Asset Pricing
Chapter III. Making Choice in Risky Situations

June 20, 2006
3.2 Choosing Among Risky Prospects: Preliminaries

- A future risky cash flow is modelled as a random variable
- State-by-state dominance => incomplete ranking
- « riskier »

<table>
<thead>
<tr>
<th>Table 3.1: Asset Payoffs ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>Investment 1</td>
</tr>
<tr>
<td>Investment 2</td>
</tr>
<tr>
<td>Investment 3</td>
</tr>
</tbody>
</table>
### Table 3.2: State Contingent ROR \((r)\)

<table>
<thead>
<tr>
<th></th>
<th>(\theta = 1)</th>
<th>(\theta = 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Investment 1</td>
<td>5%</td>
<td>20%</td>
</tr>
<tr>
<td>Investment 2</td>
<td>-50%</td>
<td>60%</td>
</tr>
<tr>
<td>Investment 3</td>
<td>5%</td>
<td>60%</td>
</tr>
</tbody>
</table>

\(E_r_1 = 12.5\% ; \sigma_1^2 = \frac{1}{2} (5 - 12.5)^2 + \frac{1}{2} (20 - 12.5)^2 = (7.5)^2\), or \(\sigma_1 = 7.5\%\)

\(E_r_2 = 5\% ; \sigma_2 = 55\%\) (similar calculation)

\(E_r_3 = 32.5\% ; \sigma_3 = 27.5\%\)

- Investment 1 **mean-variance dominates** 2
- Investment 3 does not **m-v dominate** 1!
Table 3.3: State-Contingent Rates of Return

<table>
<thead>
<tr>
<th></th>
<th>$\theta = 1$</th>
<th>$\theta = 2$</th>
</tr>
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<tbody>
<tr>
<td>Investment 4</td>
<td>3%</td>
<td>5%</td>
</tr>
<tr>
<td>Investment 5</td>
<td>2%</td>
<td>8%</td>
</tr>
</tbody>
</table>

$\pi_1 = \pi_2 = \frac{1}{2}$
$ER_4 = 4\%; \sigma_4 = 1\%$
$ER_5 = 5\%; \sigma_5 = 3\%$

- What is the trade-off between risk and expected return?
- Investment 4 has a higher **Sharpe ratio** than investment 5
3.3 A Prerequisite: Choice Theory Under Certainty

- **A.1** Every investor possesses such a preference relation and it is complete.
- **A.2** This preference relation satisfies the fundamental property of *transitivity*.
- **A.3** Investor’s preference relations are "relative" stable over time.
- **A.4** The preference $\succeq$ relation is *continuous*.

$$a \succeq b$$
Theorem (3.1)

Assumptions **A.1** - **A.4** are sufficient to guarantee the existence of a continuous, time invariant, real valued utility function \( u \), such that for any two objects of choice (consumption bundles of goods and services; amounts of money, etc.) \( a \) and \( b \),

\[
 a \geq b \text{ if and only if } \quad u(a) \geq u(b).
\]
Under uncertainty, ranking bundles of goods involves more than pure elements of taste or preferences. Particularly true when objects of choice are assets. New Chapter on Asset management: Ch.14

Table 3.4: Forecasted Price per Share in One Period

<table>
<thead>
<tr>
<th></th>
<th>State 1</th>
<th>State 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>IBM</td>
<td>$100</td>
<td>$150</td>
</tr>
<tr>
<td>Royal Dutch</td>
<td>$90</td>
<td>$160</td>
</tr>
</tbody>
</table>

Current Price of both assets is $100
### Table 3.5: Forecasted Price per Share in One Period

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<td>$100</td>
<td>$90</td>
</tr>
<tr>
<td>Royal Dutch</td>
<td>$150</td>
<td>$200</td>
</tr>
<tr>
<td><strong>Current Price of both assets is $100</strong></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 3.6: Forecasted Price per Share in One Period

<table>
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<td>Royal Dutch</td>
<td>$90</td>
<td>$160</td>
</tr>
<tr>
<td>Sony</td>
<td>$90</td>
<td>$150</td>
</tr>
<tr>
<td></td>
<td>Current Price of all assets is $100</td>
<td></td>
</tr>
</tbody>
</table>

The expected utility theorem: hypotheses under which the preference ranking may be represented by a utility function combining linearly probabilities and preferences on ex-post payoffs.
\( \tilde{p}_{IBM} \succeq \tilde{p}_{RDP} \)

if and only if there exists a real valued function \( U \) for which

\[
EU(\tilde{p}_{IBM}) = \pi_1 U(p_{IBM}(\theta_1)) + \pi_2 U(p_{IBM}(\theta_2)) \\
> \pi_1 U(p_{RDP}(\theta_1)) + \pi_2 U(p_{RDP}(\theta_2)) = EU(\tilde{p}_{RDP})
\]

More generally, the utility of any asset \( A \) with payoffs \( p_A(\theta_1), p_A(\theta_2), ..., p_A(\theta_N) \) in the \( N \) possible states of nature with probabilities \( \pi_1, \pi_2, ..., \pi_N \) can be represented by

\[
U(A) = EU(p_A(\theta_i)) = \sum_{i=1}^{N} \pi_i U(p_A(\theta_i))
\]
Allais Paradox

Four lotteries: \[ L^1 = (10,000,0,0.1) \quad L^2 = (15,000,0,0.09) \]
\[ L^3 = (10,000,0,1) \quad L^4 = (15,000,0,0.9) \]

By the structure of compound lotteries: \[ L^1 = (L^3, L^0, 0.1) \]
\[ L^2 = (L^4, L^0, 0.1) \] where \( L^0 = (0,0,1) \)

By the independence axiom, the ranking between \( L^1 \) and \( L^2 \)
on the one hand, and \( L^3 \) and \( L^4 \) on the other, should thus be identical.

\[ L^2 \succ L^1, \]
\[ L^3 \succ L^4, \]
Prospect Theory

\[ U(Y) = \begin{cases} 
\frac{(|Y - \overline{Y}|)^{1-\gamma_1}}{1-\gamma_1}, & \text{if } Y \geq \overline{Y} \\
-\lambda(\overline{Y} - Y)^{1-\gamma_2}, & \text{if } Y \leq \overline{Y}
\end{cases} \]
3.1 Introduction
3.2 Choosing Among Risky Prospects: Preliminaries
3.3 A Prerequisite: Choice Theory Under Certainty
3.4 Choice Theory Under Uncertainty: An Introduction
3.5 Allais Paradox
3.6 Prospect Theory
3.7 Key concepts and ideas

Fig 2.2 Utility function for Prospect Theory

Parameter values: $\bar{P} = 1000; \gamma_1 = \gamma_2 = 0.5, \lambda = 5$
Key concepts and ideas

- **Dominance**: state by state; mean-variance
- The need for a representation of the risk-return trade-off in individual preferences
- **Expected utility theory** and some of its limitations