Asset Pricing

Chapter VI. Risk Aversion and Investment Decisions, Part II: Modern Portfolio Theory

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6.1 Introduction
6.2 More About Utility Functions
6.3 Opportunity Set in the Mean-Variance Space
6.4 The Optimal Portfolio: A Separation Theorem
6.5 Conclusions

\[ \max \{ a_1, a_2, \ldots, a_N \} \]
\[ EU(Y_0(1 + r_f) + \sum_{i=1}^{N} a_i(\bar{r}_i - r_f)) \]
\[ = \max \{ w_1, w_2, \ldots, w_N \} \]
\[ EU(Y_0(1 + r_f) + \sum_{i=1}^{N} w_i Y_0(\bar{r}_i - r_f)) \]  (1)

\[ \max \{ w_1, w_2, \ldots, w_N \} \]
\[ EU \left\{ Y_0 \left[ (1 + r_f) + \sum_{i=1}^{N} w_i(\bar{r}_i - r_f) \right] \right\} = EU \left\{ Y_0 [1 + \bar{r}_P] \right\} = EU \left\{ \bar{y}_1 \right\} \]  (2)
More About Utility Functions

- **Step 1: The consumption-savings decision**
  - $C_0 + S_0 = Y_0$

- **Step 2: The Portfolio Problem**
  - $N$ risky assets with $(1 - \sum_{i=1}^N w_i)(Y_0 - C_0)$
  - $(w_1(Y_0 - C_0), w_2(Y_0 - C_0), .., w_N(Y_0 - C_0))$

- **Step 3: Tomorrow’s Consumption Choice**
  - $U(c_1, c_2, .....c_n)$

\[
p_1 \theta c_{1\theta} + .. + p_m \theta c_{m\theta} \leq Y_{\theta}.
\]

\[
Y_{\theta} = (Y_0 - C_0) \left[ (1 + r_f) + \sum_{i=1}^N w_i(r_i \theta - r_f) \right]
\]
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Backward Induction

\[ U(Y_\theta) \equiv_{def} \max_{(c_{1\theta}, \ldots, c_{m\theta})} u(c_{1\theta}, \ldots, c_{m\theta}) \]

s.t. \( p_{1\theta} c_{1\theta} + \ldots + p_{m\theta} c_{m\theta} \leq Y_\theta. \)

\[ \max \left\{ w_1, w_2, \ldots, w_N \right\} \]

\[ EU(\tilde{Y}) = \sum_\theta \pi_\theta U(Y_\theta). \]
From Utility on wealth to utility on return:

$$\max EU(\tilde{Y}) = \max EU((Y_0 - C_0)(1 + \tilde{r}_P)) =_{def} \max E\hat{U}(\tilde{r}_P)$$

"Mean-Variance Utility Function"
Taylor series approximation:

\[ EU(\tilde{Y}) = U\left[ E(\tilde{Y}) \right] + U'\left[ E(\tilde{Y}) \right] \left[ \tilde{Y} - E(\tilde{Y}) \right] \]

\[ + \frac{1}{2} U''\left[ E(\tilde{Y}) \right] \left[ \tilde{Y} - E(\tilde{Y}) \right]^2 + H_3 \] (3)

Computing expected utility using this approximation:

\[ EU(\tilde{Y}) = U\left[ E(\tilde{Y}) \right] + U'\left[ E(\tilde{Y}) \right] \left[ E(\tilde{Y}) - E(\tilde{Y}) \right] = 0 \]

\[ + \frac{1}{2} U''\left[ E(\tilde{Y}) \right] E\left[ \tilde{Y} - E(\tilde{Y}) \right]^2 + EH_3 \]

\[ = U\left[ E(\tilde{Y}) \right] + \frac{1}{2} U''\left[ E(\tilde{Y}) \right] \sigma^2(\tilde{Y}) + EH_3 \]
Description of the Opportunity Set in the Mean-Variance Space: The Gains form Diversification and the Efficient Frontier

"The main idea of this section is the following: The expected return to a portfolio is the weighted average of the expected returns of the assets composing the portfolio. The same result is not generally true for the variance: the variance of a portfolio is generally smaller than the weighted average of the variances of individual asset returns corresponding to this portfolio. Therein lies the gain from diversification".
The typical investor likes expected return $\mu_R$ and dislikes standard deviation $\sigma_R$

Recall that an asset (or portfolio) A is said to mean-variance dominate an asset (or portfolio B) if $\mu_A \geq \mu_B$ while $\sigma_A \geq \sigma_B$

Define the **efficient frontier** as the locus of all non-dominated portfolios in the mean-standard deviation space.

By definition, no ("rational") mean-variance investor would choose to hold a portfolio not located on the efficient frontier.
The Efficient Frontier: Two perfect correlated risky assets

\[ \sigma_R = w_1 \sigma_1 + (1 - w_1) \sigma_2. \]

\[ \mu_R = \bar{r}_1 + \frac{\bar{r}_2 - \bar{r}_1}{\sigma_2 - \sigma_1} (\sigma_R - \sigma_1), \]

\[ \begin{align*}
\mu_R &= \bar{r}_1 + \frac{\bar{r}_2 - \bar{r}_1}{\sigma_2 - \sigma_1} (\sigma_R - \sigma_1), \\
\end{align*} \]
The Efficient Frontier: Two imperfectly correlated risky assets

![Diagram showing the efficient frontier for two imperfectly correlated risky assets. The diagram includes points A, B, C, and D, with axes for mean return ($\mu_R$) and standard deviation ($\sigma_R$).]
The Efficient Frontier: Two perfectly negative correlated risky assets

\[
\frac{\sigma_2}{\sigma_1 + \sigma_2} \bar{r}_1 + \frac{\sigma_1}{\sigma_1 + \sigma_2} \bar{r}_2
\]

\[
\frac{\sigma_2}{\sigma_1 + \sigma_2} \bar{r}_1 + \frac{\sigma_1}{\sigma_1 + \sigma_2} \bar{r}_2 - \frac{\bar{r}_1 - \bar{r}_2}{\sigma_1 + \sigma_2} \sigma_R
\]
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Two perfect correlated risky assets
Two imperfectly correlated risky assets
Two perfectly negative correlated risky assets
One Risky and One Risk Free Asset
One Risk Free Asset and "N" Risky Assets
Portfolio optimization problem

One Risky and One Risk Free Asset

\[
\begin{align*}
\mu &= \bar{r}_2 - \bar{r}_1 \\
\sigma &= \sigma_2 \\
\end{align*}
\]
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One Risk Free Asset and "N" Risky Assets
The portfolio optimization problem:

\[
\min_{\mathbf{w} \text{'s}} \sum_i \sum_j w_i w_j \sigma_{ij}
\]

subject to

\[
\sum_i w_i \bar{r}_i = \mu
\]

\[
\sum_i w_i = 1
\]
A Separation Theorem

The Optimal Portfolios of Two Differently Risk Aversion Investors
6.5 Conclusions

- MPT: A normative interpretation
- Spelling out the information requirements for portfolio analysis
- Using historical returns for asset allocation purposes
- Static vs. Dynamic portfolio allocation
Key concepts

- Backward induction; indirect utility
- Gains from diversification
- Efficient Frontier
- Optimal Portfolio
- Two fund or separation theorem