+\ln(Y-1000))\right)$$
$$P=Y-\exp(.)$$
where P is the insurance premium.
$$P(Y=10000)=50.13$$
$$P(Y=100000)=0.50$$
The utility function is DARA, so the outcome (smaller premium associated with higher wealth) was expected.

4.11.

Case 1	Case 2	Case 3
$\sigma_a > \sigma_b$	$\sigma_a = \sigma_b$	$\sigma_a < \sigma_b$
$E_a = E_b$	$E_a > E_b$	$E_a < E_b$

Case 1: cannot conclude with FSD, but B SSD A

Case 2: A FSD B, A SSD B

Case 3: cannot conclude (general case)

4.12. a. $U(Y-CE)=EU(Y-L(\theta))$
$$\frac{(10{,}000+CE)^{-.2}}{-.2}=.10\frac{(10{,}000-1{,}000)^{-.2}}{-.2}+.20\frac{(10{,}000-2{,}000)^{-.2}}{-.2}$$

$$+.35\frac{(10{,}000-3{,}000)^{-2}}{-.2}+.20\frac{(10{,}000-5{,}000)^{-2}}{-.2}+.15\frac{(10{,}000-6{,}000)^{-2}}{-.2}$$

$$(10{,}000+CE)^{-2}=.173846$$

$$(10{,}000+CE)^{-2}=\frac{1}{.173846}=5.752$$

CE = -3702.2

$$E\tilde{L}=-\{.1(1000)+.2(2000)+.35(3000)+.2(5000)+.15(6000)\}$$
$$=-3450$$

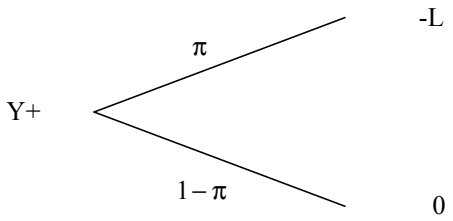
$$CE(\tilde{z},y)=E(\tilde{z})-\Pi(y,\tilde{z})$$

$$\Pi(y,\tilde{z})=252.2$$

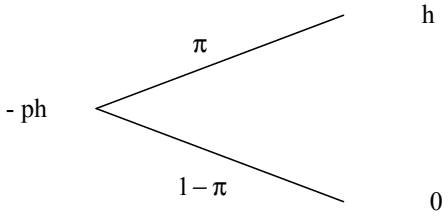
If the agent were risk neutral the CE = - 3450

b. If $U'(y)>0, U''(y)>0$, the agent loves risk. The premium would be negative here.

4.13. Current Wealth :



Insurance Policy :



Certainly $p \le 1$

a. Agent solves

$$\max_h \pi\ln(y-ph-L+h)+(1-\pi)\ln(y-ph)$$

The F.O.C. is

$$\frac{\pi(1-p)}{y-L+h(1-p)}=\frac{p(1-\pi)}{y-ph}\text{, which solves for}$$

$$h=Y\left(\frac{\pi}{p}\right)-\left(\frac{1-\pi}{1-p}\right)(Y-L)$$

Note : if p = 0, h = ∞ ; if $\pi=1, ph=Y$.

b. expected gain is $ph-\pi L$

c. $$ph=p\left(Y\left(\frac{\pi}{p}\right)-\left(\frac{1-\pi}{1-p}\right)(Y-L)\right)=\pi L$$

$$\Rightarrow p=\pi$$

d. $$h=Y\left(\frac{\pi}{p}\right)-\left(\frac{1-\pi}{1-p}\right)(Y-L)$$

$$h = Y\left(\frac{\pi}{\pi}\right) - \left(\frac{1-\pi}{1-\pi}\right)(Y - L) = L.$$

The agent will perfectly insure. None ; this is true for all risk averse individuals.

4.14.

a. $\underline{\tilde{x}, \pi(\tilde{x})}$ $\underline{\tilde{z}, \pi(\tilde{z})}$

$E\tilde{x} = -10(.1) + 5(.4) + 10(.3) + 12(.2) = -1 + 2 + 3 + 2.4 = 6.4$

$E\tilde{z} = .2(2) + 3(.5) + 4(.2) + 30(.1) = .4 + 1.5 + .8 + 3 = 5.7$

$\sigma_{\tilde{x}}^2 = .1(-10 - 6.4)^2 + .4(5 - 6.4)^2 + .3(10 - 6.4)^2 + .2(12 - 6.4)^2$

$= 26.9 + .78 + 3.9 + 6.27$

$= 37.85$

$\sigma_{\tilde{x}} = 6.15$

$\sigma_{\tilde{z}}^2 = .2(2 - 5.7)^2 + .5(3 - 5.7)^2 + .2(4 - 5.7)^2 + .1(30 - 5.7)^2$

$= 2.74 + 3.65 + .58 + 59.04$

$= 66.01$

$\sigma_{\tilde{z}} = 8.12$

There is mean variance dominance in favor of $\tilde{x}$:

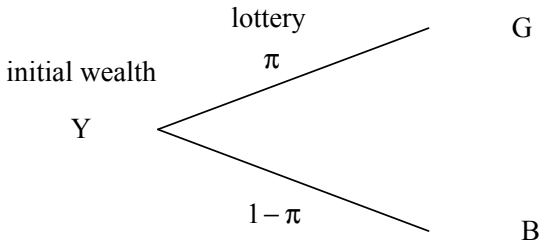
$E\tilde{x} > E\tilde{z}$ and $\sigma_{\tilde{x}} < \sigma_{\tilde{z}}$. The latter is due to the large outlying payment of 30.

b. 2nd order stochastic dominance :

r	$F_{\tilde{x}}(r)$	$\int_0^r F_x(t)dt$	$F_{\tilde{z}}(r)$	$\int_0^r F_z(t)dt$	$\int_0^r [F_x(t) - F_z(t)]dt$
-10	.1	.1	0	0	.1
-9	.1	.2	0	0	.2
-8	.1	.3	0	0	.3
-7	.1	.4	0	0	.4
-6	.1	.5	0	0	.5
-5	.1	.6	0	0	.6
-4	.1	.7	0	0	.7
-3	.1	.8	0	0	.8
-2	.1	.9	0	0	.9
-1	.1	1.0	0	0	1.0
0	.1	1.1	0	0	1.1
1	.1	1.2	0	0	1.2
2	.1	1.3	.2	.2	1.1
3	.1	1.4	.7	.9	.5
4	.1	1.5	.9	1.8	-.3

Since the final column is not of uniform sign, we cannot make any claim about relative 2nd order SD.

4.15.



a. If he already owns the lottery, P_s must satisfy

$$U(Y+P_s) = \pi U(Y+G) + (1-\pi)U(Y+B)$$
$$\text{or } P_s = U^{-1}\big(\pi U(Y+G) + (1-\pi)U(Y+B)\big) - Y.$$

b. If he does not own the lottery, the maximum he would be willing to pay, P_b, must satisfy :

$$U(Y) = \pi U(Y - P_b + G) + (1-\pi)U(Y - P_b + B)$$

c. Assume now that $\pi = ½, G = 26, B = 6, Y = 10$. Find P_s, P_b.

P_b satisfies :

$$U(10) = ½\,U(10 - P_b + 26) + ½\,U(10 - P_b + 6)$$
$$10^{½} = ½(36 - P_b)^{½} + ½(16 - P_b)^{½}$$
$$6.32 = (36 - P_b)^{½} + (16 - P_b)^{½}$$
$$P_b \approx 13.5$$

P_s satisfies :

$$(10 + P_s)^{½} = ½(10+26)^{½} + ½(10+6)^{½}$$
$$= ½(6) + ½(4) = 5$$
$$10 + P_s = 25$$
$$P_s = 15$$

Clearly, $P_b < P_s$. If the agent already owned the asset his minimum wealth is 10 + 6 =16. If he is considering buying, its wealth is 10. In the former case, he is less risk averse and the lottery is worth more.
If the agent is risk neutral, $P_s = P_b = \pi G + (1-\pi)B$. To check it out, assume U(x) = x :

P_s : $U(Y+P_s) = \pi U(Y+G) + (1-\pi)U(Y+B)$

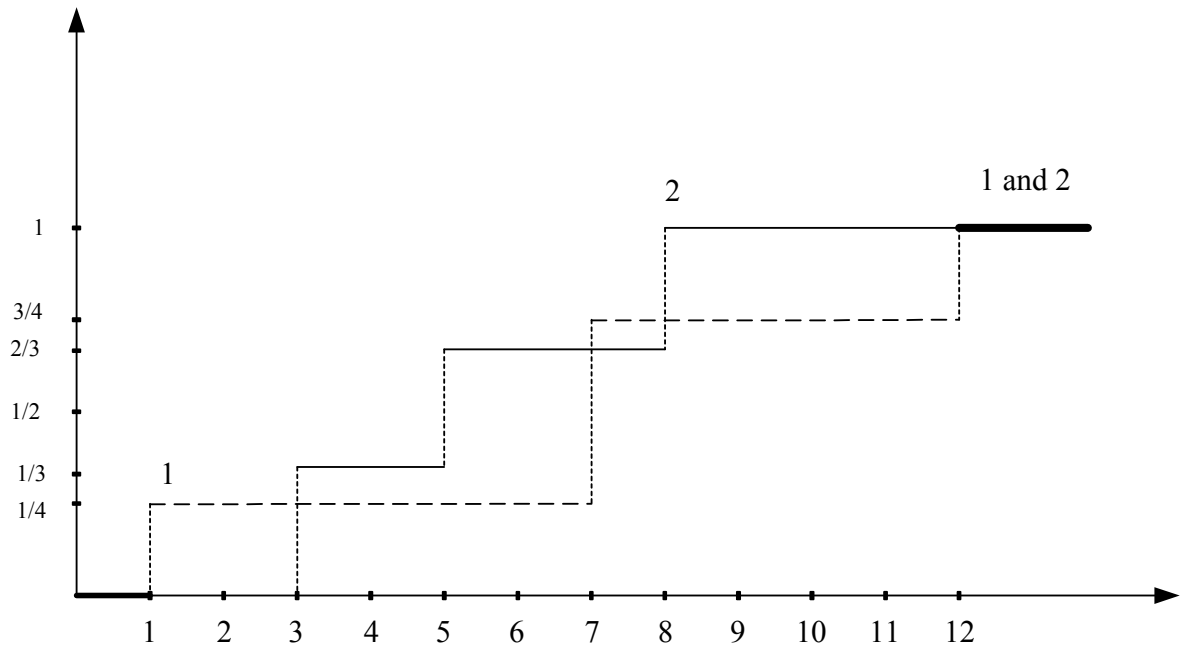
$$Y + P_s = \pi(Y+G) + (1-\pi)(Y+B) = Y + \pi G + (1-\pi)B$$
$$P_s = \pi G + (1-\pi)B$$

P_b : $U(Y) = \pi U(Y - P_b + G) + (1-\pi)U(Y - P_b + B)$

$$Y = \pi(Y - P_b + G) + (1-\pi)(Y - P_b + B)$$
$$P_b = \pi G + (1-\pi)B$$

4.16. Mean-variance: $Ex_1 = 6.75$, $(\sigma_1)^2 = 15.22$; $Ex_2 = 5.37$, $(\sigma_2)^2 = 4.25$; no dominance.

FSD: No dominance as the following graph shows:



SSD:

x	$\int_0^x f_1(t)dt$	$\int_0^x F_1(t)dt$	$\int_0^x f_2(t)dt$	$\int_0^x F_2(t)dt$	$\int_0^x [F_1(t)-F_2(t)]dt$
0	0	0	0	0	0
1	.25	.25	0	0	.25
2	.25	.50	0	0	.50
3	.25	.75	0	0	.75
4	.25	1	.33	.33	.67
5	.25	1.25	.33	.66	.67
6	.25	1.50	.66	1.32	.18
7	.50	2	.66	1.98	.02
8	.50	2.50	1	2.98	-.48

There is no SSD as the sign of $\int_0^x [F_1(t)-F_2(t)]dt$ is not monotone.

Using Expected utility. Generally speaking, one would expect the more risk averse individuals to prefer investment 2 while less risk averse agents would tend to favor investment 1.

Chapter 5

5.6. a.

Scenario	(z_1, z_2)	π	(c_1, c_2)	E(c)	Var(c)
1	(20, 80)	1/5	(35, 65)	59	144
2	(38, 98)	1/2	(44, 74)	59	225
3	(30, 90)	1/3	(40, 70)	60	200

b.

- Mean-Variance analysis:

1 is preferred to 2 (same mean, but lower variance)
3 is preferred to 2 (higher mean and lower variance)
1 and 3 can not be ranked with standard mean-variance analysis

- Stochastic Dominance:

No investment opportunity FSD one of the other investments.
Investment 1 SSD Scenario 2 (mean preserving spread).

- Expected Utility (with assumed U)

3>1>2

c.

Scenario 1

$$EU(a) = \frac{1}{5}[-\exp(-A[50-.6a50])] + \frac{4}{5}[-\exp(-A[50+.6a50])]$$

$$\frac{\partial EU(a)}{\partial a} = \frac{1}{5}(.6A50)[-\exp(-A[50-.6a50])] - \frac{4}{5}(.6A50)[-\exp(-A[50+.6a50])] = 0$$

$$a = -\frac{\ln\left(\frac{1}{4}\right)}{(1.2A50)} = .5$$

The scenarios 2 and 3 can be solved along the same lines.

5.7. a. $\max_{x_1,x_2} EU(x_1 z_1 + x_2 \tilde{z}_2)$

s.t. $p_1 x_1 + p_2 x_2 \le Y_0$

Since we assume (maintained assumption) U'( )>0, $p_1 x_1 + p_2 x_2 = Y_0$, and $x_1 = \frac{Y_0 - p_2 x_2}{p_1}$

The problem may thus be written :

$$\max_{x_2} EU\left[\left(\frac{Y_0 - p_2 x_2}{p_1}\right) z_1 + x_2 \tilde{z}_2\right]$$

The necessary and sufficient F.O.C. (under the customary assumption) is :

$$EU'\left[\left(\frac{Y_0 - p_2 x_2}{p_1}\right) z_1 + x_2 \tilde{z}_2\right]\left(\tilde{z}_2 - \frac{p_2}{p_1} z_1\right) = 0$$

b. Suppose $U(y) = a - be^{-AY}$; the above equation becomes :

$$AbE\left\{\exp\left[\left(\frac{Y_0 - p_2 x_2}{p_1}\right) z_1 + x_2 \tilde{z}_2\right]\left(\tilde{z}_2 - \frac{p_2}{p_1} z_1\right)\right\} = 0$$

equivalently,

$$AbE\left\{\exp\left[\frac{Y_0 z_1}{p_1}\right]\exp\left[\frac{-p_2 x_2}{p_1}z_1 + x_2\tilde{z}_2\right]\left(\tilde{z}_2 - \frac{p_2}{p_1}z_1\right)\right\} = 0$$

The first term which contains Y_0 can be eliminated from the equation. The intuition for this result is in the fact that the stated utility is CARA, that is, the rate of absolute risk aversion is constant and independent of the initial wealth.

5.8. The problem with linear mean variance utility is

$$\max(y_0 - s) + \delta\left[sE\tilde{x} - \tfrac{1}{2}\chi s^2\sigma_x^2\right]$$

$$\text{FOC} \quad -1 + \delta E\tilde{x} - sx\sigma_x^2 = 0$$

$$\text{or} \quad s = \frac{\delta E(\tilde{x}) - 1}{\chi\sigma_x^2}$$

Clearly s is inversely related with σ_x^2. For a given $E\tilde{x}$ (x_B is a mean-preserving spread of x_A),

$$s_A > s_B$$

Chapter 7

7.7. A ray in R^2 is defined by $y - y_1 = n(x - x_1)$. Rewrite this in the following way $y = n(x - x_1) + y_1$ and apply it to the problem:

$$E(r_P) = \frac{E(r_P) - r_f}{\sigma_P}(\sigma_P - 0) + r_f .$$

This can be maximized with respect to the Sharpe ratio. Of course, we get $\sigma_P = 0$; i.e. the slope is infinite. Now we constrain x to be $x = \sigma_P = \sqrt{\frac{C}{D}\left(E(r_P) - \frac{A}{C}\right)^2 + \frac{1}{C}}$. Inserting this back leads to $E(r_P) = \theta\sqrt{\frac{C}{D}\left(E(r_P) - \frac{A}{C}\right)^2 + \frac{1}{C}} + r_f$ where θ is the Sharpe ratio. From this it is easy to solve for σ_P^2 and the Sharpe ratio. (A, B, C, and D are the notorious letters defined in Chapter 6.)

7.8. 1) Not possible to say without further knowledge of preferences. The reason is that with both risk-free and risky returns higher, there is what is called a 'wealth effect': with given amount of initial wealth to invest, the end-of-period is unambiguously expected to be higher: the investor is 'richer'. But we have not made any assumption as to whether at higher wealth level he/she will be more or less risk averse. The CAPM does not require to specify IARA or CARA or DARA utility functions (although we know that we could build the model on a quadratic (IARA) utility function, this is not the only route.)
2) Here it is even more complicated; the efficient frontier is higher: there is a wealth effect, but it is also steeper: there is also a substitution effect. Everything else equal, the risky portfolio is more attractive. It is more likely that an investor will select a riskier optimal portfolio in this situation, but one cannot rule out that the wealth effect dominates and at the higher expected end-of-period wealth the investor decides to invest more conservatively.

7.9. Questions about the Markowitz model and the CAPM.

a. If it were not, one could build a portfolio composed of two efficient portfolios that would not be itself efficient. Yet, the new portfolio's expected return would be higher than the frontier portfolio with the same standard deviation, in violation of the efficiency property of frontier portfolios.

b. With a lower number of risky assets, one expects that the new frontier will be contained inside the previous one as diversification opportunities are reduced.

c. The efficient frontier is made of three parts, including a portion of the frontier. Note that borrowers and lenders do not take positions in the same "market portfolio".

d. Asset A is a good buy: it pays on average a return that exceeds the average return justified by its beta. If the past is a good indication of the pattern of future returns, buying asset A offers the promise of an extra return compared to what would be fair according to the CAPM. What could expect that in the longer run many investors will try to exploit such an opportunity and that, as a consequence, the price of asset A will increase with the expected return decreasing back to the SML level.

7.10. "If" part has been shown in Chapter 6.

"Only if" : start with Vw=ae+b**1**; premultiply by V^{-1}

$w_P = aV^{-1}e + bV^{-1}\iota = aA\left(\frac{V^{-1}e}{A}\right) + bC\left(\frac{V^{-1}\iota}{C}\right)$ where $\frac{V^{-1}e}{A}$, and $\frac{V^{-1}\iota}{C}$ are frontier portfolios with means $\frac{B}{A}$, and $\frac{A}{C}$ respectively. Since aA+bC=1 (Why?) the result follows.

7.11. We build a portfolio with P and the MVP, with minimum variance. Then, the weights a and (1-a) must satisfy the condition

$$\underbrace{\min}_{a}\left\{a^2 \times \sigma_P^2 + 2 \times a \times (1-a) \times \mathrm{cov}(r_P, r_{MVP}) + (1-a)^2 \times \sigma_{MVP}^2\right\}.$$

The FOC is

$$2 \times a \times \sigma_P^2 + 2 \times (1 - 2 \times a) \times \mathrm{cov}(r_P, r_{MVP}) - 2 \times (1-a) \times \sigma_{MVP}^2 = 0 .$$

Since MVP is the minimum variance porfolio, a=0 must satisfy the condition, which simplifies to $\mathrm{cov}(r_P, r_{MVP}) = \sigma_{MVP}^2$.

7.12. For any portfolio on the frontier we have

$$\sigma^2(\tilde{r}_p) = \frac{C}{D}\left(E(\tilde{r}_P) - \frac{A}{C}\right)^2 + \frac{1}{C} .$$

where A, B, C, and D are our notorious numbers. Additionally, we know that $E(\tilde{r}_{zcp}) = \frac{A}{C} - \frac{D/C^2}{E(r_p) - A/C}$. Since the zero covariance portfolio is also a frontier portfolio we have

$$\sigma^2(\tilde{r}_{zcp}) = \frac{C}{D}\left(\frac{D/C^2}{E(r_p) - A/C}\right)^2 + \frac{1}{C} .$$

Now, we need to have $\sigma^2(\tilde{r}_p) = \sigma^2(\tilde{r}_{zcp})$. This leads to

$$\frac{C}{D}\left(E(r_P) - \frac{A}{C}\right)^2 + \frac{1}{C} = \frac{C}{D}\left(\frac{D/C^2}{E(r_p) - A/C}\right)^2 + \frac{1}{C}$$

$$E(r_P) - \frac{A}{C} = \frac{D/C^2}{E(r_p) - A/C} .$$

$$E(r_P) - \frac{A}{C} = \frac{\sqrt{D}}{C}$$

$$E(r_P) = \frac{A}{C} + \frac{\sqrt{D}}{C}$$

Given $E(r_P)$, we can use (6.15) from chapter 6 to find the portfolio weights.

7.13. a. As shown in Chapter 5, we find the slope of the mean-variance frontier and utilize it in the equation of the line passing through the origin. If we call the portfolio we are seeking "p", then it follows that $E(r_P) = \frac{D + A^2C}{C^2A}$ where A, B, C, and D are our notorious numbers. The zero-covariance portfolio of p is such that $E(r_{zcp}) = 0$.

$$E(r_{zcp}) = \frac{A}{C} - \frac{D/C^2}{E(r_p) - A/C} = 0$$

$$\Leftrightarrow E(r_p) = \frac{D}{CA} + \frac{A}{C} = 0.1733$$

b. We need to compare the weights of the portfolio p with $w_T = \frac{V^{-1}(e - r_f \iota)}{A - r_f C}$. The two portfolios should differ because we are comparing the tangency points of two different lines on the same mean-variance frontier. Note also that the intercepts are different:

$$E(r_{zcp}) = 0.05$$

$$\Leftrightarrow E(r_p) = \frac{D/C^2}{[A/C - 0.05]} + \frac{A}{C} = 0.1815$$

7.14. a. True
b. False: the CAPM holds even with investors have different rates of risk aversion. It however requires that they are all mean-variance maximizers.
c. True. Only the mean and variance of a portfolio matters. They have no preference for the third moment of the return distribution. Portfolio including derivative instruments may exhibit highly skewed return distribution. For non-quadratic utility investors the prescriptions of the CAPM should, in that context, be severely questioned.

Chapter 8

8.9. a.

The optimization problem of the first agent is

$MaxEU(c)$ s.t. $q_1c_{11} + q_2c_{12} = q_1e_{11} + q_2e_{12}$.

The FOC's are,

$$\pi\frac{1}{c_{11}} = \lambda q_1$$

$$(1-\pi)\frac{1}{c_{12}} = \lambda q_2$$

$$q_1c_{11} + q_2c_{12} = q_1e_{11} + q_2e_{12}$$

where λ is the Lagrange multiplier of the problem. Clearly, if we define $c_{11} = y_1, c_{12} = y_2$ we have

$$\frac{q_1}{q_2} = \frac{\pi}{(1-\pi)}\frac{y_2}{y_1}.$$

A-D can be derived as follows

$$\frac{q_1}{q_2} = \frac{\pi}{(1-\pi)}\frac{c_{12}}{c_{11}} = \frac{\pi}{(1-\pi)}\frac{c_{22}}{c_{21}} = \frac{\pi}{(1-\pi)}\frac{c_{12}}{c_{11}} + \frac{\pi}{(1-\pi)}\frac{c_{22}}{c_{21}} = \frac{\pi}{(1-\pi)}\frac{e_2}{e_1}.$$

Using $1 = q_1 + q_2$ and after some manipulation we get

$$q_1 = \frac{\pi e_2}{(1-\pi)e_1 + \pi e_2}$$
$$q_2 = \frac{(1-\pi)e_1}{(1-\pi)e_1 + \pi e_2}.$$

b.

If $\pi = (1-\pi)$ A-D prices are

$$q_1 = \frac{e_2}{e_1 + e_2}$$
$$q_2 = \frac{e_1}{e_1 + e_2}.$$

The price of the risky asset is

$$P_2 = \frac{1}{2}q_1 + 2q_2.$$

Now we insert A-D prices and since endowments are

$$e_1 = Q_1 + \frac{1}{2}Q_2$$

$$e_2 = Q_1 + 2Q_2$$

the pricing formula

$$P_2 = \frac{5Q_1 + 4Q_2}{4Q_1 + 5Q_2}$$

follows.

Chapter 10

10.5. The dividends (computed on the face value, 1000) are d_1 =80, d_2 =65. The ratio is d_1/d_2; buying 1 unit of bond 1 and selling d_1/d_2 units of bond 2, we can build a 5-yr zero-coupon bond with following payoffs:

	Price	Maturity Value
Bond 1	-1300	+1080
Bond 2	$+1200d_1/d_2$	$1065d_1/d_2$
	176.92	-230.77

The price of the 5-yr A-D security is then 176.92/230.77 = 23/30.

10.6. a. r_1 : $950 = \frac{1000}{(1+r_1)}; (1+r_1) = \frac{1000}{950}; r_1 = .05263$

r_2 : $880 = \frac{1000}{(1+r_2)^2}; (1+r_2) = \left(\frac{1000}{880}\right)^{1/2}; r_2 = .0660$

r_3 : $780 = \frac{1000}{(1+r_3)^3}; (1+r_3) = \left(\frac{1000}{780}\right)^{1/3}; r_3 = .0863$

b. We need the forward rates ${}_1f_2$ and ${}_2f_1$.

(i) $(1+r_1)(1+{}_1f_2)^2 = (1+r_3)^3$

$$(1+{}_1f_2) = \left[\frac{(1.0863)^3}{(1.05263)}\right]^{\frac{1}{2}} = 1.1035$$

$$(1+r_2)^2(1+{}_2f_1) = (1+r_3)^3$$

$$(1+{}_2f_1) = \frac{(1.0863)^3}{(1.0660)^2} = 1.1281$$

So ${}_1f_2$ = .1035, and ${}_2f_1$ = .1281.

The $CF_{T=3} = (1.25M)(1+{}_2f_1) + 1M(1+{}_1f_2)^2$

$= (1.25M)(1.1281) + 1M(1.1035)^2$

= 1.410 M + 1.2177 M

= 2.6277 M.

c. To lock in the ${}_2f_1$ applied to the 1.25M, consider the following transactions :

We want to replicate

	t=0	1	2	3
			-1.25M	1.410M

Short : 1250 2 yr bond

Long : 1410 3 yr bond

Consider the corresponding cash flows :

	t=0	1	2	3
Short : (1250)(880)= +1,100,000			-1,250,000	
Long : -(1410)(780)= - 1,100,000				1,410,000
Total	0		-1,250,000	1,410,000

To lock in the ${}_1f_2$ (compounded for two periods) applied to the 1M ; consider the following transactions :

t=0	1	2	3
	-1M		1.2177M

Short : 1000 1 yr bonds.
Long : 1217.7 3 yr bonds.

Consider the corresponding cash flows :

	t=0	1	2	3
Short : (1000)(950)=	950,000	-1,000,000		
Long : -(1217.7)(780)=	- 950,000			+1,217,700
Total	0	-1,000,000		+1,217,700

The portfolio that will allow us to invest the indicated cash flows at the implied forward rates is :

Short : 1000, 1 yr bond
Short : 1000, 2 yr bond
Long : 1410 + 1217.7 = 2627.7, 3 yr bond

10.7. If today's state is state 1, to get \$1.- for sure tomorrow using Arrow-Debreu prices, I need to pay $q_{11} + q_{12}$; thus

$$q_{11} + q_{12} = \frac{1}{1+r_1^1} = .9 \qquad \text{(i)}$$

Similarly, if today's state is state 2:

$$q_{21} + q_{22} = \frac{1}{1+r_2^1} = .8 \qquad \text{(ii)}$$

Given the matrix of Arrow-Debreu prices

$$q = \begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix}; \; q^2 = \begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix}\begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix}.$$

To get \$1.- for sure two periods from today, I need to pay

$$q_{11}q_{11} + q_{12}q_{21} + q_{11}q_{12} + q_{12}q_{22} = \frac{1}{\left(1+r_1^2\right)^2} = .78 \qquad \text{(iii)}$$

If state 2 today

$$q_{21}q_{11} + q_{22}q_{21} + q_{21}q_{12} + q_{22}q_{22} = \frac{1}{\left(1+r_2^2\right)^2} = .68 \qquad \text{(iv)}$$

The 4 equations (i) to (iv) can be solved for the 4 unknown Arrow-Debreu prices as per

$q_{11} = 0.6$
$q_{22} = 0.3$
$q_{21} = 0.4$
$q_{22} = 0.4$

Chapter 13

13.6.

a. From the main APT equation and the problem data, one obtains the following system :

$12.0 = Er_A = \lambda_0 + \lambda_1\ 1 + \lambda_2\ 0.5$ (i)
$13.4 = Er_B = \lambda_0 + \lambda_1\ 3 + \lambda_2\ 0.2$ (ii)
$12.0 = Er_C = \lambda_0 + \lambda_1\ 3 - \lambda_2\ 0.5$ (iii)

This system can easily be solved for
$\lambda_0 = 10$; $\lambda_1 = 1$; $\lambda_2 = 2$

Thus, the APT tells us that

$Er_i = 10 + 1\ b_{i1} + 2\ b_{i2}$

b. (i) If there is a risk free asset one must have
$\lambda_0 = r_f = 10$
(ii) Let P_i be the pure factor portfolio associated with factor i. One has $\lambda_i = \bar{r}_{Pi} - r_f$. Furthermore if the CAPM holds one should have

$$\lambda_i = \bar{r}_{Pi} - r_f = \beta_{Pi}(\bar{r}_M - r_f).$$

Thus

$$\lambda_1 = 1 = \beta_{P1} 4 \rightarrow \beta_{P1} = \tfrac{1}{4}, \text{ and}$$
$$\lambda_2 = 2 = \beta_{P2} 4 \rightarrow \beta_{P2} = \tfrac{1}{2}.$$

13.7.

Expected APT Return	Expected Returns
$r_A = 4\% + .8(8\%) = 10.4\%$	10.4%
$r_B = 4\% + 1(8\%) = 12.0\%$	12.0%
$r_C = 4\% + 1.2(8\%) = 13.6\%$	13.6%

The Expected return of B is less than what is consistent with the APT.

This provides an arbitrage opportunity. Consider a combination of A and C that gives the same factor sensitivity as B.

$$w_A(.8) + \underbrace{w_C}_{1-w_A}(1.2) = 1.00$$
$$\Rightarrow w_A = w_C = 1/2$$

What is the expected return on this portfolio:

$$E\left(r_{\frac{1}{2}A,\frac{1}{2}C}\right) = \tfrac{1}{2}E(\tilde{r}_A) + \tfrac{1}{2}E(\tilde{r}_C)$$
$$= \tfrac{1}{2}(10.4\%) + \tfrac{1}{2}(13.6\%)$$
$$= 12\%$$

This clearly provides an arbitrage opportunity: short portfolio B, buy a portfolio of ½ A and ½ C.

13.8.

That diversifiable risk is not priced has long been considered as the main lesson of the CAPM. While defining systematic risk differently (in terms of the consumption portfolio rather than the market portfolio), the CCAPM leads to the same conclusion. So does the APT with possibly yet another definition for systematic risk, at least in the case where the market portfolio risk does not encompass the various risk factors identified by the APT. The Value additivity theorem seen in Chapter 7 proves that Arrow-Debreu pricing also leads to the same implication. The equivalence between risk neutral prices and Arrow-Debreu prices in complete markets guarantees that the same conclusion follows from the martingale pricing theory. When markets are incomplete, some risks that we would understand as being diversifiable may no longer be so. In those situations a reward from holding these risks may be forthcoming.

Chapter 15

15.6. a. Agent problems :

Agent 1 : $\max_{Q_1^1, Q_2^1} \ln(e_1 - p_1Q_1^1 - p_2Q_2^1) + \frac{3}{4}\ln Q_1^1 + \frac{1}{4}\ln Q_2^1$

Agent 2 : $\max_{Q_1^2, Q_2^2} \ln(e_2 - p_1Q_1^2 - p_2Q_2^2) + \frac{3}{4}\ln Q_1^2 + \frac{1}{4}\ln Q_2^2$

FOCs :

Agent 1 :

(i) $Q_1^1 : \dfrac{p_1}{e_1 - p_1Q_1^1 - p_2Q_2^1} = \dfrac{3}{4Q_1^1}$

(ii) $Q_2^1 : \dfrac{p_2}{e_1 - p_1Q_1^1 - p_2Q_2^1} = \dfrac{1}{4Q_2^1}$

(i) and (ii) together imply $p_2Q_2^1 = \frac{1}{3}p_1Q_1^1$. This is not surprising ; agent 1 places a higher subjective probability on the first state. Taking this into account, (i) implies $p_1Q_1^1 = \frac{3}{8}e_1$ and thus $p_2Q_2^1 = \frac{1}{8}e_1$.

The calculations thus far are symmetric and (iii) and (iv) solve for $p_1Q_1^2 = \frac{1}{8}e_1$ while $p_2Q_2^2 = \frac{3}{8}e_2$.

Now suppose there is 1 unit of each security available for purchase.

a) $Q_1^1 + Q_1^2 = 1$

b) $Q_2^1 + Q_2^2 = 1$

Substituting the above demand function into these equations gives :

a) $\left(\dfrac{3}{8p_1}\right)e_1 + \left(\dfrac{1}{8p_1}\right)e_2 = 1$

$p_1 = \frac{3}{8}e_1 + \frac{1}{8}e_2$

b) $\left(\dfrac{3}{8p_2}\right)e_1 + \left(\dfrac{1}{8p_2}\right)e_2 = 1$

$p_2 = \frac{1}{8}e_1 + \frac{3}{8}e_2$

If $e_2 > e_1$, then $p_2 > p_1$;

If $e_2 < e_1$, then $p_1 > p_2$

b. There is now only one security

t=1	t=2	
$-p^b$	θ_1	θ_2
	1	1

<u>Agent Endowments</u>

	t=1	t=2 θ_1	t=2 θ_2
Agent 1	e_1	a_1	a_1
Agent 2	e_2	a_2	a_2

Now, there can be trade here even with this one asset, if, say $e_1 = 0$, $a_1 > 0$, $e_2 > 0$, $a_2 = 0$ to take an extreme case.

If $e_1 = e_2 = 0$, then there will be no trade as the security payoff do not allow the agents to tailor their consumption plans to their subjective probabilities.

c. Suppose we introduce a risky asset

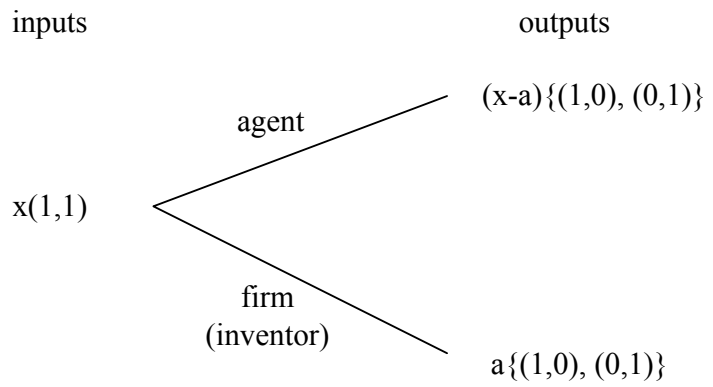
t=0	t=1	
	θ_1	θ_2
-p	z_1	z_2

where $z_1 \neq z_2, z_1, z_2 > 0$.

Combinations of this security and the riskless one can be used to construct the state claims. This will be welfare improving relative to the case where only the riskless asset is traded. The final equilibrium outcome and the extent to which each agents welfare is improved will depend upon the relative endowments of the risky security assigned to each agent ; and the absolute total quantity bestowed on the economy.

d. The firm can convert x units of (1,1) into x units of {(1,0), (0,1)}. These agents (relative to having only the riskless asset) would avail themselves of this technology, and then trade the resultant claims to attain a more preferred consumption state.

Furthermore, the agents would be willing to pay for such a service in the following sense :



Clearly, if a = x, the agents would ignore the inventor. However, each agent would be willing to pay something. Assuming the inventor charges the same a to each agent, the most he could charge would be that a at which one of the agents were no better off ex ante than if he did not trade.

Suppose the inventor could choose to convert x, 2x, 3x, …, nx securities (x understood to be small). The additional increment he could charge would decline as n increased.