Setting: An Arrow Debreu Economy

General Equilibrium

- Supply = Demand
- With production (if desired)
- Static but Multi-period
- No restrictions on preferences

The most general of the theories we shall consider
1. Two dates: 0, 1. This set-up, however, is fully generalizable to multiple periods.

2. N possible states of nature at date 1, which we index by $\theta = 1, 2, \ldots, N$ with probabilities $\pi_\theta$

3. One perishable (non storable) consumption good

4. K agents, indexed $k = 1, \ldots, K$, with preferences:

$$U^k_0 \left(c^k_0\right) + \delta^k \sum_{\theta=1}^{N} \pi_\theta U^k \left(c^k_\theta\right);$$

5. In addition, agent k’s endowment is described by the vector $\{e^k_0, (e^k_\theta)_{\theta=1,2,\ldots,N}\}$ Possibly:

$$u^k(c^k_0, c^k_{\theta_1}, c^k_{\theta_2}, \ldots, c^k_{\theta_N}).$$
Traded Securities

- Exclusively Arrow-Debreu securities (contingent claims): Security $\theta$ priced $q_{\theta}$ promises delivery of one unit of commodity tomorrow if state $\theta$ is realized and nothing otherwise.

- How to secure one unit of consumption tomorrow?
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8.3 Pareto Optimality and risk Sharing

8.4 Implementing Pareto Optimal Allocations: On the Possibility of

8.5 Arrow-Debreu pricing: concluding remarks

\[
\max \ U_k^0(c_0^k) + \delta^k \sum_{\theta=1}^{N} \pi_{\theta} U_k^0(c_{\theta}^k)
\]

s.t.
\[
c_0^k + \sum_{\theta=1}^{N} q_{\theta} c_{\theta}^k \leq e_0^k + \sum_{\theta=1}^{N} q_{\theta} e_{\theta}^k
\]
\[c_0^k, c_1^k, \ldots, c_N^k \geq 0\]

Equilibrium is a set of prices \((q_1, \ldots, q_N)\) such that:

1. at those prices \((c_0^k, \ldots, c_N^k)\) solve problem \((P)\), for all \(k\), and

2. \[\sum_{k=1}^{K} c_0^k = \sum_{k=1}^{K} e_0^k, \sum_{k=1}^{K} C_{\theta}^k = \sum_{k=1}^{K} e_{\theta}^k, \text{ for every } \theta\]
### Table 8.1. Endowments and Preferences in Our Reference Example

<table>
<thead>
<tr>
<th>Agents</th>
<th>Endowments</th>
<th>Preferences</th>
</tr>
</thead>
<tbody>
<tr>
<td>t = 0</td>
<td>t = 1</td>
<td></td>
</tr>
<tr>
<td>(\theta_1)</td>
<td>(\theta_2)</td>
<td>(\frac{1}{2} c_0^1 + 0.9 \left( \frac{1}{3} \ln (c_1^1) + \frac{2}{3} \ln (c_1^2) \right))</td>
</tr>
<tr>
<td>Agent 1</td>
<td>10 1 2</td>
<td>(\frac{1}{2} c_0^2 + 0.9 \left( \frac{1}{3} \ln (c_1^2) + \frac{2}{3} \ln (c_2^2) \right))</td>
</tr>
<tr>
<td>Agent 2</td>
<td>5 4 6</td>
<td></td>
</tr>
</tbody>
</table>
The respective agent problems are:

Agent 1:

$$\max \frac{1}{2} (10 + 1q_1 + 2q_2 - c_1^1 q_1 - c_2^1 q_2) + 0.9 \left( \frac{1}{3} \ln (c_1^1) + \frac{2}{3} \ln (c_2^1) \right)$$
S.t. $$c_1^1 q_1 + c_2^1 q_2 \leq 10 + q_1 + 2q_2$$ and $$c_1^1, c_2^1 \geq 0$$

Agent 2:

$$\max \frac{1}{2} (5 + 4q_1 + 6q_2 - c_1^2 q_1 - c_2^2 q_2) + 0.9 \left( \frac{1}{3} \ln (c_1^2) + \frac{2}{3} \ln (c_2^2) \right)$$
S.t. $$c_1^2 q_1 + c_2^2 q_2 \leq 5 + 4q_1 + 6q_2$$ and $$c_1^2, c_2^2 \geq 0$$

Note: $$q_1, q_2$$ solve «key finance problem»!

Agent 1:

$$c_1^1 : \frac{q_1}{2} = 0.9 \left( \Pi \theta \right) \frac{1}{c_1^1}$$
$$c_2^1 : \frac{q_2}{2} = 0.9 \left( \frac{2}{3} \right) \frac{1}{c_2^1}$$

Agent 2:

$$c_1^2 : \frac{q_1}{2} = 0.9 \left( \frac{1}{3} \right) \frac{1}{c_1^2}$$
$$c_2^2 : \frac{q_2}{2} = 0.9 \left( \frac{2}{3} \right) \frac{1}{c_2^2}$$
$$c_1^1 + c_2^1 = 8$$
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Checking Pareto Optimality
Interior vs. Corner Solution

\[ q_\theta = \frac{\delta \pi_\theta \frac{\partial U^k}{\partial c^k_\theta}}{\frac{\partial U^k_0}{\partial c^k_0}}, \quad k, \theta = 1, 2 \]  \hspace{1cm} (1)

\[
\text{price of gold if state } \theta \text{ is realized} = \frac{MU^k_\theta}{MU^k_0},
\]

\[
c_1^1 = c_1^2 = 2.5
\]
\[
c_2^1 = c_2^2 = 4
\]

\[
q_1 = 2 \left( 0.9 \right) \left( \frac{1}{3} \right) \left( \frac{1}{c_1^1} \right) = 2 \left( 0.9 \right) \left( \frac{1}{3} \right) \left( \frac{1}{2.5} \right) = \left( 0.9 \right) \left( \frac{1}{3} \right) \left( \frac{4}{5} \right) = 0.24
\]

\[
q_2 = 2 \left( 0.9 \right) \left( \frac{2}{3} \right) \left( \frac{1}{c_2^1} \right) = 2 \left( 0.9 \right) \left( \frac{2}{3} \right) \left( \frac{1}{4} \right) = \left( 0.9 \right) \left( \frac{2}{3} \right) \left( \frac{4}{8} \right) = 0.3
\]
Table 8.2: Post-Trade Equilibrium Consumptions

<table>
<thead>
<tr>
<th></th>
<th>$t = 0$</th>
<th>$t = 1$</th>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agent 1:</td>
<td>9.04</td>
<td>2.5</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Agent 2:</td>
<td>5.96</td>
<td>2.5</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>15.00</td>
<td>5.0</td>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>

It is Pareto optimal? H1 and H2 are satisfied!
Checking Pareto Optimality

\[
\begin{align*}
\max_{\{c_0^1, c_1^1, c_2^1\}} & \quad u^1(c_0^1, c_1^1, c_2^1) + \lambda u^2(c_0^2, c_1^2, c_2^2) \\
\text{s.t.} & \quad c_0^1 + c_2^2 = 15; \quad c_1^1 + c_2^2 = 5; \quad c_1^1 + c_2^2 = 8, \\
& \quad c_0^1, c_1^1, c_2^1, c_0^2, c_1^2, c_2^2 \geq 0
\end{align*}
\]

\[
\frac{u_0^1}{u_0^2} = \frac{u_1^1}{u_1^2} = \frac{u_2^1}{u_2^2} = \lambda \tag{2}
\]

\[
\frac{1/2}{1/2} = \frac{(0.9)^{1/3} \frac{1}{c_1^2}}{(0.9)^{1/3} \frac{1}{c_1^2}} = \frac{(0.9)^{2/3} \frac{1}{c_1^2}}{(0.9)^{2/3} \frac{1}{c_1^2}}
\]
Box 8-1 Interior vs. Corner Solution

\[ q_0 \frac{\partial U_0^k}{\partial c_0^k} \leq \delta \pi_\theta \frac{\partial U^k}{\partial c^k_\theta}, \text{ if } c_0^k > 0, \text{ and } k, \theta = 1, 2 \]  (3)
If one security \( (q_1) \) only (incomplete markets):

**Table 8.3 : The Post-Trade Allocation**

<table>
<thead>
<tr>
<th></th>
<th>( t = 0 )</th>
<th>( t = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_1 )</td>
<td>9.64</td>
<td>2.5</td>
</tr>
<tr>
<td>( \theta_1 )</td>
<td>2.5</td>
<td>2</td>
</tr>
<tr>
<td>Agent 1:</td>
<td>5.36</td>
<td>2.5</td>
</tr>
<tr>
<td>Agent 2:</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>Total</td>
<td>15.00</td>
<td>5.0</td>
</tr>
</tbody>
</table>

- Agent 1: 5.51 instead of 5.62
- Agent 2: 4.03 instead of 4.09
Table 8.4: The New Endowment Matrix

<table>
<thead>
<tr>
<th></th>
<th>$t = 0$</th>
<th>$t = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\theta_1$</td>
<td>$\theta_2$</td>
</tr>
<tr>
<td>Agent 1</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>Agent 2</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>
### Table 8.5: Agents’ Utility In The Absence of Trade

<table>
<thead>
<tr>
<th>State-Contingent Utility</th>
<th>Expected Utility in Period 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_1 )</td>
<td>( \theta_2 )</td>
</tr>
<tr>
<td>Agent 1: ( \ln(1) = 0 )</td>
<td>( \ln(5) = 1.609 )</td>
</tr>
<tr>
<td>Agent 2: ( \ln(5) = 1.609 )</td>
<td>( \ln(1) = 0 )</td>
</tr>
</tbody>
</table>

### Table 8.6: The Desirable Trades And Post-Trade Consumptions

<table>
<thead>
<tr>
<th>Date 1</th>
<th>Endowments Pre-Trade</th>
<th>Consumption Post-Trade</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \theta_1 )</td>
<td>( \theta_2 )</td>
</tr>
<tr>
<td>Agent 1</td>
<td>1</td>
<td>5 [( \downarrow 2 )]</td>
</tr>
<tr>
<td>Agent 2</td>
<td>5 [( \uparrow 2 )]</td>
<td>1</td>
</tr>
</tbody>
</table>

- Post-trade allocation is Pareto Optimal
- After trade, EU in period 1 is approx. 1.1 for both agents.
Table 8.7: Another Set of Initial Allocations

<table>
<thead>
<tr>
<th></th>
<th>( t = 0 )</th>
<th>( t = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agent 1</td>
<td>( \theta_1 )</td>
<td>( \theta_2 )</td>
</tr>
<tr>
<td>Agent 1</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>Agent 2</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 8.8: Plausible Trades And Post-Trade Consumptions

<table>
<thead>
<tr>
<th>Date 1</th>
<th>Endowments Pre-trade</th>
<th>Consumption Post-trade</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \theta_1 )</td>
<td>( \theta_2 )</td>
</tr>
<tr>
<td>Agent 1</td>
<td>1</td>
<td>3 [↓1]</td>
</tr>
<tr>
<td>Agent 2</td>
<td>5 [↑1]</td>
<td>3</td>
</tr>
</tbody>
</table>

- Post-trade allocation is PO
- Features perfect risk sharing or full mutual insurance (no aggregate risk)
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\[
\frac{u_0^1}{u_0^2} = \frac{u_1^1}{u_1^2} = \frac{u_2^1}{u_2^2} = \lambda 
\]  

(4)

\[
\frac{u_1^1}{u_1^2} = \frac{u_2^1}{u_2^2} = \frac{u_1^1}{u_2^1} = \frac{u_2^2}{u_2^1} 
\]  

(5)

1. If one of the two agents is fully insured, the other must be as well.

2. More generally, if the MRS are to differ from \( q \), given that they must be equal between them, the low consumption - high MU state must be the same for both agents and similarly for the high consumption - low MU state.

3. If there is aggregate risk, however, the above reasoning also implies that, at a Pareto optimum, it is shared "proportionately" among agents with same risk tolerance.

4. Finally, if agents are differentially risk averse, in a Pareto optimal allocation the less risk averse will typically provide some insurance services to the more risk averse.

5. More generally, optimal risk sharing dictates that the agent most tolerant of risk bears a disproportionate share of it.
Implementing Pareto Optimal Allocations: On the Possibility of Market failure

(i) Agent 1 solves: max \((4 - q_Qz_Q^1) + \left[\frac{1}{2}\ln(1 + z_Q^1) + \frac{1}{2}\ln(5)\right]\)
\[\text{s.t. } q_Qz_Q^1 \leq 4\]

(ii) Agent 2 solves: max \((4 - q_Qz_Q^2) + \left[\frac{1}{2}\ln(5 + z_Q^2) + \frac{1}{2}\ln(1)\right]\)
\[\text{s.t. } q_Qz_Q^2 \leq 4\]

Assuming an interior solution, the FOCs are

(i)’ : \(-q_Q + \frac{1}{2} \left(\frac{1}{1+z_Q^1}\right) = 0;\)

(ii)’ : \(-q_Q + \frac{1}{2} \left(\frac{1}{5+z_Q^2}\right) = 0 \Rightarrow \frac{1}{1+z_Q^1} = \frac{1}{5+z_Q^2};\)
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Table 8.9: Market Allocation When Both Securities Are Traded

<table>
<thead>
<tr>
<th></th>
<th>( t = 0 )</th>
<th>( t = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \theta_1 )</td>
<td>( \theta_2 )</td>
</tr>
<tr>
<td>Agent 1</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>Agent 2</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 8.10: The Net Gains From Trade
Expected Utility Levels and Net Trading Gains
(Gain to issuer in bold)

<table>
<thead>
<tr>
<th></th>
<th>No Trade</th>
<th>Trade Only ( Q )</th>
<th>Trade Both ( Q ) and ( R )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>EU</td>
<td>EU</td>
<td>( \Delta EU^{(i)} )</td>
</tr>
<tr>
<td>Agent 1</td>
<td>4.8047</td>
<td>5.0206</td>
<td>0.2159</td>
</tr>
<tr>
<td>Agent 2</td>
<td>4.8047</td>
<td>4.883</td>
<td>\textbf{0.0783}</td>
</tr>
<tr>
<td>Total</td>
<td>0.2942</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
• The «father» of all asset pricing relationships
• A reference
• Conceptual import
• Hard to identify pure state of nature
• Static