Asset Pricing

Chapter IX. The Consumption Capital Asset Pricing Model

June 20, 2006
The Representative Agent Hypothesis and its Notion of Equilibrium

9.2.1 An infinitely lived Representative Agent
- Avoid terminal period problem
- Equivalence with finite lives if operative bequest motive

9.2.2 On the Concept of a «No Trade» Equilibrium
- Positive net supply: the representative agent willingly hold total supply
- Zero net supply: at the prevailing price, supply = demand = 0
Recursive Trading: many periods; investment decisions are made one period at a time, taking due account of their impact on the future state of the world

- One perfectly divisible share
- Dividend = economy’s total output
- Output arises exogenously and stochastically (fruit tree)
- Stationary stochastic process
Table 9.1: Three-State Probability Transition Matrix

Output in Period $t + 1$

$$
\begin{bmatrix}
Y^1 & Y^2 & Y^3 \\
Y^1 & \pi_{11} & \pi_{12} & \pi_{13} \\
Y^2 & \pi_{21} & \pi_{22} & \pi_{23} \\
Y^3 & \pi_{31} & \pi_{32} & \pi_{33}
\end{bmatrix} = \mathbb{T}
$$

$$G(Y_{t+1} \mid Y_t) = \text{Prob} \left( Y_{t+1} \leq Y^j, \mid Y_t = Y^i \right).$$

- Lucas fruit tree
- Grafting an aggregate output process
- Rational expectations economy: knowledge of the economic structure and the stochastic process
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Probability Transition Matrix
Interpreting the Exchange Equilibrium
Interpreting the CCAPM
The Formal Consumption CAPM

\[
\max_{\{z_{t+1}\}} E \left( \sum_{t=0}^{\infty} \delta^t U(\tilde{c}_t) \right) \\
\text{s.t. } c_t + p_t z_{t+1} \leq z_t Y_t + p_t z_t \\
\quad z_t \leq 1, \forall t
\]

F.O.C:

\[
U_1(c_t)p_t = \delta E_t \left\{ U_1(\tilde{c}_{t+1}) \left( \tilde{p}_{t+1} + \tilde{Y}_{t+1} \right) \right\} \\
U_1(c_t(Y^i))p_t(Y^i) = \delta \sum_j U_1(c_{t+1}(Y^j))(p_{t+1}(Y^j) + Y^j) \pi_{ij}
\]
Definition of an equilibrium

For the entire economy to be in equilibrium, it must, therefore, be true that:

(i) $z_t = z_{t+1} = z_{t+2} = ... \equiv 1$, in other words, the representative agent owns the entire security;

(ii) $c_t = Y_t$, that is, ownership of the entire security entitles the agent to all the economy’s output and,

(iii) $U_1(c_t)p_t = \delta E_t \left\{ U_1(\tilde{c}_{t+1}) (\tilde{p}_{t+1} + \tilde{Y}_{t+1}) \right\}$, or, the agents’ holdings of the security are optimal given the prevailing prices.

Substituting (ii) into (iii) informs us that the equilibrium price must satisfy:

$$U_1(Y_t)p_t = \delta E_t \left\{ U_1(\tilde{Y}_{t+1})(\tilde{p}_{t+1} + \tilde{Y}_{t+1}) \right\}$$  \hspace{2cm} (2)
\[ p_{h,t} U_1 (c_t) = \delta E_t \left\{ U_1 (\tilde{c}_{t+1}) (\tilde{p}_{h,t+1} + \tilde{Y}_{h,t+1}) \right\} \quad (3) \]

\[ p_t = E_t \sum_{\tau=1}^{\infty} \delta^\tau \left[ \frac{U_1(\tilde{Y}_{t+\tau})}{U_1(Y_t)} \tilde{Y}_{t+\tau} \right] , \quad (4) \]

- Discounting at the IMRS of the representative agent!
- Assume risk neutrality

\[ p_t = E_t \sum_{\tau=1}^{\infty} \delta^\tau \left[ \tilde{Y}_{t+\tau} \right] = E_t \sum_{\tau=1}^{\infty} \left[ \frac{\tilde{Y}_{t+\tau}}{(1 + r_f)^\tau} \right] , \quad (5) \]
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Interpreting the Exchange Equilibrium

\[ 1 + r_{j,t+1} = \frac{p_{j,t+1} + Y_{j,t+1}}{p_{j,t}} \]

\[ 1 = \delta E_t \left\{ \frac{U_1(\tilde{c}_{t+1})}{U_1(c_t)} (1 + \tilde{r}_{j,t+1}) \right\} \]  \hspace{1cm} (6)

\[ q_t^b U_1(c_t) = \delta E_t \left\{ U_1(\tilde{c}_{t+1}) 1 \right\} \]

\[ \frac{1}{1 + r_{f,t+1}} = q_t^b = \delta E_t \left\{ \frac{U_1(\tilde{c}_{t+1})}{U_1(c_t)} \right\} \]  \hspace{1cm} (7)

Link between discount factor and risk-free rate in a risk neutral world.
\[ 1 = \delta E_t \left\{ \frac{U_1(\tilde{c}_{t+1})}{U_1(c_t)} \right\} E_t \left\{ 1 + \tilde{r}_{j,t+1} \right\} + \delta \text{COV}_t \left\{ \frac{U_1(\tilde{c}_{t+1})}{U_1(c_t)}, \tilde{r}_{j,t+1} \right\} \]

or, rearranging,

\[ \frac{1 + \tilde{r}_{j,t+1}}{1 + r_{f,t+1}} = 1 - \delta \text{COV}_t \left( \frac{U_1(\tilde{c}_{t+1})}{U_1(c_t)}, \tilde{r}_{j,t+1} \right) \], or

\[ \tilde{r}_{j,t+1} - r_{f,t+1} = -\delta \left( 1 + r_{f,t+1} \right) \text{COV}_t \left( \frac{U_1(\tilde{c}_{t+1})}{U_1(c_t)}, \tilde{r}_{j,t+1} \right) \]
Risk premium is large for those securities paying high returns when consumption is high (MU is Low) and low returns when consumption is low.

Intuition not far from CAPM, but CCAPM adopts consumption smoothing perspective

The key to an asset’s value is its covariation with the MU of consumption rather than the MU of wealth.
One step further: towards a CAPM equation

Let $U(c_t) = ac_t - \frac{b}{2}c_t^2$

$U_1(c_t) = a - bc_t$

Marginal utility is inversely proportional to $c_t$

$$
\bar{r}_{j,t+1} - r_{f,t+1} = -\delta (1 + r_{f,t+1}) \text{cov}_t \left( \tilde{r}_{j,t+1}, \frac{a - b\tilde{c}_{t+1}}{a - bc_t} \right)
$$

$$
= -\delta (1 + r_{f,t+1}) \frac{1}{a - bc_t} \text{cov}_t \left( \tilde{r}_{j,t+1}, \tilde{c}_{t+1} \right) (-b), \text{ or }
$$

$$
\bar{r}_{j,t+1} - r_{f,t+1} = \frac{\delta b (1 + r_{f,t+1})}{a - bc_t} \text{cov}_t \left( \tilde{r}_{j,t+1}, \tilde{c}_{t+1} \right). \quad (10)
$$
The Formal Consumption CAPM

\[
\bar{r}_{c,t+1} - r_{f,t+1} = \left[ \frac{\delta b (1 + r_{f,t+1})}{a - bc_t} \right] \text{cov}_t (\bar{r}_{c,t+1}, \bar{c}_{t+1}). \tag{11}
\]

"c" denotes portfolio most correlated with consumption

\[
\frac{\bar{r}_{j,t+1} - r_{f,t+1}}{\bar{r}_{c,t+1} - r_{f,t+1}} = \frac{\text{cov}_t (\bar{r}_{j,t+1}, \bar{c}_{t+1})}{\text{cov}_t (\bar{r}_{c,t+1}, \bar{c}_{t+1})}, \text{ or }
\]

\[
\frac{\bar{r}_{j,t+1} - r_{f,t+1}}{\bar{r}_{c,t+1} - r_{f,t+1}} = \frac{\text{cov}_t (\bar{r}_{j,t+1}, \bar{c}_{t+1})}{\text{var}(\bar{c}_{t+1})}, \text{ or }
\]

\[
\frac{\bar{r}_{j,t+1} - r_{f,t+1}}{\bar{r}_{c,t+1} - r_{f,t+1}} = \frac{\beta_{j,c_t}}{\beta_{c,c_t}} \left[ \bar{r}_{c,t+1} - r_{f,t+1} \right]. \tag{12}
\]

\[
\bar{r}_{j,t+1} - r_{f,t+1} = \beta_{j,c_t} \left( \bar{r}_{c,t+1} - r_{f,t+1} \right). \tag{13}
\]
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Finite states

\[ U_1(c(s)) \, q(s_{t+1} = s'; s_t = s) = \delta U_1(c(s')) \, \text{prob}(s_{t+1} = s'; s_t = s), \]

or

\[ q(s_{t+1} = s'; s_t = s) = \delta \frac{U_1(c(s'))}{U_1(c(s))} \, \text{prob}(s_{t+1} = s'; s_t = s). \]

Continuum of states

\[ q(s'; s) = \delta \frac{U_1(c(s'))}{U_1(c(s))} \, f(s'; s) \]

\[ q(s'; s) = \delta f(s'; s) = \delta \pi_{ss}. \]
N-period claims
N period risk free zero discount bond:

\[ q_t^{bN}(s) = \delta^N \sum_{s'} \frac{U_1(c(s'))}{U_1(c(s))} \text{prob} \left( s_{t+N} = s'; s_t = s \right) \]  

Value future cash flows revisited:

\[ p_t = E_t \sum_{\tau=1}^{\infty} \delta^\tau \left[ \frac{U_1(c_{t+\tau})}{U_1(c_t)} Y_{t+\tau} \right] \]

\[ = \sum_{\tau=1}^{\infty} \sum_{s'} \delta^\tau \left[ \frac{U_1(c_{t+\tau}(s'))}{U_1(c_t)} Y_{t+\tau}(s') \right] \text{prob} \left( s_{t+\tau} = s'; s_t = s \right) \]

\[ = \sum_{\tau} \sum_{s'} q^\tau(s', s) Y_{t+\tau}(s'), \]  

Discounting at the IMRS = valuing at A-D prices!
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\[ p_t = \sum_{\tau=1}^{\infty} \frac{E_t[Y_{t+\tau}]}{(1 + r_{f,t+\tau})^\tau} \left\{ 1 + \frac{\text{cov}(U_1(\tilde{c}_{t+\tau}), \tilde{Y}_{t+\tau})}{E_t[U_1(\tilde{c}_{t+\tau})]E_t[\tilde{Y}_{t+\tau}]} \right\}. \]  
(16)

\[ p_t = \sum_{\tau=1}^{\infty} \frac{Y_{t+\tau}}{(1 + r_{f,t+\tau})^\tau}. \]  
(17)
Table 9.3: Properties of U.S. Asset Returns

<table>
<thead>
<tr>
<th></th>
<th>U.S. Economy</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(a)</td>
</tr>
<tr>
<td>( r )</td>
<td>6.98</td>
</tr>
<tr>
<td>( r_f )</td>
<td>.80</td>
</tr>
<tr>
<td>( r - r_f )</td>
<td>6.18</td>
</tr>
</tbody>
</table>

(a) Annualized mean values in percent;
(b) Annualized standard deviation in percent.

Source: Data from Mehra and Prescott (1985).
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\[ U(c) = \frac{c^{1-\gamma}}{1-\gamma} \cdot \]

\[ \frac{U_1(c_{t+1})}{U_1(c_t)} = \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma}. \]

\[ 1 = \delta E_t \left\{ \left( \frac{\tilde{c}_{t+1}}{c_t} \right)^{-\gamma} R_{t+1} \right\} = \delta \bar{x}^{-\gamma} \bar{R}, \]
\[ p_t = vY_t \]

\[ vY_t = \delta E_t \left\{ \left( v\tilde{Y}_{t+1} + \tilde{Y}_{t+1} \right) \frac{U_1(\tilde{c}_{t+1})}{U_1(c_t)} \right\}. \]

\[ v = \delta E \left\{ (v+1) \frac{\tilde{Y}_{t+1}}{Y_t} \tilde{x}_{t+1}^{-\gamma} \right\}. \]

\[ v = \delta E \left\{ (v+1)\tilde{x}^{1-\gamma} \right\} = \frac{\delta E \left\{ \tilde{x}^{1-\gamma} \right\}}{1 - \delta E \left\{ \tilde{x}^{1-\gamma} \right\}}. \]

\[ R_{t+1} = 1 + r_{t+1} = \frac{p_{t+1} + Y_{t+1}}{p_t} = \frac{v + 1}{v} \frac{Y_{t+1}}{Y_t} = \frac{v + 1}{v} x_{t+1}. \]

\[ E_t \left( \tilde{R}_{t+1} \right) = E \left( \tilde{R}_{t+1} \right) = \frac{v + 1}{v} E \left( \tilde{x}_{t+1} \right) = \frac{E \left( \tilde{x} \right)}{\delta E \left\{ \tilde{x}^{1-\gamma} \right\}}. \]
\[ R_{f,t+1} \equiv \frac{1}{q_t^b} = \left[ \delta E_t \left\{ \frac{U_1(\tilde{c}_{t+1})}{U_1(c_t)} \right\} \right]^{-1} = \frac{1}{\delta} \frac{1}{E \{ \tilde{x} - \gamma \}}, \quad (20) \]

\[ \frac{E \left( \tilde{R}_{t+1} \right)}{R_f} = \frac{E \{ \tilde{x} \} E \{ \tilde{x} - \gamma \}}{E \{ \tilde{x}^{1-\gamma} \}} = \exp \left[ \gamma \sigma_x^2 \right], \quad (21) \]

\[ \ln (ER) - \ln (R_f) = \gamma \sigma_x^2. \quad (22) \]

\[ \frac{\ln (ER) - \ln (E_{r_f})}{\sigma_x^2} = \frac{1.0698 - 1.008}{(.0357)^2} = 50.24 = \gamma. \]

\[ 2(.00123) = .002 = (\ln(ER) - \ln(E_{r_f}) \approx ER - E_{r_f}) \quad (23) \]
\[ p(s_t) = E_t[m_{t+1}(\tilde{s}_{t+1})X_{t+1}(\tilde{s}_{t+1}); s_t], \quad (24) \]

\[ m_{t+1}(\tilde{s}_{t+1}) = \frac{\delta U_1(c_{t+1}(\tilde{s}_{t+1}))}{U_1(c_t)}. \]

\[ p_t = E_t[\tilde{m}_{t+1}\tilde{X}_{t+1}] . \quad (25) \]

\[ 1 = E_t[\tilde{m}_{t+1}\tilde{R}_{t+1}], \]

\[ 1 = E[\tilde{m}\tilde{R}] \]
\[ E[\tilde{m}(\tilde{R}_i - \tilde{R}_j)] = 0, \]

or

\[ E[\tilde{m}\tilde{R}_{i-j}] = 0, \]

\[ E\tilde{m}E\tilde{R}_{i-j} + \text{cov}(\tilde{m}, \tilde{R}_{i-j}) = 0, \]

or

\[ E\tilde{m}E\tilde{R}_{i-j} + \rho(\tilde{m}, \tilde{R}_{i-j})\sigma_m\sigma_{R_{i-j}} = 0, \]

or

\[ \frac{E\tilde{R}_{i-j}}{\sigma_{R_{i-j}}} + \rho(\tilde{m}, \tilde{R}_{i-j}) \frac{\sigma_m}{E\tilde{m}} = 0, \]

or

\[ \frac{E\tilde{R}_{i-j}}{\sigma_{R_{i-j}}} = -\rho(\tilde{m}, \tilde{R}_{i-j}) \frac{\sigma_m}{E\tilde{m}}. \]
\[
\frac{\sigma_m}{E\tilde{m}} > \frac{|E\tilde{R}_{i-j}|}{\sigma_{R_{i-j}}}.
\]

\[
\frac{\sigma_m}{E\tilde{m}} > \frac{|E(\tilde{r}_M - r_f)|}{\sigma_{r_M-r_f}} = \frac{0.062}{0.167} = 0.37.
\]

\[
E\tilde{m} = \delta \exp(-\gamma \mu_x + \frac{1}{2} \gamma^2 \sigma_x^2) = 0.99(0.967945) = 0.96 \text{ for } \gamma = 2.
\]