Measuring Risk Aversion

4.1 Measuring Risk Aversion
4.2 Interpreting the Measures of Risk Aversion
4.4 Risk Premium and Certainty Equivalence
4.5 Assessing an Investor’s Level of Relative Risk Aversion
4.6 The Concept of Stochastic Dominance
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Utility function
Indifference Curves

Asset Pricing
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Indifference Curves

Utility function

\[ EU(c) = k \]

\[ EU(c) = k_2 \]

\[ EU(c) = k_1 \]

\[ c^*_1 \]

\[ c_1 \]

\[ (c^*_1 + c_1)/2 \]

\[ (c^*_2 + c_2)/2 \]

\[ c^*_2 \]

\[ I_1 \]

\[ I_2 \]
Arrow-Pratt measures of risk aversion and their interpretations

(i) absolute risk aversion \( = - \frac{U''(Y)}{U'(Y)} \equiv R_A(Y) \)

(ii) relative risk aversion \( = - \frac{YU''(Y)}{U'(Y)} \equiv R_R(Y) \).
Absolute risk aversion $= -\frac{U''(Y)}{U'(Y)} \equiv R_A(Y)$

$$\pi(Y, h) \approx 1/2 + (1/4)hR_A(Y),$$ (1)
Relative risk aversion $= -\frac{YU''(Y)}{U'(Y)} \equiv R_R(Y)$. 

$$\pi(Y, \theta) \approx \frac{1}{2} + \frac{1}{4}\theta R_R(Y).$$  (2)
Theorem ((4.1) Jensen’s Inequality)

Let $g(\ )$ be a concave function on the interval $(a, b)$, and $\tilde{x}$ be a random variable such that $\text{Prob}\{\tilde{x} \in (a, b)\} = 1$. Suppose the expectations $E(\tilde{x})$ and $Eg(\tilde{x})$ exist; then

$$E [g(\tilde{x})] \leq g [E(\tilde{x})].$$

Furthermore, if $g(\ )$ is strictly concave and $\text{Prob}\{\tilde{x} = E(\tilde{x})\} \neq 1$, then the inequality is strict.


\[ EU(Y + \tilde{Z}) = U(Y + CE(Y, \tilde{Z})) \]

\[ = U(Y + E\tilde{Z} - \Pi(Y, \tilde{Z})) \]
Certainty Equivalent and Risk Premium: An illustration

Jensen's Inequality
Certainty Equivalent
4.5 Assessing an Investor’s Level of Relative Risk Aversion

\[
\frac{(Y + CE)^{1-\gamma}}{1 - \gamma} = \frac{1}{2} \frac{(Y + 50,000)^{1-\gamma}}{1 - \gamma} + \frac{1}{2} \frac{(Y + 100,000)^{1-\gamma}}{1 - \gamma}
\]  

Assuming zero initial wealth (\(Y = 0\)), we obtain the following sample results (clearly, \(CE > 50,000\)):

- \(\gamma = 0\) \(\Rightarrow\) \(CE = 75,000\) (risk neutrality)
- \(\gamma = 1\) \(\Rightarrow\) \(CE = 70,711\)
- \(\gamma = 2\) \(\Rightarrow\) \(CE = 66,667\)
- \(\gamma = 5\) \(\Rightarrow\) \(CE = 58,566\)
- \(\gamma = 10\) \(\Rightarrow\) \(CE = 53,991\)
- \(\gamma = 20\) \(\Rightarrow\) \(CE = 51,858\)
- \(\gamma = 30\) \(\Rightarrow\) \(CE = 51,209\)

Given a current wealth of \(Y = $100,000\) and a degree of risk aversion of \(\gamma = 5\), the equation results in a \(CE = $66,532\).
In this section we show that the postulates of Expected Utility lead to a definition of two alternative concepts of dominance which are weaker and this of wider application than the concept of state-by-state dominance. These are of interest because they circumscribe the situations in which rankings among risky prospects are preference-free, i.e., can be defined independently of the specific trade-offs (between return, risk and other characteristics of probability distributions) represented by an agent’s utility function.
### Table 4.1: Sample Investment Alternatives

<table>
<thead>
<tr>
<th>Payoffs</th>
<th>10</th>
<th>100</th>
<th>2000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prob $Z_1$</td>
<td>.4</td>
<td>.6</td>
<td>0</td>
</tr>
<tr>
<td>Prob $Z_2$</td>
<td>.4</td>
<td>.4</td>
<td>.2</td>
</tr>
</tbody>
</table>

$EZ_1 = 64, \sigma_{Z_1} = 44$

$EZ_2 = 444, \sigma_{Z_2} = 779$
Definition 4.1: **First Order Stochastic Dominance (FSD)**

Let $F_A(\tilde{x})$ and $F_B(\tilde{x})$, respectively, represent the cumulative distribution functions of two random variables (cash payoffs) that, without loss of generality assume values in the interval $[a, b]$. We say that $F_A(\tilde{x})$ **first order stochastically dominates** ($FSD$) $F_B(\tilde{x})$ if and only if $F_A(x) \leq F_B(x)$ for all $x \in [a, b]$. 
First Order Stochastic Dominance: A More General Representation
Theorem (4.2)

Let $F_A(\tilde{x})$, $F_B(\tilde{x})$, be two cumulative probability distributions for random payoffs $\tilde{x} \in [a, b]$. Then $F_A(\tilde{x})$ FSD $F_B(\tilde{x})$ if and only if $E_A U(\tilde{x}) \geq E_B U(\tilde{x})$ for all non-decreasing utility functions $U(\ )$. 
### Table 4.2: Two Independent Investments

<table>
<thead>
<tr>
<th>Payoff</th>
<th>Prob.</th>
<th>Payoff</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.25</td>
<td>1</td>
<td>0.33</td>
</tr>
<tr>
<td>5</td>
<td>0.50</td>
<td>6</td>
<td>0.33</td>
</tr>
<tr>
<td>9</td>
<td>0.25</td>
<td>8</td>
<td>0.33</td>
</tr>
</tbody>
</table>
Second Order Stochastic Dominance Illustrated

Asset Pricing
Definition 4.2: **Second Order Stochastic Dominance** Let $F_A(\tilde{x})$, $F_B(\tilde{x})$, be two cumulative probability distributions for random payoffs in $[a, b]$. We say that $F_A(\tilde{x})$ **second order stochastically dominates** ($SSD$) $F_B(\tilde{x})$ if and only if for any $x$:

$$\int_{-\infty}^{x} [F_B(t) - F_A(t)] \, dt \geq 0.$$  

(with strict inequality for some meaningful interval of values of $t$).
Theorem (4.3)

Let \( F_A(\tilde{x}) \), \( F_B(\tilde{x}) \), be two cumulative probability distributions for random payoffs \( \tilde{x} \) defined on \([a, b]\). Then, \( F_A(\tilde{x}) \) SSD \( F_B(\tilde{x}) \) if and only if \( E_A U(\tilde{x}) \geq E_B U(\tilde{x}) \) for all nondecreasing and concave \( U \).
4.7 More or less risky $\equiv$ mean preserving spread

$$E^A(x) = \int xf_A(x)dx = \int xf_B(x)dx = E^B(x)$$

\[ f_A(x) \]

\[ f_B(x) \]
Theorem (4.4)

Let $F_A(\cdot)$ and $F_B(\cdot)$ be two distribution functions defined on the same state space with identical means. If this is true, the following statements are equivalent:

(i) $F_A(\tilde{x})$ SSD $F_B(\tilde{x})$

(ii) $F_B(\tilde{x})$ is a mean preserving spread of $F_A(\tilde{x})$ in the sense of Equation

$$\tilde{x}_B = \tilde{x}_A + \tilde{z}$$

(6)
Key Concepts

- Absolute and relative measures of risk aversion
- Certainty equivalence and risk premium
- Stochastic dominance and the reason for searching for the broadest concept of dominance