MODEL UNCERTAINTY AND OPTIMAL MONETARY POLICY

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Abstract

This dissertation characterizes desirable interest-rate rules for the conduct of monetary policy in small forward-looking macroeconomic models that can be derived from first principles.

The first chapter characterizes optimal monetary policy with commitment and proposes an interest-rate rule that implements the optimal plan. Such a rule involves a “super-inertial” interest rate. Simple Taylor rules are then compared to price-level targeting rules (called Wicksellian rules). It is argued that appropriate Wicksellian rules perform generally better than optimal Taylor rules, because they result in a lower welfare loss, a lower variability of inflation and of nominal interest rates, by introducing desirable history dependence in monetary policy. Moreover, unlike optimal Taylor rules, which may yield an indeterminate equilibrium, Wicksellian rules do in general result in a determinate equilibrium.

The second chapter proposes a general method based on a property of zero-sum two-player games to derive robust optimal monetary policy rules – the best rules among those that yield an acceptable performance in a specified range of models – when the true model is unknown, and model uncertainty is viewed as uncertainty about parameters of the structural model. The method is applied to characterize robust optimal “Taylor rules” in a simple forward-looking model. While it is commonly believed that monetary policy should be less responsive when there is parameter uncertainty, it is shown that robust optimal Taylor rules prescribe in general a stronger response of the interest rate to fluctuations in inflation and the output gap than is the case in the absence of uncertainty. Thus model uncertainty does not necessarily justify a relatively small response of actual monetary policy.

The third chapter extends chapter two by characterizing a robust optimal policy rule in a flexible class, when the policymaker faces uncertainty about parameters of
the structural model and the nature of shock processes. As in chapter two, the robust rule involves a stronger response of the interest rate to fluctuations in inflation and the output gap than is the case in the absence of uncertainty, under a reasonable calibration of the model. However uncertainty may amplify the degree of “super-inertia” required by optimal monetary policy.
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Chapter 1

Optimal Interest-Rate Rules in a Forward-Looking Model, and Inflation Stabilization versus Price-Level Stabilization
1 Introduction

Recent studies of monetary policy have emphasized the importance of history dependence for the conduct of monetary policy. Woodford (1999c, 1999d, 2000b), has shown that when agents are forward-looking, it is optimal for policymakers not only to respond to current shocks and the current state of the economy, but that it is desirable to respond to lagged variables as well. Committing to a monetary policy of this kind allows the central bank to affect the private sector’s expectations appropriately. This in turn improves the performance of monetary policy because the evolution of the policymaker’s goal variables depends not only upon its current actions, but also upon how the private sector foresees future monetary policy.

In this paper, we investigate the implications of this history dependence for desirable monetary policy rules (or instrument rules) in a simple forward-looking model. We first propose an interest-rate feedback rule that implements the optimal plan. This policy rule is characterized by a strong response of the current interest rate to lagged interest rates – in fact a response to the lagged interest that is greater than one. Such super-inertial policy rules have been advocated by Rotemberg and Woodford (1999), and Woodford (1999c), for their ability to affect expectations of the private sector appropriately. However Taylor (1999b) has criticized super-inertial rules on robustness grounds.1 The concern for robustness has led many authors to focus on very simple policy rules (see, e.g., Taylor, 1999a, McCallum, 1988, 1999, and Levin et al., 1999a). We therefore turn to very simple

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1Taylor (1999b) shows that super-inertial rules perform poorly in non-rational expectations, backward-looking, models. This should not be surprising since super-inertial rules rely precisely on the private sector’s forward-looking behavior.
policy rules that are not super-inertial. In particular, we compare the performance of standard Taylor rules to interest-rate rules that involve responses to price-level variations (called hereafter Wicksellian rules).

Previous studies have usually found that inflation targeting is more desirable than price-level targeting. For example Lebow et al. (1992), and Haldane and Salmon (1995) show that price-level targeting rules result in a higher short-run variability of inflation and output.\(^2\) The intuition for this result is simple: in the face of an unexpected temporary rise in inflation, price-level targeting requires the policymaker to bring inflation below the target in later periods, in order for the price level to return to its path. With nominal rigidities, fluctuations in inflation result in turn in fluctuations in output. In contrast, with inflation targeting, the drift in the price level is accepted so that there is no need to generate a deflation in subsequent periods. Thus price-level targeting is a “bad idea” according to this conventional view because it would “add unnecessary short term fluctuations to the economy” (Fischer, 1994, p. 282), while it would only provide a small gain in long-term price predictability in the US (McCallum, 1999).

However, when agents are forward-looking and have rational expectations, this conventional result is likely to be reversed. We argue that appropriate Wicksellian rules perform generally better than optimal Taylor rules in the model considered. Wicksellian rules result in a lower welfare loss, a lower variability of inflation and of nominal interest rates, by introducing desirable history dependence in monetary policy. In the face of

\(^2\)In contrast, Fillion and Tetlow’s (1994) simulations indicate that price-level targeting results in lower inflation variability but higher output variability than under inflation targeting. They however provide little explanation for that result.
a temporary increase in inflation, forward-looking agents expect relatively low inflation in subsequent periods under price-level targeting, as the policymaker will have to bring inflation below trend. This in turn dampens the initial increase in inflation, lowers the variability of inflation and rises welfare. Williams (1999) confirms this result by simulating the large-scale FRB/US model under alternative simple interest-rate rules. He reports that price-level targeting rules result in lower inflation and output variability than inflation targeting rules for a large set of parameter values, under rational expectations.\(^3\)

Moreover, Wicksellian rules have the desirable property of resulting in general in a determinate equilibrium, unlike optimal Taylor rules, which may result in an indeterminate equilibrium.

We assume that the policymaker credibly commits to a policy rule for the entire future. This approach, which has been advocated by McCallum (1988, 1999), Taylor (1993, 1999a), and Woodford (1999c, 1999d) among others, allows the policymaker to achieve a better performance of monetary policy by taking advantage not only of the gains from commitment made popular by Kydland and Prescott (1977), but also of the effect of a credible commitment on the way the private sector forms expectations of future variables. The policy rules that we derive are time-consistent if policymakers take the “timeless perspective” proposed by Woodford (1999d). Another branch of the recent literature assumes instead that policymakers cannot credibly commit and that monetary policy is conducted under full discretion. These studies generally compare the effects

\(^3\)In contrast, price-level targeting rules perform worse than inflation-targeting rules when the expectations channel described above is shut off, i.e., when expectations are formed according to a forecasting model (VAR) based on a regime of inflation targeting, and independent of the policy rule actually followed.
of a regime in which the policymaker is assigned a loss function that involves inflation variability (called inflation targeting), to a regime in which the loss function involves price-level variability (price-level targeting). Svensson (1999b) and Dittmar et al. (1999) show that when the central bank acts under discretion, and the perturbations to output are sufficiently persistent, price-level targeting results in lower inflation variability than inflation targeting. (However under commitment, Svensson (1999b) still obtains the conventional result that price-level targeting is responsible for a higher variability of inflation.) While these authors use a Neoclassical Phillips curve, Vestin (2000), and Dittmar and Gavin (2000) show that these results hold more generally in a simple model with a “New Keynesian” supply equation — a simplified version of the model presented below. Specifically, they show that when the central bank acts under discretion, price-level targeting results in a more favorable trade-off between inflation and output gap variability relative to inflation targeting, even when perturbations to output are not persistent.4

The rest of the paper is organized as follows. Section 2 reviews the model. Section 3 describes the optimal response of endogenous variables to perturbations, under the optimal plan and the plan in which history dependence is not allowed. Section 4 first determines a policy rule that implements the optimal plan. It then compares optimal Taylor rules to optimal Wicksellian rules. Section 5 concludes.

4Kiley (1998) emphasizes that price-level targeting results in more expected variation in output relative to inflation targeting, in a model with a “New Keynesian” supply equation.
2 A Simple Optimizing Model

This section reviews a simple macroeconomic model that can be derived from first principles, and that has been used in many recent studies of monetary policy (see Appendix 6.1 for details). The behavior of the private sector is summarized by two structural equations, an intertemporal IS equation and a “New Keynesian” aggregate supply equation.

The intertemporal IS equation, which relates spending decisions to the interest rate, is given by

\[ Y_t - g_t = E_t (Y_{t+1} - g_{t+1}) - \sigma^{-1} (i_t - E_t \pi_{t+1}), \]

where \( Y_t \) denotes (detrended) real output, \( \pi_t \) is the quarterly inflation rate, \( i_t \) is the nominal interest rate (all three variables expressed in percent deviations from their values in a steady-state with zero inflation and constant output growth), and \( g_t \) is an exogenous variable representing autonomous variation in spending such as government spending.

This equation can be obtained by performing a log-linear approximation to the representative household’s Euler equation for optimal timing of consumption, and using the market clearing condition on the goods market. The parameter \( \sigma > 0 \) represents the inverse of the intertemporal elasticity of substitution. According to (1), consumption depends not only on the real interest rate, but also on expected future consumption.

\(^5\)The model is very similar to the one presented in Woodford (1999c). Variants of this model have been used in a number of other recent studies of monetary policy such as Kerr and King (1996), Bernanke and Woodford (1997), Goodfriend and King (1997), Rotemberg and Woodford (1997, 1999), Kiley (1998), Clarida et al. (1999), and McCallum and Nelson (1999a, 1999b). Derivations of the structural equations from first principles can be found in Woodford (1996, 1999b, 2000a).

\(^6\)The latter equation states that in equilibrium output is equal to consumption plus government expenditures.
This is because when they expect to consume more in the future, households also want
to consume more in the present, in order to smooth their consumption. Note finally that
by iterating (1) forward, one obtains
\[ Y_t - g_t = -\sigma^{-1} E_t \sum_{j=0}^{\infty} (i_{t+j} - \pi_{t+1+j}). \]

This reveal that aggregate demand depends not only upon current short-term real interest
rates, but also upon expected long-term real rates, which are determined by expected
future short-term rates. It is important to note that current output is therefore affected
by the private sector’s beliefs about future monetary policy.

It is assumed that prices are sticky, and that suppliers are in monopolistic competi-
tion. It follows that the aggregate supply equation, which can be viewed as a log-linear
approximation to the first-order condition for the suppliers’ optimal price-setting deci-
sions, is of the form
\[ \pi_t = \kappa (Y_t - Y^n_t) + \beta E_t \pi_{t+1}, \]
where \( Y^n_t \) represents the natural rate of output, i.e., the equilibrium rate of output under
perfectly flexible prices. The parameter \( \kappa > 0 \) can be interpreted as a measure of the
speed of price adjustment, and \( \beta \in (0, 1) \) denotes the discount factor of the representative
household. The natural rate of output is a composite exogenous variable that depends in
general on a variety of perturbations such as productivity shocks, shifts in labor supply,
but also fluctuations in government expenditures and shifts in preferences (see Woodford,
1999b). Whenever the natural rate of output represents fluctuations in those variables,
then it corresponds also to the efficient rate of output, i.e., the rate of output that would
maximize the representative household’s welfare in the absence of distortions such as
Here, however, we allow the natural rate of output to differ from the efficient rate. We assume exogenous time variation in the degree of inefficiency of the natural rate of output. Such variation could be due for instance to exogenous variation in the degree of market power of firms, i.e., the desired markup. In this case, we show in Appendix 6.1 that the percentage deviation of the efficient rate of output \( Y_t^e \) from the natural rate \( Y_t^n \) (the equilibrium level under flexible prices) is given by

\[
Y_t^e - Y_t^n = (\omega + \sigma)^{-1} \mu_t, \tag{3}
\]

where \( \mu_t \) represents the percent deviation of desired markup from the steady-state, and \( \omega > 0 \) is the elasticity of an individual firm’s real marginal cost with respect to its own supply, evaluated at the steady-state. As we will evaluate monetary policy in terms of deviations of output from the \textit{efficient} rate, it will be convenient to define the “output gap” as

\[
x_t \equiv Y_t - Y_t^e.
\]

Using this, we can rewrite the structural equations (1) and (2) as

\[
x_t = E_t x_{t+1} - \sigma^{-1} (i_t - E_t \pi_{t+1} - r_t^e) \tag{4}
\]

\[
\pi_t = \kappa x_t + \beta E_t \pi_{t+1} + u_t, \tag{5}
\]

where

\[
r_t^e \equiv \sigma E_t \left[ (Y_{t+1}^e - Y_t^e) - (g_{t+1} - g_t) \right]
\]

\[
u_t \equiv \kappa (Y_t^e - Y_t^n) = \kappa (\omega + \sigma)^{-1} \mu_t.
\]
In (4), $r_e^t$ denotes the “efficient” rate of interest, i.e., the equilibrium real interest rate that would equate output to the efficient rate of output $Y_e^t$. It is the real interest rate that would prevail in equilibrium in the absence of distortions. Since the nominal interest rate enters the structural equations only through the interest rate gap $(i_t - E_t \pi_{t+1} - r_e^t)$, monetary policy is expansive or restrictive only insofar as the equilibrium real interest rate is below or above the efficient rate. Note that if the equilibrium real interest rate was perfectly tracking the path of the efficient rate, the output gap would be zero at all times, so that equilibrium output would vary in tandem with the efficient rate of output, while inflation would fluctuate only in response to the shocks $u_t$.

We call the exogenous disturbance $u_t$ in (5) an “inefficient supply shock” since it represents a perturbation to the natural rate of output that is not efficient. Time variation in the degree of efficiency of the natural rate of output is understood as representing exogenous time variation in the desired markup on the goods market, but it could alternatively be interpreted as time variation in distortionary tax rates or exogenous variation in the degree of market power of workers on the labor market. We prefer to call $u_t$ an “inefficient supply shock” rather than a “cost-push shock” as is often done in the literature (see, e.g., Clarida et al., 1999), because perturbations that affect inflation by changing costs of production do not necessarily affect the degree of inefficiency of the natural rate of output. Indeed, cost shocks (due, e.g., to perturbations to energy prices) may well shift the efficient rate of output as well as the natural rate of output. Such shocks are therefore represented in our model by changes in the output gap $x_t$, rather than disturbances to $u_t$.

As we will also be interested in describing the evolution of the log of the price level
We now turn to the goal of monetary policy. We assume that the policymaker seeks to minimize the following loss criterion

\[
L_0 = E_0 \left\{ (1 - \beta) \sum_{t=0}^{\infty} \beta^t \left[ \pi_t^2 + \lambda_x (x_t - x^*)^2 + \lambda_i (i_t - i^*)^2 \right] \right\},
\]

where \( \lambda_x, \lambda_i > 0 \) are weights placed on the stabilization of the output gap and the nominal interest rate, \( \beta \in (0, 1) \) is the discount factor mentioned above, and where \( x^* \geq 0 \) and \( i^* \) represent some optimal levels of the output gap and the nominal interest rate. This loss criterion can be viewed as a second-order Taylor approximation to the lifetime utility function of the representative household in the underlying model (see Woodford 1999b). The presence of the interest rate variability reflects both welfare costs of transactions first mentioned by Friedman (1969), and the fact that the nominal interest rate has a lower bound at zero. The approximation of the utility function allows us furthermore to determine the relative weights \( \lambda_x, \lambda_i \), and the parameters \( x^*, i^* \) in terms of the parameters of the underlying model.8

---

7 According to Friedman (1969), the welfare costs of transactions are eliminated only if nominal interest rates are zero in every period. Assuming that the deadweight loss is a convex function of the distortion, it is not only desirable to reduce the level, but also the variability of nominal interest rates.

8 Woodford (1999b) abstracts from inefficient supply shocks. His derivation of the loss criterion from first principles is essentially unaffected by the introduction of inefficient supply shocks. One difference, however, is that the loss function involves the deviation of output from its efficient rate, and not from its natural rate. Moreover, when the desired markup, \( \mu_t \), is exogenously time varying, the parameters of the loss function are function of the steady-state markup \( \bar{\mu} \) instead of some constant value \( \mu \).
We will assume that the policymaker chooses monetary policy in order to minimize the unconditional expectation $E[L_0]$ where the expectation is taken with respect to the stationary distribution of the shocks. It follows that optimal monetary policy is independent of the initial state. As Woodford (1999d) explains, such a policy is furthermore time-consistent if the central bank adopts a “timeless perspective”, i.e., if it chooses “the pattern of behavior to which it would have wished to commit itself to at a date far in the past, contingent upon the random events that have occurred in the meantime ” (Woodford, 1999d).

The inefficient supply shock is responsible for a trade-off between the stabilization of inflation on one hand, and the output gap on the other hand. Indeed, in the face of an increase in $u_t$, the policymaker could completely stabilize the output gap by letting inflation move appropriately, or he could stabilize inflation, by letting the output gap decrease by the right amount, but he could not keep both inflation and the output gap constant. By how much he will let inflation and the output gap vary depends ultimately on the weight $\lambda_x$. In the absence of inefficient supply shocks, however, the policymaker could in principle completely stabilize both inflation and the output gap by letting the interest rate track the path of the efficient rate of interest, $r_t^e$ (which incidentally is equal to the natural rate of interest in the absence of inefficient supply shocks, as $Y_t^e = Y_t^n$). But when $\lambda_i > 0$ in (7), welfare costs associated to fluctuations in the nominal interest rate introduce a tension between stabilization of inflation and the output gap on one hand, and stabilization of the nominal interest rate on the other hand.

In the rest of the paper, we will characterize optimal monetary policy for arbitrary positive values of the parameters. At times however we will focus on a particular parame-
trization of the model, using the parameter values estimated by Rotemberg and Woodford (1997) for the U.S. economy, and summarized in Table 1. While the econometric model of Rotemberg and Woodford (1997) is more sophisticated than the present model, their structural equations correspond to (1) and (2) when conditioned upon information available two quarters earlier in their model. The weights $\lambda_x$ and $\lambda_i$ are calibrated as in Woodford (1999c), using the calibrated structural parameters and the underlying micro-economic model. Rotemberg and Woodford (1997) provide estimated time-series for the disturbances $Y^n_t$ and $g_t$. They do however not split the series for the natural rate of output in an efficient component $Y^n_t$, and an inefficient component. For simplicity, we calibrate the variance of $r^n_t$ by assuming that all shifts in the aggregate supply equation are efficient shifts, so that the variance of the efficient rate of interest is the same as the variance of the natural rate of interest reported in Woodford (1999c). Inversely, we calibrate $\text{var} (u_t)$ by assuming that all shifts in the aggregate supply equation are due to inefficient shocks. By definition of $u_t$, this upper bound for $\text{var} (u_t)$ is given by $\text{var} (u_t) = \kappa^2 \text{var} (Y^n_t)$.

Table 1: “Calibrated” Parameter values

<table>
<thead>
<tr>
<th></th>
<th>$\beta$</th>
<th>$\sigma$</th>
<th>$\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Structural parameters</td>
<td>0.99</td>
<td>0.1571</td>
<td>0.0238</td>
</tr>
<tr>
<td>Shock processes</td>
<td>$\rho_r$</td>
<td>$\rho_u$</td>
<td>$\text{var} (r^n_t)$</td>
</tr>
<tr>
<td></td>
<td>0.35</td>
<td>0.35</td>
<td>13.8266</td>
</tr>
<tr>
<td>Loss function</td>
<td>$\lambda_x$</td>
<td>$\lambda_i$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.048</td>
<td>0.236</td>
<td></td>
</tr>
</tbody>
</table>
3 Optimal Responses to Perturbations

In this section, we characterize the optimal response of endogenous variables to the two kinds of perturbations relevant in the model presented above: disturbances to the efficient rate of interest, and inefficient supply shocks. We generalize the results of Woodford (1999c) by introducing inefficient supply shocks. Clarida et al. (1999) and Woodford (1999d) have also characterized the optimal plan in the presence of inefficient supply shocks, but they assume that $\lambda_i = 0$ in the loss function, while we let $\lambda_i > 0$.

3.1 Optimal Plan

The optimal plan is characterized by the stochastic processes of endogenous variables $\{\pi_t, x_t, i_t\}$ that minimize the unconditional expectation of the loss criterion (7) subject to the constraints (4) and (5) at all dates. It specifies the entire future state-contingent evolution of endogenous variables as of date zero. As will become clearer in the next section, this corresponds to a plan to which the policymaker is assumed to commit for the entire future. Following Currie and Levine (1993) and Woodford (1999c), we write the policymaker’s Lagrangian as

$$
\mathcal{L} = E \left\{ E_0 \sum_{t=0}^{\infty} \beta^t \left[ \frac{1}{2} \pi_t^2 + \lambda_x (x_t - x^*)^2 + \lambda_i (i_t - i^*)^2 \right] \ight. \\
+ \phi_{1t} \left[ x_t - x_{t+1} + \sigma^{-1} (i_t - \pi_{t+1} - r_t^e) \right] + \phi_{2t} \left[ \pi_t - \kappa x_t - \beta \pi_{t+1} - u_t \right] \} \right\}. \tag{8}
$$

The first-order necessary conditions with respect to $\pi_t, x_t, \text{and } i_t$ are

$$
\pi_t - (\beta \sigma)^{-1} \phi_{1t-1} + \phi_{2t} - \phi_{2t-1} = 0 \tag{9}
$$

$$
\lambda_x (x_t - x^*) + \phi_{1t} - \beta^{-1} \phi_{1t-1} - \kappa \phi_{2t} = 0 \tag{10}
$$

$$
\lambda_i (i_t - i^*) + \sigma^{-1} \phi_{1t} = 0 \tag{11}
$$

13
at each date \( t \geq 0 \), and for each possible state. In addition, we have the initial conditions

\[
\phi_{1,-1} = \phi_{2,-1} = 0 \tag{12}
\]

indicating that the policymaker has no previous commitment at time 0.

The optimal plan is a bounded solution \( \{\pi_t, x_t, i_t, \phi_1t, \phi_2t\}_{t=0}^{\infty} \) to the system of equations (4), (5), (9) – (11) at each date \( t \geq 0 \), and for each possible state, together with the initial conditions (12). We first determine the steady-state values of the endogenous variables of interest that satisfy the previous equations at all dates in the absence of perturbations. They are given by

\[
\begin{align*}
\piop &= \frac{\lambda_i \istar}{\lambda_i + \beta}, \\
xop &= \frac{1 - \beta}{\kappa} \frac{\lambda_i \istar}{\lambda_i + \beta}.
\end{align*} \tag{13}
\]

Note that the optimal steady-state inflation is independent of \( \istar \). It follows that when the optimal nominal interest rate \( i^* = 0 \), steady-state inflation is zero in the optimal plan, whether the steady-state output level is inefficient (\( \istar > 0 \)) or not. However, in general when \( \piop \neq 0 \), the log price level follows a deterministic trend

\[
\hat{p}_t = \piop + \hat{p}_{t-1}.
\]

To characterize the optimal responses to perturbations, we define \( \hat{\pi}_t \equiv \pi_t - \piop \), \( \hat{x}_t \equiv x_t - xop \), \( \hat{i}_t \equiv i_t - iop \), and \( \hat{p}_t \equiv p_t - pop \), and rewrite the equations above in terms of deviations from the steady-state values. We note that the same equations (4), (5), (9) – (11), and (6) hold now in terms of the hatted variables, but without the constant terms. Using (11) to substitute for the interest rate, we can rewrite the dynamic system (4), (5),
(9), and (10) in matrix form as

\[ E_t \begin{bmatrix} z_{t+1} \\ \phi_t \end{bmatrix} = M \begin{bmatrix} z_t \\ \phi_{t-1} \end{bmatrix} + me_t, \quad (14) \]

where \( z_t \equiv [\hat{\pi}_t, \hat{x}_t]' \), \( \phi_t \equiv [\hat{\phi}_{1t}, \hat{\phi}_{2t}]' \), \( e_t \equiv [\hat{r}_t, \hat{u}_t]' \), and \( M \) and \( m \) are matrices of coefficients. Following Blanchard and Kahn (1980), this dynamic system has a unique bounded solution (given a bounded process \( \{e_t\} \)) if and only if the matrix \( M \) has exactly two eigenvalues outside the unit circle. Investigation of the matrix \( M \) reveals that if a bounded solution exists, it is unique.\(^9\) In this case the solution for the endogenous variables can be expressed as

\[ q_t = D\phi_{t-1} + \sum_{j=0}^{\infty} d_j E_t e_{t+j}, \quad (15) \]

where \( q_t \equiv [\hat{\pi}_t, \hat{x}_t, \hat{i}_t, \hat{p}_t]' \), and the Lagrange multipliers follow the law of motion

\[ \phi_t = N\phi_{t-1} + \sum_{j=0}^{\infty} n_j E_t e_{t+j} \quad (16) \]

for some matrices \( D, N, d_j, n_j \) that depend upon the parameters of the model. Woodford (1999c) has emphasized that in the optimal plan, the endogenous variables should depend not only upon expected future values of the disturbances, but also upon the predetermined variables \( \phi_{t-1} \). This dependence indicates that optimal monetary policy should involve inertia in the interest rate, regardless of the possible inertia in the exogenous shocks.

### 3.2 Optimal Non-Inertial Plan

To evaluate more directly the importance of the history dependence in the optimal plan, we compare the latter to a plan in which policy is prevented from responding to lagged

\(^{9}\)The matrix \( M \) has two eigenvalues with modulus greater than \( \beta^{-1/2} \) and two with modulus smaller than this.
variables. Following Woodford (1999c), we call this plan the optimal non-inertial plan. As we will see in the next section, this plan is furthermore interesting because it can be implemented by a simple Taylor rule.

For simplicity, we assume that the exogenous shocks follow univariate stationary AR(1) processes

\[ r_t^e = \rho_r r_{t-1}^e + \varepsilon_{rt} \]  
\[ u_t = \rho_u u_{t-1} + \varepsilon_{ut} \]

where the disturbances \( \varepsilon_{rt}, \varepsilon_{ut} \) are unforecastable one period in advance, and \( \text{E}(\varepsilon_{rt}) = \text{E}(\varepsilon_{ut}) = 0, \Omega_r \equiv \text{E}(\varepsilon_{rt}^2) > 0, \Omega_u \equiv \text{E}(\varepsilon_{ut}^2) > 0, \) and \( 0 < \rho_r, \rho_u < 1 \). In this case, the equilibrium evolutions of the endogenous variables in the optimal non-inertial plan can be described by

\[ \pi_t = \pi^{ni} + \pi_r r_t^e + \pi_u u_t, \quad x_t = x^{ni} + x_r r_t^e + x_u u_t, \quad i_t = i^{ni} + i_r r_t^e + i_u u_t, \]

where \( \pi^{ni}, x^{ni}, i^{ni} \) are the steady-state values of the respective variables in this optimal equilibrium, and \( \pi_r, \pi_u, \) and so on, are the optimal equilibrium response coefficients to fluctuations in \( r_t^e \) and \( u_t \). For the solution (19) to correspond to an equilibrium, the coefficients \( \pi^{ni}, \pi_r, \pi_u, \) and so on, need to satisfy the structural equations (4) and (5) at each date, and for every possible realization of the shocks. These coefficients need therefore to satisfy the following feasibility restrictions, obtained by substituting (19)
into the structural equations (4) and (5):

\[(1 - \beta) \pi^{ni} - \kappa x^{ni} = 0 \]  
\[\pi^{ni} - i^{ni} = 0 \]  
\[(1 - \rho_r) x_r + \sigma^{-1} (i_r - \rho_r \pi_r - 1) = 0 \]  
\[(1 - \beta \rho_r) \pi_r - \kappa x_r = 0 \]  
\[(1 - \rho_u) x_u + \sigma^{-1} (i_u - \rho_u \pi_u) = 0 \]  
\[(1 - \beta \rho_u) \pi_u - \kappa x_u - 1 = 0. \]  

Similarly, substituting (19) into (7), and using \(E(r^e_t u_t) = 0\), we can rewrite the loss function as

\[E[L_0] = \left[ (\pi^{ni})^2 + \lambda_x (x^{ni} - x^*)^2 + \lambda_i (i^{ni} - i^*)^2 \right] + (\pi_r^2 + \lambda_x x_r^2 + \lambda_i i_r^2) \var{r^e_t} + (\pi_u^2 + \lambda_x x_u^2 + \lambda_i i_u^2) \var{u_t}. \]

To determine the optimal non-inertial plan, we choose the equilibrium coefficients that minimize the loss \(E[L_0]\) subject to the restrictions (20) – (25). The steady-state of the optimal non-inertial equilibrium is given by

\[i^{ni} = \pi^{ni} = \frac{(1 - \beta) \kappa^{-1} \lambda_x x^* + \lambda_i i^*}{1 + (1 - \beta)^2 \kappa^{-2} \lambda_x + \lambda_i}, \quad x^{ni} = \frac{1 - \beta}{\kappa} \frac{(1 - \beta) \kappa^{-1} \lambda_x x^* + \lambda_i i^*}{1 + (1 - \beta)^2 \kappa^{-2} \lambda_x + \lambda_i} \]  

and the optimal response coefficients to fluctuations in \(r^e_t\) and \(u_t\) in the optimal non-inertial equilibrium are given by

\[\pi_r = \frac{\lambda_i (\sigma \gamma_r - \rho_r \kappa) \kappa}{h_r}, \quad \pi_u = \frac{\lambda_i \sigma (\sigma \gamma_u - \rho_u \kappa) (1 - \rho_u) + \lambda_x (1 - \beta \rho_u)}{h_u} \]  
\[x_r = \frac{\lambda_i (\sigma \gamma_r - \rho_r \kappa) (1 - \beta \rho_r)}{h_r}, \quad x_u = \frac{-\kappa - \rho_u \lambda_i (\sigma \gamma_u - \rho_u \kappa)}{h_u} \]  
\[i_r = \frac{\lambda_x (1 - \beta \rho_r)^2 + \kappa^2}{h_r}, \quad i_u = \frac{\sigma \kappa (1 - \rho_u) + \lambda_x (1 - \beta \rho_u) \rho_u}{h_u} \]  

\[17\]
where

$$\gamma_j \equiv (1 - \rho_j) (1 - \beta \rho_j) > 0$$

$$h_j \equiv \lambda_i (\sigma \gamma_j - \rho_j \kappa)^2 + \lambda_x (1 - \beta \rho_j)^2 + \kappa^2 > 0,$$

and where $j \in \{r, u\}$.

It is clear from (29) that both $i_r$ and $i_u$ are positive for any positive weights $\lambda_i, \lambda_x$.

Thus the optimal non-inertial plan involves an adjustment of the nominal interest rate in the direction of the perturbations. Equations (27) and (28) reveal that the response coefficients $\pi_r, x_r$ are positive if and only if

$$\frac{\sigma}{\kappa} > \frac{\rho_r}{(1 - \beta \rho_r) (1 - \rho_r)},$$

that is, whenever the fluctuations in the efficient rate are not too persistent (relative to the ratio $\frac{\sigma}{\kappa}$). Thus when (30) holds, a positive shock to the efficient rate stimulates aggregate demand, so that both the output gap and inflation increase. In the special case in which the interest rate does not enter the loss function ($\lambda_i = 0$), or when the persistence of the perturbations is such that $\sigma (1 - \beta \rho_r) (1 - \rho_r) = \rho_r \kappa$, we obtain $\pi_r = x_r = 0$ and $i_r = 1$.

As a result, in the absence of inefficient supply shocks, the central bank optimally moves the interest rate by the same amount as the efficient rate in order to stabilize the output gap and inflation completely.

When the disturbances to the efficient rate are sufficiently persistent ($\rho_r$ large enough but still smaller than 1) for the inequality (30) to be reversed, inflation and the output gap decrease in the face of an unexpected positive shock to the efficient rate in the optimal non-inertial plan. Even if the nominal interest rate increases less than the natural
rate, optimal monetary policy is restrictive in this case, because the real interest rate 

\( (i_t - E_t \pi_{t+1}) \) is higher than the efficient rate of interest \( r^e_t \).

### 3.3 Description of Impulse Responses and Moments

We now illustrate the properties of the optimal plan and the optimal non-inertial plan by looking at the response of endogenous variables to an unexpected disturbance to the efficient rate of interest or to an unexpected inefficient supply shock, when we adopt the calibration summarized in Table 1.

**Shock to** \( r^e_t \): Figure 1a plots the optimal response of the interest rate, inflation, the output gap, and the price level to an unexpected temporary increase in the efficient rate of interest (or equivalently the natural rate of interest, as it is assumed that there is no inefficient supply shock) when \( \rho_r = 0.10 \). Such an increase in \( r^e_t \) may reflect an exogenous increase in demand (represented by \( g_t \)) and/or an adverse supply shock represented by a decrease in \( Y^e_t \). One period after the shock, the efficient rate of interest is expected to be back at its steady-state value; its expected path is indicated by a dotted line in the upper panel.

As discussed above, the nominal interest rate increases by less than the natural rate of interest in the optimal non-inertial plan (dashed lines), in order to dampen the variability of the nominal interest rate (which enters the loss function). Monetary policy is therefore relatively expansionary so that inflation and the output gap increase at the time of the shock. In later periods however, these variables return to their initial steady-state in the

[10] The responses of all variables are reported in annual terms. Therefore, the responses of \( i_t \) and \( \pi_t \) are multiplied by 4.
optimal non-inertial plan, as the perturbation vanishes.

In contrast, in the optimal plan illustrated by solid lines, the short-term interest rate is more inertial than the efficient rate. Inertia in monetary policy is especially desirable here because it induces the private sector to expect future restrictive monetary policy, hence future negative output gaps which in turn have a disinflationary effect already when the shock hits the economy. Thus the expectation of an inertial policy response allows the policymaker to offset the inflationary impact of the shock by raising the short-term interest rate by less than in the optimal non-inertial plan. Note that at the time of the shock, the price level is expected to decline as a result of future restrictive monetary policy. Although the perturbation is purely transitory, the price level is expected to end up at a slightly lower level in the future, in the optimal plan.

Figures 1b and 1c illustrate the impulse responses of the same variables when the shock to the efficient rate of interest is more persistent. The path that the efficient rate is expected to follow is described by an AR(1) process with a coefficient of autocorrelation of 0.35, and 0.9. In the optimal non-inertial plan, the nominal interest rate remains above steady-state as long as the shock is expected to affect the economy, but as in the purely transitory case, the interest rate increases by less than the efficient rate of interest. Note that inflation and the output gap decline below steady-state on impact when $\rho_r = 0.9$, as the equilibrium real interest rate is higher than the efficient real interest rate in this case (condition (30) is violated). In the optimal plan, the nominal interest rate increases initially by less than in the optimal non-inertial plan, but is expected to be higher than the efficient rate in later periods. Again, as people expect monetary policy to remain tight in the future, the output gap is expected to be negative in the future. This removes
pressure on inflation already at the time of the shock, and the price level is expected to end up at a lower level in the future.

**Shock to** $u_t$. We now turn to the effects of an unexpected inefficient supply shock. Figure 2a illustrates the optimal response of endogenous variables to a purely transitory rise in $u_t$, that is when the latter follows the process (18) with $\rho_u = 0$. In the optimal non-inertial plan (dashed lines), the policymaker is expected to stabilize the variables at their steady state in future periods, after the shock has disappeared. At the time of the (adverse) shock, it is optimal to raise the nominal interest rate in order to reduce output (gap), and therefore to remove some inflationary pressure. In the optimal plan (solid lines), however, it is optimal to maintain the output gap below steady state for several periods, even if the disturbance is purely transitory. This generates the expectation of a slight deflation in later periods and thus helps dampening the initial increase in inflation. The last panel confirms that the price level initially rises with the adverse shock but then declines back to almost return to its initial steady-state level. In fact the new steady-state price level is slightly below the initial one. The optimal interest rate that is consistent with the paths for inflation and the output gap hardly deviates from the steady-state. It is however optimal to slightly raise the interest rate, and to maintain it above steady-state for several periods, to achieve the desired deflation in later periods.

Figures 2b and 2c illustrate the optimal responses when $u_t$ follows the same process but with coefficient of autocorrelation of 0.35 and 0.9. They reveal that the mechanisms described above still work, even though the response of each variable is more sluggish.

**Moments.** The previous figures reveal that, in the optimal non-inertial plan, the effects of perturbations on inflation, output gap and the interest rate last only as long as
the shocks last. In contrast, in the optimal plan, the effects of disturbances last longer. Yet the loss is lower in the optimal plan, as the variability of inflation and the interest rate is reduced by allowing the policy to respond to past variables. This can be seen from Table 2, which reports the policymaker’s loss, $E[L_0]$, in addition to the following measure of variability

$$V[z] = E\left\{ E_0 \left[ (1 - \beta) \sum_{t=0}^{\infty} \beta^t z_t^2 \right] \right\}$$

for the four endogenous variables, $\pi, x, i, \text{and } p$, where the unconditional expectation is taken over all possible histories of the disturbances. Note that the loss $E[L_0]$ is a weighted sum of $V[\pi], V[x], \text{and } V[i]$ with weights being the ones of the loss (7). The table reports the statistics in the case in which $x^* = i^* = 0$, so that the steady state is the same for each plan (and is zero for each variable). The statistics measure therefore the variability of each variable around its steady state, and the column labeled with $E[L_0]$ indicates the loss due to temporary disturbances in excess of the steady-state loss.\(^{11}\)

A comparison of the statistics in Table 2 for the optimal plan and the optimal non-inertial plan reveals that there are substantial gains from history dependence in monetary policy. For instance, when $\rho_r = \rho_u = 0.35$, as in the baseline calibration, the loss is 1.28 in the optimal plan, while it is 2.63 in the optimal non-inertial plan. The welfare gains due to inertial monetary policy are primarily related to a lower variability of inflation and of the nominal interest rate.

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\(^{11}\)All statistics in Table 2 are reported in annual terms. The statistics $V[\pi], V[i], \text{and } E[L_0]$ are therefore multiplied by 16. Furthermore, the weight $\lambda_x$ reported in Table 1 is also multiplied by 16 in order to represent the weight attributed to the output gap variability (in annual terms) relative to the variability of annualized inflation and of the annualized interest rate.
4 Optimal Policy Rules

So far, we have characterized how the endogenous variables should respond to perturbations in order to minimize the welfare loss. We haven’t said anything about how monetary policy should be conducted, especially if the shocks are not observed by policymakers. To this issue, we now turn. Following recent studies of monetary policy (see, e.g., Taylor, 1999a), we characterize monetary policy in terms of interest-rate feedback rules. Specifically, we assume that the policymaker commits credibly at the beginning of period 0 to a policy rule that determines the nominal interest rate as a function of present and possibly past observable variables, at each date $t \geq 0$.

First, we propose a simple monetary policy rule that implements the optimal plan, even when the perturbation are not observed. We then determine optimal policy rules in restricted families: we compute optimal Taylor rules and then optimal Wicksellian rules, i.e., interest-rate rules that respond to deviations of the price level from some deterministic trend. Finally, we compare the performance of Taylor rules and Wicksellian rules.

4.1 Commitment to an Optimal Rule

We now turn to the characterization of an optimal monetary policy rule. As will become clear below, there exists a unique policy rule of the form

$$i_t = \psi_\pi \pi_t + \psi_x (x_t - x_{t-1}) + \psi_{i1} i_{t-1} + \psi_{i2} i_{t-2} + \psi_0$$

(31)
at all dates \( t \geq 0 \), that is fully optimal, i.e., that implements the optimal plan described in the previous section.\(^{12}\) To obtain the optimal rule, we solve (11) for \( \phi_{1t} \) and (10) for \( \phi_{2t} \), and use the resulting expressions to substitute for the Lagrange multipliers in (9). This yields

\[
i_t = \frac{\kappa}{\lambda_t \sigma} \pi_t + \frac{\lambda_x}{\lambda_t \sigma} (x_t - x_{t-1}) + \left( 1 + \frac{\kappa}{\beta \sigma} + \beta^{-1} \right) i_{t-1} - \beta^{-1} i_{t-2} - \frac{\kappa i^*_t}{\beta \sigma}.
\]

(32)

This is an equilibrium condition that relates the endogenous variables in the optimal plan. It can alternatively be viewed as an optimal rule, provided that it results in a unique bounded equilibrium. Using (13), we can rewrite the same policy rule in terms of hatted variables, by dropping the constant \( \frac{\kappa i^*_t}{\beta \sigma} \). The dynamic system obtained by combining (4), (5), and (32), has the property of system (14) that, if any bounded solution exists, it is unique.\(^{13}\) Moreover as we show in Appendix 6.2.1, (32) is the unique optimal policy rule in the family (31), at least in the baseline parametrization.

Notice that this rule makes no reference to either the efficient rate of interest \( r^e_t \) or the inefficient supply shock \( u_t \). It achieves the minimal loss regardless of the stochastic process that describes the evolution of the exogenous disturbances, provided that the latter are stationary (bounded). The policymaker achieves the optimal equilibrium by

\(^{12}\) As we evaluate monetary policy regardless of specific initial conditions, the policy rule is assumed to be independent of the values the endogenous variables might have taken before it was implemented. Specifically, we assume that the policymaker considers the initial values as satisfying \( i_{-2} = i_{-1} = x_{-1} = 0 \), whether they actually do or not. Equivalently, we could assume that the policy rule satisfies \( i_0 = \psi_\pi \pi_0 + \psi_x x_0, \ i_1 = \psi_\pi \pi_1 + \psi_x (x_1 - x_0) + \psi_i i_0 \), and (31) at all dates \( t \geq 2 \).

\(^{13}\) The eigenvalues of this system are the same as the eigenvalues of \( M \) in (14) plus one eigenvalue equal to zero. As there is one predetermined variable more than in (14), this system yields a unique bounded equilibrium, if it exists.
setting the interest rate according to (32) even if the natural rate and the inefficient supply shock depend upon a large state vector representing all sorts of perturbations such as productivity shocks, autonomous changes in aggregate demand, labor supply shocks, etc. Another advantage of this family of policy rules is that it includes recent descriptions of actual monetary policy such as the one proposed by Judd and Rudebusch (1998). If we would allow for a broader family of policy rules than (31), then other interest-rate feedback rules may implement the same optimal plan. In the case in which there are no inefficient supply shocks, Woodford (1999c), for example, proposes a rule in which the interest rate depends upon current and lagged values of the inflation rate as well as lagged interest rates. While his rule makes no reference to the output gap, it is dependent upon the driving process of the efficient rate of interest.

Equation (32) indicates that to implement the optimal plan, the central bank should relate the interest rate positively to fluctuations in current inflation, in changes of the output gap, and in lagged interest rates. While it is doubtful that the policymaker knows the current level of the output gap with great accuracy, the change in the output gap may be known with greater precision. For example, Orphanides (1998) shows that subsequent revisions of U.S. output gap estimates have been quite large (sometimes as large as 5.6 percentage points), while revisions of estimates of the quarterly change in the output gap have been much smaller.

Note finally that the interest rate should not only be inertial in the sense of being positively related to past values of the interest rate, it should be super-inertial, as the
lagged polynomial for the interest rate in (32)

\[ 1 - \left( 1 + \frac{\kappa}{\beta \sigma} + \beta^{-1} \right) L + \beta^{-1} L^2 = (1 - z_1 L) (1 - z_2 L) \]

involves a root \( z_1 > 1 \) while the other root \( z_2 \in (0, 1) \). A reaction greater than one of the interest rate to its lagged value has initially been found by Rotemberg and Woodford (1999) to be a desirable feature of a good policy rule in their econometric model with optimizing agents. As explained further in Woodford (1999c), it is precisely such a super-inertial policy rule that the policymaker should follow to bring about the optimal responses to shocks when economic agents are forward-looking. Because of a root larger than one, the optimal policy requires an explosively growing response of the interest rate to deviations of inflation and the output gap from the target (which is 0).

This is illustrated in Figure 3 which displays the response of the interest rate to a sustained 1 percent deviation in inflation (upper panel) or the output gap (lower panel) from target. In each panel, the solid line represents the optimal response in the baseline case. The corresponding coefficients of the optimal policy rule are reported in the upper right panel of Table 2.\(^\text{14}\) For comparison, the last panel of Table 2 reports the coefficients derived from Judd and Rudebusch’s (1998) estimation of actual Fed reaction functions between 1987:3 and 1997:4, along with the statistics that such a policy would imply if the model provided a correct description of the actual economy.\(^\text{15}\) As shown on Table 2, the estimated historical rule in the baseline case involves only slightly smaller responses to

\(^{14}\text{The coefficients } \psi_y \text{ reported here are multiplied by 4, so that the response coefficients to output gap, and to annualized inflation are expressed in the same units. (See footnote 11.)}\)

\(^{15}\text{The estimated historical policy rule refers to regression A for the Greenspan period in Judd and Rudebusch (1998).}\)
fluctuations in inflation and the output gap than the optimal rule. However the estimated response to lagged values of the interest rate is sensibly smaller that the optimal one. As a result, the estimated historical rule involves a non-explosive response of the interest rate to a sustained deviation in inflation or the output gap, represented by the dashed-dotted lines in Figure 3.

While optimal policy would involve an explosive behavior of the interest rate in the face of a sustained deviation of inflation or the output gap, such a policy is perfectly consistent with a stationary rational expectations equilibrium, and a low variability of the interest rate in equilibrium. (In Table 2, $V[z]$ is always smaller when the interest rate is set according to the optimal flexible rule, than when it is set according to the estimated historical rule or the optimal Taylor rule to be discussed below.) In fact, the interest rate does not explode in equilibrium because (as appears clearly in Figures 1 and 2) the current and expected future optimal levels of the interest rate counteract the effects of an initial deviation in inflation and the output gap by generating subsequent deviations with the opposite sign of these variables.

While the policy rule (32) allows the policymaker to achieve the lowest possible loss, recent research has given considerable attention to even simpler policy rules (see, e.g., contributions collected in Taylor, 1999a). In addition, super-inertial rules have been criticized on robustness grounds. In fact, the ability of super-inertial rules to perform well depends critically on the assumption that each agent knows the model of the economy, and that the private sector understands the way monetary policy will be conducted in the future. As Taylor (1999b) reports, these rules perform poorly in models which involve
no rational expectations and no forward-looking behavior.\textsuperscript{16} We therefore turn to very simple policy rules that are not super-inertial.

4.2 Commitment to a Standard Taylor Rule

We proceed with the standard “Taylor rule” made popular by Taylor (1993), and satisfying

\[ i_t = \psi_{\pi} \pi_t + \psi_x x_t + \psi_0, \]  

(33)

at all dates \( t \geq 0 \), where \( \psi_{\pi}, \psi_x, \) and \( \psi_0 \) are policy coefficients. For simplicity, we assume again that the law of motion of the shocks is given by (17) and (18). Using (33) to substitute for the interest rate in the structural equations (4) and (5), we can rewrite the resulting difference equations as follows

\[ E_t z_{t+1} = Az_t + a e_t, \]  

(34)

where \( z_t \equiv [\pi_t, x_t, 1]' \), and \( e_t \equiv [r^e_t, u_t]' \) and \( A \) and \( a \) are matrices of coefficients. Since both \( \pi_t \) and \( x_t \) are non-predetermined endogenous variables at date \( t \), and \( \{e_t\} \) is assumed to be bounded, the dynamic system (34) admits a unique bounded solution if and only if \( A \) has exactly two eigenvalues outside the unit circle.\textsuperscript{17} If we restrict our attention to

\textsuperscript{16}However, Levin et al. (1999a) show that rules that have a coefficient of one on the lagged interest rate perform well across models. Moreover, in Giannoni (2001c), it is shown that a variant of the super-inertial rule discussed above is robust to uncertainty about the parameters of the model and the stochastic process of the shocks.

the case in which $\psi_\pi, \psi_x \geq 0$, then it is shown in Appendix 6.2.2 that the policy rule (33) results in a determinate equilibrium if and only if

$$\psi_\pi + \frac{1 - \beta}{\kappa} \psi_x > 1. \quad (35)$$

In this case, we can solve (34) for $z_t$. Using (33) to determine also the equilibrium evolution of the interest rate, one realizes that the equilibrium inflation, output gap, and nominal interest rate are in fact given by expressions of the form (19). It follows that the optimal Taylor rule is the rule that implements the optimal equilibrium of the form (19), i.e., the optimal non-inertial plan characterized by (26) – (29).

The optimal Taylor rule can be obtained by substituting the solution (19) into (33). This yields three restrictions upon the coefficients of the policy rule

$$i_r = \psi_\pi \pi_r + \psi_x x_r \quad (36)$$

$$i_u = \psi_\pi \pi_u + \psi_x x_u \quad (37)$$

$$i_{ni} = \psi_\pi \pi_{ni} + \psi_x x_{ni} + \psi_0. \quad (38)$$

Notice that if all supply shocks are efficient, so that all disturbances can be represented by the efficient (or natural) rate of interest, the constraint (37) is not relevant. As (36) and (38) form a system of two equations in three unknown coefficients $\psi_\pi, \psi_x, \text{ and } \psi_0$, there exist many Taylor rules that implement the optimal non-inertial plan in this case (see Giannoni, 2001b, for more details). However in general, when we allow for both perturbations to the efficient rate of interest and inefficient supply shock, there is a unique optimal Taylor rule. Solving the first two restrictions for the policy coefficients
\( \psi_\pi, \psi_x \), yields

\[
\begin{align*}
\psi_\pi &= \frac{x_u i_r - i_u x_r}{x_u \pi_r - \pi_u x_r} \\
\psi_x &= \frac{\pi_r i_u - i_r \pi_u}{x_u \pi_r - \pi_u x_r}
\end{align*}
\]

provided that \( x_u \pi_r - \pi_u x_r \neq 0 \). Finally, using the expressions (27) – (29) to substitute for the coefficients \( \pi_r, x_r, \ldots \), characterizing the optimal non-inertial equilibrium, we obtain the coefficients of the optimal Taylor rule

\[
\begin{align*}
\psi_\pi &= \frac{(\kappa - \rho_u \lambda_i \eta_u) (\xi_r (1 - \beta \rho_r) + \kappa^2) + (\sigma \kappa (1 - \rho_u) + \rho_u \xi_u) \lambda_i \eta_r (1 - \beta \rho_r)}{\lambda_i \eta_r ((\kappa - \rho_u \lambda_i \eta_u) \kappa + (\lambda_i \sigma \eta_u (1 - \rho_u) + \xi_u) (1 - \beta \rho_r))} \\
\psi_x &= \frac{(\lambda_i \sigma \eta_u (1 - \rho_u) + \xi_u) (\xi_r (1 - \beta \rho_r) + \kappa^2) - \lambda_i \eta_r \kappa (\sigma \kappa (1 - \rho_u) + \rho_u \xi_u)}{\lambda_i \eta_r ((\kappa - \rho_u \lambda_i \eta_u) \kappa + (\lambda_i \sigma \eta_u (1 - \rho_u) + \xi_u) (1 - \beta \rho_r))}
\end{align*}
\] (39) (40)

where \( \xi_j = \lambda_x (1 - \beta \rho_j) > 0 \), \( \eta_j = \sigma \gamma_j - \rho_j \kappa \), and \( j \in \{r, u\} \). Note that these expressions are well defined provided that \( \lambda_i > 0 \) and \( \sigma \gamma_r - \rho_r \kappa \neq 0 \). (The limiting case in which \( \lambda_i = 0 \) is discussed below). Finally, the constant \( \psi_0 \) is obtained by solving (38), using the optimal values for \( \psi_\pi \) and \( \psi_x \), and the steady-state expressions (26).

While the optimal Taylor rule depends in a complicated way on all parameters of the model and the degree of persistence of the perturbations, it is interesting to note that it is completely independent of the variability of the disturbances. Table 2 reports the optimal coefficients (39) and (40) for different degrees of persistence of the perturbations, using the calibration summarized in Table 1. These coefficients are displayed in Figure 4. The white region of Figure 4 indicates the set of policy rules that result in a unique bounded equilibrium. In contrast, the gray region indicates combinations \( (\psi_\pi, \psi_x) \) that
result in indeterminacy of the equilibrium.\textsuperscript{18}

Figure 4 reveals for example that when both shocks are purely transitory ($\rho_r = \rho_u = 0$), the “optimal” Taylor rule lies in the region of indeterminacy. In fact, the “optimal” coefficients $\psi_\pi, \psi_x$, while positive, are not large enough to satisfy (35). This means that for any bounded solution $\{z_t\}$ to the difference equation (34), there exists another bounded solution of the form

$$z'_t = z_t + v\xi_t$$

where $v$ is an appropriately chosen (nonzero) vector, and the stochastic process $\{\xi_t\}$ may involve arbitrarily large fluctuations, which may or may not be correlated with the fundamental disturbances $r_t^e$ and $u_t$. It follows that the dynamic system (34) admits a large set of bounded solutions, including solutions that involve arbitrarily large fluctuations of inflation and the output gap. The policymaker should therefore not use the “optimal” Taylor rule whenever it lies in the region of indeterminacy, as it might result in an arbitrarily large value of the loss criterion (7). Note from Figure 4 that the problem of indeterminacy arises not only when $\rho_r = \rho_u = 0$, but also in some cases when the disturbances are more persistent (e.g., when $\rho_r = 0.35$ and $\rho_u = 0$, or when $\rho_r = \rho_u = 0.9$).

To get some intuition about the optimal Taylor rule, let us consider the special case in which both perturbations have the same degree of persistence, i.e., $\rho_r = \rho_u \equiv \rho$. In

\textsuperscript{18}For the determination of the boundaries of the region of determinacy, see Appendix 6.2 of Giannoni (2001b).
This case, (39) and (40) reduce to

\[
\begin{align*}
\psi_\pi &= \frac{\kappa}{\lambda_i (\sigma (1 - \rho) (1 - \beta \rho) - \rho \kappa)} \\
\psi_x &= \frac{\lambda_x (1 - \beta \rho)}{\lambda_i (\sigma (1 - \rho) (1 - \beta \rho) - \rho \kappa)}
\end{align*}
\]

It is easy to see that the optimal coefficient on inflation, \( \psi_\pi \), increases when the aggregate supply curve becomes steeper, to prevent a given output gap to create more inflation. Similarly the optimal coefficient on output gap, \( \psi_x \), increases when \( \lambda_x \) increases, as the policymaker is more willing to stabilize the output gap. In addition, the optimal Taylor rule becomes more responsive to both inflation and output gap fluctuations, when the weight \( \lambda_i \) decreases, as the policymaker is willing to let the interest rate vary more, and when the intertemporal IS curve becomes flatter (\( \sigma \) is smaller), as shocks to the efficient rate of interest have a larger impact on the output gap and inflation.

### 4.2.1 The Importance of \( \lambda_i > 0 \)

We have assumed throughout that the policymaker cares about the variability of the nominal interest rate. As mentioned above however, the optimal Taylor rule (as well as the optimal rule (32)) is not well defined when \( \lambda_i = 0 \). To see why, we return to the characterization of the optimal non-inertial equilibrium. When \( \lambda_i = 0 \), (27) – (29) reduce to

\[
\begin{align*}
\pi_r &= 0, \quad \pi_u = \frac{\lambda_x (1 - \beta \rho_u)}{\lambda_x (1 - \beta \rho_u)^2 + \kappa^2} \\
x_r &= 0, \quad x_u = -\frac{\lambda_x (1 - \beta \rho_u)^2 + \kappa^2}{\kappa} \\
i_r &= 1, \quad i_u = \frac{\sigma \kappa (1 - \rho_u) + \lambda_x (1 - \beta \rho_u) \rho_u}{\lambda_x (1 - \beta \rho_u)^2 + \kappa^2}
\end{align*}
\]
Because, $i_r = 1$, the nominal interest rate is set in a way to completely offset the perturbations to the efficient rate of interest, so that fluctuations to inflation and the output gap only come from inefficient supply shocks. But as both inflation and the output gap remain unaffected by a shift in $r^e_t$ in the optimal equilibrium, the policymaker cannot extract any information about the current level of the efficient rate of interest from observable variables. There is therefore no Taylor rule that determines the optimal response of the nominal interest rate.

In contrast, as inefficient supply shocks imply a trade-off between the stabilization of inflation and the output gap, the optimal interest rate is set so as to equate the marginal loss on both dimensions. The optimal response to $u$ can be expressed as an optimal response to fluctuations in inflation and the output gap by using (37). Substituting $\pi_u, x_u, \text{ and } i_u$ with the above expressions in (37), we obtain

$$\psi_\pi = \left( \frac{\sigma \kappa (1 - \rho_u)}{\lambda_x (1 - \beta \rho_u)} + \rho_u \right) + \psi_x \frac{\kappa}{\lambda_x (1 - \beta \rho_u)}.$$  

There are many combinations $\psi_\pi, \psi_x$ that achieve the optimal response to the inefficient supply shock. In particular, if we choose $\psi_x = 0$, the optimal response of the interest rate in the optimal non-inertial plan is given by

$$i_t = \left( \frac{\sigma \kappa (1 - \rho_u)}{\lambda_x (1 - \beta \rho_u)} + \rho_u \right) \pi_t + r^e_t.$$  

Note that this rule determines the interest rate such that inflation and the output gap are insulated from disturbances to the efficient rate of interest (this results in an equilibrium in which $\pi_r = x_r = 0$). However, for this rule to be implemented in practice, one needs

\[\text{It remains to be checked that the response to inflation is large enough to yield a determinate equilibrium.}\]
to know the efficient interest rate \( r^e \). This is therefore not a Taylor rule. Clarida et al. (1999) propose a similar interest-rate rule that determines the optimal nominal interest rate in the case \( \lambda_i = 0 \). Their rule however specifies the interest rate as a function of expected future inflation \( E_t \pi_{t+1} \) instead of \( \pi_t \).

### 4.3 Commitment to a Simple Wicksellian Rule

In the previous section, we have emphasized the gains from history dependence in monetary policy, and (keeping aside the problem of indeterminacy) we have argued that standard Taylor rules do not have this desirable property. We now turn to an alternative very simple rule that is history dependent. It is given by

\[
i_t = \psi_p (p_t - \bar{p}_t) + \psi_x x_t + \psi_0
\]

at all dates \( t \geq 0 \), where \( \bar{p}_t \) is some deterministic trend for the (log of the) price-level, satisfying

\[
\bar{p}_t = \bar{p}_{t-1} + \bar{\pi},
\]

where \( \bar{\pi} \) is some constant. Following Woodford (1998a, 1999a) we will call such a rule a Wicksellian rule, after Wicksell (1907).\(^{20}\) The price level depends by definition not only on current inflation but also on all past rates of inflation. It follows that the rule (41) introduces history dependence in monetary policy, as it forces the policymaker to compensate any shock that might have affected inflation in the past. While rules of this form are as simple as standard Taylor rules, they have received considerably less attention in recent studies of monetary policy. One reason may be because it is widely believed

\(^{20}\)Wicksell (1907) argued that “price stability” could be obtained by letting the interest rate respond positively to fluctuations in the price level.
that such rules would result in a larger variability of inflation (and the output gap), as the policymaker would respond to an inflationary shock by generating a deflation in subsequent periods. However, as we show below, this is not true when agents are forward-looking, and it is understood that the policymaker commits to a rule of the form (41). Although the policymaker and the private sector do not care about the price level per se, as the latter does not enter the loss criterion (7), we shall argue that a Wicksellian rule has desirable properties for the conduct of monetary policy.

To see in what sense a policy rule of the form (41) introduces history dependence, we now turn to the characterization of the equilibrium that obtains if the policymaker commits to (41) for the entire future. Consider a steady state in which in which inflation, the output gap and the nominal interest rate take respectively the constant values $\pi^{wr}, x^{wr}, i^{wr}$. Substituting the latter into the structural equations (4) and (5), we obtain

$$i^{wr} = \pi^{wr},$$
$$x^{wr} = \frac{1 - \beta}{\kappa} \pi^{wr}.$$

As usual, we define the deviations from the steady state as $\hat{\pi}_t \equiv \pi_t - \pi^{wr}$, $\hat{x}_t \equiv x_t - x^{wr}$, and $\hat{i}_t \equiv i_t - i^{wr}$, and we also let $\hat{p}_t \equiv p_t - \bar{p}_t$ be the (percentage) deviation of the price level from its trend. Again, the structural equations (4) and (5) hold in terms of the
hatted variables, and the policy rule (41) may be written as\(^{21}\)

\[ \dot{i}_t = \psi_p \hat{p}_t + \psi_x \hat{x}_t. \]  

(44)

Using this to substitute for \(\dot{i}_t\) in the intertemporal IS equation, we can rewrite (4), (5), and (6) in matrix form as

\[ E_t z_{t+1} = \hat{A} z_t + \hat{a} e_t, \]  

(45)

where \(z_t \equiv [\hat{\pi}_t, \hat{x}_t, \hat{p}_{t-1}]', e_t \equiv [r_t^e, u_t]', \) and \(\hat{A}\) and \(\hat{a}\) are matrices of coefficients. The difference equation (45) admits a unique bounded solution if and only if \(\hat{A}\) admits exactly two unstable eigenvalues. It is shown in appendix 6.2.3 that sufficient conditions for the policy rule (44) to result in a determinate equilibrium are given by

\[ \psi_p > 0, \text{ and } \psi_x \geq 0. \]  

(46)

Assuming again that the law of motion of the disturbances is given by (17) and (18), one realizes that the equilibrium obtained by combining the policy rule (41) with the structural equations (4) and (5) is of the form

\[ \hat{z}_t = z_t \dot{r}_t^e + z_u u_t + z_p \hat{p}_{t-1} \]  

(47)

for any variable \(z_t \in \{\hat{\pi}_t, \hat{x}_t, \hat{p}_t\}\), where \(z_r, z_u, z_p\) are equilibrium response coefficients to fluctuations in \(r_t^e, u_t,\) and \(p_{t-1}\). (Of course, in levels we have \(z_t = z^{wr} + \hat{z}_t\), where \(z^{wr}\)

\(^{21}\)To obtain (44), we make an implicit assumption on the coefficient \(\psi_0\) which has no effect on the welfare analysis that follows. First, note from (41) that \(\hat{p}_t\) must be constant in the steady state. For convenience, we set this constant to zero. The optimal policy coefficient \(\psi_0\) is the only coefficient affected by this normalization, but this has no effect on optimal monetary policy. Comparing (41) and (44) one can see that \(\psi_0\) is implicitly given by \(\psi_0 = \dot{\pi}^{wr} - \psi_x \pi^{wr}\). Note also from the definition of inflation that \(\pi_t = p_t - p_{t-1} = \hat{p}_t - \hat{p}_{t-1} + \pi\). Hence, in the steady state, we have \(\pi^{wr} = \pi\).
represents the steady-state values of the respective variables in this optimal equilibrium.)

Using this, and noting that $E \{ E_0 (1 - \beta) \sum_{t=0}^{\infty} \beta^t \hat{z}_t \} = 0$, we can write the loss criterion (7) as

$$E[L_0] = \left( (\pi^{wr})^2 + \lambda_x (x^{wr} - x^*)^2 + \lambda_i (i^{wr} - i^*)^2 \right) + E[\hat{L}_0].$$

where

$$E[\hat{L}_0] \equiv E \left\{ E_0 (1 - \beta) \sum_{t=0}^{\infty} \beta^t [\bar{\pi}_t^2 + \lambda_x \bar{x}_t^2 + \lambda_i \bar{i}_t^2] \right\}.$$  \hspace{1cm} (48)

The optimal steady state is then found by minimizing the first term in brackets in the previous expression subject to (42) and (43). Since this is the same problem as the one encountered for the optimal non-inertial plan, we have

$$\pi^{wr} = \pi^{ni}, \quad x^{wr} = x^{ni}, \quad \text{and} \quad i^{wr} = i^{ni}. \hspace{1cm} (49)$$

where $\pi^{ni}, x^{ni},$ and $i^{ni}$ are given in (26).

To determine the optimal equilibrium responses to disturbances, we note, as in the optimal non-inertial plan, that the solution (47) may only describe an equilibrium if the coefficients $z_r, z_u, z_p$ satisfy the structural equations (4) and (5) at each date, and for every possible realization of the shocks. These coefficients need therefore to satisfy the following feasibility restrictions, obtained by substituting (47) into the structural equations (4), (5), and using (6):

$$x_r (1 - \rho_r) - x_p p_r + \sigma^{-1} \left( i_r + (1 - p_p - \rho_r) p_r - 1 \right) = 0 \hspace{1cm} (50)$$

$$x_u (1 - \rho_u) - x_p p_u + \sigma^{-1} \left( i_u + (1 - p_p - \rho_u) p_u \right) = 0 \hspace{1cm} (51)$$

$$x_p - x_p p_p + \sigma^{-1} \left( i_p + (1 - p_p) p_p \right) = 0 \hspace{1cm} (52)$$

$$(\beta \rho_r + \beta p_p - 1 - \beta) p_r + \kappa x_r = 0 \hspace{1cm} (53)$$
Similarly, substituting the solution (47) into the policy rule (44) yields

\[ i_r = \psi_p r + \psi_x x \tag{56} \]
\[ i_u = \psi_u p + \psi_u x \tag{57} \]
\[ i_p = \psi_p p + \psi_x x. \tag{58} \]

Using (56) and (57), we can then determine the policy coefficients \( \psi_p \) and \( \psi_x \), to obtain

\[ \psi_p = \frac{x_u i_r - i_u x_r}{x_u p_r - x_r p_u} \tag{59} \]
\[ \psi_x = \frac{p_r i_u - i_r p_u}{x_u p_r - x_r p_u} \tag{60} \]

provided that \( x_u p_r - x_r p_u \neq 0 \). Substituting (59) and (60) into (58), we obtain

\[ i_p - \frac{x_u i_r - i_u x_r}{x_u p_r - x_r p_u} p - \frac{p_r i_u - i_r p_u}{x_u p_r - x_r p_u} x = 0, \tag{61} \]

which is an additional constraint that must be satisfied by the equilibrium coefficients, for the structural equations and the policy rule to be satisfied at each date and in every state.

Finally using (6), the solution (47), and the laws of motion (17) and (18), we can rewrite the loss (48) as

\[
E \left[ \tilde{L}_0 \right] = \text{var} (r_t^e) \left( (p_t^2 + \lambda_x x_t^2 + \lambda_i i_t^2) + (p_r (p_p - 1) + \lambda_x x_r x_p + \lambda_i i_r i_p) \cdot \frac{2 \beta \rho_p r}{1 - \beta \rho_p p} \right) + \text{var} (u_t) \left( (p_u^2 + \lambda_x x_u^2 + \lambda_i i_u^2) + (p_u (p_p - 1) + \lambda_x x_u x_p + \lambda_i i_u i_p) \cdot \frac{2 \beta \rho_u u}{1 - \beta \rho_u p} \right) + \left( (p_p - 1)^2 + \lambda_x x_p^2 + \lambda_i i_p^2 \right) \times \left( \text{var} (r_t^e) \cdot \frac{\beta p_t^2}{1 - \beta p_t^2} + \text{var} (u_t) \cdot \frac{\beta p_u^2}{1 - \beta p_u^2} \right). \tag{62} \]
The optimal equilibrium resulting from a Wicksellian rule (41) is therefore characterized by the optimal steady state (49), and the optimal response coefficients $p_r, p_u$, and so on, that minimize the loss function (62) subject to the constraints (50) – (55) and (61). The coefficients of the optimal Wicksellian rule that are consistent with that equilibrium are in turn determined by (59) and (60). In general, the coefficients of optimal Wicksellian rule are complicated functions of the parameters of the model. Moreover, unlike those of the optimal Taylor rule, they are also function of the variance of the shocks. Rather than trying to characterize analytically the optimal Wicksellian rule, we proceed with a numerical investigation of its properties and its implications for equilibrium inflation, output gap and the nominal interest rate.

4.3.1 Optimal Responses to Perturbations under a Wicksellian Rule

When monetary policy is conducted according to a Wicksellian rule, the equilibrium evolution of inflation, output gap and the interest rate is history dependent, because it depends on the lagged price level (see (47)). We now argue that appropriate Wicksellian rules involve the kind of history dependence that is desirable for monetary policy.

**Shock to $r^e_t$**. Figures 1a to 1c represent with a dashed-dotted line the response of endogenous variables to an unexpected temporary increase in the efficient rate of interest when monetary policy is set according to the optimal Wicksellian rule. Figures 1a and 1b reveal that the responses of endogenous variables are more persistent under the optimal Wicksellian rule than in the optimal non-inertial plan (dashed lines) which results from the optimal Taylor rule. The dashed-dotted lines lie in general between the dashed and the solid lines. Commitment to an optimal Wicksellian policy allows the policymaker...
to achieve a response of endogenous variables that is closer to the optimal plan than is the case with the optimal Taylor rule. One particularity of the equilibrium resulting from a Wicksellian policy, of course, is that the price level is stationary. This feature turns out to affect the response of endogenous variables in particular when shocks are very persistent as in Figure 1c. In fact, the mere expectation of future deflation under the optimal plan and the optimal non-inertial plan already depresses inflation when the shock hits the economy, and is expected to keep inflation below steady-state for several periods. In contrast, under optimal Wicksellian policy, both inflation and the price level rise strongly on impact, but they are expected to return progressively to their initial steady-state.

**Shock to** $u_t$. In Figures 2a to 2c we represent with a dashed-dotted line the response of endogenous variables to an unexpected temporary inefficient supply shock under optimal Wicksellian policy. A striking feature of optimal Wicksellian policy in Figures 2a and 2b is that the interest rate has to rise importantly in order for the response of inflation to match the optimal response. While this of creates a significant drop in output (gap), the welfare loss is only moderately affected by the recession, given the low weight $\lambda_x$ of our calibration. When inefficient supply shocks are very persistent, however, it is optimal to decrease the nominal interest rate on impact. This is because the expectation that the price level will need to return to its initial steady state in the future depresses the economy already at the time of the shock. In contrast, both under the optimal plan and the optimal non-inertial plan, the price level is expected to end up at a higher level in the future.
4.4 A Comparison of Taylor Rules and Wicksellian Rules

While the optimal Wicksellian rule results in inertial responses of the endogenous variables to exogenous disturbances, unlike the optimal Taylor rule, it is not clear a priori how these rules perform in terms of welfare. To gain some intuition, we first consider an analytical characterization in a special case. We then proceed with a numerical investigation of the more general case.

4.4.1 A Special Case

To simplify the analysis, we consider the special case in which the short-term aggregate supply equation is perfectly flat so that \( \kappa = 0 \), and both shocks have the same degree of serial correlation \( \rho \). In this case, we can solve for equilibrium inflation using (5), and we obtain

\[
\pi_t = \beta \mathbb{E}_t \pi_{t+1} + u_t = \sum_{j=0}^{\infty} \beta^j \mathbb{E}_t u_{t+j} = \sum_{j=0}^{\infty} (\beta \rho)^j u_t = (1 - \beta \rho)^{-1} u_t.
\]

Inflation is perfectly exogenous in this case. The best the policymaker can do is therefore to minimize the variability of the output gap and the interest rate.

Using (39) and (40), we note that the optimal Taylor rule reduces in this case to

\[
\hat{i}_t = \frac{\lambda_x}{\lambda_i \sigma (1 - \rho)} \hat{x}_t.
\]

The optimal Taylor rule involves no response to inflation. Since inflation is cannot be affected by monetary policy in this case, it would be desirable to respond to inflation, only if this would help dampening fluctuations in the output gap and the interest rate in the present or in the future. However, since the Taylor rule is non inertial, the equilibrium endogenous variables depend only on contemporaneous shocks (see (19)). It follows that
one cannot reduce the variability of future output gaps and interest rates by responding to current shocks in inflation. Responding to contemporaneous fluctuations in inflation would only make the interest rate and the output gap more volatile.

In contrast, with a Wicksellian rule, both the policymaker’s response to price-level fluctuations in the present, and the belief that he will respond in the same way to price-level fluctuations in the future have an effect on the expected future path of the output gap and the interest rate. We can establish the following result.

**Proposition 1** When \( \kappa = 0 \) and \( \rho_y = \rho_u \equiv \rho > 0 \), the Wicksellian rule

\[
\hat{i}_t = \frac{\rho \lambda_x (1 - \beta) (1 - \beta \rho^2)}{(1 - \rho) (\lambda_i \sigma^2 (1 - \beta \rho^2) (1 - \beta \rho) + \lambda_x (1 + \beta \rho))} \hat{p}_t + \frac{\lambda_x}{\lambda_i \sigma (1 - \rho)} \hat{x}_t
\]  

(63)
results in a unique bounded equilibrium, and achieves a lower loss than the one resulting from the optimal Taylor rule.

**Proof.** See Appendix 6.2.4.

Equation (63) is not the optimal Wicksellian rule, as the latter would in general depend also on the variance of each shock, but it is a relatively simple policy rule that performs well. A corollary of proposition 1 is of course that the optimal Wicksellian rule achieves a lower loss than the one resulting from the optimal Taylor rule, provided that it results in a determinate equilibrium. Note finally, that in the limit, as \( \rho \to 0 \), the rule (63) (as well as the optimal Wicksellian rule) do not respond to price level fluctuations, as the latter are not expected to last.
4.4.2 General Case: A Numerical Investigation

In the more general case in which $\kappa > 0$, and we allow for arbitrary degrees of serial correlation of the shocks, the analytical characterization is substantially more complicated. However a numerical investigation suggests again that appropriate Wicksellian rules perform better than the optimal Taylor rule in terms of the loss criterion (7). Using the calibration of Table 1, and for various degrees persistence of the disturbances, Table 2 reveals that the loss is systematically lower with the optimal Wicksellian rule than it is with the optimal Taylor rule. For instance, when $\rho_r = \rho_u = .35$, the loss is 1.67 with the Wicksellian rule, compared to 2.63 with the Taylor rule, and 1.28 with the fully optimal rule.\(^{22}\) This relatively good performance of the Wicksellian rules is due to the low variability of inflation and the nominal interest rate. On the other hand, the output gap is in general more volatile under the optimal Wicksellian rule. Of course the variability of the price level is much higher for fully optimal rules and optimal Taylor rules, but this does not affect the loss criterion.

The optimal Wicksellian rules are reported in Table 2. They are also plotted in Figure 5. In contrast to the optimal Taylor rules represented in Figure 4, the optimal Wicksellian rules are less sensitive to the different assumptions about serial correlation of the disturbances. In fact the points are more concentrated in Figure 5 than they are in Figure 4. In addition, the optimal Wicksellian rules satisfy conditions (46) as they lie in the positive orthant. Thus optimal Wicksellian rules result in a determinate equilibrium,

\(^{22}\) Recall that Table 2 indicates the losses due to fluctuations around the steady state. However, since the steady states are the same for the optimal Taylor rule and the optimal Wicksellian rule, the comparison of statistics is also relevant for levels of the variables, for any values $x^*, i^*$. 

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unlike optimal Taylor rules, in some cases.

Finally, to assess the robustness of the results, we represent in Figures 6a – 6f, the statistics and policy rules for different parameter values. Figures 6a – 6c plot the statistics and optimal policy rules for different degrees of serial correlation of the shocks. Figure 6d plots the same variables for different assumptions about the variance of the efficient rate of interest. Figure 6e does the same for different assumptions about the variance of inefficient supply shocks. Finally, Figure 6f represents the plots for different assumptions about $\lambda_x$. Discontinued lines indicate that the policy rule results in an indeterminate equilibrium.

These figures confirm that the variability of inflation and the variability of the price level are lower under the optimal Wicksellian rule than under either the optimal Taylor rule or the optimal rule. In general, the variability of the output gap is higher with the Wicksellian rule than with either the Taylor rule or the optimal rule, unless $\rho_u$ is very large and $\rho_r$ is small. In general, the variability of the nominal interest rate implied by the Wicksellian rule is higher than the one implied by the optimal rule, but (much) smaller than the one implied by the Taylor rule, unless $\rho_r$ is very large, say around 0.9. In all figures, the loss $E[L_0]$ is lower with the Wicksellian rule than with the Taylor rule, and it is only slightly higher with the Wicksellian rule than with the optimal rule.

5 Conclusion

In this paper, we have characterized optimal monetary policy with commitment in a simple, standard, forward-looking model. We have proposed a monetary policy rule that
implements the optimal plan. This rule requires the interest rate to be related positively to fluctuations in current inflation, in changes of the output gap, and in lagged interest rates. Moreover, the optimal rule is super-inertial, in the sense that it requires the interest rate to vary by more than one for one to past fluctuations of the interest rate. Rotemberg and Woodford (1999) and Woodford (1999c) have previously advocated super-inertial rules for their ability to affect the private sector’s expectations appropriately.

However the concern for robustness of the conclusions to various modelling assumption has led many authors to focus on very simple policy rules. We have therefore also analyzed simple families of policy rules such as Taylor rules and Wicksellian rules. We have shown that appropriate Wicksellian rules perform generally better than optimal Taylor rules, because they result in a lower welfare loss, a lower variability of inflation and of nominal interest rates, by introducing desirable history dependence in monetary policy. Moreover, unlike optimal Taylor rules, which may result in indeterminacy of the equilibrium, Wicksellian rules do in general result in a determinate equilibrium.

While Wicksellian rules require policymakers to bring prices back to their initial trend path, it is in general not globally optimal for the price level to do so. Indeed, Figures 1a to 1c reveal for instance that in the optimal plan, the price level eventually ends up below its initial trend path in the face of fluctuations in the efficient rate of interest. Nevertheless, appropriate Wicksellian rules achieve a lower welfare loss than optimal Taylor rules because they involve dynamic responses of the economy to exogenous perturbations that are in general closer to the optimal ones. In a model similar to the one presented here, Barnett and Engineer (2000) argue that inflation targeting is in general optimal “in the sense that optimal policy with commitment displays price-level drift”.

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But, this is because they define inflation targeting in a very general way. As Boivin (2000) remarks, Barnett and Engineer’s (2000) definitions of inflation targeting and price-level targeting imply that price-level targeting would be optimal only in “non-generic” cases in which optimal policy displays a stationary price path. Thus in our framework also, they would have found that inflation targeting is optimal. In our opinion however, the relevant question is not whether optimal policy displays price-level drift or not. Instead, we believe that a more relevant question is whether an appropriate interest-rate rule that systematically responds to price-level variations preforms better than an interest-rate rule that systematically responds to inflation variations or not. We have shown that the answer is yes in the model considered.
6 Appendix

6.1 Underlying Model

This appendix explains in more details the underlying structure of the model presented in section 2.

6.1.1 Intertemporal IS Equation

We assume that there exists a continuum of households indexed by $j$ and distributed uniformly on the $[0, 1]$ interval. Each household $j$ seeks to maximize its lifetime expected utility given by

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[ u\left( C_j^t; \xi_t \right) - v\left( h_t(j); \xi_t \right) \right] \right\},$$

(64)

where $\beta \in (0, 1)$ is a discount factor, $h_t(j)$ is the amount of labor that household $j$ supplies at date $t$, and $C_j^t$ is an index of the household’s consumption of each of the differentiated goods defined by

$$C_j^t \equiv \left[ \int_0^1 c_j^t(z) \frac{\theta_t - 1}{\theta_t} dz \right]^{\frac{\theta_t}{\theta_t - 1}}.$$  

(65)

While (65) is similar to the Dixit and Stiglitz (1977) index, we let the elasticity of substitution between goods, $\theta_t > 1$, vary exogenously over time. As we will see below, a time-varying elasticity of substitution implies time variation in desired markup. We assume that each household specializes in the supply of one type of labor, and that each type of labor is supplied by an equal number of households. The stationary vector $\xi_t$ represents disturbances to preferences. For each value of $\xi$, the function $u(\cdot; \xi)$ is increasing and concave, while $v(\cdot; \xi)$ is increasing and convex.
Optimal behavior on the part of each household requires first an optimal allocation of consumption spending across differentiated goods at each date, for given level of overall expenditure. Optimal consumption of good \( z \) is given by the usual expression

\[
c^j_t (z) = C^j_t \left( \frac{p_t(z)}{P_t} \right)^{-\theta_t},
\]

where the price index is defined as

\[
P_t \equiv \left[ \int_0^1 p_t(z)^{1-\theta_t} dz \right]^{1-\theta_t}.
\]

It follows that the demand for good \( z \) is given by

\[
y^j_t (z) = \tilde{Y}^j_t \left( \frac{p_t(z)}{P_t} \right)^{-\theta_t},
\]

where \( \tilde{Y}^j_t = C_t \equiv \int_0^1 C^j_t dj \) represents aggregate demand at date \( t \).

We assume that financial markets are complete so that risks are efficiently shared. As a result, all households face an identical intertemporal budget constraint, and choose identical state-contingent plans for consumption. We may therefore drop the index \( j \) on this variable. The optimal intertemporal allocation of consumption is then given by the familiar Euler equation

\[
\frac{1}{1 + \tilde{i}_t} = E_t \left\{ \frac{\beta u_c \left( C_{t+1}; \xi_{t+1} \right)}{\bar{u}_c \left( C_t; \xi_t \right)} \frac{P_t}{P_{t+1}} \right\},
\]

where \( \tilde{i}_t \) denotes the nominal interest rate on a one-period riskless bond purchased in period \( t \). The optimal supply of labor is given by

\[
\frac{v_h(h_t(j); \xi_t)}{u_c(C_t; \xi_t)} = \frac{W_t(j)}{P_t},
\]

where \( W_t(j) \) is the nominal wage of labor \( j \).
We will consider log-linear approximations of these relationships about the steady state where the exogenous disturbances take the values $\xi_t = 0$, and where there is no inflation, i.e., $P_t/P_{t-1} = 1$. We let $\bar{Y}$ and $\bar{i}$ be the constant values of output and nominal interest rate in that steady state, and define the percent deviations $Y_t \equiv \log \left( \frac{\bar{Y}_t}{\bar{Y}} \right)$, $i_t \equiv \log (1 + \bar{i}_t/1 + \bar{i})$, $\pi_t \equiv \log (P_t/P_{t-1})$. Performing a log-linear approximation to (69), and using the market clearing condition, $\bar{Y}_t = C_t$ to substitute for consumption, we obtain the “intertemporal IS equation” (1), where $\sigma \equiv -\frac{u_c}{u_e} > 0$ denotes the inverse of the intertemporal elasticity of substitution, and where $g_t \equiv \frac{u_c}{u_e} \sigma \xi_t$ represents exogenous demand shifts.

### 6.1.2 Natural Rate of Output and Efficient Rate of Output

On the firm’ side, we assume that each good $z \in [0, 1]$ is produced using a production function $y_t(z) = f(h_t(z))$ where $f' > 0$, $f'' < 0$, and capital is fixed so that labor of type $z$ is the only variable input for firm $z$. The real marginal cost of supplying goods $y_t(z)$ obtained by differentiating the total real variable cost $W_t(z) (f^{-1}(y_t(z)))/P_t$ is

$$s_t(z) = \frac{W_t(z)}{P_t f'(f^{-1}(y_t(z)))}.$$  

Combining this with (70) we can write the real marginal cost function as

$$s_t(z) \equiv s \left( y_t(z), \bar{Y}_t; \xi_t \right)$$

where

$$s \left( y, \bar{Y}; \xi \right) \equiv \frac{v_h \left( f^{-1}(y); \xi \right)}{u_c \left( \bar{Y}; \xi \right) f'(f^{-1}(y))}.$$  

(71)

If prices were perfectly flexible, each firm would maximize profits

$$\Pi_t(z) \equiv p_t(z) y_t(z) - W_t(z) h_t(z) = p_t(z)^{1-\theta_t} P_t^{\theta_t} \bar{Y}_t - W_t(z) f^{-1} \left( p_t(z)^{-\theta_t} P_t^{\theta_t} \bar{Y}_t \right),$$  

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at each period by setting prices according to

\[ p_t(z) = \tilde{\mu}_t s_t(z) P_t, \]

where the time-varying desired markup is defined as \( \tilde{\mu}_t \equiv \frac{\theta_t}{\tilde{\mu}_t}. \) Here we have used the fact that since there are infinitely many suppliers in this economy, each supplier has only a negligible effect on the index of aggregate output \( \tilde{Y}_t \) or the price index \( P_t. \) Using (68), we note that the relative supply of good \( z \) must in turn satisfy

\[ \left( \frac{y_t(z)}{\tilde{Y}_t} \right)_{-1/\theta_t} = \tilde{\mu}_t \cdot s \left( y_t(z), \tilde{Y}_t, \xi_t \right). \]

Because \( s \) is increasing in its first argument, this equation admits a unique solution for \( y_t(z), \) so that each supplier must offer the same quantity \( \tilde{Y}_t^n \) in equilibrium, i.e., the solution to

\[ s \left( \tilde{Y}_t^n, \tilde{Y}_t^n, \xi_t \right) = \tilde{\mu}_t^{-1}. \]  

(72)

We will refer to \( \tilde{Y}_t^n \) as the “natural rate of output”, that is the equilibrium output that would obtain with perfectly flexible prices. Log-linearizing (72) about the steady-state mentioned above, where in addition \( \theta_t = \bar{\theta}, \tilde{\mu}_t = \bar{\mu}, \) \( p_t(z) / P_t = 1, y_t(z) / \tilde{Y}_t = 1, \) yields

\[ Y_t^n = \frac{1}{\omega + \sigma} \left( \frac{u_c \xi_t}{u_c} - \frac{v_h \xi_t}{v_h} - \mu_t \right), \]  

(73)

where \( Y_t^n = \log \left( \tilde{Y}_t^n / \tilde{Y} \right), \) \( \mu_t = \log (\tilde{\mu}_t / \bar{\mu}), \) and \( \omega \equiv \left( \frac{v_h}{u_h} - \frac{\bar{m}}{\bar{P}} \right) \frac{\tilde{Y}}{\tilde{P}} > 0 \) represents the elasticity of each firm’s real marginal cost with respect to its own supply. Note that the natural rate of output depends upon the exogenous real perturbations, so that it is completely independent of monetary policy.

When \( \tilde{\mu}_t \) is exogenously time varying, variations in the natural rate of output \( \tilde{Y}_t^n \) differ in general from fluctuations of the efficient rate of output, \( \tilde{Y}_t^e, \) i.e., the equilibrium
rate of output that would obtain in the absence of distortions due to market power. The efficient rate of output solves

\[ s \left( \hat{Y}_t^e, \hat{Y}_t^e; \xi_t \right) = 1. \] (74)

In steady state, it reaches the constant level of output, \( Y^* \), which satisfies \( s (Y^*, Y^*; 0) = 1 \). Log-linearizing (74) around \( \hat{Y}_t^e = Y^* \), and solving for \( Y_t^e = \log \left( \hat{Y}_t^e / Y^* \right) \), we obtain

\[ Y_t^e = \frac{1}{\omega + \sigma} \left( \frac{u_c^t}{u_c^{t-1}} \xi_t - \frac{v_h^t}{v_h^{t-1}} \xi_t \right). \] (75)

Comparing (75) with (73), we note that the deviation \( Y_t^e - Y_t^p \) is given by (3).

### 6.1.3 Aggregate Supply Equation

In order for monetary policy to have real effects, we assume some price stickiness. Specifically, we assume as in Calvo (1983) that only a fraction \( 1 - \alpha \) of suppliers may change their prices at the end of any given period, regardless of the time elapsed since the last change. Since each supplier faces the same demand function, each supplier that changes its price in period \( t \) chooses the same optimal price \( p_t^* \). It follows that the price index (67) satisfies

\[ P_t = \left[ (1 - \alpha) p_t^{1-\theta_t} + \alpha P_{t-1}^{1-\theta_t} \right]^{1-\theta_t}. \]

Log-linearizing this yields

\[ \pi_t = \frac{1 - \alpha}{\alpha} \hat{p}_t^* \] (76)

where \( \hat{p}_t^* \equiv \log \left( p_t^* / P_t \right) \). A supplier that changes its price in period \( t \) chooses its new price \( p_t^* \) to maximize the expected present discounted value of future profits given by

\[ E_t \left\{ \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \left[ p_t (z)^{1-\theta_T} P_T^{\theta_T} \hat{Y}_T - W_T (z) f^{-1} \left( p_t (z)^{-\theta_T} P_T^{\theta_T} \hat{Y}_T \right) \right] \right\}, \]
where the stochastic discount factor satisfies $Q_{t,T} = \frac{\beta^{T-t} u_c(\hat{Y}_T; \xi_T) P_t}{u_c(Y_t; \xi_t) P_T}$. This discount factor is adjusted for the fact that the price chosen at date $t$ remains in effect at date $T$ with probability $\alpha^{T-t}$. Using (71), we can write the first-order condition to this problem as

$$0 = E_t \left\{ \sum_{T=t}^{\infty} (\alpha \beta)^{T-t} \left[ u_c \left( \hat{Y}_T; \xi_T \right) \hat{Y}_T P_T^{\theta_T} \hat{p}_t^{\theta_T} (\theta_T - 1) \right] \times \left[ \frac{\hat{p}_t^*}{P_T} - \hat{\mu}_T \left( \hat{Y}_T P_T^{\theta_T} \hat{p}_t^{\theta_T}, \hat{Y}_T; \xi_T \right) \right] \right\}.$$ 

Log-linearizing this equation, and solving for $\hat{p}_t^* \equiv \log (p_t^*/P_t)$, we obtain

$$\hat{p}_t^* = \frac{1 - \alpha \beta}{1 + \omega \theta} \sum_{T=t}^{\infty} (\alpha \beta)^{T-t} E_t [\hat{s}_T + \mu_T] + \sum_{T=t+1}^{\infty} (\alpha \beta)^{T-t} E_t \pi_{T+1},$$

where $\hat{s}_t$ is the deviation from its steady-state value of the log of the average real marginal cost, i.e., the real marginal cost for a good with output $y_t(z) = \hat{Y}_t$. Quasi-differentiating the last equation, and using (76), we get

$$\pi_t = \zeta (\hat{s}_t + \mu_t) + \beta E_t \pi_{T+1}$$

(77)

where $\zeta \equiv \frac{(1 - \alpha)(1 - \alpha \beta)}{\alpha (1 + \omega \theta)} > 0$. Log-linearizing the real marginal cost function (71), and using (73), we can express the deviation of the log of the real marginal cost of firm $z$ from its steady-state value, $\hat{s}_t(z) \equiv \log (s_t \mu)$ as

$$\hat{s}_t(z) = \omega \tilde{y}_t(z) + \sigma Y_t - (\omega + \sigma) Y^n_t - \mu_t,$$

where $\tilde{y}_t(z) \equiv \log (y_t(z) / \hat{Y})$. It follows that the deviation of average marginal cost satisfies

$$\hat{s}_t = (\omega + \sigma) (Y_t - Y^n_t) - \mu_t.$$ 

Combining this with (77) yields finally an aggregate supply function of the form (2), where $\kappa \equiv \zeta (\omega + \sigma) > 0$. 

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6.2 Proofs


In this appendix, we show that (32) is the unique optimal policy rule in the family (31), for at least the baseline parametrization, when the exogenous perturbations evolve according to $e_{t+1} = Te_t + e_{t+1}$, where $E_t e_{t+j} = 0$, $\forall j > 0$. We know that there exists at least one policy rule in the family (31) that is consistent with the optimal plan. We need to show that there is no more than one such policy rule.

First, note that with the above law of motion for the shocks, the solution (15), (16) reduces to

$$
\begin{bmatrix}
\hat{\pi}_t \\
\hat{x}_t \\
\hat{i}_t
\end{bmatrix}
= 
\begin{bmatrix}
k'_\pi \\
k'_x \\
k'_i
\end{bmatrix}
\xi_t
$$

(78)

$$
\xi_{t+1} = \Xi \xi_t + \theta e_{t+1},
$$

(79)

where $\xi_t = [\phi'_t, e'_{t+1}]'$, and $\Xi, \theta$ are matrices, and $k_\pi, k_x, k_i$ are vectors. Using (78) and (79) to substitute for the endogenous variables in (31), we can rewrite (31) as

$$
0 = \left\{ [\psi_x k'_\pi + \psi_x k'_x - k'_i] \theta \right\} e_t + \left\{ [(\psi_x k'_\pi + \psi_x k'_x - k'_i) \Xi + (\psi_{i1} k'_i - \psi_x k'_x)] \theta \right\} e_{t-1} + \left\{ (\psi_x k'_x + \psi_x k'_x - k'_i) \Xi^2 + (\psi_{i1} k'_i - \psi_x k'_x) \Xi + \psi_{i2} k'_i \right\} \xi_{t-2} + \left\{ \psi_{0} + \psi_{0, \pi} \psi_{0} + (\psi_{i1} + \psi_{i2} - 1) i^{op} \right\}.
$$

The policy rule (31) is consistent with the optimal plan (for each possible realization of the exogenous shocks) if and only if each term in curly brackets is a zero vector. These
restrictions can be written as

\[
\begin{align*}
\vartheta' k_\pi \psi_\pi + \vartheta' k_x \psi_x &= \vartheta' k_i \\
\vartheta' \Xi' k_\pi \psi_\pi + \vartheta' (\Xi' - I) k_x \psi_x + \vartheta' k_i \psi_{i1} &= \vartheta' \Xi' k_i \\
\Xi'^2 k_\pi \psi_\pi + \Xi' (\Xi' - I) k_x \psi_x + \Xi' k_i \psi_{i1} + k_i \psi_{i2} &= \Xi'^2 k_i \\
\pi^{op} \psi_\pi + i^{op} \psi_{i1} + i^{op} \psi_{i2} + \psi_0 &= i^{op},
\end{align*}
\]

or, in matrix form, as

\[ C \psi = B, \]

where \( \psi = [\psi_\pi, \psi_x, \psi_{i1}, \psi_{i2}, \psi_0]' \). As \( \psi \) is a vector of dimension 5, and the rank of \( C \) is 5 in the baseline parametrization (and for all other parameter combinations that we tried), there is no more than one vector \( \psi \) that satisfies the previous equation. Thus, the policy rule (32) is the unique optimal policy rule in the family (31), for (at least) the baseline parametrization.

### 6.2.2 Determinacy with a Taylor Rule

Here we establish conditions for determinacy of the rational expectations equilibrium when monetary policy is set according to a Taylor rule.

**Proposition 1** When \( \sigma, \kappa > 0, 0 < \beta < 1 \), and the policy coefficients satisfy \( \psi_\pi, \psi_x \geq 0 \), equations (4), (5) and (33) result in a unique bounded rational expectations equilibrium \( \{\pi_t, x_t, i_t\} \), if and only if

\[
\psi_\pi + \frac{1 - \beta}{\kappa} \psi_x > 1.
\]

**Proof.** First, note that for any policy coefficients \( \psi_\pi, \psi_x, \) and \( \psi_0 \), the system formed by (4), (5) and (33) results in a unique steady state \( \bar{\pi}, \bar{x}, \) and \( \bar{i} \) obtained by solving (4),
(5) and (33), maintaining the shocks $r_t^e = u_t = 0$ at all times. This steady state is given by

$$\bar{\pi} = -\psi_0 \left( \psi_\pi + \frac{1-\beta}{\kappa} \psi_x - 1 \right)^{-1}, \quad \text{and} \quad \bar{x} = -\frac{1-\beta}{\kappa} \psi_0 \left( \psi_\pi + \frac{1-\beta}{\kappa} \psi_x - 1 \right)^{-1}.$$ 

The structural equations (4), (5) and the Taylor rule (33) can then be rewritten in terms of deviations from the steady state. The Taylor rule becomes

$$\hat{\pi}_t = \psi_\pi \hat{\pi}_t + \psi_x \hat{x}_t,$$  

where $\hat{\pi}_t \equiv \pi_t - \bar{\pi}$, $\hat{\pi}_t \equiv \pi_t - \bar{\pi}$, and $x_t \equiv x_t - \bar{x}$. Using this to substitute for the interest rate in the structural equations (4) and (5), we can rewrite the resulting difference equations as follows

$$E_t \begin{bmatrix} \hat{\pi}_{t+1} \\ \hat{x}_{t+1} \end{bmatrix} = \tilde{A} \begin{bmatrix} \hat{\pi}_t \\ \hat{x}_t \end{bmatrix} + \bar{a} \begin{bmatrix} r_t^e \\ u_t \end{bmatrix}$$  

where the matrix $\tilde{A}$ is given by

$$\tilde{A} = \begin{bmatrix} \beta^{-1} & -\kappa \beta^{-1} \\ \sigma^{-1} (\psi_\pi - \beta^{-1}) & (1 + \sigma^{-1} (\frac{\kappa}{\pi} + \psi_x)) \end{bmatrix}.$$  

The characteristic polynomial associated to $\tilde{A}$ is

$$P(X) = X^2 + a_1 X + a_0$$  

where

$$a_0 \equiv \beta^{-1} + \frac{\psi_x + \kappa \psi_\pi}{\beta \sigma}, \quad a_1 \equiv -\left( 1 + \beta^{-1} + \frac{\kappa + \beta \psi_x}{\beta \sigma} \right).$$  

Since $\hat{\pi}_t$ and $\hat{x}_t$ are non-predetermined variables, the system (82) admits a unique bounded solution $\{\hat{\pi}_t, \hat{x}_t\}$ if and only if the characteristic polynomial $P(X)$ has both roots outside
the unit circle. Following proposition 1 of Woodford (2000c), the characteristic polynomial \( P(X) \) has both roots outside the unit circle if and only if either (case 1)

\[
a_0 > 1, \quad P(-1) > 0, \quad \text{and} \quad P(1) > 0
\]

or (case 2)

\[
P(-1) < 0, \quad \text{and} \quad P(1) < 0.
\]

Using the expressions for \( a_0 \) and \( a_1 \), we note that when \( \psi_\pi \) and \( \psi_x \) are non-negative, we have \( a_0 > 1 \), and

\[
P(-1) = (\kappa + 2\sigma (1 + \beta) + \psi_x (1 + \beta) + \kappa \psi_x) / (\beta \sigma) > 0.
\]

It follows that when \( \psi_\pi, \psi_x \geq 0 \), the polynomial \( P(X) \) has both roots outside the unit circle if and only if

\[
P(1) = \frac{\kappa \psi_\pi + \psi_x (1 - \beta) - \kappa}{\beta \sigma} > 0.
\]

Thus when \( \psi_\pi, \psi_x \geq 0 \), (82) admits a unique bounded solution if and only if (80) holds.

Using (81), \( \hat{t} \) is uniquely determined whenever \( \hat{\pi}_t \), and \( \hat{x}_t \) are. Therefore the Taylor rule results in a determinate equilibrium if and only if (80) holds.

\[ \]
Proof. First recall that using (44) to substitute for $\hat{i}_t$ in the intertemporal IS equation, we can rewrite (4), (5) and (6) (expressed in terms of hatted variables) in matrix form as:

$$
E_t \begin{bmatrix} \hat{\pi}_{t+1} \\ \hat{x}_{t+1} \\ \hat{p}_t \\ \hat{p}_{t-1} \end{bmatrix} = \hat{A} \begin{bmatrix} \hat{\pi}_t \\ \hat{x}_t \\ \hat{p}_t \\ \hat{p}_{t-1} \end{bmatrix} + \hat{a} \begin{bmatrix} r^e \\ u_t \end{bmatrix}
$$

(84)

where the matrix $\hat{A}$ is given by

$$
\hat{A} = \begin{bmatrix}
\beta^{-1} & -\kappa \beta^{-1} & 0 \\
\sigma^{-1} (\psi_p - \beta^{-1}) & \left(1 + \sigma^{-1} \left(\frac{\kappa}{\beta \sigma} + \psi_x\right)\right) & \psi_p \sigma^{-1} \\
1 & 0 & 1
\end{bmatrix}
$$

The characteristic polynomial associated to $\hat{A}$ is

$$
P(X) = X^3 + A_2 X^2 + A_1 X + A_0
$$

where

$$
A_0 = -\frac{\sigma + \psi_x}{\beta \sigma}, \\
A_1 = \frac{\kappa + \sigma \beta + 2 \sigma + \psi_x \beta + \kappa \psi_p + \psi_x}{\beta \sigma}, \\
A_2 = -2 \sigma \beta + \sigma + \kappa + \psi_x \beta
$$

The system (84) results in a determinate equilibrium if and only if the characteristic polynomial $P(X)$ admits two roots outside and one root inside the unit circle. Using proposition 2 of Woodford (2000c), $P(X)$ has one root inside the unit circle and two roots outside if

$$
P(1) > 0, \quad P(-1) < 0, \quad \text{and} \quad |A_2| > 3.
$$

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Assume that $\psi_p$ and $\psi_x$ satisfy (83). This implies

\[
\begin{align*}
P(1) &= \frac{\kappa}{\beta \sigma} \psi_p > 0 \\
P(-1) &= -\frac{2 \kappa + 4 \sigma (1 + \beta) + \kappa \psi_p + 2 (1 + \beta) \psi_x}{\beta \sigma} < 0 \\
|A_2| &= 2 + \beta^{-1} + \frac{\kappa + \psi_x \beta}{\beta \sigma} > 3.
\end{align*}
\]

Hence $P(X)$ has exactly 2 roots outside the unit circle, and (84) results in a determinate equilibrium.

6.2.4 Proof of Proposition 1

In this proof, we compute the loss criterion (7) for the optimal Taylor rule, and for some particular Wicksellian rule that will turn out to be (63). We then show that the loss resulting from the optimal Taylor rule is higher than the one resulting from the particular Wicksellian rule. First note that since the optimal steady-state is the same for both families of rules, it is sufficient to compare the loss $E[\hat{L}_0]$ resulting from deviations from the steady-state.

6.2.4.1 Loss for optimal Taylor rule. When $\kappa = 0$, and $\rho_r = \rho_u = \rho$ the equilibrium resulting from the optimal Taylor rule, i.e., the optimal non-inertial equilibrium characterized in (27) – (29), reduces to

\[
\begin{align*}
\pi_r &= 0, \quad \pi_u = (1 - \beta \rho)^{-1} \\
x_r &= \frac{\lambda_i \sigma (1 - \rho) (1 - \beta \rho)^2}{h}, \quad x_u = \frac{\lambda_i \sigma (1 - \rho) (1 - \beta \rho) \rho}{h} \\
i_r &= \frac{\lambda_x (1 - \beta \rho)^2}{h}, \quad i_u = \frac{\lambda_x (1 - \beta \rho) \rho}{h}
\end{align*}
\]
where \( h \equiv \lambda_i \sigma^2 (1 - \rho)^2 (1 - \beta \rho)^2 + \lambda_x (1 - \beta \rho)^2 > 0 \). It results from (6) that \( p_r = 0 \), \( p_u = (1 - \beta \rho)^{-1} \) and \( p_p = 1 \), in this equilibrium. Using these expressions to substitute for the equilibrium coefficient in the loss function (62), we obtain\(^{23}\)

\[
E \left[ L^\text{tr}_0 \right] = \frac{\lambda_i \lambda_x}{\lambda_i \sigma^2 (1 - \rho)^2 + \lambda_x} \text{var} (r^*_t) + \frac{\lambda_x (1 + \lambda_i \rho^2) + \lambda_i \sigma^2 (1 - \rho)^2}{\left( \lambda_i \sigma^2 (1 - \rho)^2 + \lambda_x \right) (1 - \beta \rho)^2} \text{var} (u_t). \tag{85}
\]

### 6.2.4.2 Loss for some particular Wicksellian rule.

In the case in which \( \kappa = 0 \) and \( \rho_r = \rho_u = \rho > 0 \), the restrictions (50) – (55) and (61) constraining the equilibrium resulting from any Wicksellian rule (44) can be solved in terms of \( x_r, x_u \) to yield:

\[
p_r = 0, \quad p_u = \frac{1}{1 - \beta \rho}, \quad p_p = 1 \tag{86}
\]

\[
x_p = \frac{x_u (1 - \beta \rho) - x_r \rho}{1 + x_r \sigma \rho} \tag{87}
\]

\[
i_r = 1 - x_r \sigma (1 - \rho), \quad i_u = \rho \frac{(1 - x_r \sigma (1 - \rho)) (1 + x_u \sigma (1 - \beta \rho))}{(1 + x_r \sigma \rho) (1 - \beta \rho)}, \quad i_p = 0. \tag{88}
\]

There is also a second solution which is not admissible as it involves \( p_p = \beta^{-1} > 1 \), hence an explosive price level (in terms of deviations from a trend). Thus the only admissible solution is (86) – (88). Consider now an equilibrium in which \( x_r \) and \( x_u \) satisfy

\[
x_r = \frac{\lambda_i \sigma (1 - \rho)}{\lambda_i \sigma^2 (1 - \rho)^2 + \lambda_x} \tag{89}
\]

\[
x_u = \lambda_i \sigma \rho \frac{\left( \lambda_i \sigma^2 (1 - \rho)^2 + \lambda_x \right) \beta (1 - \beta \rho^2) - \lambda_x \rho (1 - \beta)}{(1 - \beta \rho) \Delta} \tag{90}
\]

where

\[
\Delta \equiv \left( \lambda_i \sigma^2 (1 - \rho)^2 + \lambda_x \right) \left( \lambda_i \sigma^2 (1 - \beta \rho^2) (1 - \beta \rho) + \lambda_x (1 + \beta \rho) \right),
\]

\(^{23}\)Note that the equilibrium characterized here is a special case of the equilibrium (47) used to derive the loss function (62).
and where (86) – (90) are used to compute the remaining coefficients. The latter are given by (86) and

\[
x_p = -\lambda_i \sigma^2 \rho \frac{(1 - \rho)^2 + \lambda_x (1 - \beta \rho^2) (1 - \beta)}{\Delta}
\]

\[
i_r = \frac{\lambda_x}{\lambda_i \sigma^2 (1 - \rho)^2 + \lambda_x}
\]

\[
i_u = \rho \lambda_x \frac{(1 - \beta \rho^2) (1 - \rho)}{(1 - \beta \rho)} (1 - \bar{\gamma}) + \lambda_x (1 + \beta \rho)
\]

\[
i_p = 0
\]

Expressions (89) and (90 do in general not correspond to the values of \(x_r\) and \(x_u\) that would minimize the loss criterion (62). (In fact they would minimize (62) in the special case \(\Omega_u = 0\).) As a result this equilibrium is not the optimal equilibrium that might be obtained with a Wicksellian rule. We focus here on a suboptimal equilibrium because it is easier to characterize analytically. We will call this equilibrium a “quasi-optimal equilibrium”. One implication of course is that the resulting loss, \(E[\hat{L}^q_{wr}]\), cannot be smaller than the one obtained in the optimal equilibrium, \(E[\hat{L}^o_{wr}]\), so that

\[
E[\hat{L}^q_{wr}] \geq E[\hat{L}^o_{wr}].
\]

The Wicksellian rule of the form (44) that implements this quasi-optimal equilibrium is obtained by using (59) and (60). Substituting for the above equilibrium coefficients in (59) and (60) yields:

\[
\psi_p = \frac{\rho \lambda_x (1 - \beta) (1 - \beta \rho^2)}{(1 - \rho)(\lambda_i \sigma^2 (1 - \beta \rho^2) (1 - \beta\rho) + \lambda_x (1 + \beta \rho))}
\]

\[
\psi_x = \frac{\lambda_x}{\lambda_i \sigma (1 - \rho)}
\]

The Wicksellian rule considered is therefore (63). Notice that since \(\psi_p > 0\) and \(\psi_x > 0\), it
follows from proposition 2 (in Appendix 6.2.3) that this rule results in a unique bounded equilibrium.

Next, substituting the above equilibrium coefficients in the loss criterion (62), we obtain

\[
E \left[ \hat{L}_{0}^{qwr} \right] = \text{var} \left( r_{t}^{q} \right) \left( \lambda_{r} x_{r}^{2} + \lambda_{t} t_{t}^{2} \right) + \text{var} \left( u_{t} \right) \times \left( \frac{1}{(1 - \beta \rho)^2} + \lambda_{x} \left( x_{u}^{2} + 2 x_{u} x_{p} \frac{\beta \rho}{(1 - \beta \rho)^2} + x_{p}^{2} \frac{\beta (1 + \beta \rho)}{(1 - \beta \rho)^3 (1 - \beta)} \right) + \lambda_{t} t_{u}^{2} \right) \\
= \text{var} \left( r_{t}^{r} \right) \frac{\lambda_{t} \lambda_{x}}{\lambda_{t} \sigma^{2} (1 - \rho)^2 + \lambda_{x}} + \text{var} \left( u_{t} \right) \times \left( \frac{1}{(1 - \beta \rho)^2} + \lambda_{t} \lambda_{x} \rho^{2} \frac{\lambda_{t} \sigma^{2} \beta (1 - \rho)^2 (1 - \beta \rho^2) + \lambda_{x} \left( 1 - \beta^2 \rho^2 \right)}{(1 - \beta \rho)^3 \Delta} \right). \tag{91}
\]

6.2.4.3 Comparing the losses. Substracting finally (91) from (85), we obtain after some algebraic manipulations:

\[
E \left[ \hat{L}_{0}^{tr} \right] - E \left[ \hat{L}_{0}^{qwr} \right] = \frac{\lambda_{x} \lambda_{r} \sigma^{2} (1 - \beta \rho^{2})^{2} (1 - \beta \rho^{2})}{(1 - \beta \rho) \left( \lambda_{r}^{2} \sigma^{4} (1 - \beta \rho^2) (1 - \beta \rho^2) + \lambda_{t} \sigma^{2} \lambda_{u} \rho^{2} (1 - \beta \rho^2) + 2 (1 - \beta \rho^2) (1 - \rho) + \lambda_{x}^{2} (1 + \beta \rho) \right)} > 0.
\]

Thus

\[
E \left[ \hat{L}_{0}^{tr} \right] > E \left[ \hat{L}_{0}^{qwr} \right] \geq E \left[ \hat{L}_{0}^{wr} \right],
\]

which completes the proof. ■
Table 2: Statistics and Optimal Policy Rules

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Notes: the gray cells denote cases in which the policy rule results in an indeterminate equilibrium. The estimated historical rule is taken from Judd and Rudebusch (1998).
Figure 1a: Impulse response functions to a temporary shock to $r_0^e$ ($\rho_r = 0$)

Notes: solid lines (—) correspond to the optimal plan; dashed lines (---) correspond to the optimal non-inertial plan; dashed-dotted lines (---) correspond to the optimal Wicksellian rule; the dotted line (·) indicates the expected path of $r_t^e$. 

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Figure 1b: Impulse response functions to a temporary shock to $r^e_0$ ($\rho_e = 0.35$)

Notes: solid lines (—) correspond to the optimal plan; dashed lines (---) correspond to the optimal non-inertial plan; dashed-dotted lines (----) correspond to the optimal Wicksellian rule; the dotted line (··) indicates the expected path of $r^e_t$. 

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Figure 1c: Impulse response functions to a temporary shock to $r_0^e$ ($\rho_r = 0.9$)

Notes: solid lines (—) correspond to the optimal plan; dashed lines (---) correspond to the optimal non-inertial plan; dashed-dotted lines (– –) correspond to the optimal Wicksellian rule; the dotted line (· ·) indicates the expected path of $r_t^e$. 

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Figure 2a: Impulse response functions to a temporary shock to $u_0$ ($\rho_u = 0$)

Notes: solid lines (---) correspond to the optimal plan; dashed lines (---) correspond to the optimal non-inertial plan; dashed-dotted lines (-- · --) correspond to the optimal Wicksellian rule.
Figure 2b: Impulse response functions to a temporary shock to $u_0$ ($\rho_u = 0.35$)

Notes: solid lines (—) correspond to the optimal plan; dashed lines (—−) correspond to the optimal non-inertial plan; dashed-dotted lines (− ⋅ −) correspond to the optimal Wicksellian rule.
Figure 2c: Impulse response functions to a temporary shock to $u_0$ ($\rho_u = 0.9$)

Notes: solid lines (—) correspond to the optimal plan; dashed lines (---) correspond to the optimal non-inertial plan; dashed-dotted lines (---) correspond to the optimal Wicksellian rule.
Response to inflation

Response to the output gap

Figure 3: Response to a permanent increase in inflation and in the output gap
Figure 4: Optimal Taylor rules in $(\psi, \psi_x)$ space

Note: gray region indicates region of equilibrium indeterminacy.
Figure 5: Optimal Wicksellian rules in \((\psi_p, \psi_x)\) space
Figure 6a: Statistics and policy rules as functions of $\rho_u$ ($\rho_r = 0$)

Notes: solid lines (—) correspond to the optimal rule; dashed lines (---) correspond to the optimal Taylor rule; dashed-dotted lines (----) correspond to the optimal Wicksellian rule. Other coefficients are set as following: $\text{var}(r^e) = 13.8266$, $\text{var}(u) = 0.16652$. 
Figure 6b: Statistics and policy rules as functions of $\rho_u$ ($\rho_r = 0.35$)

Notes: solid lines (—) correspond to the optimal rule; dashed lines (—–) correspond to the optimal Taylor rule; dashed-dotted lines (—.-) correspond to the optimal Wicksellian rule. Other coefficients are set as following: $\text{var}(\epsilon) = 13.8266$, $\text{var}(u) = 0.16652$. 

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Figure 6c: Statistics and policy rules as functions of $\rho_u$ ($\rho_r = 0.9$)

Notes: solid lines (—) correspond to the optimal rule; dashed lines (--) correspond to the optimal Taylor rule; dashed-dotted lines (¡¢¡) correspond to the optimal Wicksellian rule. Other coefficients are set as following: $\text{var}(r^e) = 13.8266$, $\text{var}(u) = 0.16652$. 
Figure 6d: Statistics and policy rules as functions of var($r^e$)

Notes: solid lines (—) correspond to the optimal rule; dashed lines (---) correspond to the optimal Taylor rule; dashed-dotted lines (---) correspond to the optimal Wicksellian rule. Other coefficients are set as following: $\rho_r = 0.35$, $\rho_u = 0.35$, var($u$) = 0.16652.
Figure 6e: Statistics and policy rules as functions of var(u)

Notes: solid lines (—) correspond to the optimal rule; dashed lines (---) correspond to the optimal Taylor rule; dashed-dotted lines (····) correspond to the optimal Wicksellian rule. Other coefficients are set as following: $\rho_r = 0.35$, $\rho_u = 0.35$, $\text{var}(r^e) = 13.8266$. 

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Figure 6f: Statistics and policy rules as functions of $\lambda_x$

Notes: solid lines (---) correspond to the optimal rule; dashed lines (----) correspond to the optimal Taylor rule; dashed-dotted lines (---) correspond to the optimal Wicksellian rule. Other coefficients are set as following: $\rho_r = 0.35$, $\rho_u = 0.35$, $\text{var}(r^c) = 13.8266$, $\text{var}(u) = 0.16652$. 
Chapter 2

Does Model Uncertainty Justify Caution? Robust Optimal Monetary Policy in a Forward-Looking Model
1 Introduction

A considerable recent literature has sought to characterize desirable monetary policies in terms of interest-rate feedback rules, i.e., guides for setting at each period the policy instrument, such as the Federal funds rate in the U.S., in response to economic conditions. Many computations of optimal policy rules in the context of one or another econometric model — such as those collected in Taylor (1999a) — imply that an optimal rule would involve stronger responses of the Federal funds rate to fluctuations in inflation (and perhaps also in output) than are implied by estimated Fed reaction functions, or by Taylor’s (1993) much-discussed characterization of recent Fed policy. However the specific equations favored by various authors are still significantly different. This raises the question of how a policy rule should be selected in the face of uncertainty about the correct model of the economy.

A common intuition first proposed by Brainard (1967) is that parameter uncertainty should lead one to choose a more “cautious” policy: policymakers should compute the optimal change of their instrument as if they knew the functioning of the economy with certainty, and then move their instrument by less (see Blinder, 1998).\(^1\) Some commentators have therefore proposed that the strong responses of the instrument required by optimal policy in the context of an econometric model depend upon assuming that estimated model coefficients are known to be true, whereas taking proper account of one’s actual uncertainty about the true coefficients should justify gentler responses, perhaps

\(^1\)This result holds in Brainard’s model in particular when the exogenous disturbances and the parameters that relate the instrument of policy to the target variable are not too strongly correlated.
closer to current policy.²

This paper seeks to formally evaluate this argument. We seek to characterize optimal monetary policy rules that are robust to uncertainty about the proper model of the economy when all of the models considered are similar, though not identical. This can be modeled as uncertainty about the parameters that numerically specify the economic model. Uncertainty of this kind necessarily exists in practice, as researchers don’t know with certainty all parameters of their model. In contrast to the standard Bayesian approach followed by, e.g., Brainard (1967), Chow (1975), Clarida et al. (1999), Rudebusch (2000), we assume that the policymaker has multiple priors about the probability distribution of the true model, and that he is uncertainty-averse. It results that the best policy rule is a robust optimal monetary policy rule of the kind advocated recently by Sargent (1999), Hansen and Sargent (1999, 2000b), Stock (1999) and Onatski and Stock (2001). Such a rule is designed to avoid an especially poor performance of monetary policy in the event of an unfortunate parameter configuration, and guarantees to yield an acceptable performance in the specified range of models.³

We propose a method to characterize robust optimal policy rules in a broad class of models. While most studies of optimal policy in the face of model uncertainty focus on backward-looking models, we use our method to determine robust optimal policy rules in a simple forward-looking macroeconomic model. As in Woodford (1996, 1999c), the model

³A number of other recent papers have also looked for policy rules that work well across a range of models, though they do not try to actually find the optimal robust rule (see for example McCallum, 1988, 1999; Levin et al., 1999a; Christiano and Gust, 1999; and Taylor, 1998, 1999b).
is composed of a monetary policy rule and two structural equations – an intertemporal IS equation and an aggregate supply equation – that are based on explicit microeconomic foundations. Because it can be derived from first principles, the model is not subject to the famous Lucas (1976) critique for the evaluation of policy. An important property of this model is that the policymaker faces a trade-off between the stabilization of inflation and the output gap on one hand, and the nominal interest rate on the other hand.

A comparison of the robust optimal rule to the optimal policy in the absence of uncertainty allows us to determine whether Brainard’s (1967) result generalizes to the class of models considered here. In contrast to the “conventional wisdom,” we obtain that robust optimal monetary policy commands in general a stronger response of the interest rate to fluctuations in goal variables such as inflation and output gap than is the case in the absence of uncertainty. In fact, model uncertainty affects the trade-off facing the policymaker in a way that places more weight on the stabilization of inflation and the output gap, and relatively less weight on the stabilization of the nominal interest rate. This is because the robust optimal rule, which is designed to perform well in those instances in which exogenous shocks have particularly large effects on the goal variables, requires the interest rate to respond by enough to guarantee that exogenous perturbations have only a limited effect on the economy. It is therefore far from clear that model uncertainty can provide a justification for the kind of policies implied by estimates
of current policy.\footnote{Aoki (1998) derives the optimal time-consistent policy in a model very similar to ours when the structural parameters are known with certainty but when inflation and output are subject to measurement errors. He shows that measurement errors lead to less active monetary policy. Orphanides (1998) obtains a similar conclusion in a different model.}

Similar results have been obtained recently with a different approach and in other frameworks, by Sargent (1999), Stock (1999), Hansen and Sargent (2000b), Söderström (2000a), Kasa (2001), and Onatski and Stock (2001). While Sargent (1999) applies robust control theory to a backward-looking model, Hansen and Sargent (2000b) apply it to an optimal monetary policy problem that is similar (except for the nature of model uncertainty) to the one treated here. However they specify a broad, nonparametric set of additive model perturbations that represent deviations of the model actually used from the true model, and bound uncertainty in terms of a bound upon the possible size of this additive term. We assume instead uncertainty about the values of coefficients of the linear equations of the structural model. This type of uncertainty seems to us more intuitive, and it seems more likely that modelers should be able to quantify their degree of confidence in that way. We furthermore obtain an analytical characterization of the robust optimal policy, which helps us to clarify the circumstances under which robust optimal policy is more aggressive than the policy obtained in the absence of model uncertainty. Stock (1999) and Onatski and Stock (2001) study a type of uncertainty that is similar to ours. These authors, however, determine robust optimal rules in the backward-looking model of Rudebusch and Svensson (1999), while we consider a forward-looking model. Kasa (2001) also seeks to characterize robust policies in a forward-looking
model, but uses a frequency domain approach instead of a time domain approach.\(^5\)

The rest of the paper is organized as follows. Section 2 reviews the baseline model in the absence of model uncertainty. In section 3, we introduce model uncertainty and explain how this affects the objective of monetary policy. We next propose a solution procedure to derive the robust optimal policy rule in a general class of models. In section 4, we apply our solution procedure to characterize analytically robust optimal “Taylor rules” in the model of section 2. We conclude in section 5.

2 Monetary Policy in a Simple Optimizing Model with Known Parameters

This section reviews the monetary policy design problem in a formal model that can be derived from first principles, when the model parameters are known with certainty. Our baseline framework is taken from Woodford (1996, 1999b, 1999c).\(^6\) We first describe the model that characterizes the behavior of the private sector, and then turn to monetary policy.

\(^5\)Following a Bayesian approach, Söderström (2000a) shows that uncertainty about the persistence of inflation induces the policymaker to respond more aggressively to shocks, in the model due to Svensson (1997).

\(^6\)This model is similar to other small dynamic macroeconomic models that have been used in recent studies of monetary policy such as Kerr and King (1996), Bernanke and Woodford (1997), Goodfriend and King (1997), Kiley (1998), McCallum and Nelson (1999a, 1999b), and Clarida et al. (1999). It is also a simplified version of the econometric model of Rotemberg and Woodford (1997, 1999).
2.1 A Simple Structural Model

Apart from the monetary policy rule to be discussed below, Woodford’s model consists of two structural equations that can be derived as log-linear approximations to equilibrium conditions of an underlying dynamic general equilibrium model with sticky prices. The intertemporal IS equation, which relates spending decisions to the interest rate, is given by

\[ x_t = E_t x_{t+1} - \sigma^{-1} (i_t - E_t \pi_{t+1} - r^n_t), \]  

(1)

and the aggregate supply equation (or expectational Phillips curve) is given by

\[ \pi_t = \kappa x_t + \beta E_t \pi_{t+1}, \]  

(2)

where \( x_t \) denotes the output gap (defined as the deviation of output from its natural level, i.e., the equilibrium level of output under flexible prices), \( \pi_t \) is the inflation rate, and \( i_t \) is the deviation of the short-term nominal interest rate from its steady-state value.\(^7\) The composite exogenous disturbance \( r^n_t \) represents Wicksell’s “natural rate of interest”, i.e., the real interest rate that equates output to its natural level, or alternatively the interest rate that would prevail in equilibrium under flexible prices (see Blinder, 1998, chap. 2, and Woodford, 1999b, 1999c). Perturbations to the natural rate of interest represent all non-monetary disturbances that affect inflation and the output gap. For instance, a temporary increase in \( r^n_t \) could reflect a temporary exogenous increase in aggregate demand, or alternatively, a temporary decrease in the natural level of output. Moreover, as both interest rates enter the structural equations only through the “interest-rate gap”\(^7\)

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\(^7\)All three variables represent percent deviations from their values in a steady state with zero inflation and constant output growth.
(\(i_t - E_t \pi_{t+1}\) - \(r^p_t\), non-monetary perturbations affect inflation and output gap only if the interest rate controlled by the central bank is such that the real interest rate, \(i_t - E_t \pi_{t+1}\), departs from the natural rate of interest.

While (1) can be viewed as a log-linear approximation to the representative household’s Euler equation for optimal timing of consumption in the presence of complete financial markets, (2) can be interpreted as a log-linear approximation to the first-order condition for the supplier’s optimal price-setting decision. All variables are assumed to be bounded.\(^8\) The structural parameters \(\sigma\) and \(\kappa\) are both positive by assumption. The parameter \(\sigma\) represents the inverse of the intertemporal elasticity of substitution (\(-\sigma\) is the slope of the intertemporal IS curve), and \(\kappa\), which is the slope of the short run aggregate supply curve, can be interpreted as a measure of the speed of price adjustment. Finally, \(\beta \in (0, 1)\) may be interpreted as the time discount factor of the price-setters, which is assumed to be the same as the discount factor of the representative household.

Rotemberg and Woodford (1997) have shown that an estimated model similar to the one considered here (but slightly more complicated) provides a very good description of the actual behavior of inflation, output, and the quarterly average of the Federal funds rate in the U.S. between 1979 and 1995, in that it is able to replicate accurately the responses of the three endogenous variables to a monetary shock.\(^9\) Their estimated structural parameters are given in Table 1 below. They will be used here, as in Woodford

\(^8\)The structural equations (1), (2) provide an accurate approximation to the exact equilibrium conditions in the underlying model only when we restrict our attention to small perturbations around the steady state.

\(^9\)Moreover, an aggregate supply curve of the form (2) has found some empirical support in Roberts (1995), Sbordone (1998), and Gali and Gertler (1999).
(1999c), to “calibrate” the model in the baseline case.

Table 1: “Calibrated” Parameter Values

<table>
<thead>
<tr>
<th>(a) Certainty case: baseline model</th>
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<tbody>
<tr>
<td>Structural parameters</td>
</tr>
<tr>
<td>$\beta$     0.99</td>
</tr>
<tr>
<td>$\sigma$    0.1571 (0.0328)</td>
</tr>
<tr>
<td>$\kappa$    0.0238 (0.0035)</td>
</tr>
<tr>
<td>Shock process</td>
</tr>
<tr>
<td>$\rho$      0.35</td>
</tr>
<tr>
<td>sd($r^n$)   3.718</td>
</tr>
<tr>
<td>Loss function</td>
</tr>
<tr>
<td>$\lambda_x$ 0.0483</td>
</tr>
<tr>
<td>$\lambda_i$ 0.2364</td>
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<table>
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<tr>
<th>(b) Parameter uncertainty</th>
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<tr>
<td>lower bound upper bound</td>
</tr>
<tr>
<td>$\sigma$ 0.0915 0.2227</td>
</tr>
<tr>
<td>$\kappa$ 0.0168 0.0308</td>
</tr>
</tbody>
</table>

Note: standard errors are in parenthesis

2.2 Optimal Monetary Policy

We now turn to the objective of monetary policy. Researchers have traditionally assumed that policymakers should seek to minimize a weighted average of some measure of variability of inflation and of the output gap (see, e.g., Walsh, 1998, chap. 8; Woodford, 1999b; and Clarida et al., 1999, for a recent discussion). In the model considered above, the policymaker can in fact perfectly stabilize inflation and the output gap by setting $i_t = r^n_t$ in every period, so that the interest rate perfectly tracks the exogenous fluctuations in the natural rate of interest. However it may be undesirable to vary the nominal
interest rate as much as the natural rate of interest.\textsuperscript{10} For instance, Friedman (1969) has argued that high nominal interest rates involve welfare costs of transactions. Since it is plausible that the deadweight loss is a convex function of the distortion (see Woodford, 1990, 1999b), it may not only be desirable to reduce the level, but also the variability of nominal interest rates. We shall accordingly assume the following loss criterion\textsuperscript{11}

\[ L_0 = E_0 \left\{ (1 - \beta) \sum_{\tau=0}^{\infty} \beta^\tau \left[ \pi_t^2 + \lambda_x x_t^2 + \lambda_i i_t^2 \right] \right\} \]  

where \( \lambda_x, \lambda_i > 0 \) are weights that the policymaker places on the stabilization of the output gap and the nominal interest rate, and where \( \beta \in (0, 1) \) is the discount factor mentioned above.\textsuperscript{12}

\begin{itemize}
  \item Rotemberg and Woodford (1997) estimate that the standard deviation of the natural rate has been almost ten times as large as the standard deviation of the Federal funds rate, from 1979 to 1995.
  \item Note that the welfare costs due to monetary frictions mentioned above justify the presence of the interest rate in the loss function even if they have no effect on the structural equations, that is even if, e.g., utility is additively separable in real balances, consumption, and goods supply (see Woodford, 1999b).
  \item A similar loss function can also be obtained by performing a second-order Taylor approximation to the expected utility of the representative household in the model that has been used to derive (1) and (2) (see Woodford, 1999b, 1999c). The interest rate’s presence in the loss function results, e.g., from the approximation of transaction frictions modeled by the presence of real balances in the utility function. A similar term appears when one takes into account the fact that the nominal interest rate faces a lower bound at zero. There are additional reasons, from which we abstract, that make volatile interest rates undesirable. Williams (1999), for example, argues that policymakers may dislike reversals in the direction of policy because they fear that such actions would be misinterpreted by the public as mistakes on the part of the monetary authority. Finally, variable interest rates may decrease potential output through higher costs of capital, as a large variance in expected short-term rates has been observed to raise the term premium (Tinsley, 1999).
\end{itemize}
An implication of this loss criterion is that an equilibrium with complete stabilization of inflation and the output gap is not fully efficient. In fact, exogenous fluctuations in the natural rate of interest require variations in the nominal interest rate to stabilize inflation and the output gap. Hence, welfare costs associated to fluctuations in the nominal interest rate introduce a tension between stabilization of inflation and the output gap on one hand, and stabilization of the nominal interest rate on the other hand.

Following recent studies of monetary policy (see for example Taylor, 1999a), we characterize monetary policy in terms of interest-rate feedback rules. Specifically, we assume that the policymaker commits credibly at the beginning of period 0 to a policy rule of the form

\[ i_t = P_t (\pi_t, \pi_{t-1}, ..., x_t, x_{t-1}, ..., i_{t-1}, i_{t-2}, ..., r^n_t, r^n_{t-1}, ...) \] (4)

for each date \( t \geq 0 \). The policymaker’s problem is to determine the functions \( P_t (\cdot) \), \( t = 0, 1, 2, ... \) to minimize the loss \( E[L_0] \) subject to the structural equations (1) and (2). As the objective is quadratic and the constraints are linear in all variables, we may without loss of generality restrict our attention to linear functions \( P_t (\cdot) \).

We denote by \( \psi \) the vector of coefficients that completely characterizes \( \{P_t (\cdot)\}_{t=0}^{\infty} \), and we simply call \( \psi \) a “policy rule”. In practice however we shall always assume that policy rules \( \psi \) are drawn from some finite-dimensional linear space \( \Psi \subseteq \mathbb{R}^n \). We denote by \( \theta = [\theta_1, \theta_2, ..., \theta_m]' \) the finite-dimensional vector of structural parameters of the model, and by \( \Theta \subseteq \mathbb{R}^m \) the set of possible vectors \( \theta \). We also write \( q_t = [\pi_t, x_t, i_t]' \) for the vector of endogenous variables at date \( t \), and \( q \) for the stochastic process \( \{q_t\}_{t=0}^{\infty} \), specifying \( q_t \) at each date as a function of the history of exogenous shocks until that date.

To be feasible, the stochastic process \( q \) needs to satisfy the structural equations (1)
and (2) at all dates \( t \). These can be written compactly as

\[
\tilde{S}(q, \theta) = 0. \tag{5}
\]

Similarly, the restrictions imposed by the commitment to (4) at all dates \( t \geq 0 \) can be written as

\[
\tilde{P}(q, \psi) = 0. \tag{6}
\]

We assume that both \( \tilde{S}(q, \theta) \) and \( \tilde{P}(q, \psi) \) are linear in \( q \). Similarly, the loss function (3) can be denoted by \( L_0(q, \theta) \).\(^{13}\)

A rational expectations equilibrium is then defined as a stochastic process \( q(\psi, \theta) \) satisfying both (5) and (6). In general, many different policy rules may result in the same equilibrium. Some rules may also yield many different equilibria, in which case the set of equilibria always includes some with arbitrarily large fluctuations of the endogenous variables.\(^{14}\) The latter equilibria are therefore arbitrarily bad under the assumed loss criterion, and the policy rules that allow them to occur cannot be optimal in the class of rules \( \tilde{\Psi} \).\(^{15}\) We restrict therefore our attention to a subset \( \Psi \subseteq \tilde{\Psi} \) of policy rules that result in a unique bounded rational expectations equilibrium, and let \( q(\psi, \theta) \) denote this

\(^{13}\)The second argument in \( L_0(q, \theta) \) allows for the possibility that coefficients of the loss function such as \( \lambda_x, \lambda_i \) be functions of elements of the parameter vector \( \theta \).


\(^{15}\)Our concern for choosing a policy rule that doesn’t allow for the worst possible equilibrium to occur is consistent with the approach to robust policy analysis proposed below.
equilibrium. We consider only bounded equilibrium processes, as the structural equations (1) and (2) would not provide a reasonable approximation of the true equilibrium conditions in the underlying model if the endogenous variables were not bounded.

The optimal monetary policy rule that is optimal relative to the subset of rules $\Psi$ can in turn be defined as follows.

**Definition 1** In the case of known structural parameters $\theta$, let $\Psi$ be a set of policy rules such that there is a unique bounded equilibrium. Then an optimal monetary policy rule is a vector $\psi^0$ that solves

$$\min_{\psi \in \Psi} \mathbb{E}[L_0(q(\psi, \theta), \theta)]$$

where the unconditional expectation is taken over all possible histories of the disturbances $\{r^n_t\}$.

Note that the stochastic process $q(\psi, \theta)$ is a specification of the endogenous variables for all possible initial conditions $r^n_0$, as well as for all possible realizations of the exogenous shocks. Since the unconditional expectation in definition 1 is also taken over the exogenous initial states $r^n_0$, monetary policy is evaluated here without reference to any particular initial conditions.

We shall determine such optimal policy rules in section 4. First, in the next section, we describe how the introduction of uncertainty about the structural parameters alters the objective of monetary policy, and we propose a general method for finding the optimal policy rule in the presence of parameter uncertainty.
3 Model Uncertainty and Robust Optimal Monetary Policy: General Framework

In the previous section, the parameters that specify the model are supposed to be constant and known with certainty by all economic agents. The only uncertainty is due to exogenous perturbations to the natural rate of interest. In reality however, central banks and researchers do not know the parameters of their models nor the exogenous disturbances with certainty. They can extract estimates of model parameters from their data sets, but as long the sample is finite, there is no way one can be sure about the value of most structural parameters. This parameter uncertainty may well have an effect on the optimal monetary policy rule. It is precisely this effect that we analyze here.

The underlying framework we have in mind is the same as the one in the model mentioned above, except that the private sector (or representative household) can be one of many different types. We assume that the household’s type is determined in period 0; the household knows its type, but the central bank does not.\textsuperscript{16}

We assume that the policymaker commits at the beginning of period 0 to a policy rule $\psi$ (or equivalently to functions $P_t(\cdot)$ for each date $t \geq 0$). We assume that the commitment to such a rule is credible, and in particular that the policymaker does not revise it at later dates using additional information he might have gathered about unknown model parameters.\textsuperscript{17} Given the (publicly known) policy rule and the struc-

\textsuperscript{16}In contrast, in Sargent (1999), and Hansen and Sargent (1999, 2000b), both the policymaker and the private sector face similar uncertainty with respect to the correct model.

\textsuperscript{17}Note that this formulation also allows for policy rules that involve learning for at least some time on the part of the policymaker, as $\psi$ is only restricted to be finite-dimensional. In the application of section
tural equations describing the behavior of the representative household (which knows the structural parameters), a rational expectations equilibrium can be determined. Since the policymaker does not know the true parameter vector, however, he does not know which equilibrium will realize (for given exogenous disturbances).

3.1 Objective of Monetary Policy with Model Uncertainty

To characterize parameter uncertainty, we assume that the vector $\theta$ of structural parameters lies in a given (known) compact set $\Theta \subset \mathbb{R}^m$, and that the distribution of $\theta$ is unknown. Instead of assuming a particular prior distribution over $\Theta$ and deriving the policy rule that minimizes the expected loss, as is usually done in the standard Bayesian approach, we let the policymaker consider many probability measures over $\Theta$, including the possibility that any given element $\theta \in \Theta$ holds with certainty. Moreover, we assume that the policymaker has aversion towards uncertainty in the sense axiomatized by Gilboa and Schmeidler (1989). It results from Gilboa and Schmeidler (1989) that if the policymaker has multiple priors on $\Theta$, and his preferences satisfy uncertainty aversion in addition to the axioms of standard expected utility theory, the policymaker’s problem is to minimize his loss in the worst-case scenario, i.e., when the prior distribution is the worst distribution in the set of possible distributions. The optimal policy rule is then

4, we shall however restrict our attention to a family of simple rules that involves no learning.

18Since we allow for priors such as any given element $\theta \in \Theta$ holding with certainty, the worst-case scenario, for a given policy rule $\psi^*$, is the parameter vector $\theta^*$ that maximizes the loss $E[\mathcal{L}_0(q(\psi^*, \theta), \theta)]$ on $\Theta$. Note that the worst case described here does not need to be at all close to the absolute worst-case situation which involves an arbitrarily large loss for the policymaker. Indeed, by choosing a set $\Theta$ that is sufficiently small, the worst-case scenario can be made arbitrarily close to the best-case scenario.
the robust rule defined as following.

**Definition 2** Let $\Psi$ be a set of policy rules such that there is a unique bounded equilibrium process $q(\psi, \theta)$ for all $\psi \in \Psi, \theta \in \Theta$. In the case of parameter uncertainty, a **robust optimal monetary policy rule** is a vector $\psi^*$ that solves

$$\min_{\psi \in \Psi} \left\{ \max_{\theta \in \Theta} E[L_0(q(\psi, \theta), \theta)] \right\}$$

(7)

where the unconditional expectation is taken over all possible histories of the disturbances $\{r^n_t\}$.\(^{19}\)

Given that the unknown vector of structural parameters is in $\Theta$, the policymaker can guarantee that the loss is no higher than the one obtained in the following “minmax” equilibrium.

**Definition 3** A **minmax equilibrium** is a bounded rational expectations equilibrium $q^* = q(\psi^*, \theta^*)$, where $\psi^* \in \Psi$ is a robust optimal monetary policy rule and $\theta^*$ maximizes the loss $E[L_0(q(\psi^*, \theta), \theta)]$ on the constraint set $\Theta$.

### 3.2 Robust Optimal Policy Rule: Solution Method

The method that we propose to characterize the robust optimal policy rule can, in principle, be applied to any model in which the feasibility constraints can be expressed as in

\(^{19}\)Note that there is no loss of generality in restricting $\Psi$ to be a set of rules such that $q(\psi, \theta)$ is uniquely defined for all $\psi \in \Psi$, and $\theta \in \Theta$. For if the set of policy rules was a larger set $\hat{\Psi}$, and the policymaker chose a rule $\hat{\psi}$ in $\hat{\Psi}$ but not in $\Psi$, the maximum loss would always be arbitrarily large, so that $\hat{\psi}$ could not possibly be a robust optimal rule.
(5), possible policy rules may be parametrized as in (6), and the robust optimal monetary policy rule solves (7). It is, therefore, not limited to the model presented in section 2. This method is based on the relation between the solution to problem (7) and the equilibrium of a zero-sum two-player game.

Consider the game of pure strategies \( \Gamma = \langle \{ P, N \}, \langle \Psi, \Theta \rangle, (-L(\psi, \theta), L(\psi, \theta)) \rangle \), where \( L(\psi, \theta) \equiv E[L_0(q(\psi, \theta), \theta)] \). In this game, the policymaker (\( P \)) chooses the policy rule \( \psi^* \in \Psi \) to minimize his loss, \( L(\psi, \theta) \), knowing that a malevolent Nature tries to hurt him as much as possible. The other player, Nature (\( N \)), chooses the vector of structural parameters \( \theta^* \in \Theta \) to maximize the policymaker’s loss, knowing that the policymaker is going to minimize it. The Nash equilibrium (NE) of this game is a profile of strategies \( (\psi^*, \theta^*) \) such that

\[
\psi^* \in \arg \max_{\psi \in \Psi} \{-L(\psi, \theta^*)\} = \arg \min_{\psi \in \Psi} L(\psi, \theta^*) \tag{8}
\]

\[
\theta^* \in \arg \max_{\theta \in \Theta} L(\psi^*, \theta) \tag{9}
\]

We shall look for a profile of strategies \( (\psi^*, \theta^*) \) that solves both (8) and (9). If such a profile exists, then the following property of zero-sum games guarantees that the policy rule \( \psi^* \) obtained in the NE is the robust optimal policy rule that we are seeking to determine.

**Proposition 4** Suppose that \( \Gamma \) has a NE. The profile \( (\psi^*, \theta^*) \) is a NE of \( \Gamma \) if and only if the action of each player is a maxminimizer, i.e.,

\[
\psi^* \in \arg \max_{\psi \in \Psi} \left\{ \min_{\theta \in \Theta} (-L(\psi, \theta)) \right\} = \arg \min_{\psi \in \Psi} \left\{ \max_{\theta \in \Theta} L(\psi, \theta) \right\}
\]

\[
\theta^* \in \arg \max_{\theta \in \Theta} \left\{ \min_{\psi \in \Psi} L(\psi, \theta) \right\}.
\]
Proof. See Osborne and Rubinstein (1994), proposition 22.2, parts (a) and (c).

Rather than defining conditions (5) and (6) for general stochastic processes $q$, it is typically convenient to restrict attention to a particular linear subspace of processes that satisfy additional linear constraints besides (5) and (6). These additional constraints do not exclude outcomes that might result from policies in $\bar{\Psi}$, but they restrict all of the equilibria resulting from policies $\psi \in \bar{\Psi}$ so that no optimal plan is infeasible given the class of policies $\bar{\Psi}$ considered. (For example, in section 4, it is assumed that the interest rate is set according to a standard Taylor rule; this implies that an optimal plan cannot be feasible if any of the endogenous variables depends upon lagged variables.)\textsuperscript{20} It is then convenient to parametrize this subspace of possible processes by an alternative parameter vector $f$. The stochastic process corresponding to any parameters $f$ is given by $q(f)$. The restrictions (5) and (6) may then be rewritten as

\begin{align*}
S(f, \theta) &= 0 \quad (10) \\
P(f, \psi) &= 0, \quad (11)
\end{align*}

and the vector $f$ that solves (10) and (11) is given by $f(\psi, \theta)$.

We shall consider in turn the policymaker’s problem (8), for any given vector of structural parameters $\theta \in \Theta$, and Nature’s problem (9). Instead of solving the policymaker’s problem directly, it is convenient to proceed in two steps, as in Woodford (1999c): first, we determine the vector $f^*(\theta)$ parametrizing the feasible equilibrium $q(f^*(\theta))$ that minimizes the loss criterion for any given $\theta \in \Theta$, and second, we look for a policy rule $\psi^*(\theta)$.

\textsuperscript{20}Note however that these constraints do not necessarily eliminate all of the equilibria that result from policies in $\bar{\Psi}\backslash\Psi$, i.e., equilibria that are unbounded or that result from policy rules that allow for multiple rational expectations equilibria (see discussion below).
in the set $\Psi$ that implements this optimal equilibrium. Formally, we first determine $f^* (\theta)$ to minimize

$$\hat{L} (f, \theta) \equiv E [L_0 (q (f), \theta)]$$

subject to the restrictions (10) imposed by the structural equations for any $\theta \in \Theta$. The policymaker’s Lagrangian can thus be written as

$$\mathcal{L}^P (f, \phi; \theta) = \hat{L} (f, \theta) + \phi \cdot \mathbf{S} (f, \theta)$$

where $\phi$ is a row vector of Lagrange multipliers. The solution $f^* (\theta)$ and the optimal Lagrange multipliers $\phi^* (\theta)$ solve the first-order necessary conditions

$$\frac{\partial \hat{L} (f^* (\theta), \theta)}{\partial f} + \phi^* (\theta) \cdot \frac{\partial \mathbf{S} (f^* (\theta), \theta)}{\partial f} = \mathbf{0},$$

and the constraints

$$\mathbf{S} (f^* (\theta), \theta) = \mathbf{0}$$

for all $\theta \in \Theta$. In (14), $\frac{\partial \mathbf{S}}{\partial f}$ refers to the Jacobian matrix with $ij$-element $\frac{\partial \mathbf{S}_{ij}}{\partial f}$. Equations (14) and (15) allow us to determine the optimal equilibrium $q (f^* (\theta))$ for any given $\theta$.

In the second step, we look for a policy rule $\psi^* (\theta) \in \Psi$ that satisfies

$$\mathbf{P} (f^* (\theta), \psi^* (\theta)) = \mathbf{0}.$$ 

If such a policy exists in $\Psi$ (so that it results in a unique equilibrium), then it implements the optimal equilibrium parametrized by $f^* (\theta)$. As made clear in the following lemma,

\[\text{If } q (f) \text{ is linear in } f, \text{ then the objective function } \hat{L} (f, \theta) \text{ is convex in } f \text{ as } E [L_0] \text{ is convex in } q, \text{ and the constraints are all linear in } f. \text{ Thus, the first-order conditions are also sufficient to guarantee that } f^* (\theta) \text{ achieves the desired minimum. However, if } q (f) \text{ is nonlinear in } f, \text{ second-order conditions are necessary to identify local minima, and numerical methods can be used to determine which of these is a global minimum of } \hat{L} (f, \theta).\]
such a policy is the policymaker’s best response to the vector of structural parameters \( \theta \).

In particular, given an equilibrium vector \( \theta^* \), the policy rule \( \psi^* (\theta^*) \) solves (8), and hence is part of a NE.

**Lemma 5** Suppose that \( f^* (\theta) \) minimizes (12) subject to (10) for any given \( \theta \in \Theta \), and that there exists \( \psi^* (\theta) \in \Psi \) that solves (16) for all \( \theta \in \Theta \). Then \( \psi^* (\theta) \in \arg \min_{\psi \in \Psi} L (\psi, \theta) \).

**Proof.** See Appendix 6.1.1. ■

Although policies satisfying (16) are necessarily in the class \( \tilde{\Psi} \) of policy rules – as \( f \) is a parametrization of the subspace of possible processes resulting from policies in \( \tilde{\Psi} \) – they need not be in the set \( \Psi \) of policies that result in a unique bounded equilibrium. Therefore, we shall need to verify that the obtained policy rules is indeed in \( \Psi \).

To characterize the equilibrium structural parameters, we consider \( \theta^* \) that solves (9), or equivalently,

\[
\max_{\theta \in \Theta} \hat{L} (f (\psi^*, \theta), \theta) \tag{17}
\]

for a given policy rule \( \psi^* \in \Psi \). Let us form the Lagrangian for Nature

\[
\mathcal{L}^N (\theta, \mu_1, \mu_2; \psi^*) = \hat{L} (f (\psi^*, \theta), \theta) - \mu_1 \cdot (\theta - \bar{\theta}) + \mu_2 \cdot (\theta - \bar{\theta}) \tag{18}
\]

where \( \mu_1, \mu_2 \) are row vectors of Lagrange multipliers and \( \bar{\theta}, \bar{\theta} \) are some finite vectors satisfying \( [\bar{\theta}, \bar{\theta}] = \Theta \). From Kuhn-Tucker’s theorem, we know that necessary conditions

\footnote{It may happen that all policies that allow for the optimal equilibrium to occur are in \( \tilde{\Psi} \) but not in \( \Psi \), so that they all result in an indeterminate equilibrium. In such situations one could determine policy rules that implement a constrained optimal equilibrium which satisfies additional restrictions upon \( f \) such that all possible policies are in \( \Psi \).}
for $\theta^* \in \Theta$ to solve this problem are given by the first-order conditions

$$
\frac{d\hat{L}(f(\psi^*, \theta^*), \theta^*)}{d\theta} = \mu_1^* - \mu_2^*,
$$

(19)

the complementary slackness conditions

$$
\mu_1^* \cdot (\theta^* - \theta) = 0, \quad \mu_2^* \cdot (\theta^* - \theta) = 0,
$$

(20)

and the requirement that all elements of $\mu_1^*$, $\mu_2^*$ be non-negative. In general, it is difficult
to compute the left-hand side of (19) directly, as it involves differentiation of the vector
$f$ with respect to $\theta$, and it requires knowledge of the optimal policy rule $\psi^*$. Thus, it will
be convenient to rewrite (19), using the following lemma.

Lemma 6 If $\psi^*(\theta)$ is a best response to $\theta$, then

$$
\frac{d\hat{L}(f(\psi^*(\theta), \theta), \theta)}{d\theta} = \frac{\partial L^P(f^*(\theta), \phi^*(\theta); \theta)}{\partial \theta},
$$

(21)

where $L^P(f, \phi; \theta)$ is the policymaker’s Lagrangian (13).

Proof. See Appendix 6.1.2. ■

This lemma implies that the derivative of the loss function at the NE $(\psi^*, \theta^*)$ can be
computed by partially differentiating the policymaker’s Lagrangian (13) with respect to
$\theta$, and setting $\theta, f, \phi$ at their equilibrium values $\theta^*, f^* \equiv f^*(\theta^*), \phi^* \equiv \phi^*(\theta^*)$. Note that
we don’t need to differentiate $f$ with respect to $\theta$ any more. We can thus write (19) as

$$
Z(\theta^*) = \mu_1^* - \mu_2^*,
$$

(22)

where we define

$$
Z(\theta) \equiv \frac{\partial L^P(f^*(\theta), \phi^*(\theta); \theta)}{\partial \theta}.
$$

(23)
Then, the Kuhn-Tucker conditions can be written entirely in terms of $\theta^*, \mu_1^*$, and $\mu_2^*$, which is particularly useful as they no longer require knowledge of the optimal policy rule $\psi^*$.

As (17) is not a concave problem, in general, the first-order conditions (22) and the complementary slackness conditions (20) are necessary but not sufficient to guarantee that the resulting parameter vector $\theta^*$ maximizes the loss criterion. These conditions are useful, however, in restricting the set of possible solution candidates. They allow us to determine a local NE $(\psi^*, \theta^*)$, that is, a situation in which each player’s strategy is at least locally a best response to the other player’s strategy. The following lemma states formally that in a local NE, Nature chooses the highest possible value of a parameter when the loss is increasing in that parameter, while it chooses the lowest possible parameter value when the loss is decreasing.

**Lemma 7** Let $\theta_i^* \in [\theta_i, \bar{\theta}_i]$ be the $i$-th element of $\theta^*$, and let $Z_i^* = \partial L^P (f^*, \phi^*; \theta^*) / \partial \theta_i$ be the corresponding element of $Z (\theta^*)$, for $i = 1, ..., m$. If $\theta^*$ is part of a local NE $(\psi^*, \theta^*)$, then

$$
\theta_i^* = \begin{cases} 
\theta_i, & \text{if } Z_i^* < 0 \\
\bar{\theta}_i, & \text{if } Z_i^* > 0.
\end{cases}
$$

If $Z_i^* = 0$, then $\theta_i^*$ can be any value in $[\theta_i, \bar{\theta}_i]$ that is consistent with $Z_i^* = 0$.

**Proof.** See Appendix 6.1.3. 

To verify that $(\psi^*, \theta^*)$ is not only a local but also a global NE, we need to check that the solution candidate $\theta^*$ is indeed Nature’s best response to the policymaker’s optimal policy $\psi^*$ on the whole constraint set $\Theta$. This can be done by verifying numerically that
there is no vector $\theta^i \in \Theta$ such that

$$L \big( \psi^*, \theta^i \big) > L \big( \psi^*, \theta^* \big)$$

(24)

given the policy rule $\psi^*$.

In summary, our solution strategy involves the four following steps.

1. Optimal equilibrium for given $\theta$. We determine the parametrization $f^* (\theta)$ of the equilibrium process that solves the policymaker’s problem, minimizing the loss (12) subject to (10) for any given $\theta \in \Theta$.

2. Candidate minmax equilibrium. We construct the vector $Z (\theta)$ obtained by partially differentiating the policymaker’s Lagrangian (13) with respect to $\theta$, and use lemma 7 to determine a candidate worst-case parameter vector $\theta^*$. Using the results of step 1, we determine the vector $f^* (\theta^*)$ parametrizing the candidate minmax equilibrium $q \big( f^* (\theta^*) \big)$.

3. Robust optimal policy rule. We look for a policy rule $\psi^*$ that implements the candidate minmax equilibrium, i.e., that solves $P \left( f^* (\theta^*), \psi^* \right) = 0$. We verify that $\psi^* \in \Psi$, i.e., that the policy rule results in a unique bounded equilibrium process $q \left( \psi^*, \theta \right)$ for all $\theta \in \Theta$.

4. Existence of global NE. We verify that $(\psi^*, \theta^*)$ is a global NE by checking that the candidate worst-case parameter vector $\theta^*$ maximizes the loss $L \left( \psi^*, \theta \right)$ on the whole constraint set $\Theta$, i.e., that there is no vector $\theta^i \in \Theta$ satisfying (24) given the policy rule $\psi^*$.
While this final step requires calculation of the loss at all points on a grid intended to cover the entire constraint set $\Theta$, we note that this is simpler, in practice, than a brute-force evaluation of the objective (7) at all points on a grid covering $\Psi$ would have been. First, in our applications, $\Theta$ is a low-dimensional set, whereas we may wish to allow for complex families of possible policy rules. Second, it is not necessary to solve a maximization problem at each grid point in order to evaluate $L(\psi^*, \theta)$ for the candidate Nash equilibrium policy $\psi^*$. Finally, it is not necessary to consider an extremely fine grid in order to obtain an accurate approximation to the robust optimal policy rule. This is because the candidate policy $\psi^*$ has already been computed in step 3; the grid search is merely a check that the conjectured NE involves globally, and not just locally, optimal behavior on the part of Nature. For this it suffices that all regions of the constraint set $\Theta$ be given at least minimal attention. If one finds no evidence of other choices $\theta^l$ that are nearly as good as $\theta^*$ (except other choices near $\theta^*$ itself), there is no practical need for a fine grid search.

We have argued above that for given $\theta^*$, steps 1 and 3 yield a policy rule $\psi^*$ that solves (8). We have also shown that for given policy rule $\psi^*$, step 2 yields a parameter vector $\theta^*$ that solves (9), provided that step 4 is verified. Hence, a profile $(\psi^*, \theta^*)$ that is consistent with steps 1 to 4 is a NE of the game $\Gamma$, and proposition 4 guarantees that $\psi^*$ is the desired robust optimal policy rule. However, if we find a $\theta^l \in \Theta$ satisfying (24), then $\theta^*$ cannot be an equilibrium vector of structural parameters, and $(\psi^*, \theta^*)$ is not a global NE, so that $\psi^*$ may or may not be the robust optimal policy rule. Note that there need not exist any NE, even though a robust optimal policy rule should still exist. However, in applications, a global NE will often exist when parameters take reasonable
values, as we show below.

4 Robust Optimal Taylor Rules

In this section, we use the method presented above to characterize robust optimal “Taylor rules” in the framework of section 2. Formally, we restrict \( \Psi \) to the class of policy rules
\[
\psi = [\psi_\pi, \psi_\chi]'
\]satisfying
\[
i_t = \psi_\pi \pi_t + \psi_\chi x_t
\]
at all dates \( t \geq 0 \).

Policies of this form have received considerable attention in recent research (see, e.g., contributions collected in Taylor, 1999a), especially after being proposed by Taylor (1993). They are called non-inertial policies, as they involve no response to lagged variables. We seek to determine the optimal coefficients \( \psi_\pi \) and \( \psi_\chi \) in the model of section 2, assuming that the two critical structural parameters \( \sigma \) and \( \kappa \) that specify the slope of the IS and the aggregate supply equations are known only to be in given intervals \([\underline{\sigma}, \bar{\sigma}]\) and \([\underline{\kappa}, \bar{\kappa}]\) respectively, where \( 0 < \underline{\sigma} < \bar{\sigma} < \infty \), and \( 0 < \underline{\kappa} < \bar{\kappa} < \infty \).

To keep the analysis as simple as possible, we abstract from uncertainty about the intercept of these curves so that the steady-state level of endogenous variables is assumed to be known, and the policy rules specify percent deviations of the interest rate from

\[\text{It is probably not very realistic to assume that the central bank can observe the output gap in the current period. However, as shown below, an interest rate rule that responds only to deviations in observed inflation would be sufficient to implement the optimal non-inertial plan in this model. As a result we could set } \psi_\chi = 0 \text{ without any loss of generality.}\]

\[\text{Taylor (1993) has argued that such a rule with } \psi_\pi = 1.5 \text{ and } \psi_\chi = .5 \text{ constitutes an appropriate description of U.S. monetary policy under chairman Greenspan. In Taylor (1993), however, the output gap is constructed as the percent deviation of real output from a trend, rather than our variable } x.\]
the known steady state. We choose not to consider uncertainty about the time discount factor \( \beta \), as there is substantial theoretical and empirical evidence that it corresponds to a number slightly below one (say, 0.99). Furthermore, for simplicity and clarification of the mechanisms at hand, we suppose that the weights \( \lambda_x \) and \( \lambda_i \) that characterize the policymaker’s preferences are known to the policymaker.

Apart from its popularity, this simple class of policy rules is of interest as it allows a simple analytical characterization of robust optimal policy.\(^{25}\) However, as explained in Woodford (1999c), policymakers who choose optimal actions by disregarding their past actions and past states of the economy, do not achieve the best equilibrium when the private sector is forward-looking. The characterization of robust optimal rules of a more general form – in particular, rules that would implement the best equilibrium if the parameters were known with certainty – is taken up in Giannoni (2001c).\(^{26}\)

Following our solution strategy, we determine first the equilibrium processes for the endogenous variables (inflation, output, and the interest rate) that achieve the lowest value of the loss criterion (3) for a given parameter vector \( \theta = [\sigma, \kappa]' \). Second, we

\(^{25}\)Onatski and Stock (2001) also determine robust optimal Taylor rules. They however perform their analysis in the backward-looking model of Rudebusch and Svensson (1999), and use numerical methods instead of the solution strategy proposed above.

\(^{26}\)While we could, in principle, allow for families of rules that involve learning on the part of the policymaker, we abstract from this issue here. Note, however, that as long as the set of parameters \( \Theta \) is not affected by the learning process, and \( \theta \) cannot be inferred with certainty, the rule without learning is optimal at least in the \textit{weak} sense in which other rules (that involve learning) are not better in the worst case. The rule without learning may however be suboptimal according to a \textit{stronger} notion of robustness, as one could possibly find a rule with learning that is equally good at \( \theta^* \), but performs better for other values of \( \theta \). We leave this issue for further research.
characterize the minmax equilibrium process by determining the structural parameters that obtain in the NE. Finally, we look for a policy rule that implements the minmax equilibrium.

4.1 Optimal Equilibrium Process for Given Parameters

To characterize the class of possible optimal plans corresponding to policy rules \( \psi \in \tilde{\Psi} \), we use (25) to substitute for the interest rate in the structural equations (1) and (2), and rewrite the resulting difference equations in matrix form as follows

\[
E_t z_{t+1} = Az_t + ar^n_t,
\]

where \( z_t \equiv [\pi_t, x_t]' \). Since both \( \pi_t \) and \( x_t \) are non-predetermined endogenous variables at date \( t \), and the process \( \{r^n_t\} \) is assumed to be bounded, the dynamic system (26) admits a unique bounded solution if and only if both eigenvalues of \( A \) lie outside the unit circle, as explained by Blanchard and Kahn (1980). If we restrict our attention to the usual case in which \( \psi_\pi \) and \( \psi_x \) are non-negative, then it is shown in Appendix 6.2 that the policy rule results in a determinate equilibrium if and only if

\[
\psi_\pi + \frac{1 - \beta}{\kappa} \psi_x > 1.
\]  

(27)

When the structural parameters are unknown and \( \kappa \in [\underline{\kappa}, \bar{\kappa}] \), the parameter \( \kappa \) is replaced by \( \bar{\kappa} \) in (27).27

---

27The equilibrium may also be determinate when \( \psi_\pi \) and \( \psi_x \) are negative. This is, however, critically due to the discrete-time version of the model. In the continuous-time limit, negative values for either coefficient of the policy rule results in indeterminacy of the equilibrium.
For simplicity, we consider the case in which the exogenous shocks $r^n_t$ follow an autoregressive process

$$r^n_t = \rho r^n_{t-1} + \varepsilon_t$$  \hspace{1cm} (28)

where $0 \leq \rho < 1$ and $\{\varepsilon_t\}$ is a martingale difference sequence of perturbations. As optimal policy rules necessarily result in a unique bounded equilibrium (see definitions in section 3), (26) can be solved forward. Using (28), one realizes that possible optimal plans corresponding to $\Psi$ are of the form

$$\pi_t = f_\pi r^n_t, \quad x_t = f_x r^n_t, \quad i_t = f_i r^n_t$$  \hspace{1cm} (29)

where $f = [f_\pi, f_x, f_i]'$ is the vector of response coefficients that parametrizes the equilibrium process. The feasibility restrictions on the response coefficients corresponding to (10), obtained by substituting (29) into the structural equations (1) and (2), are

$$\begin{align*}
(1 - \rho) f_x + \sigma^{-1} (f_i - \rho f_\pi - 1) &= 0 \\
(1 - \beta \rho) f_\pi - \kappa f_x &= 0.
\end{align*}$$  \hspace{1cm} (30) \hspace{1cm} (31)

To solve the policymaker’s problem, we choose the plan of the form (29) and consistent with (30) – (31) to minimize the loss criterion $E[L_0]$. Because we consider non-inertial plans, we may as well minimize

$$\hat{L}(f, \theta) = f_\pi^2 + \lambda_x f_x^2 + \lambda_i f_i^2$$

subject to the constraints (30) – (31). The policymaker’s Lagrangian is

$$\mathcal{L}^P(f, \phi; \theta) = (f_\pi^2 + \lambda_x f_x^2 + \lambda_i f_i^2) + \phi_1 [(1 - \rho) f_x + \sigma^{-1} (f_i - \rho f_\pi - 1)]$$

$$+ \phi_2 [(1 - \beta \rho) f_\pi - \kappa f_x].$$
The response coefficients parametrizing the optimal feasible equilibrium, for given parameter vector \( \theta \), are given by

\[
\begin{align*}
    f^\pi_i (\theta) &= \lambda_i [\sigma (1 - \rho) (1 - \beta \rho) - \rho \kappa] \frac{\kappa}{h} \\
    f^x_i (\theta) &= \lambda_i [\sigma (1 - \rho) (1 - \beta \rho) - \rho \kappa] \frac{(1 - \beta \rho)}{h} \\
    f^i_i (\theta) &= \frac{\lambda_x (1 - \beta \rho)^2 + \kappa^2}{h} 
\end{align*}
\]

(32) (33) (34)

where

\[
    h \equiv \lambda_i [\sigma (1 - \rho) (1 - \beta \rho) - \rho \kappa]^2 + \lambda_x (1 - \beta \rho)^2 + \kappa^2.
\]

It is clear from (34) that \( 0 < f^i_i (\theta) \leq 1 \) for any vector \( \theta \in \Theta \), and any positive weights \( \lambda_i, \lambda_x \). Thus, the optimal non-inertial plan involves an adjustment of the nominal interest rate in the same direction as the perturbation to the natural interest rate, but in general by less than the natural rate. Equations (32) and (33) reveal that the response coefficients \( f^\pi_i (\theta), f^x_i (\theta) \) are positive if and only if

\[
    \frac{\sigma}{\kappa} > \frac{\rho}{(1 - \rho \beta) (1 - \rho)},
\]

(35)

that is, whenever the fluctuations in the natural rate are not too persistent (relative to the ratio \( \frac{\rho}{\sigma} \)). Thus, when (35) holds, a positive shock to the natural rate stimulates both the output gap and inflation. In the special case where the interest rate does not enter the loss function (\( \lambda_i = 0 \)), or when the persistence of the perturbations is such that \( \sigma (1 - \rho) (1 - \beta \rho) = \rho \kappa \), we obtain \( f^\pi_i (\theta) = f^x_i (\theta) = 0 \) and \( f^i_i (\theta) = 1 \); the central bank optimally moves the interest rate by the same amount as the natural rate in order to stabilize the output gap and inflation completely. In contrast, when the disturbances to the natural rate are sufficiently persistent (\( \rho \) large enough but still smaller than 1)
for the inequality (35) to be reversed, inflation and the output gap decrease in the face of an unexpected positive shock to the natural rate in the optimal non-inertial plan \( (f^*_\pi(\theta), f^*_{\pi_x}(\theta) < 0) \). Even if the nominal interest rate increases less than the natural rate, optimal monetary policy is restrictive in this case, because the real interest rate \( (i_t - E_t\pi_{t+1}) \) is higher than the natural rate of interest \( r^n_t \).

Following Woodford (1999c), we calibrate the baseline model using the parameter values estimated by Rotemberg and Woodford (1997). The baseline calibration is reported in Table 1.\(^{28}\) For these parameter values, (35) holds, and it continues to hold for any parameter values that are close to these, so that an increase in \( r^n_t \) raises both the output gap and inflation in the optimal non-inertial plan.

### 4.2 Equilibrium Structural Parameters and Minmax Equilibrium

To characterize the minmax equilibrium associated to the class of non-inertial Taylor rules, we need to determine the parameter vector \( \theta^* = [\sigma^*, \kappa^*]' \) that maximizes the policymaker’s loss on the given constraint set \( \Theta \), i.e., when \( \sigma^* \in [\underline{\sigma}, \bar{\sigma}] \) and \( \kappa^* \in [\underline{\kappa}, \bar{\kappa}] \).

As in step 2 of our solution procedure, we compute

\[
Z^*_1 \equiv \frac{\partial \mathcal{L}^P(f^*(\theta^*), \phi^*_{\pi_x}(\theta^*); \theta^*)}{\partial \sigma} = -\phi^*_1(\theta^*) [f^*_1(\theta^*) - \rho f^*_\pi(\theta^*) - 1] (\sigma^*)^{-2} \tag{36}
\]

\[
Z^*_2 \equiv \frac{\partial \mathcal{L}^P(f^*(\theta^*), \phi^*_{\pi_x}(\theta^*); \theta^*)}{\partial \kappa} = -\phi^*_2(\theta^*) f^*_x(\theta^*) \tag{37}
\]

From lemma 6, we know that \( Z^*_1 \) and \( Z^*_2 \) correspond to the slopes of the loss function with respect to \( \sigma \) and \( \kappa \) respectively, evaluated at the candidate NE \( (\psi^*, \theta^*) \). It follows

\(^{28}\)I am grateful to Thomas Laubach for providing me with the estimated standard errors for \( \sigma \) and \( \kappa \). These were computed for the Rotemberg and Woodford (1997) model, using the estimation method explained in Amato and Laubach (1999).
from lemma 7 that at a local NE:

\[
\sigma^* = \begin{cases} 
\sigma, & \text{if } Z_1^* < 0 \\
\bar{\sigma}, & \text{if } Z_1^* > 0 
\end{cases}
\]

\[
\kappa^* = \begin{cases} 
\bar{\kappa}, & \text{if } Z_2^* < 0 \\
\kappa, & \text{if } Z_2^* > 0 
\end{cases}
\]

When \( Z_1^* = 0 \), \( \sigma^* \) can be any value in \([\sigma, \bar{\sigma}]\) that is consistent with \( Z_1^* = 0 \). Similarly, when \( Z_2^* = 0 \), \( \kappa^* \) can be any value in \([\kappa, \bar{\kappa}]\) that is consistent with \( Z_2^* = 0 \). Intuitively, this means that at a local NE, Nature chooses a high value – in fact the highest possible value – for \( \sigma^* \) or \( \kappa^* \) when the loss is increasing in the respective structural parameter (i.e., \( Z \) is positive), while it chooses a low value for \( \sigma^* \) or \( \kappa^* \) when it is decreasing. Recall that because Nature’s problem is non-concave, this characterization is not sufficient to determine the parameter vector \( \theta^* \) that maximizes \( \hat{L}(f(\psi^*, \theta), \theta) \) globally. It allows us however to determine all possible solution candidates.

The candidate non-inertial minmax equilibrium \( q^* \equiv q^*(\theta^*) \), is characterized in the following proposition.

**Proposition 8** When \( \Psi \) is restricted to the class of Taylor rules (satisfying (25)), and the structural parameters \( \sigma \in [\underline{\sigma}, \bar{\sigma}] \) and \( \kappa \in [\underline{\kappa}, \bar{\kappa}] \) are uncertain, where \( \underline{\sigma}, \underline{\kappa} > 0 \) and \( \bar{\sigma}, \bar{\kappa} < \infty \), then the structural parameters \( \sigma^*, \kappa^* \) that are part of a local NE, and the candidate non-inertial minmax equilibrium \( q^* = \{\pi_t, x_t, i_t\} \) are characterized by

\[
\sigma^* = \begin{cases} 
\sigma, & \text{if } \eta < \sigma / \bar{\kappa} \\
\bar{\sigma}, & \text{if } \sigma / \bar{\kappa} < \eta 
\end{cases}
\]

\[
\kappa^* = \begin{cases} 
\bar{\kappa}, & \text{if } \eta < \sigma / \bar{\kappa} \\
\kappa, & \text{if } \sigma / \bar{\kappa} < \eta 
\end{cases}
\]

where

\[
\eta \equiv \frac{\rho}{(1-\rho)(1-\beta\rho)}, \tag{38}
\]

and

\[
\pi_t = f^*_\pi(\theta^*) r^n_t, \quad x_t = f^*_x(\theta^*) r^n_t, \quad i_t = f^*_i(\theta^*) r^n_t, \tag{39}
\]
with equilibrium response coefficients

\[
\begin{align*}
    f_\pi^* (\theta^*) & = \lambda \left[ \sigma^* (1 - \rho) (1 - \beta \rho) - \rho \kappa^* \right] \frac{\kappa^*}{h^*} \\
    f_x^* (\theta^*) & = \lambda \left[ \sigma^* (1 - \rho) (1 - \beta \rho) - \rho \kappa^* \right] \frac{(1 - \beta \rho)}{h^*} \\
    f_i^* (\theta^*) & = \frac{\lambda x (1 - \beta \rho)^2 + \kappa^*^2}{h^*}
\end{align*}
\]

(40) (41) (42)

where \( h^* = \lambda \left[ \sigma^* (1 - \rho) (1 - \beta \rho) - \rho \kappa^* \right]^2 + \lambda x (1 - \beta \rho)^2 + \kappa^*^2 \).

When \( \sigma / \kappa \leq \eta \leq \sigma / \kappa \), a NE is obtained for any combination of structural parameters \( \sigma^*, \kappa^* \) satisfying \( \sigma^* / \kappa^* = \eta \). In this case, the equilibrium response coefficients are

\[
\begin{align*}
    f_\pi^* (\theta^*) & = f_x^* (\theta^*) = 0, \quad \text{and} \quad f_i^* (\theta^*) = 1.
\end{align*}
\]

Proof. See Appendix 6.1.4. ■

When \( \rho \) is small enough (such that \( \eta < \sigma / \kappa \)), the worst situation for the policymaker is achieved when \( \kappa \) is made as large as possible, and \( \sigma \) is made as small as possible. To understand this, recall that the output gap and inflation depend upon the interest rate and the natural rate only through the real interest rate differential \( (i_t - \mathbb{E}_t \pi_{t+1}) - r^n_t \). On one hand, a lower \( \sigma \) and a higher \( \kappa \) imply stronger effects of the perturbations to the natural rate on the output gap and inflation. On the other hand, they render monetary policy more effective, as changes in \( i_t \) have a stronger effect on \( \pi_t \) and \( x_t \). When \( \eta < \sigma / \kappa \), the real interest rate moves less than the natural rate in the optimal non-inertial plan, so that the first effect dominates. Thus, for a given real interest rate differential, a lower \( \sigma \) and a higher \( \kappa \) are responsible for larger fluctuations of inflation and the output gap, and make the policymaker worse off. These larger changes in inflation and output gap induce the central bank to move its interest rate closer to the natural rate. Since the
policymaker also dislikes variability in the interest rate, though, he does not change the interest rate by enough to cancel the effect of a perturbation to the natural rate. In contrast, when the perturbations are so persistent that $\eta > \sigma / \kappa$, the worst situation is obtained when $\kappa^* = \kappa$, $\sigma^* = \sigma$. Finally, when the persistence of the perturbations is such that $\sigma / \kappa \leq \eta \leq \sigma / \kappa$, the response of the interest rate in the minmax equilibrium is given by $f_t^* (\theta^*) = 1$, which completely neutralizes the shocks to the natural rate of interest. As a result, inflation and the output gap remain at their steady-state level whether the economy is affected by shocks or not.

In the baseline parametrization $\eta = .824$. As long as the upper bound for $\kappa$ is less than .191 (i.e., eight times the baseline value), and the lower bound for $\sigma$ is above .824$\bar{\kappa}$, the condition $\eta < \sigma / \kappa$ is satisfied. Thus, if the baseline parametrization is an appropriate approximation of the true model of the economy, and the uncertainty about the structural parameters is small enough, the worst case situation is obtained when $\kappa^* = \kappa$, $\sigma^* = \sigma$.

### 4.3 Determining Robust Optimal Taylor Rules

As in step 3 of our solution strategy, we now determine a candidate robust optimal Taylor rule $\psi^*$ that implements the non-inertial minmax equilibrium characterized in proposition 8. It is convenient, for technical reasons that will become clear below, to rewrite (25) as

$$i_t = \frac{1}{\psi^*_\pi} \pi_t + \frac{\hat{\psi}_x}{\psi^*_\pi} x_t$$

(43)

where $\hat{\psi}_\pi = 1/\psi_\pi$ and $\hat{\psi}_x = \psi_x/\psi_\pi$, and to determine a robust optimal policy rule $\hat{\psi}^* = [\hat{\psi}^*_\pi, \hat{\psi}^*_x]'$ instead of $\psi^*$. In (43), we assume that $\hat{\psi}_\pi$ and $\hat{\psi}_x$ are finite real numbers
so that \( \hat{\psi} \) can be used to characterize any rule \( \psi \in \hat{\Psi} \) except those in which \( \psi_\pi = 0 \).\(^{29}\)

Using the solution (39) to eliminate the endogenous variables in (43), we obtain

\[
\hat{\psi}_\pi f^*_i (\theta^*) r^n_t = \left( f^*_\pi (\theta^*) + \hat{\psi}_x f^*_x (\theta^*) \right) r^n_t.
\]

Any policy rule \( \hat{\psi}^* \) resulting in a unique bounded equilibrium and satisfying

\[
\hat{\psi}_\pi f^*_i (\theta^*) = f^*_\pi (\theta^*) + \hat{\psi}^*_x f^*_x (\theta^*)
\]

implies the candidate non-inertial minmax equilibrium for all exogenous paths of the natural rate of interest.\(^{30}\) Substituting \( f^*_\pi (\theta^*) \), \( f^*_x (\theta^*) \), \( f^*_i (\theta^*) \) using (40)–(42), and solving for \( \hat{\psi}^*_\pi \) yields

\[
\hat{\psi}^*_\pi = \frac{\kappa^* + (1 - \beta \rho) \hat{\psi}^*_x}{(\kappa^*)^2 + \lambda_x (1 - \beta \rho) \lambda_i [\sigma^* (1 - \rho) (1 - \beta \rho) - \rho \kappa^*]}
\]

where \( \sigma^* \) and \( \kappa^* \) are determined in proposition 8.

Whenever \( \eta \) is sufficiently small (so that \( \eta < \sigma^*/\bar{\kappa} \)) or large (so that \( \eta > \sigma^*/\bar{\kappa} \)), it results from (45) that the coefficients of the robust optimal Taylor rule \( \hat{\psi}^*_\pi = 1/\hat{\psi}^*_\pi \) and \( \hat{\psi}^*_x = \hat{\psi}^*_x/\hat{\psi}^*_\pi \) satisfy

\[
\psi^*_\pi = \frac{(\kappa^*)^2 + \lambda_x (1 - \beta \rho)^2}{\lambda_i \kappa^* [\sigma^* (1 - \rho) (1 - \beta \rho) - \rho \kappa^*]} - \psi^*_x \frac{(1 - \beta \rho)}{\kappa^*}.
\]

In fact, any vector \( \psi^* = [\psi^*_\pi, \psi^*_x]' \) satisfying (46) implements the candidate non-inertial minmax equilibrium provided that it results in a unique bounded equilibrium. This is the case if \( \psi^*_\pi + \psi^*_x (1 - \beta) / \bar{\kappa} > 1 \), when we restrict ourself to policies with non-negative

\(^{29}\)The rules in which \( \psi_\pi = 0 \) are not interesting here as the results shown below indicate that optimal rules of the form (43) don’t involve values for \( \hat{\psi}_\pi \) that are extremely large.

\(^{30}\)Note that (44) corresponds to \( P \left( f^*(\theta^*), \psi^* \right) = 0 \) in the general terminology of section 3.
coefficients. Note that by setting \( \psi^*_x = 0 \), (46) determines the policy rule that implements the candidate non-inertial minmax equilibrium without any knowledge of the output gap.

When the uncertainty and the persistence in the perturbations to the natural rate of interest are such that \( \sigma / \kappa \leq \eta \leq \bar{\sigma} / \bar{\kappa} \), we have \( \hat{\psi}^*_\pi = 0 \) (recall that \( \sigma^* / \kappa^* = \eta \), and that the equilibrium response coefficients are \( f^*_\pi (\theta^*) = f^*_\pi (\theta^*) = 0 \), and \( f^*_i (\theta^*) = 1 \) in this case). The optimal interest rate responds as much as possible to inflation (and output gap deviations if \( \hat{\psi}^*_x \neq 0 \)), so that equilibrium inflation and output gap remain at their steady-state. We shall let \( \psi^*_x \to +\infty \) in this case.\(^{31}\)

To verify that the rule \( \hat{\psi}^* \) and the equilibrium parameter vector \( \theta^* \) determine a global NE, hence that the corresponding rule \( \psi^* \) is a robust optimal Taylor rule, we need to verify, as in step 4 of our solution method, that the structural parameters \( \sigma^*, \kappa^* \) are Nature’s best responses to \( \hat{\psi}^* \) on the whole constraint set \( \Theta \). (Recall that lemma 7 gives necessary but not sufficient conditions for \( \theta^* \) to maximize Nature’s objective.) In the numerical example considered here, we assume parameter uncertainty corresponding to the 95\% confidence intervals for \( \sigma \) and \( \kappa \). As \( \eta = 0.824 < \sigma / \kappa \), proposition 8 guarantees that in the local NE, Nature chooses \( \sigma^* = \underline{\sigma} \) and \( \kappa^* = \bar{\kappa} \). Figure 1 is a contour plot of the loss measure \( E[L_0] \) as a function of the structural parameters \( \sigma \) and \( \kappa \) in the specified set \( \Theta \), when the policy rule is \( \hat{\psi}^* \), the policymaker’s best response to \( \theta^* = [\underline{\sigma}, \bar{\kappa}]^\prime \), setting \( \psi_x = 0.5 \).\(^{32}\) The figure reveals that \( \theta^* \) (i.e., the lower right corner) is not only part of a

\(^{31}\)We would obtain the same minmax equilibrium by letting \( \psi^*_\pi \to -\infty \). However, as mentioned in footnote 27, even if the equilibrium is determinate in this case, it would be indeterminate in the continuous-time version of this model.

\(^{32}\)The statistic \( E[L_0] \) as well as all statistics in Table 2 are reported in annual terms. The statistics \( V[\pi] \), \( V[i] \), and \( E[L_0] \) are therefore multiplied by 16. Furthermore, the weight \( \lambda_x \) reported in Table
local but also a global NE, as it maximizes the loss on the whole set $\Theta$.

We now compare the non-inertial minmax equilibrium and the robust optimal Taylor rule with their counterpart in the absence of parameter uncertainty.

4.4 Comparing Equilibria and Taylor Rules in Certainty and Uncertainty Case

When the structural parameters are known by the policymaker, the optimal equilibrium response coefficient of the interest rate to the natural rate of interest, $f_i^0 = f_i^*(\theta^0)$, satisfies (34) with the vector of structural parameters equal to the true (known) value $\theta^0$. A comparison of the optimal equilibrium response coefficients reveals that the policymaker lets the interest rate respond more strongly to exogenous perturbations in the minmax equilibrium than in the certainty case, regardless of the degree of persistence in the perturbations. The following proposition states this result formally.

**Proposition 9** Let $f_i^0 = f_i^*(\theta^0)$ (defined in (34)) be the optimal response coefficient of the interest rate in the optimal non-inertial plan, when the parameters $\sigma_0 \in (\underline{\sigma}, \bar{\sigma})$ and $\kappa_0 \in (\underline{\kappa}, \bar{\kappa})$ are known with certainty. Let $f_i^* = f_i^*(\theta^*)$ (defined in proposition 8) be the corresponding response coefficient in the minmax equilibrium when the parameters $\sigma \in [\underline{\sigma}, \bar{\sigma}]$ and $\kappa \in [\underline{\kappa}, \bar{\kappa}]$ are uncertain. Let $\sigma, \kappa, \lambda_x, \lambda_i > 0$, and $\bar{\sigma}, \bar{\kappa} < \infty$. If $\frac{\sigma_0}{\lambda_0} \neq \eta$, 1 is also multiplied by 16 in order to represent the weight attributed to the output gap variability (in annual terms) relative to the variability of annualized inflation and of the annualized interest rate. The coefficients $\psi_x$ reported here are multiplied by 4 so that the response coefficients to the output gap and to annualized inflation are expressed in the same units.

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then

\[ 0 < f_i^0 < f_i^* \leq 1. \]

In the special case where \( \frac{\sigma_0}{\kappa_0} = \eta \), then \( f_i^0 = f_i^* = 1 \).

**Proof.** See Appendix 6.1.5. ■

The result of proposition 9 is illustrated in Figure 2 for the baseline model (see Table 1). Figure 2 displays impulse responses of all three endogenous variables (interest rate, inflation, and output gap) to an unexpected temporary increase in the natural rate of interest. In the upper panel, the dotted line represents the exogenous path of the natural rate. The solid line represents the impulse response of the interest rate in the optimal non-inertial plan, in the absence of uncertainty. The dashed line plots the corresponding impulse response in the minmax equilibrium. It appears clearly that the interest rate reacts more strongly in the presence of uncertainty than when parameters are known.

When the perturbations to the natural rate of interest are sufficiently transitory (so that \( \eta < \sigma/\bar{\kappa} \)) as is the case in Figure 2, the worst case arises when \( \sigma \) is as low as possible, while \( \kappa \) is as high as possible, which implies that positive shocks to the natural rate have a larger stimulating effect on inflation and the output gap than is the case in the absence of parameter uncertainty.\(^{33}\) Thus the policymaker who seeks to dampen fluctuations in inflation and output gap increases the interest rate by more in the minmax equilibrium than in the certainty case, so that the interest rate moves closer to the natural rate in the minmax equilibrium. The remaining panels confirm that the stronger reaction of the interest rate dampens the effect of the shock upon inflation and the output gap in the presence of parameter uncertainty.

\(^{33}\) Unless \( \sigma_0 = \bar{\sigma} \) and \( \kappa_0 = \bar{\kappa} \), which we have ruled out in proposition 9.
A comparison of optimal Taylor rules in the presence and absence of parameter uncertainty yields a similar result summarized in the following proposition. Note that in the certainty case, any optimal Taylor rule $\psi^0 = [\psi^0_\pi, \psi^0_x]'$ satisfies an equation of the form (46), but in which the vector of structural parameters $\theta^*$ is replaced with the known vector $\theta^0 = [\sigma_0, \kappa_0]'$.

**Proposition 10** Let $\psi^0 = [\psi^0_\pi, \psi^0_x]' \in \Psi$ be a Taylor rule that implements the optimal non-inertial plan given some coefficient $\psi_x$, when the parameters $\sigma_0 \in (\underline{\sigma}, \bar{\sigma})$ and $\kappa_0 \in (\kappa, \bar{\kappa})$ are known with certainty. Let $\psi^* = [\psi^*_\pi, \psi^*_x]' \in \Psi$ be the robust optimal Taylor rule given the same $\psi_x$, when the parameters $\sigma \in [\underline{\sigma}, \bar{\sigma}]$ and $\kappa \in [\kappa, \bar{\kappa}]$ are uncertain. Suppose $\eta \neq \frac{\sigma_0}{\kappa_0}$. If

$$\psi^0_\pi, \psi^*_\pi, \psi^*_x \geq 0$$

and

$$\psi^*_x > -\frac{\kappa_0 \bar{\kappa} (\sigma_0 \bar{\kappa} - \kappa_0 \underline{\sigma}) + \lambda_x (1 - \beta \rho)^2 \left( \eta \left( \bar{\kappa}^2 - \kappa_0^2 \right) + \kappa_0 \sigma_0 - \bar{\kappa} \sigma \right)}{\lambda_i (1 - \rho) (1 - \beta \rho)^2 (\sigma_0 - \eta \kappa_0) (\underline{\sigma} - \eta \bar{\kappa}) (\bar{\kappa} - \kappa_0)}$$

(47)

then

$$\psi^*_\pi > \psi^0_\pi.$$ 

**Proof.** See Appendix 6.1.6. ■

Proposition 10 states that for given (and sufficiently large) response to the output gap $\psi_x$, the policymaker should respond more strongly to inflation deviations in the presence of parameter uncertainty than when parameters are known. Such a stronger reaction to inflation deviations is exactly what is required to make the interest rate move more closely to the natural rate of interest in the presence of uncertainty, and to prevent shocks.
from having too large an effect on inflation and the output gap in the worst case.\textsuperscript{34}

To illustrate this result, we represent in Figure 3 policies that implement the optimal non-inertial plan for the baseline parametrization of the model. The solid line represents the \textit{optimal} Taylor rules in the baseline case, i.e., the combinations \((\psi^0_\pi, \psi^0_x)\) satisfying an equation similar to (46), in which the parameter vector \(\theta^*\) is replaced with the baseline vector \(\theta^0\). The dashed-dotted line plots the corresponding \textit{robust optimal} policies – the combinations \((\psi^*_\pi, \psi^*_x)\) satisfying (46) – in the presence of the parameter uncertainty given in Table 1. The white region indicates the set of policy rules that result in a determinate equilibrium for \textit{any} value of the parameters in the assumed region \(\Theta\) (see Appendix 6.2). In contrast, the gray region indicates combinations \((\psi^*_\pi, \psi^*_x)\) that result in indeterminacy of the equilibrium for at least one value of the parameters \(\sigma, \kappa\) in \(\Theta\). Thus, only optimal policies in the white region may satisfy step 3 of the solution method. The circled star indicates the coefficients of the rule proposed by Taylor (1993) as a good approximation of recent U.S. monetary policy. Figure 3 clearly shows that whenever monetary policy involves a response to the output gap that is strong enough, the optimal response to inflation is larger in the presence of uncertainty than when the parameters are known.

\textsuperscript{34}By assuming \(\psi^0_\pi, \psi^*_\pi, \psi_x \geq 0\) and \(\eta \neq \sigma_0/\kappa_0\) in proposition 10, we implicitly restrict our attention to situations in which the persistence of the perturbations is small enough for (35) to hold. If instead the shocks to the natural rate are very persistent (so that \(\eta > \sigma/\kappa\), corresponding to \(\rho > .76\), then a result similar to proposition (10) holds when \(\psi_x\) is large enough, but in this case the optimal response to inflation is more negative in the presence of uncertainty, i.e., \(\psi^*_x < \psi^0_x < 0\). (Recall that the optimal response coefficient \(f^*_\pi(\theta) < 0\) when \(\sigma/\kappa < \eta\). In this case however, the optimal policy may yield an indeterminate equilibrium (see footnote 27). This is in fact an example of a situation in which there may be no Taylor rule that implements the optimal equilibrium: there may be no \(\psi^*(\theta)\) in \(\Psi\) that solves (16).
as predicted by proposition 10. In fact, the line representing robust optimal policies is steeper and has a higher intercept than the corresponding line representing optimal policy rules in the absence of uncertainty. Condition (47) guarantees that \( \psi_x \) lies above the intersection point of the two lines.\(^{35}\) We have focused on the effect of uncertainty on the response of the interest rate to inflation for a given response to the output gap. As should be clear from Figure 3, the presence of uncertainty calls also for a larger response to the output gap, for any given \( \psi_\pi \).

Another aspect of the stronger reaction of the interest rate in the presence of model uncertainty is presented in Table 2. This table reports optimal Taylor rules (when \( \psi_x \) is set equal to .5), the policymaker’s loss criterion, and the following measure of variability

\[
V[z] \equiv \mathbb{E} \left\{ \mathbb{E}_0 \left[ (1 - \beta) \sum_{t=0}^{\infty} \beta^t z_t^2 \right] \right\}
\]

for all three endogenous variables. The latter statistic determines the contribution of each endogenous variable to the loss measure \( \mathbb{E}[L_0] \). Indeed, \( \mathbb{E}[L_0] \) is just a weighted sum of \( V[\pi] \), \( V[x] \) and \( V[i] \) with weights being those in the loss function (3).\(^{36}\) The lines of Table 2 corresponding to the baseline case are indicated by \( (\psi^0, \theta^0) \). In contrast, \( (\psi^*, \theta^0) \) denotes the case in which the central bank faces uncertainty and follows the robust optimal policy \( \psi^* \), but the actual structural parameters are equal to their values in the baseline case, \( \theta^0 \). Comparing these two lines again confirms that the central bank lets the interest rate move by more in the presence of uncertainty. The more aggressive monetary policy is then responsible for a decrease in the variability of inflation and the

\(^{35}\)Note that (47) is satisfied for all \( \psi_x \geq 0 \), whenever \( \eta (\kappa_x^2 - \kappa_0^2) + \kappa_0 \sigma_0 - \bar{z} \geq 0 \), as all other terms in the fraction in the right-hand side of (47) are positive.

\(^{36}\)See footnote 32.
output gap, but an increase in the volatility of $i$. Overall, switching from the baseline policy rule to the robust optimal rule raises the loss $E[L_0]$ from 2.28 to 2.46 when the true parameters are the ones of the baseline model. However, if the unknown parameters are not at the baseline value, but reach a less favorable combination, such as the worst case $\theta^*$, the advantage of following the robust policy rule is clear: the maximum loss is reduced from 3.30 to 3.02.

Table 2: Optimal Taylor Rules and Statistics

<table>
<thead>
<tr>
<th>Policy Statistics</th>
<th>$\psi_\pi$</th>
<th>$\psi_x$</th>
<th>$V[\pi]$</th>
<th>$V[x]$</th>
<th>$V[i]$</th>
<th>$E[L_0]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\psi^0, \theta^0)$</td>
<td>2.217</td>
<td>0.5</td>
<td>0.211</td>
<td>9.923</td>
<td>6.720</td>
<td>2.279</td>
</tr>
<tr>
<td>$(\psi^0, \theta^*)$</td>
<td>2.217</td>
<td>0.5</td>
<td>0.414</td>
<td>11.640</td>
<td>9.809</td>
<td>3.295</td>
</tr>
<tr>
<td>$(\psi^*, \theta^0)$</td>
<td>8.294</td>
<td>0.5</td>
<td>0.069</td>
<td>3.240</td>
<td>9.455</td>
<td>2.461</td>
</tr>
<tr>
<td>$(\psi^<em>, \theta^</em>)$</td>
<td>8.294</td>
<td>0.5</td>
<td>0.098</td>
<td>2.767</td>
<td>11.782</td>
<td>3.017</td>
</tr>
</tbody>
</table>

Note: $\psi_x$ is arbitrarily set at 0.5

5 Conclusion

This paper proposes a general method based on a property of zero-sum two-player games to derive robust optimal monetary policy rules – the best rules among those that yield an acceptable performance in a specified range of models – when the true model is unknown. Model uncertainty is viewed as uncertainty about the true structural parameters that numerically specify the model. The method is applied to characterize robust optimal rules in a standard forward-looking macroeconomic model that can be derived from first principles.

While it is commonly believed among economists and central bankers that monetary
policy should be less responsive when there is uncertainty about model parameters, we have shown that the opposite is likely to be true in the model considered when the two key structural parameters – the slopes of the intertemporal IS curve and the aggregate supply curve – are subject to uncertainty: the robust optimal Taylor rule requires the interest rate to respond more strongly in general to fluctuations in inflation or the output gap than is the case in the absence of uncertainty.\textsuperscript{37} Yet the policymaker is cautious in our framework – in fact he is even more cautious than in Brainard’s model, as he cares very much about situations in which monetary policy would perform poorly. In contrast to Brainard’s analysis, however, caution induces the policymaker to be more responsive.

The model has the property that the policymaker faces a trade-off between the stabilization of inflation and the output gap on one hand, and the nominal interest rate on the other. In the presence of model uncertainty, the robust policymaker seeks to limit the welfare losses, especially in those bad outcomes in which exogenous perturbations (to the natural rate of interest) have a large effect on inflation and the output gap, i.e., when the aggregate supply curve is particularly steep and the intertemporal IS curve is particularly flat. Model uncertainty therefore affects the trade-off facing the policymaker by increasing the weight given to inflation and output gap stabilization relative to the weight given to interest rate stabilization. A more aggressive policy allows the central bank to stabilize inflation and the output gap around their target values more effectively, and guarantees that welfare losses will be contained.

\textsuperscript{37}Giannoni (2000c) shows that this result generalizes to more flexible policy rules that allow for responses of the interest rate to lagged variables, in the model of section 2.
6 Appendix

6.1 Proofs

6.1.1 Proof of Lemma 5

First note that since \( \psi^* (\theta) \in \Psi \), the latter policy rule results in a unique bounded equilibrium. Suppose as a way of contradiction that there exists a policy rule \( \psi^\dagger (\theta) \in \Psi \), \( \psi^\dagger (\theta) \neq \psi^* (\theta) \), satisfying \( L (\psi^\dagger (\theta), \theta) < L (\psi^* (\theta), \theta) \). By definition of \( L (\cdot) \) and \( \hat{L} (\cdot) \) we have \( L (\psi, \theta) = \hat{L} (f (\psi, \theta), \theta) \) for all \( \psi \in \Psi, \theta \in \Theta \), so that \( \hat{L} (f (\psi^* (\theta), \theta), \theta) < \hat{L} (f (\psi^\dagger (\theta), \theta), \theta) = \hat{L} (f^* (\theta), \theta) \). But then \( f^* (\theta) \) cannot minimize (12) subject to (10).

\[ \square \]

6.1.2 Proof of Lemma 6

Because \( f^* (\theta) \) satisfies (15), and \( \psi^* (\theta) \) solves (16), we have \( f^* (\theta) = f (\psi^* (\theta), \theta) \), for all \( \theta \in \Theta \). Using this and totally differentiating \( \hat{L} (f, \theta) \), we obtain

\[
\frac{d\hat{L} (f (\psi^* (\theta), \theta), \theta)}{d\theta'} = \frac{d\hat{L} (f^* (\theta), \theta)}{d\theta'} = \frac{\partial \hat{L} (f^* (\theta), \theta)}{\partial f'} \cdot \frac{df^* (\theta)}{d\theta'} + \frac{\partial \hat{L} (f^* (\theta), \theta)}{\partial \theta'}.
\]

(48)

Note that any solution \( f (\psi, \theta) \) to (10) and (11) must always satisfy \( S (f (\psi, \theta), \theta) = 0 \) for all \( \psi \in \tilde{\Psi}, \theta \in \Theta \). Differentiating these constraints with respect to \( \theta \), and evaluating the resulting expression at \( \psi^* (\theta) \) yields

\[
0 = \frac{dS (f (\psi^* (\theta), \theta), \theta)}{d\theta'} = \frac{dS (f^* (\theta), \theta)}{d\theta'}
= \frac{\partial S (f^* (\theta), \theta)}{\partial f'} \cdot \frac{df^* (\theta)}{d\theta'} + \frac{\partial S (f^* (\theta), \theta)}{\partial \theta'}.
\]
Premultiplying this by the vector of Lagrange multipliers $\phi^* (\theta)$ associated to the constraints (10) in the policymaker’s problem, we obtain

$$0 = \phi^* (\theta) \cdot \left( \frac{\partial S (f^* (\theta), \theta)}{\partial f'} \cdot \frac{df^* (\theta)}{d\theta'} + \frac{\partial S (f^* (\theta), \theta)}{\partial \theta'} \right).$$

Adding this to (48) on both sides and rearranging yields

$$\frac{d\hat{L} (f (\psi^* (\theta), \theta), \theta)}{d\theta'} = \left( \frac{\partial \hat{L} (f^* (\theta), \theta)}{\partial f'} + \phi^* (\theta) \cdot \frac{\partial S (f^* (\theta), \theta)}{\partial f'} \right) \frac{df^* (\theta)}{d\theta'}$$

$$+ \frac{\partial \hat{L} (f^* (\theta), \theta)}{\partial \theta'} + \phi^* (\theta) \cdot \frac{\partial S (f^* (\theta), \theta)}{\partial \theta'}.$$

Using (14) to eliminate the first term on the right-hand side yields (21), where we note that in the partial derivative $\partial \mathcal{L}^P (f^* (\theta), \phi^* (\theta); \theta) / \partial \theta'$, we maintain $f^* (\theta)$ and $\phi^* (\theta)$ constant. ■

### 6.1.3 Proof of Lemma 7

If the local NE is such that $Z_i^* > 0$, then (22) implies $\mu_{i1}^* - \mu_{i2}^* > 0$, where $\mu_{i1}^*$ and $\mu_{i2}^*$ are the $i$-th elements of $\mu_1^*$ and $\mu_2^*$ respectively. Since the multipliers satisfy $\mu_{i1}, \mu_{i2} \geq 0$ for all $i = 1, \ldots, m$, we must have $\mu_{i1}^* > 0$. In this case, (20) implies $\theta_i^* = \bar{\theta}_i$. Alternatively, if the local NE is such that $Z_i^* < 0$ then $\theta_i^* = \underline{\theta}_i$. If $Z_i^* = 0$, then (22) implies that $\mu_{i1}^* = \mu_{i2}^*$. Suppose as a way of contradiction that $\mu_{i1}^* \neq 0$. Then we know from (20) that $\theta_i^* = \bar{\theta}_i$.

But as $\mu_{i1}^* = \mu_{i2}^*$, we also have $\mu_{i1}^* \neq 0$, and thus $\theta_i^* = \underline{\theta}_i$. Since $\theta_i^*$ cannot equal $\bar{\theta}_i$ and $\underline{\theta}_i$ in the same time we must have $\mu_{i1}^* = \mu_{i2}^* = 0$. So when $Z_i^* = 0$ in an NE, then $\theta_i^*$ can be any value in $[\underline{\theta}_i, \bar{\theta}_i]$ that is consistent with $Z_i^* = 0$. ■
6.1.4 Proof of Proposition 8

Given the equilibrium parameter vector \( \theta^* = [\sigma^*, \kappa^*]' \), the minmax equilibrium is by definition the process \( q^*(\theta^*) \equiv q(\psi^*(\theta^*), \theta^*) \). It is thus of the form (29) where the response coefficients \( f^*_\pi (\theta^*), f^*_x (\theta^*), f^*_i (\theta^*) \), are given by (32) – (34), evaluating the parameter vector at \( \theta^* \).

To determine \( \theta^* \), we need to determine the sign of \( Z_1^* \) and \( Z_2^* \). Using the first-order conditions to the policymaker’s problem, we can express the equilibrium Lagrange multiplier associated to (30) as \( \phi^*_1 (\theta^*) = -\lambda_i \sigma^* f^*_i (\theta^*) \). Combining this with (36), and using (40), (42) to solve for \( f^*_\pi (\theta^*), f^*_i (\theta^*) \) we obtain

\[
Z_1^* = \left( \eta - \frac{\sigma^*}{\kappa^*} \right) \chi_1
\]

where \( \chi_1 > 0 \), and \( \eta \equiv \rho (1 - \rho)^{-1} (1 - \beta \rho)^{-1} \). Similarly, using the first-order conditions to the policymaker’s problem, we can express the equilibrium Lagrange multiplier associated to (31) as \( \phi^*_2 (\theta^*) = \frac{\lambda_x}{\kappa} f^*_x (\theta^*) - \frac{\sigma^* \lambda_i (1 - \rho)}{\kappa} f^*_i (\theta^*) \). Using this to substitute for \( \phi^*_2 (\theta^*) \) in (37), and using (41), (42) to solve for \( f^*_x (\theta^*), f^*_i (\theta^*) \) we get

\[
Z_2^* = - \left( \eta - \frac{\sigma^*}{\kappa^*} \right) \chi_2
\]

where \( \chi_2 > 0 \).

We need to consider three cases.

1. **Case 1**: \( \eta < \sigma/\kappa \). In this case, \( \eta < \frac{\sigma^*}{\kappa^*} \) for all \( \sigma^*, \kappa^* \) in the allowed set \( \Theta \). Equations (49), (50) imply that for any given structural parameters \( \sigma^*, \kappa^* \), the policymaker’s best response, \( \psi^* \), is such that \( Z_1^* < 0 \) and \( Z_2^* > 0 \). By lemma 7, Nature’s best response is then \( \theta^* = [\sigma, \kappa]' \) in a local NE.
2. **Case 2**: $\bar{\sigma}/\bar{\kappa} < \eta$. Symmetrically, $\eta > \frac{\bar{\sigma}}{\bar{\kappa}}$ for all $\sigma^*, \kappa^*$ in $\Theta$. Equations (49), (50) imply that for any $\sigma^*, \kappa^*$, the policymaker’s best response, $\psi^*$, is such that $Z_1^* > 0$ and $Z_2^* < 0$. By lemma 7, Nature’s best response is then $\theta^* = [\bar{\sigma}, \bar{\kappa}]'$ in a local NE.

3. **Case 3**: $\underline{\sigma}/\underline{\kappa} \le \eta \le \bar{\sigma}/\bar{\kappa}$. We need to consider three situations.

   (a) Suppose first that we have a local NE in which Nature chooses some $\sigma^*, \kappa^*$ such that $\sigma^*/\kappa^* < \eta$. We know from (49), (50) that the policymaker’s best response is such that $Z_1^* > 0$ and $Z_2^* < 0$. But lemma 7 guarantees that Nature chooses $\sigma^* = \bar{\sigma}$ and $\kappa^* = \bar{\kappa}$ in this case. As $\eta \le \underline{\sigma}/\underline{\kappa}$, we obtain a contradiction, and such $\sigma^*, \kappa^*$ cannot be part of a NE.

   (b) Suppose alternatively that we have a local NE in which Nature chooses some $\sigma^*, \kappa^*$ such that $\sigma^*/\kappa^* > \eta$. We know from (49), (50) that the policymaker’s best response is such that $Z_1^* < 0$ and $Z_2^* > 0$. But lemma 7 guarantees that Nature chooses $\sigma^* = \bar{\sigma}$ and $\kappa^* = \bar{\kappa}$ in this case. As $\underline{\sigma}/\underline{\kappa} \le \eta$, such $\sigma^*, \kappa^*$ cannot be part of a NE.

   (c) Suppose finally that we have a NE in which Nature chooses some $\sigma^*, \kappa^*$ such that $\sigma^*/\kappa^* = \eta$. We know from (49), (50) that the policymaker’s best response is such that $Z_1^* = Z_2^* = 0$. Lemma 7 in turn says that Nature may choose any $\sigma^*, \kappa^*$ in $\Theta$ that is consistent with $Z_1^* = Z_2^* = 0$, i.e., that satisfies $\sigma^*/\kappa^* = \eta$.

Thus when $\underline{\sigma}/\underline{\kappa} \le \eta \le \bar{\sigma}/\bar{\kappa}$, Nature’s best response is given by any vector $[\sigma^*, \kappa^*]' \in \Theta$ satisfying $\sigma^*/\kappa^* = \eta$, in a local NE. In this case, (40) – (42) imply that the minmax equilibrium is characterized by $f^*_x(\theta^*) = f^*_x(\theta^*) = 0$, and $f^*_i(\theta^*) = 1$. ■
6.1.5 Proof of Proposition 9

First observe that since both $h$ and $h^\ast$ in (34) and (42) are non-negative, we have $0 < f_i^0 \equiv f_i^\ast (\theta^0) \leq 1$, and $0 < f_i^\ast \equiv f_i^\ast (\theta^\ast) \leq 1$. We now show that $f_i^0 \leq f_i^\ast$, and that $f_i^0 < f_i^\ast$ when $\sigma_0/\kappa_0 \neq \eta$. We need to consider three cases.

1. Case 1: $\eta < \sigma/\kappa$. In this case, proposition 8 implies $\kappa^\ast = \kappa > \kappa_0$, $\sigma^\ast = \sigma < \sigma_0$.

   Note that $\eta < \sigma/\kappa$ can be rewritten as
   \[ \sigma \chi - \rho \kappa > 0 \]
   where $\chi \equiv (1 - \rho \beta) (1 - \rho) > 0$. Using (34) and (42), we obtain after some algebraic manipulations
   \[ f_i^\ast - f_i^0 = \lambda_i \frac{\left( (\sigma_0 \chi - \rho \kappa_0)^2 - (\sigma \chi - \rho \kappa)^2 \right) \xi + \left( \kappa^2 (\sigma_0 \chi - \rho \kappa_0)^2 - \kappa_0^2 (\sigma \chi - \rho \kappa)^2 \right)}{h \cdot h^\ast} \]
   where $\xi = \lambda_x (1 - \beta \rho)^2 > 0$. Since $(\sigma_0 \chi - \rho \kappa_0) > (\sigma \chi - \rho \kappa) > 0$, we have $(\sigma_0 \chi - \rho \kappa_0)^2 - (\sigma \chi - \rho \kappa)^2 > 0$, so that the numerator is positive. Since the denominator is also positive, we have $f_i^\ast > f_i^0$.

2. Case 2. $\bar{\sigma}/\bar{\kappa} < \eta$. In this case, proposition 8 implies $\kappa^\ast = \bar{\kappa} < \kappa_0$, $\sigma^\ast = \bar{\sigma} > \sigma_0$.

   Note that $\bar{\sigma}/\bar{\kappa} < \eta$ can be rewritten as
   \[ \bar{\sigma} \chi - \rho \bar{\kappa} < 0 \]
   Note that this implies $\sigma_0 \chi - \rho \kappa_0 < 0$. We now have
   \[ f_i^\ast - f_i^0 = \lambda_i \frac{\left( (\sigma_0 \chi - \rho \kappa_0)^2 - (\bar{\sigma} \chi - \rho \bar{\kappa})^2 \right) \xi + \left( \bar{\kappa}^2 (\sigma_0 \chi - \rho \kappa_0)^2 - \kappa_0^2 (\bar{\sigma} \chi - \rho \bar{\kappa})^2 \right)}{h \cdot h^\ast} \]
   Since $\sigma_0 \chi < \bar{\sigma} \kappa$, we have $(\sigma_0 \chi - \rho \kappa_0) \chi = \kappa (\sigma_0 \chi - \rho \kappa_0) - \kappa_0 (\bar{\sigma} \chi - \rho \bar{\kappa}) < 0$, so that $\kappa (\sigma_0 \chi - \rho \kappa_0) < \kappa_0 (\bar{\sigma} \chi - \rho \bar{\kappa})$, and $\kappa^2 (\sigma_0 \chi - \rho \kappa_0)^2 > \kappa_0^2 (\bar{\sigma} \chi - \rho \bar{\kappa})^2$. As this
implies also \((\sigma_0 \chi - \rho \kappa_0)^2 > (\bar{\sigma} \chi - \rho \bar{\kappa})^2\), the numerator is positive. Because the denominator is also positive, we have \(f_i^* > f_i^0\).

3. Case 3: \(\bar{\sigma}/\bar{\kappa} \leq \eta \leq \bar{\sigma}/\kappa\). In this case, proposition 8 implies \(\sigma^*/\kappa^* = \eta\) so that \(f_i^* = 1\). In general, when \(\frac{\sigma_0}{\kappa_0} \neq \eta\), we have \(f_i^0 < 1 = f_i^*\). In the special case where \(\frac{\sigma_0}{\kappa_0} = \eta\), we obtain \(f_i^0 = f_i^* = 1\).

6.1.6 Proof of Proposition 10

In the certainty case, any optimal Taylor rule \(\psi^0 = [\psi^0_\pi, \psi^0_x]^T\) satisfies an equation of the form (46), but where the vector of structural parameters \(\theta^*\) is replaced with the known vector \(\theta^0 = [\sigma_0, \kappa_0]^T\), i.e.,

\[
\psi^0_\pi = \frac{\kappa_0^2 + \lambda_\pi (1 - \beta \rho)^2}{\lambda_i \kappa_0 (\sigma_0 (1 - \rho) (1 - \beta \rho) - \rho \kappa_0)} - \psi^0_x \frac{(1 - \beta \rho)}{\kappa_0}. \tag{51}
\]

Assuming \(\psi^0_\pi, \psi_x \geq 0\), it results from (51) that \(\sigma_0 (1 - \rho) (1 - \beta \rho) - \rho \kappa_0 \geq 0\), or equivalently \(\sigma_0 / \kappa_0 \geq \eta\). We need to consider two cases.

1. Case 1: \(\eta < \bar{\sigma}/\bar{\kappa}\). Using (46), (51), and setting \(\psi^0_x = \psi^*_x = \psi_x\), we obtain after some algebraic manipulations

\[
\psi^*_x - \psi^0_x = \frac{(1 - \beta \rho) (\bar{\kappa} - \kappa_0)}{\bar{\kappa} \kappa_0} \times \left( \psi_x + \frac{\kappa_0 \bar{\kappa} (\sigma_0 \tilde{\kappa} - \kappa_0 \tilde{\sigma}) + \lambda_x (1 - \beta \rho)^2 (\eta (\bar{\kappa}^2 - \kappa_0^2) + \kappa_0 \sigma_0 (\sigma - \eta \bar{\kappa}) (\tilde{\kappa} - \kappa_0)}{\lambda_i (1 - \rho) (1 - \beta \rho)^2 (\sigma_0 - \eta \kappa_0) (\sigma - \eta \bar{\kappa}) (\tilde{\kappa} - \kappa_0)} \right).
\]

Since the first fraction in the right-hand side is positive, \(\psi^*_x > \psi^0_x\) if and only if (47) holds.

2. Case 2: \(\bar{\sigma}/\bar{\kappa} \leq \eta \leq \bar{\sigma}/\kappa\). In this case, \(\hat{\psi}^*_x = 0\), so that \(\hat{\psi}^*_x = 1/\hat{\psi}^*_x \rightarrow +\infty\). Since \(\psi^0_\pi\) is finite when \(\sigma_0 / \kappa_0 \neq \eta\), we have \(\psi^*_x > \psi^0_x\). ■

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6.2 Region of Determinacy for Taylor Rules

As mentioned in the text, the dynamic system (26) admits a unique bounded solution if and only if both eigenvalues of $A$ lie outside the unit circle. The characteristic polynomial of $A$ is

$$P(\gamma) = (\gamma^2 - (\sigma + \beta \sigma + \kappa + \beta \psi_x) \gamma + \sigma + \psi_x + \kappa \psi_\pi) (\beta \sigma)^{-1}.$$  

As the roots of $P(\gamma)$ can be represented by complex numbers of the form $\gamma = e^{i\nu} R$, with modulus $R$, the characteristic polynomial has one or more roots on the unit circle when $\gamma = e^{i\nu}$. Since the coefficients of $P(\gamma)$ are all real, we know that if $\gamma = e^{i\nu} R$ is a root, then its complex conjugate $\bar{\gamma} = e^{-i\nu} R$ is also a root. Using this result, we can find conditions for at least one eigenvalue of $A$ to be on the unit circle by solving

$$P(e^{i\nu}) = 0$$  
$$P(e^{-i\nu}) = 0$$  

for $\nu$ and $\psi_\pi$. The solutions are

$$\psi_\pi = 1 - \psi_x (1 - \beta) \kappa^{-1}, \quad v = 0 \quad (52)$$  
$$\psi_\pi = -2\sigma (1 + \beta) \kappa^{-1} - 1 - \psi_x (1 + \beta) \kappa^{-1}, \quad v = \pi \quad (53)$$  
$$\psi_\pi = -\sigma (1 - \beta) \kappa^{-1} - \psi_x \kappa^{-1}, \quad v = -i \ln (z) \quad (54)$$

where $z$ is a root of $z^2 - (\sigma + \beta \sigma + \kappa + \beta \psi_x) z + \beta \sigma$. The conditions involving $\psi_\pi, \psi_x$ determine the boundaries of the region of determinacy. They can be represented by lines in the $(\psi_\pi, \psi_x)$ plane (see boundaries of the gray region in Figure 3). If we restrict our attention to the case with $\psi_\pi, \psi_x \geq 0$, then only the first boundary is relevant, as it is the only one that crosses the positive orthant. Since there is only one eigenvalue of $A$
outside the unit circle when \( \psi_\pi = \psi_x = 0 \), then the same must be true for all couples \((\psi_\pi, \psi_x)\) in the positive orthant and below the boundary (52), so that the equilibrium is indeterminate in this region. In contrast, all couples above the boundary (52) result in a determinate equilibrium, as both eigenvalues are outside the unit circle.

In the presence of parameter uncertainty, the set \( \Psi \) of policies (in the positive orthant) that result in a determinate equilibrium for all parameter vectors \( \theta \in \Theta \) is the intersection of all sets above the boundary (52) when \( \sigma \) and \( \kappa \) vary in the respective intervals \([\underline{\sigma}, \bar{\sigma}]\) and \([\underline{\kappa}, \bar{\kappa}]\). Hence when \( \psi_\pi, \psi_x \geq 0 \), the region of determinacy is the set of all couples above \( \psi_\pi = 1 - \psi_x (1 - \beta) / \bar{\kappa} \).
Figure 1: Contour plot of $E[L_0]$ with robust optimal rule in $(\kappa, \sigma)$ space
Figure 2: Impulse response functions to a temporary increase in $r_0^n$ ($\rho = 0.35$)
Figure 3: Optimal Taylor rules in \((\psi_\pi, \psi_x)\) space
Chapter 3

Robust Optimal Monetary Policy in a Forward-Looking Model with Parameter and Shock Uncertainty
1 Introduction

During the last decade, economists have given increasing attention to the study of interest-rate feedback rules for the conduct of monetary policy. While some have focused on the estimation of central banks reaction functions, and the description of actual monetary policy (see, e.g., Taylor, 1993, Clarida et al. 1998, Judd and Rudebusch, 1998), others have characterized optimal policy rules in the context of particular models of the economy (see, e.g., contributions collected in Taylor 1999a). Although most of these contributions have suggested that optimal rules require stronger responses to fluctuations in economic variables than those derived from estimated reaction functions, the rules preferred by various authors still differ significantly from each other, reflecting the lack of agreement on an appropriate model of the economy. Both this lack of agreement and the inherent uncertainty about the actual functioning of the economy have recently induced researchers to characterize desirable policy rules in the face of uncertainty about the true model of the economy.

A popular idea due to Brainard (1967), and emphasized by Blinder (1998) and others, is that policymakers should be cautious in the presence of uncertainty about the true parameters of a model. By “cautious” it is meant that the instrument of monetary policy should be moved by less than in the absence of parameter uncertainty.1 Some authors have therefore suggested that optimal policy rules that take proper account of the uncertainty surrounding model parameters should be less aggressive, and thus closer

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1 As Brainard (1967) pointed out, this result holds in his setup provided that the exogenous disturbances and the parameters that relate the policy instrument to the target variable are not too strongly correlated.
to estimated policy rules. However, a number of recent studies have challenged this
conventional wisdom. For instance, in Giannoni (2001b), we argue that the opposite
result is likely to be obtained in a simple forward-looking model that has been used in
many recent studies of monetary policy. In that paper, we show that simple Taylor rules
that are robust to uncertainty about structural parameters of the model may be more
responsive to fluctuations in inflation and the output gap than the optimal Taylor in the
absence of parameter uncertainty. We compute robust optimal Taylor rules in a simple
“new synthesis” or “new Keynesian” model that contains a single composite perturbation,
and we allow for uncertainty about two critical parameters of the economy: the slopes of
both the aggregate demand and supply schedules.

We call robust optimal policy rules policy rules that perform best in the worst-case
parameter configuration, within a specified set of parameter configurations. Policy rules
of this kind have recently been advocated by Sargent (1999), Hansen and Sargent (1999,
2000a, 2000b), Stock (1999), Onatski and Stock (2001), Onatski (2000a, 2000b) and Tet-
low and von zur Muehlen (2000). Robust rules are designed to avoid an especially poor
performance of monetary policy in the event of an unfortunate parameter configuration.
They guarantee to yield an acceptable performance of monetary policy in the specified
range of models.

This paper generalizes the results obtained in Giannoni (2001b) in three important
ways. First, instead of restricting ourselves to Taylor rules, we determine a robust op-

2See Clarida et al. (1999), Estrella and Mishkin (1999), Hall et al. (1999), Martin and Salmon (1999),
others.

3Von zur Muehlen (1982) is an early study of such monetary policy rules.
timal monetary policy rule in a family of rules that is flexible enough to implement the optimal plan, if the parameters are known with certainty. Second, we allow the model to be affected by a variety of exogenous shocks, instead of assuming a single composite exogenous perturbation. We emphasize in particular the distinction between efficient and inefficient supply shocks, and consider uncertainty about the relative importance of each kind of shock. Thirdly, we consider robustness of monetary policy not only to uncertainty about critical structural parameters, but also to uncertainty about the degree of persistence in the shock processes. Moreover, we emphasize the importance of deriving the model from microeconomic foundations in order to determine precisely how the exogenous disturbances are transmitted through the economy. This turns out to be important for the determination of the worst-case parameter configuration.

The results that we obtain here are consistent with the ones obtained in Giannoni (2001b). While it is commonly believed that monetary policy should be less responsive in the presence of uncertainty, we show that the opposite is likely to be true in the model considered. In fact, for a reasonable calibration of the model, the robust optimal policy rule requires the interest rate to respond more strongly to fluctuations in inflation, in changes in the output gap, and to lagged interest rates, than in the absence of uncertainty. This result depends however critically on the way the exogenous shocks affect the economy.

The rest of the paper is organized as follows. The next subsection reviews the recent literature on robust monetary policy and emphasizes the various kinds of uncertainty considered. Section 2 reviews the method presented in Giannoni (2001b), that is used here to derive the robust optimal policy rule. Section 3 presents a simple optimizing
monetary model. While the model is similar to models presented in a number of recent studies, we briefly expose the microeconomic foundations of this model to specify precisely how exogenous disturbances affect the endogenous variables, when there is uncertainty about the structural parameters of the model. Section 4 characterizes both the optimal policy rule in the absence of uncertainty, and the robust optimal policy rule when there is uncertainty. Section 5 first tries to give an intuition for the results obtained, and then discusses the sensitivity of the results to various assumptions. Finally, section 6 concludes.

1.1 Related literature

Recently, several researchers have tried to determine policy rules that are robust to uncertainty about the correct model of the economy. One approach, first advocated by McCallum (1988, 1999), and followed by Levin et al. (1999a, 1999b), Christiano and Gust (1999), and Taylor (1999b), determines policy rules that perform well across a range of models, by simulating given rules in a number of different models. While very useful for understanding the effects of particular rules in various models, these papers do actually not determine an “optimal” rule in the face of model uncertainty. Interestingly, the rules preferred by Levin et al. (1999a), while not super-inertial, involve a coefficient near unity on the lagged interest rate.

As a second approach, some authors have characterized optimal policy in particular classes of models, taking into account uncertainty about various aspects of the model. Researchers have for instance considered uncertainty about the parameters of the model and have used Bayesian methods to determine the policy that minimizes the expected
loss, given a prior distribution on the parameters. This approach, initially started by Brainard (1967), was developed by Chow (1975), and has recently been followed by Clarida et al. (1999), Estrella and Mishkin (1999), Hall et al. (1999), Martin and Salmon (1999), Svensson (1999a), Rudebusch (2000), Sack (2000), Söderström (2000a, 2000b), and Wieland (1998), among others. Most of these studies focus on backward-looking models, and support Brainard’s result that optimal policy should be less aggressive in the face of parameter uncertainty.4

Another branch of the literature has looked for robust rules that minimize a loss criterion in some worst-case scenario, within a specified set of possible scenarios. One justification for this approach is the view that uncertainty about the true model of the economy takes the form of uncertainty in the sense of Knight (1921), i.e., a situation in which the probabilities on the alternative models are not known, so that Bayesian methods cannot be used to compute the expected loss over different models.5 Furthermore, it has been shown by Gilboa and Schmeidler (1989) that if the policymaker has multiple priors on the set of alternative models, and his preferences satisfy uncertainty aversion in addition to the axioms of standard expected utility theory, the policymaker faces a min-max problem: to minimize his loss in the worst-case scenario, i.e., when the prior distribution is the worst distribution in the set of possible distributions. Several authors

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4One notable exception is Söderström (2000a) who shows that uncertainty about the persistence of inflation induces the policymaker to respond more aggressively to shocks in the model due to Svensson (1997).

5Knight (1921) first made the distinction between “known risk,” i.e. a situation in which a distribution of outcomes is known, and “uncertainty”, i.e., a situation in which no known probability distribution exists.
such as Sargent (1999), Hansen and Sargent (2000b), Stock (1999), Kasa (2001), Onatski and Stock (2001), Onatski (2000a, 2000b) and Tetlow and von zur Muehlen (2000) have recently applied robust control theory, to derive robust monetary policies of this kind. All of these authors have however not focused on the same type of uncertainty.

For instance, Sargent (1999), Hansen and Sargent (2000b), and Kasa (2001) consider very unstructured uncertainty. They append to their equations shock terms that represent model misspecifications — i.e., deviations of the model actually used from the true model — and limit uncertainty by imposing a penalty on the statistical distance (the relative entropy) between the model used and the perturbed model. They compute robust policies by minimizing a given loss criterion in the worst-case realization of the shock process that represent misspecifications. These authors also deviate from the standard rational expectations framework by assuming that all agents are robust decision-makers. (See Hansen and Sargent (2000a) for a transparent discussion.) In contrast, Stock (1999), Onatski and Stock (2001), Onatski (2000a, 2000b), and Tetlow and von zur Muehlen (2000) consider more structured non-parametric uncertainty. They construct a non-parametric set of models around some reference model that approximates the true model of the economy, but they impose some structure on the set of possible models. They then seek to determine rules that minimize the loss for the worst possible model. These authors measure the robustness of given policy rules with the maximal size of the uncertainty set that does not include models with an indeterminate equilibrium or unstable models. While they can measure the degree of robustness of given rules, they are able to characterize the actual min-max rules only for simple types of uncertainty.

In this paper, as in Giannoni (2001b), we consider uncertainty about the parame-
ters of the structural model. While this approach limits the kind of uncertainty that may be considered, we find the parametric treatment more intuitive, transparent, than a non-parametric approach, and we believe that it allows modelers to quantify their degree of confidence more easily. This approach allows us furthermore to characterize analytically the robust rule. While most other studies, except Hansen and Sargent (2000b) and Onatski (2000b), study robust policies in backward-looking models, we consider a forward-looking model. In contrast to Hansen and Sargent (2000b), however, but as in Onatski (2000b), we maintain the rational expectations framework, by assuming that the private sector knows the true model of the economy, while the policymaker faces model uncertainty.

So far, there is no clear answer to whether robust policy rules in the presence of uncertainty should in general be more or less aggressive than optimal rules absent model uncertainty, even among the papers that use min-max objective functions.\(^6\) Sargent (1999), Stock (1999), and Onatski and Stock (2001) find that robust policy requires in most cases stronger policy responses, in the backward-looking models of Ball (1999) and Rudebusch and Svensson (1999). We obtain similar results, both in this paper and in Giannoni (2001b), for a simple forward-looking model and in the face of parameter uncertainty. In contrast, Hansen and Sargent (2000b) find that the interest rate responds less aggressively to shocks under the robust rule, in a similar forward-looking model, but with unstructured uncertainty, and when all agents are robust decision makers. Finally, Onatski (2000b) finds robust rules to be more responsive to the output gap and less

\(^6\)Onatski (2000a) shows that the results obtained with the min-max approach are very similar to those obtained with the Bayesian approach in the Brainard (1967) setting.
responsive to inflation in a model that involves both forward- and backward-looking elements. The answer to whether robust policy rules should in general be more or less aggressive than optimal rules absent model uncertainty depends critically both on the model and the type of uncertainty considered. In section 5, we discuss how changing various assumptions about the model can affect the results.

2 Uncertainty and Robust Optimal Monetary Policy

In reality, central banks and researchers do generally not know with certainty the true parameters of their model, in addition to not knowing the exogenous disturbances. In this paper, as in Giannoni (2001b), we assume that the parameters of the economic model are unknown to the policymaker, but remain constant over time. The policymaker commits credibly at the beginning of period 0 to a policy rule for the entire future. He chooses a policy rule to minimize some loss criterion $L_0$, while facing uncertainty about the true parameters of the economy. We denote by $\psi$ the vector of coefficients that completely characterizes the policy rule, and we simply call $\psi$ a “policy rule”. We assume furthermore that the policy rules $\psi$ are drawn from some finite-dimensional linear space $\hat{\Psi} \in \mathbb{R}^n$.

In contrast, agents in the private sector are assumed to know the true parameters of the economy. They act optimally, i.e., in a way to maximize their utility subject to their constraints, in every period, and in every state. Specifically, we assume that the private sector may be one of many different types. Its type is determined once and for all, before period 0, and is characterized by the finite-dimensional vector of structural parameters $\theta = [\theta_1, \theta_2, ..., \theta_m]'$ defined on the compact set $\Theta \subseteq \mathbb{R}^m$. Agents in the private sector
know the true type \( \theta \), but the central bank does not.

We write \( q_t \) for the vector of endogenous variables at date \( t \), and \( q \) for the stochastic process \( \{q_t\}_{t=0}^{\infty} \), specifying \( q_t \) at each date as a function of the history of exogenous shocks until that date. The behavior of the private sector is determined by a set of equations for each date \( t \), and each state. These may be written compactly as

\[
S(q, \theta) = 0. \tag{1}
\]

The restrictions imposed by the commitment to the policy rule at each date can in turn be written as

\[
P(q, \psi) = 0. \tag{2}
\]

A rational expectations equilibrium is then defined as a stochastic process \( q(\psi, \theta) \) satisfying the structural equations (1) and the policy rule (2), at each date, and in every state. As in Giannoni (2001b), we restrict our attention to a subset \( \Psi \) of policy rules that result in a unique bounded rational expectations equilibrium, and let \( q(\psi, \theta) \) denote this equilibrium.

When the structural parameters are known with certainty, the optimal monetary policy rule that is optimal relative to the subset of rules \( \Psi \) can be defined as follows.

**Definition 1** In the case of known structural parameters \( \theta \), let \( \Psi \) be a set of policy rules such that there is a unique bounded equilibrium. Then an **optimal monetary policy rule** is a vector \( \psi^0 \) that solves

\[
\min_{\psi \in \Psi} \mathbb{E}[L_0(q(\psi, \theta))] \tag{3}
\]
where $L_0(q)$ is the policymaker’s loss function, and the unconditional expectation is taken over all possible histories of the disturbances.\footnote{The setup presented in Giannoni (2001b) is more general than the one presented here, as it considers a loss function of the form $L_0(q, \theta)$, where the second argument allows the coefficients of the loss function to be functions of the parameter vector $\theta$.}

To characterize parameter uncertainty, we assume that the vector $\theta$ of structural parameters lies in a given (known) compact set $\Theta$, and that the distribution of $\theta$ is unknown. As argued in the previous section, it results from Gilboa and Schmeidler (1989) that if the policymaker has multiple priors on $\Theta$ (including the priors that any element $\theta \in \Theta$ holds with certainty), and his preferences satisfy uncertainty aversion in addition to the axioms of standard expected utility theory, the policymaker’s problem is to minimize his loss in the worst-case parameter configuration. The optimal policy rule is then the robust rule defined as following.

**Definition 2** Let $\Psi$ be a set of policy rules such that there is a unique bounded equilibrium process $q(\psi, \theta)$ for all $\psi \in \Psi, \theta \in \Theta$. In the case of parameter uncertainty, a robust optimal monetary policy rule is a vector $\psi^*$ that solves

$$
\min_{\psi \in \Psi} \left\{ \max_{\theta \in \Theta} \mathbb{E}[L_0(q(\psi, \theta))] \right\}
$$

where $L_0(q)$ is the policymaker’s loss function, and where the unconditional expectation is taken over all possible histories of the disturbances.

Given that the unknown parameter vector is in $\Theta$, the policymaker can guarantee that the loss is no higher than the one obtained in the following “minmax” equilibrium.
Definition 3 A **minmax equilibrium** is a bounded rational expectations equilibrium 

$q^* = q(\psi^*, \theta^*)$, where $\psi^* \in \Psi$ is a robust optimal monetary policy rule and $\theta^*$ maximizes

the loss $\mathbb{E}[L_0(q(\psi^*, \theta))]|$ on the constraint set $\Theta$.

However, the equilibrium that actually realizes (given the exogenous processes) depends upon the true value of $\theta$, and is hence unknown to the policymaker.

To characterize the robust optimal policy rule, we apply the method proposed in Giannoni (2001b). This method relates the solution to the problem (3) to a pure strategy Nash equilibrium (NE) of a zero-sum two-player game between a policymaker and a malevolent Nature. In this game, the policymaker chooses the policy rule $\psi^* \in \Psi$ to maximize his loss $L(\psi, \theta) \equiv \mathbb{E}[L_0(q(\psi, \theta))]$ knowing that a malevolent Nature tries to hurt him as much as possible. Symmetrically, Nature chooses the parameter vector $\theta^* \in \Theta$ to maximize the policymaker’s loss, knowing that the policymaker is going to minimize it. A NE of this game, $(\psi^*, \theta^*)$, involves a best response on the part both players. Moreover, since this is a zero-sum game, the equilibrium action of each player is a minmaximizer so that the equilibrium strategy $\psi^*$ is a solution to (3) (see Giannoni, 2001b, for additional details).

The solution procedure involves the four following steps to characterize the robust optimal rule $\psi^*$.

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8The method presented in Giannoni (2001b) is more general than the one summarized here, as it allows one to characterize robust optimal rules in restricted families of policy rules. As these restricted families of rules impose restrictions besides (1) and (2) on the space of possible processes, the space of possible processes is parametrized by an alternative parameter vector $f$. We don’t need to consider this complication here, as the family of policy rules that we consider below does not impose any additional restrictions besides (1) and (2).
1. Optimal equilibrium for any given parameter vector $\theta$. We determine the equilibrium process $q^*(\theta)$ that minimizes the loss $\hat{L}(q) \equiv E[L_0(q)]$ subject to the restrictions imposed by the structural equations (1) for any $\theta \in \Theta$.

2. Candidate minmax equilibrium. Using $q^*(\theta)$ from step 1, we determine numerically the candidate worst parameter vector $\theta^*$ in the allowed set, i.e., the parameter vector that maximizes $\hat{L}(q^*(\theta))$ in the set $\Theta$. The process $q^*(\theta^*)$ is the candidate minmax equilibrium.

3. Optimal policy rule. We look for a policy rule $\psi^*$ that implements the candidate minmax equilibrium, i.e., that solves $P(q^*(\theta^*), \psi^*) = 0$. We then verify that the policy rule $\psi^*$ is in $\Psi$, i.e., that it results in a unique bounded equilibrium process $q(\psi^*, \theta)$ for all $\theta \in \Theta$.

4. Check for existence of global NE. We verify that $(\psi^*, \theta^*)$ is a global NE, hence that $q(\psi^*, \theta^*)$ is indeed a minmax equilibrium, by checking that the solution candidate $\theta^*$ maximizes the loss $L(\psi^*, \theta)$ on the constraint set $\Theta$, i.e., that there is no vector $\theta^1 \in \Theta$ satisfying

$$L(\psi^*, \theta^1) > L(\psi^*, \theta^*)$$

(4)
given the policy rule $\psi^*$.

Steps 1 and 3 determine the policymaker’s best response $\psi^* = \psi^*(\theta^*)$ to a given parameter vector $\theta^*$. Step 2 and 4 insure in turn that $\theta^*$ is Nature’s best response to $\psi^*$. It follows that a profile $(\psi^*, \theta^*)$ that satisfies steps 1 to 4 is a NE, and hence that $\psi^*$ is the robust optimal rule that we are looking for. Step 4 is required to insure that the
candidate worst parameter vector computed in step 2 is indeed Nature’s best response to the robust optimal rule $\psi^*$ on the whole constraint set $\Theta$, so that $(\psi^*, \theta^*)$ is not only a local NE — i.e., a situation in which each player’s strategy is at least locally a best response to the other player’s strategy — but also a global NE. Note that a global NE may not exist, even though a robust optimal rule should still exist. However, in applications such as the one in section 4, a global NE will exist.

While steps 2 and 4 require a numerical maximization of the loss function with respect to $\theta$, on the set $\Theta$, it is simpler to characterize the robust optimal rule following the four steps mentioned here, than trying to solve (3) directly. Indeed, solving (3) would require maximizing the loss function over $\theta$ for any given policy rule $\psi$, until the robust rule $\psi^*$ is obtained. In addition, the solution procedure proposed here may allow one to obtain an analytical characterization of the robust rule as will be the case in section 4.\(^9\)

3 A Simple Optimizing Model for Monetary Policy Analysis

In this section, we review a simple optimizing model with known parameters that underlies the structural equations, and the policymaker’s objective function that will be used in the following sections. In the next section, we will determine a robust optimal policy

\(^9\)Note that if we compute the worst vector $\theta^*$ by maximizing directly $L(q(\psi^*(\theta), \theta))$ with respect to $\theta \in \Theta$, we would obtain the solution to $\max_{\theta \in \Theta} \{\min_{\psi \in \Psi} E[L_0(q(\psi, \theta))]\}$, and not necessarily the parameter vector $\theta$ that solves (3). The solution to both problems is however the same provided that it is part of a global NE. Our four-step procedure guarantees that we obtain the robust policy rule that we are looking for, provided that a global NE exists.
rule using the method just reviewed. The model is closely related to models discussed in Woodford (1996, 1999b, 1999c, 2000a), Clarida et al. (1999), and Giannoni (2001a). We first describe the model that characterizes the private sector’s behavior, and then turn to the objective of monetary policy.

3.1 Underlying Structural Model

We assume that there exists a continuum of households indexed by \( j \) and distributed uniformly on the \([0, 1]\) interval. Each household \( j \) consumes all of the goods and supplies a single differentiated good. It seeks to maximize its lifetime expected utility given by

\[
E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[ u \left( C^j_t; \xi_t \right) + \chi \left( \frac{M^j_t}{P_t}; \xi_t \right) - v \left( y_t (j); \xi_t \right) \right] \right\}
\]

(5)

where \( \beta \in (0, 1) \) is the household’s discount factor (assumed to be equal for each household), \( M^j_t \) is the amount of money balances held at the end of period \( t \), \( y_t (j) \) is the household’s supply of its good, \( C^j_t \) is an index of the household’s consumption of each of the differentiated goods defined by

\[
C^j_t = \left[ \int_0^1 c^j_t (z) \frac{\varphi_t - 1}{\varphi_t} \, dz \right]^{\frac{1}{\varphi_t - 1}},
\]

(6)

and \( P_t \) is the corresponding price index. The consumption index aggregates consumption of each good, \( c^j_t (z) \), with a constant elasticity of substitution between goods, \( \varphi_t > 1 \), at each date. In contrast to Dixit and Stiglitz (1977) however, we let the elasticity

---

of substitution vary exogenously over time. As will appear more clearly below, such perturbations to the elasticity of substitution imply time variation in the price elasticity of demand of each good, and variations of the desired markup. The stationary vector \( \xi_t \) represents disturbances to preferences. For each value of \( \xi \), the functions \( u (\cdot; \xi) \) and \( \chi (\cdot; \xi) \) are assumed to be increasing and concave, while the disutility from supplying goods, \( v (\cdot; \xi) \), is increasing and convex.

Expenditure minimization and market clearing imply that the demand for each good \( j \) is given by

\[
y_t(j) = Y_t \left( \frac{p_t(j)}{p_t} \right)^{-\varphi_t}
\]

where \( p_t(j) \) is the price of good \( j \), and \( Y_t = C_t = \int_0^1 C_t^j \, dj \) represents aggregate demand at date \( t \).

We assume that financial markets are complete so that risks are efficiently shared. It follows that all households face an identical intertemporal budget constraint, and choose identical state-contingent plans for consumption, and money balances. We may therefore drop the index \( j \) on those variables.

Each household maximizes (5) subject to its budget constraint, and the constraint that it satisfies the demand for its good (7). It follows that the optimal intertemporal allocation of consumption satisfies a familiar Euler equation of the form

\[
\frac{1}{1 + i_t} = E_t \left\{ \frac{\beta u_c (Y_{t+1}; \xi_{t+1})}{u_c (Y_t; \xi_t)} \frac{P_t}{P_{t+1}} \right\},
\]

where \( i_t \) denotes the nominal interest rate on a riskless one-period nominal bond purchased in period \( t \). We will consider a log-linear approximation of this relationship about the steady state where the exogenous disturbances take the values \( \xi_t = 0 \) and where there
is no inflation. We let \( \bar{Y} \) and \( \bar{i} \) be the constant values of output and nominal interest rate in that steady state, and define the percent deviations \( \bar{Y}_t \equiv \log (Y_t/\bar{Y}) \), \( \bar{i}_t \equiv \log \left( \frac{1+i_t}{1+i} \right) \), \( \pi_t \equiv \log (P_t/P_{t-1}) \). The log-linear approximation to (8) is

\[
\bar{Y}_t = \bar{E}_t \bar{Y}_{t+1} - \sigma^{-1} (\bar{i}_t - \bar{E}_t \bar{\pi}_{t+1}) + \sigma^{-1} \delta_t \tag{9}
\]

where \( \sigma \equiv -\frac{u_{cc} \bar{C}}{u_c} > 0 \) represents the inverse of the intertemporal elasticity of substitution, and where

\[
\delta_t \equiv \frac{u_{c\xi} u_c}{u_c} (\xi_t - \bar{E}_t \xi_{t+1}) \tag{10}
\]

represents exogenous disturbances to (9). Equation (9), which represents the demand side of the economy, is often called the “intertemporal IS equation” as it relates negatively desired expenditures to the real interest rate. We assume that \( \delta_t \) is independent of \( \sigma \).\(^{11}\)

Monetary policy has real effects in this model because prices do not respond immediately to perturbations. Specifically, we assume as in Calvo (1983) that only a fraction \( 1 - \alpha \) of suppliers may change their prices at the end of any given period, regardless of the the time elapsed since the last change. Because of monopolistic competition, each household chooses the optimal prices \( \{p_t(j)\} \), taking as given the evolution of aggregate demand and the price level, that determine the location of the demand for its product (7). Since each supplier faces the same demand function, each supplier that changes its price in period \( t \) chooses its new price \( p_t^* \) to maximize

\[
\bar{E}_t \left\{ \sum_{T=1}^{\infty} (\alpha \beta)^{T-t} \left[ \frac{u_{c} (Y_T; \xi_T) Y_T}{P_T} \right] p_t^{\xi_T} - \bar{v} \left( Y_T \left( \frac{p_t}{P_T} \right)^{\bar{\phi}_T} ; \xi_T \right) \right\} ,
\]

\(^{11}\)This is true, for instance, for any utility function of the form \( u(C, \xi) = v(C) \cdot w(\xi) \) where \( v \) and \( w \) are independent of each other, since \( \sigma = -\frac{u_{cc} \bar{C}}{u_c} \), and \( \delta_t = \frac{u_{\xi} u_c}{u_c} (\xi_t - \bar{E}_t \xi_{t+1}) \) in this case.
The first term in brackets represents the household’s utility of consumption in period $T$ given the price $p_t$ chosen in period $t$. It is the product of marginal utility of consumption at date $T$ and total revenues from sales at price $p_t$. The second term represents the household’s disutility of providing the amount of goods demanded at period $T$. The discount factor for these streams of utility is adjusted for the fact the price chosen at date $t$ remains in effect at date $T$ with probability $\alpha^{T-t}$. Log-linearizing the first-order conditions to the above problem and using the law of motion of the price level, we obtain the following aggregate supply equation

$$\pi_t = \kappa \left( \hat{Y}_t - \hat{Y}_t^n \right) + \beta E_t \pi_{t+1}, \quad (11)$$

where $\kappa > 0$, and $\hat{Y}_t^n$ represents the natural rate of output, i.e., the percentage deviations from steady-state of the level of output that would obtain with perfectly flexible prices. As further shown in the appendix of Giannoni (2001a), the natural rate of output satisfies

$$\hat{Y}_t^n = \frac{1}{\omega + \sigma} \left( \frac{u_e}{u_c} \xi_t - \frac{v_g}{v_y} \xi_t - \mu_t \right), \quad (12)$$

where $\omega > 0$ represents the elasticity of each firm’s real marginal cost with respect to its own supply and $\mu_t$ represents percent deviations of the desired markup $\varphi_t / (\varphi_t - 1)$ from steady state. Note that while the natural rate of output depends upon both supply and demand exogenous real perturbations, it is completely independent of monetary policy. Because of market power, however, steady-state level of output is inefficiently low. Furthermore, as the percent deviations of the efficient rate of output — i.e., the equilibrium rate of output that would obtain in the absence of price rigidities and market power — are given by $\hat{Y}_t^e = \frac{1}{\omega + \sigma} \left( \frac{u_e}{u_c} \xi_t - \frac{v_g}{v_y} \xi_t \right)$, exogenous time variation in the desired markup results in deviations of the natural rate of output from the efficient rate, $\hat{Y}_t^e$. 

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given by

\[ \hat{Y}_t^e - \hat{Y}_t^n = \frac{1}{\omega + \sigma} \mu_t. \]

As we will evaluate monetary policy in terms of deviations of output from its efficient level, it will be convenient to define the “output gap” as

\[ x_t \equiv \hat{Y}_t - \hat{Y}_t^e. \] (13)

Using this, we can rewrite the two structural equations (9) and (11) as

\[ x_t = E_t x_{t+1} - \sigma^{-1} (\delta_t - E_t \pi_{t+1}) + \frac{\omega}{(\omega + \sigma) \sigma} \delta_t + \frac{1}{\omega + \sigma} \epsilon_t \] \hspace{1cm} (14)

\[ \pi_t = \kappa \left( x_t + \frac{1}{\omega + \sigma} \mu_t \right) + \beta E_t \pi_{t+1}, \] \hspace{1cm} (15)

where \( \delta_t \) is the demand shock defined in (10), and where

\[ \epsilon_t \equiv \frac{\hat{\nu}}{\nu} (\xi_t - E_t \xi_{t+1}) \]

is an adverse “efficient” supply shock. We suppose that the vector of shocks \( u_t \equiv [\delta_t, \epsilon_t, \mu_t] \) satisfies \( E(u_t) = 0 \), and that these perturbations are independent of the parameters \( \sigma, \kappa, \) or \( \omega \).\(^{12}\)

As in Giannoni (2001a), we call the exogenous disturbance to the aggregate supply equation, \( \mu_t \), an “inefficient supply shock” since it represents a perturbation to the natural rate of output that is not efficient. While \( \mu_t \) represents fluctuations in the desired markup, this term may alternatively represent variations in distortionary tax rates, or variations in the degree of market power of workers. We prefer to call \( \mu_t \) an “inefficient supply shock” rather than a “cost-push shock” as is often done in the literature (see, e.g., Clarida et al., \(^{12}\) Again, \( \epsilon_t \) is independent of \( \omega \) if, for instance, the disutility of supplying goods is of the form \( v(y, \xi) = \varphi(y) \cdot \nu(\xi) \). (See footnote 11.)
1999), because perturbations that affect inflation by changing costs may well change the efficient rate of output as well as the natural rate of output. It follows that cost shocks are represented in our model by changes in \( x_t \) rather than \( \mu_t \).

Many recent studies have emphasized the role of the “natural” or “efficient” rate of interest for evaluating the stance of monetary policy (see, e.g., Blinder 1998, Woodford, 1999b, 1999c). The efficient rate of interest, i.e., the equilibrium real interest rate that would equate output to the efficient rate of output, \( \hat{Y}_t^e \), is defined here as

\[
r_t^e = \frac{\omega}{\omega + \sigma} \delta_t + \frac{\sigma}{\omega + \sigma} \varepsilon_t. \tag{16}
\]

Equation (14) can then be rewritten as

\[
x_t = E_t x_{t+1} - \sigma^{-1} (i_t - E_t \pi_{t+1} - r_t^e). \tag{17}
\]

It is clear from (16) that the efficient rate of interest depends both on demand shocks \( \delta_t \) and efficient supply shocks \( \varepsilon_t \). It follows from (17) that monetary policy is expansive or restrictive only insofar as the equilibrium real interest rate is below or above the efficient rate. If the central bank was perfectly tracking the path of \( r_t^e \), then the output gap would be zero at all times, and inflation would only depend on fluctuations in \( \mu_t \).\(^{13}\)

\(^{13}\)There is an additional first-order condition that determines the optimal holdings of monetary balances as a function of equilibrium consumption (or output), the nominal interest rate, and the price level. When monetary policy determines the nominal interest rate, as is the case here, this condition can be omitted as it has no effect on the equilibrium values of inflation, output, and nominal interest rate. The presence of real balances in the utility function (5) matters however for the determination of the loss function below.
3.2 Monetary Policy

We now turn to the objective of monetary policy. The policymaker is assumed to have the following loss function

\[
L_0 = E_0 \left\{ (1 - \beta) \sum_{t=0}^{\infty} \beta^t \left[ \pi_t^2 + \lambda_x (x_t - x^*)^2 + \lambda_i i_t^2 \right] \right\}
\]

where \( \lambda_x, \lambda_i > 0 \) are weights placed on the stabilization of the output gap and the nominal interest rate, and where \( x^* \geq 0 \) represents some optimal level of the output gap. (Note that we implicitly assume that the optimal levels of both inflation and the interest rate are zero). As in most studies of monetary policy, we assume that the policymaker seeks to stabilize fluctuations in inflation and in the output gap. We furthermore assume that he also cares about the variability of the nominal interest rate, as a result of transaction frictions. Friedman (1969) has argued that high nominal interest rates involve welfare costs of transactions. Whenever the deadweight loss is a convex function of the distortion, then it is desirable to reduce not only the level but also the variability of the nominal interest rate (see Woodford, 1990, 1999b). Such a loss criterion can finally obtained as a second-order Taylor approximation to the utility function of the household’s lifetime utility (5) in equilibrium, when the parameters are known with certainty. We will assume that the policymaker minimizes the unconditional expectation of the above loss criterion, \( E[L_0] \), where the expectation is taken with respect to the stationary distribution of the shocks. As a result, optimal policy will be independent of the initial state.

Following recent studies of monetary policy (see for example Taylor, 1999a), we characterize monetary policy in terms of interest-rate rules. Specifically, we assume that the
policymaker commits credibly at the beginning of period 0 to a feedback rule of the form

\[ i_t = P_t (\pi_t, \pi_{t-1}, ..., x_t, x_{t-1}, ..., i_{t-1}, i_{t-2}, ..., u_t, u_{t-1}, ...) \]  

(19)

for each date \( t \geq 0 \). The policymaker’s problem is to determine the functions \( P_t (\cdot) \), \( t = 0, 1, 2, ... \) to minimize the loss \( E[L_0] \) subject to the structural equations (14) and (15). As the objective is quadratic and the constraints are linear in all variables, we may without loss of generality restrict our attention to linear functions \( P_t (\cdot) \). Using the notation of section 2, we denote by \( \psi \) the finite-dimensional vector of coefficients that completely characterizes \( \{ P_t (\cdot) \}_{t=0}^{\infty} \), and we call \( \psi \) a “policy rule”.

3.3 Calibration

The model considered here is very similar to a simplified version of the econometric model that Rotemberg and Woodford (RW) (1997, 1999) have estimated for the US economy. The structural equations in RW correspond to (9) and (11) only when conditioned upon information available two quarters earlier.\(^{14}\) We will use their estimates to calibrate our

\(^{14}\)When conditioning both the intertemporal IS equation and the aggregate supply equation in RW (1997) upon information available at \( t - 2 \), we obtain

\[
E_{t-2} \hat{Y}_t = \ E_{t-2} \hat{Y}_{t+1} - \sigma^{-1} E_{t-2} \left( \hat{R}_t - \hat{\pi}_{t+1} \right) + E_{t-2} \left( \hat{G}_t - \hat{G}_{t+1} \right)
\]

\[
E_{t-2} \hat{\pi}_t = \ k E_{t-2} \left( \hat{Y}_t - \hat{Y}_S^\text{t} \right) + \beta E_{t-2} \hat{\pi}_{t+1},
\]

where \( \hat{Y}_t, \hat{\pi}_t, \hat{R}_t \) represent respectively output, inflation, and the nominal interest rate expressed as percentage deviations from steady state in RW (1997), and where \( \hat{G}_t \) is an exogenous variable representing autonomous changes in demand, and \( \hat{Y}_t^S \) represents exogenous disturbances to the aggregate supply equation. Defining \( \hat{Y}_t \equiv E_t \hat{Y}_{t+2}, \pi_t \equiv E_t \hat{\pi}_{t+2}, \ i_t \equiv E_t \hat{R}_{t+2}, \ \sigma^{-1} \frac{\hat{\omega}_t^\text{n} \epsilon_t}{\hat{\epsilon}_t} \equiv E_t \hat{G}_{t+2}, \) and \( \hat{Y}_t^\text{m} \equiv E_t \hat{Y}_{t+2}^S \), we obtain (9) and (11).
model. This will constitute our baseline parametrization. RW calibrate $\beta$, setting it at 0.99. They estimate $\sigma = .1571$, $\kappa = .0238$. The standard errors (se) for these parameters are respectively 0.0328 and 0.0035. These numbers were computed for the RW model using the estimation method explained in Amato and Laubach (1999).\(^{15}\) Finally, RW calibrate $\omega$, setting it at 0.4729. As we will consider uncertainty also about $\omega$ we will assume that the standard error is 0.0946, corresponding to 20% of the calibrated value (which is approximately in line with the uncertainty about $\sigma$ and $\kappa$). We assume that the uncertainty about the critical structural parameters is given by the approximate 95% intervals

$$
\begin{align*}
[\bar{\sigma}, \bar{\sigma}] &= [\sigma - 2se_\sigma, \sigma + 2se_\sigma] = [0.0915, 0.2227] \\
[\bar{\kappa}, \bar{\kappa}] &= [\kappa - 2se_\kappa, \kappa + 2se_\kappa] = [0.0168, 0.0308] \\
[\bar{\omega}, \bar{\omega}] &= [\omega - 2se_\omega, \omega + 2se_\omega] = [0.2837, 0.6621].
\end{align*}
$$

For simplicity, we assume that $\beta$ is known with certainty. We now turn to the calibration of the variance-covariance matrix of the exogenous disturbances. RW estimate the process for the exogenous variables $\hat{G}_t, \hat{Y}^S_t$ in their model. This process is given by

$$
\begin{align*}
\begin{bmatrix}
\hat{G}_{t+1} \\
\hat{Y}^S_{t+1}
\end{bmatrix}
&= \begin{bmatrix}
c_1 \\
c_2
\end{bmatrix} \begin{bmatrix}
\bar{Z}_{t-1} \\
\bar{d}_t
\end{bmatrix} + \begin{bmatrix}
d_1 \\
d_2
\end{bmatrix} \bar{e}_t \\
\bar{Z}_t
&= B\bar{Z}_{t-1} + U\bar{e}_t
\end{align*}
$$

where $E_t\bar{e}_{t+j} = 0$ for all $j > 0$, and the variance-covariance matrix of the state vector $\bar{Z}_t$ is $\Omega$. The variables $E_t\hat{G}_{t+2}$ and $E_t\hat{Y}^S_{t+2}$ in their model correspond respectively to $\sigma^{-1}\frac{\eta_t}{\omega}\xi_t$.

\(^{15}\)I am grateful to Thomas Laubach for providing me with these numbers.
and $\hat{Y}_t^n$ in our model. It follows that the process for $\delta_t$ is given by

$$\delta_t = \sigma E_t \left( \hat{G}_{t+2} - \hat{G}_{t+3} \right) = \sigma c_1 \left( \hat{Z}_t - E_t \hat{Z}_{t+1} \right) = \sigma c_1 \left( I - B \right) \hat{Z}_t.$$ 

Let us define the supply shock

$$s_t \equiv \mu_t + \frac{\nu \xi}{\nu \eta} \xi_t$$

We know from (12) that $s_t = \frac{\nu \xi}{\nu \eta} \xi_t - (\omega + \sigma) \hat{Y}_t^n$. It follows from the above equations that

$$s_t = \sigma E_t \hat{G}_{t+2} - (\omega + \sigma) E_t \hat{Y}_{t+2}^S = h \tilde{Z}_t$$

where $h \equiv \sigma c_1 - (\omega + \sigma) c_2$. While we can characterize the process for $s_t$, we don’t have enough information to determine the split between the efficient component $\frac{\nu \xi}{\nu \eta} s_t$, and the inefficient supply shock $\mu_t$. We therefore simply assume that $\mu_t = \nu s_t$ and $\frac{\nu \xi}{\nu \eta} s_t = (1 - \nu) s_t$, where $\nu$ is some constant between 0 and 1. It follows that the processes for the two supply shocks are given by

$$\varepsilon_t = (1 - \nu) h \left( \hat{Z}_t - E_t \hat{Z}_{t+1} \right) = (1 - \nu) h \left( I - B \right) \tilde{Z}_t,$$

and

$$\mu_t = \nu h \tilde{Z}_t.$$

As a result, the variance-covariance matrix of the vector of exogenous disturbances $u_t$ is given by

$$\mathbf{E} \left( u_t u_t' \right) = \begin{bmatrix} \sigma c_1 (I - B) \\ (1 - \nu) h (I - B) \\ \nu h \end{bmatrix} \Omega \begin{bmatrix} \sigma c_1 (I - B) \\ (1 - \nu) h (I - B) \\ \nu h \end{bmatrix}' \cdot \quad (20)$$

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We will compute the covariance matrix for different values of \( \nu \). Below we will consider uncertainty about \( \nu \), knowing only that \( \nu \) lies between 0 and 1.\(^{16}\) Finally, we will assume as in Woodford (1999c) that the three exogenous shocks follow an AR(1) process, with coefficients of serial correlation of \( \rho_\delta, \rho_\varepsilon, \rho_\mu \). Woodford (1999c) argues that the coefficient of autocorrelation of the natural rate of interest is 0.35. In contrast, Hansen and Sargent (2000b) assume that this coefficient is 0.8. We will consider as a benchmark the case in which \( \rho_\delta = \rho_\varepsilon = \rho_\mu = 0.35 \), but we will also consider the case in which there is uncertainty about the coefficients of autocorrelation, allowing their values to be anywhere in the \([0, 0.8]\) interval. The parameters are summarized in Table 1.

\(^{16}\) The assumption that both \( \mu_t \) and \( \frac{\varepsilon}{\psi} \xi_t \) are proportional to \( s_t \) may seem unappealing as it implies that these variables are perfectly correlated, as long as \( 0 < \nu < 1 \). However, as we will see below, once we consider uncertainty about \( \nu \), the variance-covariance matrix of \( u_t \) that matters is actually either the one for which \( \nu = 0 \) or the one for which \( \nu = 1 \).
4 Robust Optimality within a Flexible Class of Interest-Rate Rules

We now turn to the characterization of optimal monetary policy within a flexible class of interest-rate rules $\hat{\Psi}$ that allow the instrument to respond to past variables. We define $\hat{\Psi}$ as the set of policy rules

$$\psi = [\psi_\pi, \psi_x, \psi_{i1}, \psi_{i2}]$$

satisfying

$$i_t = \psi_\pi \pi_t + \psi_x (x_t - x_{t-1}) + \psi_{i1} i_{t-1} + \psi_{i2} i_{t-2}$$

(21)

at all dates $t \geq 0$.\(^{17}\) As will become clear below, the set $\hat{\Psi}$ is flexible enough to include a fully optimal rule in the case of any parameter vector $\theta \in \Theta$ (if the parameters were known with certainty), though it is still specific enough to contain only one rule consistent with the optimal plan in any such case. Moreover this class of rules includes recent descriptions of actual monetary policy such as the one proposed by Judd and Rudebusch (1998). We start with the characterization of the optimal plan for a given $\theta$, and propose an interest-rate rule that implements that plan. We next determine the minmax equilibrium, and the robust optimal policy rule that implements it.

4.1 Optimal Plan with Given Parameters

To characterize the optimal plan for a given parameter vector $\theta \in \Theta$, we determine the stochastic process $q^*(\theta)$ of endogenous variables that minimizes the unconditional

\(^{17}\)As we evaluate monetary policy regardless of specific initial conditions, the policy rule is assumed to be independent of the values the endogenous variables might have taken before it was implemented. Specifically, we assume that the policymaker considers the initial values as satisfying $i_{-2} = i_{-1} = x_{-1} = 0$, whether they actually do or not. Equivalently, we could assume that the policy rule satisfies $i_0 = \psi_\pi \pi_0 + \psi_x x_0$, $i_1 = \psi_\pi \pi_1 + \psi_x (x_1 - x_0) + \psi_{i1} i_0$, and (21) at all dates $t \geq 2$. 

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expectation of the loss criterion (18) subject to the constraints (14) and (15) at all dates \( t \geq 0 \), and in every state that may occur at date \( t \), i.e., for every possible history of the disturbances until that date. In terms of the notation laid out in section 2, we determine the stochastic process \( q^* (\theta) \) that minimizes the loss \( \hat{L} (q) \) subject to the restrictions (1) imposed by the structural equations (14) and (15). The policymaker’s Lagrangian can be written as

\[
\mathcal{L} = \mathbb{E} \left\{ \mathbb{E}_0 (1 - \beta) \sum_{t=0}^{\infty} \beta^t \left[ \pi_t^2 + \lambda_x (x_t - x^*)^2 + \lambda_i t^2 \right] \right. \\
+ 2 \phi_{1t} \left[ x_t - x_{t+1} + \sigma^{-1} (\hat{i}_t - \pi_{t+1}) - \frac{\omega}{\omega + \sigma} \delta_t - \frac{1}{\omega + \sigma} \varepsilon_t \right] \\
+ 2 \phi_{2t} \left[ \pi_t - \kappa \left( x_t + \frac{1}{\omega + \sigma} \mu_t \right) - \beta \pi_{t+1} \right] \left\} \right.
\]

(22)

The first-order necessary conditions with respect to \( \pi_t, x_t, \) and \( \hat{i}_t \) are

\[
\pi_t - (\beta \sigma)^{-1} \phi_{1t-1} + \phi_{2t} - \phi_{2t-1} = 0 
\]

(23)

\[
\lambda_x (x_t - x^*) + \phi_{1t} - \beta^{-1} \phi_{1t-1} - \kappa \phi_{2t} = 0 
\]

(24)

\[
\lambda \hat{i}_t + \sigma^{-1} \phi_{1t} = 0 
\]

(25)

at each date \( t \geq 0 \), and for each possible state. In addition, we have the initial conditions

\[
\phi_{1,-1} = \phi_{2,-1} = 0 
\]

(26)

indicating that the policymaker has no previous commitment at time 0. Note that since the objective function is convex in \( q \), and the constraints are linear in \( q \), (14), (15), and (23) – (25) at all dates \( t \), together with the initial condition (26) are not only necessary but also sufficient to determine the bounded optimal plan \( \{ \pi_t, x_t, \hat{i}_t, \phi_{1t}, \phi_{2t} \} \). In the steady-state, i.e., in the absence of perturbations, (14), (15), and (23) – (26) reveal that
the endogenous variables remain constant at the values
\[
\begin{align*}
\pi^{op} &= x^{op} = i^{op} = \phi_1^{op} = 0 \\
\phi_2^{op} &= -\frac{\lambda_x}{\kappa} x^*.
\end{align*}
\]

It will be convenient to replace \( \phi_{2t} \) with \( \hat{\phi}_{2t} \equiv \phi_{2t} - \phi_{2t}^{op} \) so that the constant drops out of (24). Using (25) to substitute for the interest rate, we can rewrite the dynamic system (14), (15), (23), and (24) in matrix form as
\[
E_t \begin{bmatrix}
  z_{t+1} \\
  \phi_t
\end{bmatrix} = M \begin{bmatrix}
  z_t \\
  \phi_{t-1}
\end{bmatrix} + m u_t,
\] (27)
where \( z_t \equiv [\pi_t, x_t, i_t]' \), \( \phi_t \equiv [\phi_{1t}, \hat{\phi}_{2t}]' \), \( u_t \equiv [\delta_t, \varepsilon_t, \mu_t]' \), and \( M \) and \( m \) are matrices of coefficients. Following Blanchard and Kahn (1980), this dynamic system has a unique bounded solution (given a bounded process \( \{u_t\} \)) if and only if the matrix \( M \) has exactly two eigenvalues outside the unit circle. Investigation of the matrix \( M \) reveals that if a bounded solution exists, it is unique.\(^{18} \) In this case the solution for the endogenous variables can be expressed as
\[
q_t = D\phi_{t-1} + \sum_{j=0}^{\infty} d_j E_t u_{t+j},
\] (28)
where \( q_t \equiv [\pi_t, x_t, i_t]' \), and the Lagrange multipliers follow the law of motion
\[
\dot{\phi}_t = N\phi_{t-1} + \sum_{j=0}^{\infty} n_j E_t u_{t+j},
\] (29)
for some matrices \( D, N, d_j, n_j \) that depend upon the parameters of the model. Woodford (1999c) has emphasized that in the optimal plan with given structural parameters,\(^{18} \) The matrix \( M \) has two eigenvalues with modulus greater than \( \beta^{-1/2} \) and two with modulus smaller than this.
the endogenous variables should depend not only upon expected future values of the disturbances, but also upon the predetermined variables $\phi_{t-1}$. This dependence indicates that optimal monetary policy should involve inertia in the interest rate, regardless of the possible inertia in the exogenous perturbations. In fact, as argued by Woodford (1999c), policymakers who choose optimal actions by disregarding their past actions and past states of the economy, don’t achieve the best equilibrium when the private sector is forward-looking. The central bank should realize that the evolution of its goal variables depends not only upon its current actions, but also upon how the private sector foresees future monetary policy. It should therefore act in a way that affects the response of the private sector appropriately. As will become clearer below, it can do so by committing itself to a rule of the kind (21).

Figure 1 plots with solid lines the optimal response of the interest rate, inflation, and the output gap to an unexpected demand shock, in the baseline calibration. The disturbance $\delta_t$ unexpectedly increases by 1 at date 0 and is expected to return to steady state following an AR(1) process with a coefficient of autocorrelation of $\rho_b = 0.35$. The path that the efficient rate of interest (16) is expected follow is indicated by the dashed-dotted line in the upper panel. Similar impulse responses would be generated by an adverse efficient supply shock, i.e., an increase in $\varepsilon_t$. While the policymaker could in principle completely stabilize the output gap and inflation, by tracking the path of the efficient rate of interest, it is optimal to increase the nominal interest rate by less than

\[ \text{the efficient rate of interest at the period of the shock because the policymaker also wants} \]

\[ \text{19 The impulse responses of all variables are reported in annual terms. Therefore, the responses of } i_t \text{ and } \pi_t \text{ are multiplied by 4.} \]
to dampen fluctuations in the nominal rate of interest. As monetary policy is relatively
expansionary, inflation and the output gap increase in response to the perturbation. The
short-term interest is also more inertial than the efficient rate. Inertia in monetary policy
is especially desirable here because it induces the private sector to expect future negative
output gaps which in turn have a disinflationary effect. Therefore, by acting in an inertial
way, the policymaker can offset the inflationary impact of the shock without having to
raise the short-term interest much. Qualitatively similar figures would be obtained for
different degrees of serial correlation in the perturbations.

Figure 2 displays with solid lines the optimal response of the endogenous variables
to an unexpected inefficient supply shock $\mu_t$. Specifically, we assume that the desired
markup increases unexpectedly by one percentage point at date 0, and is expected to
return to steady-state according to AR(1) process with coefficient of autocorrelation
$\rho_\mu = 0.35$. Figure 2 reveals that it is optimal to slightly raise the nominal interest rate.
This helps maintaining the output gap (and output since there is no change in $\hat{Y}_t$) below
steady state for several periods. As a result, the private sector expects a slight deflation
in the future, which removes some inflationary pressure already at the time of the shock.

4.2 Optimal Interest-Rate Rule with Given Parameters

We now turn to the determination of an optimal interest rate rule, namely the policy rule
in the family (21) that implements the optimal plan, for given structural parameters $\theta$.
We solve (25) for $\phi_{1t}$ and (24) for $\phi_{2t}$, and use the resulting expressions to substitute for
the Lagrange multipliers in (23). This yields

$$
\hat{i}_t = \frac{\kappa}{\lambda_i \sigma} \pi_t + \frac{\lambda_x}{\lambda_i \sigma} (x_t - x_{t-1}) + \left(1 + \frac{\kappa}{\beta \sigma + \beta^{-1}}\right) \hat{i}_{t-1} - \beta^{-1} \hat{i}_{t-2} \tag{30}
$$
for all $t \geq 0$. As this equilibrium condition relates the endogenous variables in the optimal plan, the policy rule

$$\psi^\ast (\theta) = \left[ \frac{\kappa}{\lambda_i \sigma} \frac{\lambda_x}{\lambda_i \sigma} \left( \frac{\lambda_x}{\lambda_i \sigma} + \frac{\kappa}{\beta \sigma} + \beta^{-1} \right) - \beta^{-1} \right]^t$$

(31)

satisfies the restrictions $P (q^\ast (\theta), \psi^\ast (\theta)) = 0$. Furthermore, since the endogenous variables entering (30) minimize the loss criterion $\hat{L} (q)$ subject to the constraints (1) in the optimal plan, the following lemma guarantees that $\psi^\ast (\theta)$ is an optimal rule for any given $\theta \in \Theta$, provided that it results in a unique bounded equilibrium.

**Lemma 4** Suppose that $q^\ast (\theta)$ minimizes $\hat{L} (q)$ subject to (1) for any given $\theta \in \Theta$, and that there exists $\psi^\ast (\theta) \in \Psi$ that solves $P (q^\ast (\theta), \psi^\ast (\theta)) = 0$ for all $\theta \in \Theta$. Then $\psi^\ast (\theta) \in \arg \min_{\psi \in \Psi} L (\psi, \theta)$.

**Proof.** First note that since $\psi^\ast (\theta) \in \Psi$, the latter policy rule results in a unique bounded equilibrium. Suppose as a way of contradiction that there exists a policy rule $\psi^\dagger (\theta) \in \Psi$, $\psi^\dagger (\theta) \neq \psi^\ast (\theta)$, satisfying $L (\psi^\dagger (\theta), \theta) < L (\psi^\ast (\theta), \theta)$. By definition of $L (\cdot)$ and $\hat{L} (\cdot)$ we have $L (\psi, \theta) = \hat{L} (q (\psi, \theta))$ for all $\psi \in \Psi, \theta \in \Theta$, so that $\hat{L} (q (\psi^\dagger (\theta), \theta)) < \hat{L} (q (\psi^\ast (\theta), \theta)) = \hat{L} (q^\ast (\theta))$. But then $q^\ast (\theta)$ cannot minimize $\hat{L} (q)$ subject to (1). $\blacksquare$

The dynamic system obtained by combining (14), (15), and (30), has the property of system (27) that, if any bounded solution exists, it is unique. Moreover it can be shown, at least in the baseline parametrization, and for all values $\theta \in \Theta$ of our example, that $\psi^\ast (\theta)$ is the unique optimal policy rule in the set $\tilde{\Psi}$ (see Appendix 6.2.1 in Giannoni, 2001a).

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20 See remarks in footnote 17.

21 The eigenvalues of this system are the same as the eigenvalues of $M$ in (27) plus one eigenvalue equal to zero. As there is one predetermined variable more than in (27), this system yields a unique bounded equilibrium, if it exists.
Notice that this rule makes no reference to any of the exogenous shocks. It achieves the minimal loss *regardless* of the processes that describe the evolution of $\delta_t$, $\varepsilon_t$, and $\mu_t$, provided that the latter processes are stationary (bounded). If we would allow for a broader class of policy rules than $\Psi$, other interest-rate feedback rules may implement the same optimal plan. Woodford (1999c), for example, proposes a rule in which the interest rate depends upon current and lagged values of the inflation rate as well as lagged interest rates in a similar model in which there is no inefficient supply shock. While his rule makes no reference to the output gap, it is dependent upon the driving process of the efficient rate of interest.

Equation (30) indicates that to implement the optimal plan, the central bank should relate the interest rate positively to fluctuations in current inflation, in *changes* of the output gap, and in lagged interest rates. While it is doubtful that the policymaker knows the current level of the output gap with great accuracy, the *change* in the output gap may be known with greater precision. For example, Orphanides (1998) shows that subsequent revisions of U.S. output gap estimates have been quite large (sometimes as large as 5.6 percentage points), while revisions of estimates of the quarterly change in the output gap have been much smaller.

Note finally that the interest rate should not only be inertial in the sense of being positively related to past values of the interest rate, it should be *super-inertial*, as the lagged polynomial for the interest rate in (30)

$$1 - \left( 1 + \frac{\kappa}{\beta \sigma} + \beta^{-1} \right) L + \beta^{-1}L^2 = (1 - z_1 L) (1 - z_2 L)$$
involves a root $z_1 > 1$ while the other root $z_2 \in (0, 1)$. A reaction greater than one of the interest rate to its lagged value has initially been found by Rotemberg and Woodford (1999) to be a desirable feature of a good policy rule in their econometric model with optimizing agents. As explained further in Woodford (1999c), it is precisely such a super-inertial policy rule that the policymaker should follow to bring about the optimal responses to shocks when economic agents are forward-looking. Because of a root larger than one, the optimal policy requires an explosively growing response of the interest rate to deviations of inflation and the output gap from target.

This is illustrated in Figure 3 which displays the response of the interest rate to a sustained 1 percent deviation in inflation (upper panel) or the output gap (lower panel) from target. In each panel, the solid line represents the optimal response in the baseline case. The corresponding coefficients of the optimal policy rule are reported in the upper panel of Table 2 (lines indicated by $\psi^0$). For comparison, the lower panel of Table 2 reports the coefficients derived from Judd and Rudebusch’s (1998) estimation of actual Fed reaction functions between 1987:3 and 1997:4.\textsuperscript{22} Table 2 reveals that the estimated historical rule in the baseline case involves only slightly smaller responses to fluctuations in inflation and the output gap than the optimal rule. However the estimated response to lagged values of the interest rate is sensibly smaller that the optimal one. As a result, the estimated historical rule involves a non-explosive response of the interest rate to a sustained deviation in inflation or the output gap, represented by the dashed-dotted lines in Figure 3.

\textsuperscript{22}The estimated historical policy rule refers to regression A for the Greenspan period in Judd and Rudebusch (1998).
Table 2 also reports the loss $E L_0$ along with the following measure of variability

$$V[z] \equiv E \left\{ E_0 \left[ (1 - \beta) \sum_{t=0}^{\infty} \beta^t \xi_t^2 \right] \right\}$$

of the endogenous variables $\pi$, $x$ and $i$, for the various policy rules and parameter configurations. The statistic $V[z]$ determines the contribution of each endogenous variable to the loss $E L_0$, as the latter measure is a weighted sum of $V[\pi]$, $V[x]$, and $V[i]$ with weights corresponding to those of the loss function (18). The lines of Table 2 indicated by $\theta^0$ report statistics evaluated using the baseline parametrization. This table indicates to what extent the optimal rule results in a lower loss than the estimated historical rule.

While optimal policy would involve an explosive behavior of the interest rate in the face of a sustained deviation of inflation or the output gap, such a policy is perfectly consistent with a stationary rational expectations equilibrium, and a low variability of the interest rate in equilibrium. (In Table 2, $V[i]$ is always smaller when the interest rate is set according to the optimal flexible rule, than when it is set according to the estimated historical rule.) In fact, the interest rate does not explode in equilibrium because (as appears clearly in Figures 1 and 2) the current and expected future optimal levels of the interest rate are sufficient to counteract the effects of an initial deviation in inflation and the output gap by generating subsequent deviations with the opposite sign of these variables.

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23 All statistics in Table 2 are expressed in annual terms. The statistics $V[\pi]$, $V[i]$, and $E[L_0]$ are therefore multiplied by 16. Furthermore, the weight $\lambda_x$ reported in Table 1 is also multiplied by 16 in order to represent the weight attributed to output gap variability (in annual terms) relative to the variability of annualized inflation and of the annualized interest rate.
4.3 Robust Optimal Policy in the Presence of Parameter Uncertainty

So far we have assumed that all parameters are known with certainty. We now determine the robust optimal policy rule that obtains when the policymaker faces uncertainty about the three structural parameters $\sigma$, $\kappa$, and $\omega$, the degrees of serial correlation of the exogenous perturbations $\rho_\delta$, $\rho_\varepsilon$, $\rho_\mu$, and the parameter $\nu$ describing the importance of the inefficient supply shocks. We consider uncertainty about the parameter vector $\theta = [\sigma, \kappa, \omega, \rho_\delta, \rho_\varepsilon, \rho_\mu, \nu]'$ within a specified set $\Theta = [\underline{\theta}, \overline{\theta}]$, where the extent of uncertainty for $\sigma$, $\kappa$, and $\omega$ is given by the approximate 95% confidence intervals mentioned in section 3.3, and reported in Table 1. We assume that the coefficients of autocorrelation lie in the $[0, 0.8]$ interval.\footnote{Alternative intervals yield similar results. It is important for our methodology, however, that the interval of uncertainty be closed.} Finally, we allow $\nu$ to lie anywhere in the $[0, 1]$ interval.

In the previous section, we have characterized the optimal equilibrium for any given parameter vector $\theta$. Following the solution method summarized in section 2, we need to determine the candidate worst-case parameter vector, i.e., the parameter vector $\theta^*$ that obtains in the candidate minmax equilibrium. Once the worst-case parameter vector is identified, it will be straightforward to determine the robust optimal rule.

4.3.1 Worst-Case Parameter Vector and Minmax Equilibrium

We now determine the worst-case parameter vector $\theta^*$, i.e., the vector $\theta$ that maximizes the loss function $\hat{L}(q^*(\theta))$ on the constraint set $\Theta$, or in other words the vector determining the minmax equilibrium $q^*(\theta^*)$. For any parameter vector $\theta \in \Theta$, the structural equations (14), (15) and the first-order conditions (23) – (25) can be written in matrix form, as in (27), and standard methods can be applied to get the solution of the form (28)
– (29). As in the certainty case, the bounded solution is unique if one exists. Equations (28) – (29) can then be used to compute the loss \( \tilde{L}(q^*(\theta)) \). Maximizing this loss function with respect to \( \theta \in \Theta \), we obtain:

\[
\theta^* = \begin{bmatrix} \underline{a}, \bar{\kappa}, \underline{w}, \bar{\rho}_k, \rho_e, \bar{\rho}_\mu, \bar{v} \end{bmatrix}'
\]

\[
= [0.0915, 0.0308, 0.2837, 0.8, -, 0.8, 1].
\]

Note that \( \rho_e \) may take any value in the allowed interval [0, 1], since the loss is maximized when \( \nu^* = 1 \), i.e., when there are no efficient supply shocks. We performed the maximization of \( \tilde{L}(q^*(\theta)) \) numerically starting from a large number of different initial values for \( \theta \), including values close to the boundaries of the set \( \Theta \). There are a few local maxima, but none of them yields a loss higher than the one implied by the parameter vector reported above.

### 4.3.2 Robust Optimal Policy Rule

Following the solution procedure reviewed in section 2, we characterize the robust optimal policy rule simply by looking at the best response \( \psi^*(\theta^*) \) to the worst-case parameter vector determined above, assuming that a global NE does exist. As (31) is the best response to any given \( \theta \), the candidate robust optimal policy rule, \( \psi^* = \psi^*(\theta^*) \) satisfies

\[
\dot{i}_t = \frac{\bar{\kappa}}{\lambda_\sigma} \pi_t + \frac{\lambda_\sigma}{\lambda_\rho} (x_t - x_{t-1}) + \left( 1 + \frac{\bar{\kappa}}{\beta_\sigma} + \beta^{-1} \right) \dot{i}_{t-1} - \beta^{-1} \dot{i}_{t-2}.
\]

The couple \((\psi^*, \theta^*)\) constitutes at least a local NE, and the policy rule satisfying (33) is a robust optimal policy rule, provided that a global NE exists.

In accordance with step 4 of our solution procedure, we verify that the couple \((\psi^*, \theta^*)\) is also a global NE. We do so by maximizing the loss function \( L(\psi^*, \theta) \equiv E[L_0(q(\psi^*, \theta))] \)
numerically with respect to $\theta \in \Theta$. This loss function results from the equilibrium obtained by combining the structural equations (14), (15) for any given $\theta$, and the candidate robust optimal policy rule $\psi^*$ satisfying (33). We repeated the maximization many times, starting each time from a different initial values for $\theta$. We obtained again that the vector $\theta^*$ defined in (32) maximizes this loss function. It follows that by choosing $\theta^*$, malevolent Nature best-responds to the policy rule $\psi^*$, and that $(\psi^*, \theta^*)$ is indeed a global NE. Because $\theta^*$ is the only equilibrium parameter vector, and $\psi^* \equiv \psi^*(\theta^*)$ is the unique best response to $\theta^*$ in the set $\tilde{\Psi}$, the profile $(\psi^*, \theta^*)$ constitutes the unique global NE, and $\psi^*$ is the unique robust optimal policy rule in the class $\tilde{\Psi}$.

Denoting by $\sigma_0 \in (\underline{\sigma}, \bar{\sigma})$ and $\kappa_0 \in (\underline{\kappa}, \bar{\kappa})$ the parameter values in the absence of model uncertainty, and noting that $\underline{\sigma} < \sigma_0$ and $\kappa_0 < \bar{\kappa}$, it is clear that the policymaker reacts more strongly to perturbations to inflation, changes in the output gap, and the lagged interest rate than is the case in the absence of uncertainty. As illustrated in Figure 3, the robust optimal rule (dashed line) involves (i) a larger response to a sustained increase in inflation, at every time, (ii) a larger response to a sustained increase in the output gap, at every time, and (iii) a faster asymptotic rate of explosion of both of these responses (thus a greater degree to which the policy rule is super-inertial).

To give a sense of the magnitude of optimal policy coefficients, we report in the middle panel of Table 2 the robust optimal rule (33) (lines indicated by $\psi^*$), in addition to the optimal policy rule (30) in the certainty case (lines $\psi^0$). It can be verified that the robust rule involves stronger responses of the interest rate to fluctuations in inflation, changes in the output gap, and the first lagged value of the interest rate. The lagged polynomial
for the interest rate can be written as

\[
\text{Baseline} : \quad 1 - 2.163L + 1.010L^2 = (1 - 1.481L)(1 - 0.682L)
\]

\[
\text{Robust} : \quad 1 - 2.350L + 1.010L^2 = (1 - 1.784L)(1 - 0.566L).
\]

As the larger root is even greater in the presence of uncertainty, the interest rate is *supер-inertial* to an even greater extent when the central bank follows the robust optimal rule. But again, the presence of a root larger than one is consistent with a stationary rational expectations equilibrium. This is illustrated for instance by the dashed lines in Figures 1 and 2, which represent the impulse responses of the endogenous variables when the policymaker follows the robust optimal policy rule (but when the true parameters are those of the baseline calibration).

Figure 4 compares the performance of the robust optimal rule \(\psi^*\) (solid line) and the rule \(\psi^0\) (dashed line) that is optimal in the absence of parameter uncertainty. Each panel plots the loss as a function of one parameter, keeping the remaining parameters at the respective worst-case values. It appears that the robust rule performs better than the baseline rule at least when the parameters are close to their worst-case value. In contrast, the upper right panel indicates that the loss is higher with the robust rule than with the baseline rule when \(\kappa\) is relatively low. The figure also reveals that if all elements of \(\theta\) with the exception of one, reach their worst-case values at the respective values in \(\theta^*\), then the remaining element of \(\theta\) also maximizes the loss at the corresponding value in \(\theta^*\). This illustrates the fact that \(\theta^*\) is a best response on the part of malevolent Nature to the robust rule \(\psi^*\). As further indicated in Table 2, while the robust optimal rule \(\psi^*\) performs only slightly better than \(\psi^0\), i.e., the optimal rule absent model uncertainty,
both $\psi^0$ and $\psi^*$ perform significantly better than the estimated historical rule.

As we discuss in the next section, note that while the parameters $\omega, \rho_\delta, \rho_\varepsilon, \rho_\mu, \nu$ don’t enter directly the policy rule (33), they may still indirectly affect the robust optimal rule by determining the worst-case parameter configuration.

5 Discussion

In this section, we first try to give some intuition for the worst-case parameter configuration. We then discuss the sensitivity of the robust optimal rule to alternative assumptions.

5.1 Worst-Case Parameter Configuration: Intuition

We now try to give some intuition about the worst-case parameter values for $\sigma, \kappa,$ and $\omega$. We do this in a simple case in which monetary policy is assumed to be non-inertial — i.e., it does not depend on lagged variables — and all shocks are i.i.d. In this case, all future variables are expected to remain at steady state in equilibrium ($E_t x_{t+1} = E_t \pi_{t+1} = 0$), so that the two structural equations (14) and (15) reduce to

\[
\hat{i}_t = -\sigma x_t + \frac{\omega}{\omega + \sigma} \delta_t + \frac{\sigma}{\omega + \sigma} \varepsilon_t \\
\pi_t = \kappa \left( x_t + \frac{1}{\omega + \sigma} \mu_t \right).
\]

These two equations are represented by respectively the lines IS and AS in Figures 5a to 5c.

Figure 5a represents the effects of a unit exogenous increase in $\delta_t$. In the case in which the parameters are known with certainty, the IS curve shifts vertically from $IS(\sigma_0, \omega_0)$ to $IS'(\sigma_0, \omega_0)$, by an amount $\frac{\omega_0}{\omega_0 + \sigma_0}$. The policymaker faces a trade-off between the stabilization of inflation and the output gap on one hand, and the interest rate on the
other hand. He could completely stabilize inflation and the output gap by raising the interest rate by \( \frac{\sigma_0}{\omega_0 + \sigma_0} \). Such a policy is however not optimal as the policymaker also cares about fluctuations in the interest rate (see loss function (18)). He acts optimally by increasing the interest rate to some level \( i_t^0 \), and letting the output gap increase to \( x_t^0 \). From the lower panel of the figure, we note that inflation, determined by the AS equation, rises to \( \pi_t^0 \). In the presence of parameter uncertainty and a demand shock, the worst case is obtained when \( \sigma \) is as low as possible, and \( \omega \) is as high as possible, so that \( \sigma^* = \sigma \) and \( \omega^* = \omega \). Equation (34) reveals that it is in this case that a given increase in \( \delta_t \) results in the highest possible upward shift of the IS curve. Furthermore, as this implies a flatter IS curve, the output gap increases by more, for given nominal interest rate. On the supply side, the worst value for \( \kappa \) is obtained when \( \kappa^* = \bar{\kappa} \), so that any given change in the output gap is associated with a large change in inflation. As a result, the policymaker who seeks to minimize the loss in the worst-case parameter configuration optimally sets the interest rate above \( i_t^0 \) in the presence of uncertainty, in order to contain the increase both in the output gap and in inflation.\(^{25}\)

Figure 5b shows that the effects of a unit exogenous increase in \( \varepsilon_t \) are qualitatively similar to those of a shock to \( \delta_t \) when parameters are known with certainty. Note that it is the IS and not the AS schedule that shifts following an efficient supply shock. The IS schedule shifts upwards to IS’ by an amount \( \frac{\sigma_0}{\omega_0 + \sigma_0} \). However, in the presence of parameter uncertainty, the worst-case value for \( \omega \) is \( \omega \), because it generates the largest shift of the IS

\(^{25}\)The fact that the nominal interest rate rises above \( i_t^0 \) in the presence of uncertainty does not necessarily mean that the policy rule involves larger responses to variables such as inflation and the output gap, since the latter variables may increase too. However, we know from (30) that the robust optimal rule requires larger responses to fluctuations in inflation and the output gap when the worst-case values for \( \sigma \) and \( \kappa \) are respectively \( \sigma \) and \( \bar{\kappa} \).
schedule. As before, the worst-case value for $\kappa$ is $\bar{\kappa}$, so that a given non-zero output gap results in the largest change in inflation. It is however not trivial to determine a priori the worst-case value for $\sigma$. Whether $\sigma^*$ is $\underline{\sigma}$, $\bar{\sigma}$, or any value in between is an empirical question. For the calibration of Table 1, it is $\bar{\sigma}$, as this is the value which is responsible for the largest upward shift of IS, even though it implies a steeper IS schedule.

Figure 5c illustrates the effects of an inefficient supply shock, which shifts the AS curve from $AS(\kappa_0)$ up to $AS'(\kappa_0)$, in the absence of parameter uncertainty. The policymaker faces a trade-off between the stabilization of inflation on one hand, and the output gap on the other hand. He acts optimally by raising the interest rate to some level $i^0_t$, so that the output gap decreases to $x^0_t$, and inflation increases to $\pi^0_t$. In the presence of parameter uncertainty, the worst case slope of the AS curve is again obtained when $\kappa^* = \bar{\kappa}$, and the horizontal shift of the AS curve is largest when $\sigma^* = \underline{\sigma}$, and $\omega^* = \underline{\omega}$. In the upper panel, however, we notice that the worst slope of the IS curve obtains when $\sigma^* = \bar{\sigma}$, so that the policymaker needs to increase the interest rate by more, to obtain a given change in the output gap.

To summarize, while the worst-case value for $\kappa$ is $\bar{\kappa}$ regardless of the shock considered, the worst-case parameter values for $\sigma$ and $\omega$ depend on the parametrization of the model, and the relative importance of the disturbances. However, as discussed in section 4, the worst case parameter configuration involves $\sigma^* = \underline{\sigma}$ and $\omega^* = \underline{\omega}$ in the model considered and with the parametrization summarized in Table 1.

Although this subsection focuses on the simple case in which shocks are transitory, the minmax equilibrium involves very persistent shocks in the more general model of the previous sections. Indeed, the worst-case values for the coefficients of serial correlations
are equal to the upper bound 0.8. Note that this is true even though the variance-
covariance matrix of the shocks (20) is given, and independent of $\rho_\delta$, $\rho_\epsilon$, and $\rho_\mu$.

5.2 Sensitivity of Robust Policy to Alternative Assumptions

Equation (33) indicates that the robust optimal rule depends critically on the worst-case
values for $\sigma$ and $\kappa$. As discussed above, the worst-case value for $\kappa$ is $\bar{\kappa}$ regardless of the
importance of the shocks considered — at least when monetary policy is non-inertial.
This induces the policymaker to let the interest rate react more strongly to fluctuations
in inflation and lagged interest rates, in the presence of parameter uncertainty. By
how much the policymaker should actually respond depends on the value attributed to $\bar{\kappa}$. Specifically, the robust rule involves larger responses to fluctuations in inflation and
lagged interest rate, the higher the upper bound on the slope of the short run aggregate
supply schedule.

As mentioned above, the worst-case value for $\sigma$ is $\underline{\sigma}$, for the calibration summarized in
Table 1. Figure 6 indicates that $\bar{\kappa}$ and $\underline{\sigma}$ remain the worst-case values for $\kappa$ and $\sigma$, under
alternative assumptions about $\omega$ and the degree of serial correlation of the perturbations,$\rho$, when $\nu$ is maintained at its worst-case value $\nu^* = 1$. This figure represents contour
plots of the loss criterion $E[L_0]$ as a function of the parameters $\kappa$ and $\sigma$, when monetary
policy is conducted according to the robust optimal rule (33). These contour plots are
produced for various values of $\omega$ and $\rho$, and we report the cases in which $\omega$ is respectively
$\underline{\omega}$, the baseline value $\omega_0$, and $\bar{\omega}$, and $\rho$ is respectively 0, 0.35, and 0.8. The star in each plot
indicates the baseline values for $\sigma$ and $\kappa$. The figure reveals that the worst-case couple
$(\kappa^*, \sigma^*)$ — indicated by a circled star — is in each case in the lower right corner, i.e.,
when \( \kappa \) is as large as possible and \( \sigma \) is as small as possible. Similar results are obtained for alternative values for \( \omega \) and \( \rho \), and for alternative values of \( \nu \), provided that \( \nu \) is larger than a critical value around 0.5. This suggests that the robust optimal rule is not affected by alternative assumptions about \( \omega \) and \( \rho \), as long as \( \nu \) is large enough.

However, if \( \nu \) is small — so that most supply shocks are efficient supply shocks \( \varepsilon_t \) — then the worst-case value for \( \sigma \) may be \( \bar{\sigma} \), as argued in subsection 5.1. For instance, if \( \nu \) is constrained to lie in the interval \([0, 0.3]\), then the worst case value for \( \nu \) is 0, as can be guessed from Figure 4 (see panel that represents the loss as a function of \( \nu \)). It follows that the worst case value for \( \sigma \) is \( \bar{\sigma} \). This, in turn, tends to make the robust rule respond less to fluctuations in inflation, output gap and lagged interest rates. Clearly, the response to fluctuations in the output gap would be smaller in the presence of uncertainty than in the certainty case, as \( \lambda_x (\lambda_i \bar{\sigma})^{-1} < \lambda_x (\lambda_i \sigma_0)^{-1} \). However, whether the robust rule responds more to fluctuations in inflation and in the lagged interest rate depends in the end on the amount of uncertainty about \( \kappa \) relative to the one about \( \sigma \). In our numerical example, if the worst case value for \( \sigma \) is \( \bar{\sigma} \), it appears from (33) that the response coefficients to fluctuations in inflation and the lagged interest rate are also smaller in the presence of uncertainty, since \( \tilde{\kappa} / \bar{\sigma} = 0.0308 / 0.2227 < 0.0238 / 0.1571 = \kappa_0 / \sigma_0 \).

To summarize, the optimal rule (30) does not depend on \( \omega, \nu \), as well as any parameters describing the shock processes. Similarly, the robust optimal policy rule (33) does not depend on \( \omega \) and the coefficients of serial correlation of the shocks, provided that \( \nu \) is large enough. However, if \( \nu \) is constrained to be small, then the worst-case value for \( \sigma \) is \( \bar{\sigma} \), and the robust optimal rule is less aggressive than the optimal rule absent parameter uncertainty.
6 Conclusion

In this paper, we have characterized a robust optimal policy rule in a simple forward-looking model, when the policymaker faces uncertainty about the parameters of the structural model and the nature of the shock processes. We have derived the structural model from first principles to determine precisely how the exogenous perturbations are transmitted to the endogenous variables.

The optimal policy rule considered here has a number of advantages with respect to simpler policy rules such as the Taylor rule. First, as it implements the optimal plan in the absence of parameter uncertainty, it achieves the lowest possible loss, and hence performs better than restricted policy rules. Second, the analytical characterization of the optimal rule allows us to identify to what extent the policy rule is sensitive to particular parameters. While the optimal Taylor rule derived in Giannoni (2001b) depends critically on the characteristics of the exogenous shock processes, the optimal rule proposed here does not depend them, in the absence of parameter uncertainty. The invariance to various specifications of the shock processes is an attractive feature of the optimal rule, especially when exogenous disturbances cannot be observed directly. The robust optimal rule depends however indirectly on the assumptions about the shock process to the extent that they affect the worst-case parameter configuration. Finally, an interesting feature of the optimal rule is that it is super-inertial, i.e., it involves response coefficients to lagged interest rates that are larger than one. As first shown in Rotemberg and Woodford (1999), and Woodford (1999c), this feature of monetary policy allows the central bank to affect

26 For a comparison of the performance of both rules in the absence of parameter uncertainty, see Giannoni (2001a).
the private sector’s expectations appropriately. We have shown that in the presence of parameter uncertainty, the robust policy may be super-inertial to an even greater extent.

Even if he responds more to perturbations, in the presence of parameter uncertainty, the policymaker is cautious in our framework. In fact he is even more cautious than in Brainard’s model, as he cares very much about worst-case situations. We presented an example in which “being cautious” does not necessarily mean “to do less”.
**Table 2: Policy Rules and Statistics**

| Optimal rules | Coefficients of policy rule | Statistics | | | | |
|---|---|---|---|---|---|
| | $\pi_t$ | $x_t$ | $x_{t-1}$ | $i_{t-1}$ | $i_{t-2}$ | $V[\pi]$ | $V[x]$ | $V[i]$ | $E[L_0]$ |
| $\nu = 0$ | $(\psi^0, \theta^0)$ | 0.641 | 0.325 | -0.325 | 2.163 | -1.010 | 0.130 | 10.599 | 1.921 | 1.097 |
| | $(\psi^0, \theta^*)$ | 0.641 | 0.325 | -0.325 | 2.163 | -1.010 | 0.408 | 24.659 | 2.116 | 2.482 |
| | $(\psi^*, \theta^0)$ | 1.424 | 0.559 | -0.559 | 2.350 | -1.010 | 0.126 | 7.334 | 2.806 | 1.144 |
| | $(\psi^*, \theta^*)$ | 1.424 | 0.559 | -0.559 | 2.350 | -1.010 | 0.366 | 5.325 | 6.635 | 2.192 |
| $\nu = 0.5$ | $(\psi^0, \theta^0)$ | 0.641 | 0.325 | -0.325 | 2.163 | -1.010 | 0.213 | 4.435 | 0.718 | 0.597 |
| | $(\psi^0, \theta^*)$ | 0.641 | 0.325 | -0.325 | 2.163 | -1.010 | 0.790 | 24.659 | 2.116 | 2.482 |
| | $(\psi^*, \theta^0)$ | 1.424 | 0.559 | -0.559 | 2.350 | -1.010 | 0.182 | 3.831 | 1.081 | 0.622 |
| | $(\psi^*, \theta^*)$ | 1.424 | 0.559 | -0.559 | 2.350 | -1.010 | 0.592 | 26.086 | 2.439 | 2.429 |
| $\nu = 1$ | $(\psi^0, \theta^0)$ | 0.641 | 0.325 | -0.325 | 2.163 | -1.010 | 0.569 | 5.759 | 0.257 | 0.908 |
| | $(\psi^0, \theta^*)$ | 0.641 | 0.325 | -0.325 | 2.163 | -1.010 | 2.431 | 88.093 | 0.724 | 6.859 |
| | $(\psi^*, \theta^0)$ | 1.424 | 0.559 | -0.559 | 2.350 | -1.010 | 0.490 | 7.057 | 0.415 | 0.929 |
| | $(\psi^*, \theta^*)$ | 1.424 | 0.559 | -0.559 | 2.350 | -1.010 | 1.833 | 97.981 | 0.848 | 6.769 |

| Estimated Historical | | | | | | |
|---|---|---|---|---|---|
| $\nu = 0$ | $(\psi, \theta^0)$ | 0.424 | 0.297 | -0.032 | 1.160 | -0.430 | 0.079 | 11.852 | 2.952 | 1.349 |
| | $(\psi, \theta^*)$ | 0.424 | 0.297 | -0.032 | 1.160 | -0.430 | 1.284 | 14.891 | 14.184 | 5.357 |
| $\nu = 0.5$ | $(\psi, \theta^0)$ | 0.424 | 0.297 | -0.032 | 1.160 | -0.430 | 0.465 | 3.737 | 1.504 | 1.001 |
| | $(\psi, \theta^*)$ | 0.424 | 0.297 | -0.032 | 1.160 | -0.430 | 11.782 | 9.122 | 17.341 | 16.322 |
| $\nu = 1$ | $(\psi, \theta^0)$ | 0.424 | 0.297 | -0.032 | 1.160 | -0.430 | 1.363 | 1.469 | 0.959 | 1.661 |
| | $(\psi, \theta^*)$ | 0.424 | 0.297 | -0.032 | 1.160 | -0.430 | 36.041 | 22.245 | 27.291 | 43.568 |

Note: The estimated historical rule refers to Judd and Rudebusch (1998).
Figure 1: Impulse response functions to a temporary shock to $\delta_0$ ($\rho_\delta = 0.35$)

Notes: solid lines (—) correspond to the baseline rule; dashed lines (---) correspond to the robust rule; the dashed-dotted line (··~) indicates the expected path of $r^e$. The parameters $\sigma = 0.1571$, $\kappa = 0.0238$, $\omega = 0.4729$ are set at their baseline values.
Figure 2: Impulse response functions to a temporary shock to $\mu_0$ ($\rho_{\mu} = 0.35$)

Notes: solid lines (—) correspond to the baseline rule; dashed lines (—–) correspond to the robust rule. The parameters $\sigma = 0.1571$, $\kappa = 0.0238$, $\omega = 0.4729$ are set at their baseline values.
Figure 3: Response to a permanent increase in inflation and in the output gap
Figure 4: Loss criterion $E[L_0]$ as a function of various parameters

Notes: solid lines (—) correspond to the robust rule; dashed lines (---) correspond to the baseline rule. The remaining parameters are set at their worst-case value: $\sigma^* = 0.0915$, $\kappa^* = 0.0308$, $\omega^* = 0.2837$, $\nu^* = 1$, $\rho_\delta^* = 0.8$, $\rho_\mu^* = 0.8$. 

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Figure 6: Contour plots of $E[L_0]$ with robust optimal rule in $(\kappa, \sigma)$ space [$\nu = 1$]
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