Optimal Consumption and Asset Allocation with Unknown Income Growth: Appendix∗

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Abstract

Recent empirical evidence supports the view that the income process has an important individual-specific growth rate component (Baker (1997), Guvenen (2007), and Huggett, Ventura, and Yaron (2007)). Moreover, the individual-specific growth component may be stochastic. Motivated by these empirical observations, I study an individual’s optimal consumption-saving and portfolio choice problem when he needs to learn about his income growth. As in standard income fluctuation problems, the individual cannot fully insure his income shocks. In addition to the standard income-risk-induced precautionary saving demand, the individual also has estimation-risk-induced precautionary saving, which is greater when belief is more uncertain. With constant unobserved income growth, changes of belief are not predictable. However, with stationary stochastic income growth, belief is no longer a martingale and mean reverts. Mean reversion of belief reduces the hedging demand and in turn mitigates the impact of estimation risk on the individual’s consumption-saving and portfolio decisions.

Keywords: Incomplete markets, precautionary saving, Kalman filter, learning, regime switching, portfolio choice, hedging, estimation risk.

JEL Classification: E2, G11, G31

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This note provides technical details for Wang (2008), which develop four consumption-saving/portfolio choices models. There are two dimensions along which the four models differ: Whether the agent knows his income growth or not and whether his income growth is constant or stochastic. The following table characterizes the differences among the four models in the key assumption. Each special case will provide new insights on the effect of learning on precautionary saving. I start with the model where the agent knows the value of his income growth parameter \( \alpha \).

This appendix contains (i) proofs for the four propositions in the main text and (ii) the details for the permanent income hypothesis for the four models. I have written the appendix in a user-friendly way, which allows the reader to work through the technical details of the paper. I thank Bob King, the editor, for valuable suggestions which substantially helped the exposition of this appendix.

1 Proofs of four propositions

Proof of Proposition 1. Consider the case where the growth parameter \( \alpha \) is constant and known. By the principle of optimality, the agent’s value function \( V(x, y) \) solves the following Hamilton-Jacobi-Bellman (HJB) equation:

\[
\beta V = \max_{c, \psi} u(c) + (rx + \psi \zeta + y - c) V_x + (\alpha - \kappa y) V_y + \frac{\psi^2 \nu^2}{2} V_{xx} + \psi \nu \sigma V_{xy} + \frac{\sigma^2}{2} V_{yy}. \tag{1}
\]

Conjecture that the agent’s value function takes the following exponential form:

\[
V(x, y) = -\frac{1}{\gamma r} \exp \left[ -\gamma r(x + g(y)) \right], \tag{2}
\]

where \( g(y) \) is a function to be determined. Note that \( V_x = \exp \left[ -\gamma r(x + g(y)) \right] \), \( V_y = g'(y)V_x \), \( V_{xx} = -\gamma r V_x \), \( V_{xy} = -\gamma rg'(y)V_x \), and \( V_{yy} = (g''(y) - \gamma rg'(y))^2 V_x \).

The first-order conditions (FOCs) with respect to consumption \( c \) and portfolio rules \( \psi \) are

\[
u'(c) = V_x(x, y) = e^{-\gamma r(x + g(y))}, \tag{3}
\]

\[
\psi = -\frac{\zeta V_x + \nu \sigma V_{xy}}{\nu^2 V_{xx}} = \frac{\zeta - \gamma r \nu \sigma g'(y)}{\gamma r \nu^2}. \tag{4}
\]
The FOC (3) implies $c = r(x + g(y))$. Substituting the above implied consumption and portfolio allocation rules into the HJB equation (1) yields the following equation:

$$0 = \frac{\beta - r}{\gamma r} + \left[(y - rg(y)) + \frac{\zeta^2 - \gamma r \rho \sigma \zeta' g'(y)}{\gamma r \nu^2}\right] + (\alpha - \kappa y) g'(y) - \frac{\gamma r \nu^2}{2} \left[\frac{\zeta - \gamma r \rho \sigma g'(y)}{\gamma r \nu^2}\right]^2 - \frac{\zeta - \gamma r \rho \sigma g'(y)}{\gamma r \nu^2} \rho \sigma \gamma r g'(y) + \frac{\sigma^2}{2} \left(g''(y) - \gamma r g'(y)^2\right)$$

(5)

Simplifying the above gives the following differential equation for $g(y)$:

$$rg(y) = y + \left(\alpha - \frac{\rho \sigma \zeta}{\nu} - \kappa y\right) g'(y) + \frac{\sigma^2}{2} g''(y) - \frac{\gamma r (1 - \rho^2)}{2} g'(y)^2 + \frac{\beta - r}{\gamma r} + \frac{\zeta^2}{2 \gamma r \nu^2}.$$  

(6)

Conjecture that $g(y)$ is an affine function in $y$. Solving gives

$$g(y; \alpha) = \frac{1}{r + \kappa} \left[y + \frac{1}{r} \left(\alpha - \frac{\rho \sigma \zeta}{\nu}\right) - \frac{\gamma \sigma^2 (1 - \rho^2)}{2(r + \kappa)}\right] + \left(\frac{\beta - r}{\gamma r^2} + \frac{\zeta^2}{2 \gamma r^2 \nu^2}\right).$$

(7)

Using the above formula, for $g(y)$, we immediately obtain the portfolio rule:

$$\psi^* = \frac{\zeta}{\gamma r \nu^2} - \frac{\rho \sigma}{\nu} \frac{1}{r + \kappa}.$$  

(8)

I verify that the transversality condition $\lim_{T \to \infty} E[e^{-rT}V(x_T, y_T)] = 0$ holds. We may also directly use the result from Model I, the general set up with known (but possibly stochastic) growth parameter $\alpha$. That is, by setting $\lambda_1 = 0$, we obtain the same results. The results derived in this section include Merton (1971), Caballero (1990), Davis and Willen (2000), Svensson and Werner (1993), and Wang (2006) as special cases.

Proof of Proposition 2. Consider the case where the growth parameter $\alpha$ is constant, but is unknown to the agent. The value $\alpha$ may take two possible values $\alpha_1$ and $\alpha_2$. Let $p(t)$ denote the time-$t$ conditional probability that income growth $\alpha$ is equal to $\alpha_1$. By the principle of optimality, the agent’s value function $J(x, y, p)$ solves the following HJB equation:

$$\beta J = \max_{c, \psi} \left[u(c) + (rx + \psi \zeta + y - c) J_x + \frac{\psi^2 \nu^2}{2} J_{xx} + (\alpha_2 + \delta p - \kappa y) J_y + \frac{\lambda \sigma^2}{2} J_{yy}\right] + \psi \rho \nu \sigma J_{xy} + \frac{\delta^2}{2 \sigma^2} (1 - p)^2 J_{pp} + \psi \rho \nu \sigma^{-1} \delta p (1 - p) J_{xp} + \delta p (1 - p) J_{yp}.$$  

(9)

Conjecture that the agent’s value function takes the following exponential form:

$$J(x, y, p) = -\frac{1}{\gamma r} \exp[-\gamma r(x + g(y; \alpha_2) + f(p))].$$

(10)
where \( g(y; \alpha_2) \) is given in (7) and \( f(p) \) is to be determined.

Using (10), we have \( J_x = \exp \left[ -\gamma t(x + g(y; \alpha_2) + f(p(t))) \right] \), \( J_y = g'(y; \alpha_2) J_x \), \( J_{xx} = -\gamma r J_x \), \( J_{xy} = -\gamma r g'(y; \alpha_2) J_x \), \( J_{yy} = \left( g''(y; \alpha_2) - \gamma r g'(y; \alpha_2)^2 \right) J_x \), \( J_p = f'(p) J_x \), \( J_{pp} = \left( f''(p) - \gamma r f'(p)^2 \right) J_x \), \( J_{xp} = -\gamma r f'(p) J_x \), and \( J_{yp} = -\gamma r f'(p) g'(y; \alpha_2) J_x \). Recall that (7) implies \( g'(y; \alpha_2) = 1/(r + \kappa) \).

The first-order conditions (FOCs) with respect to consumption \( c \) and portfolio rules \( \psi \) are

\[
\begin{align*}
\psi &= \frac{\zeta}{\gamma r \nu^2} - \frac{\rho \sigma g'(y; \alpha_2) + \rho \sigma^{-1} \delta p (1 - p)}{\nu}.
\end{align*}
\]

The FOC (11) implies \( c = r(x + g(y; \alpha_2) + f(p)) \). Substituting the implied consumption and portfolio allocation rules into the HJB equation (9) yields the following equation:

\[
\begin{align*}
0 &= \frac{\beta - r}{\gamma r} + \left( y - rg(y; \alpha_2) \right) + \frac{\zeta^2 - \gamma r \rho \sigma \zeta g'(y)}{\gamma r \nu^2} + (\alpha_2 - \kappa y) g'(y; \alpha_2) + \delta p g'(y; \alpha_2) \\
&\quad - \frac{\gamma r \nu^2}{2} \left[ \frac{\zeta - \gamma r \rho \sigma g'(y)}{\gamma r \nu^2} \right]^2 - \frac{\zeta - \gamma r \rho \sigma g'(y)}{\gamma r \nu^2} \rho \sigma \gamma r g'(y) + \frac{\sigma^2}{2} (g''(y) - \gamma r g'(y)^2) - rf(p) \\
&\quad + \frac{\delta^2 p^2 (1 - p)^2}{2r^2} - \left( \frac{\rho \sigma}{\sigma} + \frac{\gamma r}{r + \kappa} (1 - \rho^2) \right) \delta p (1 - p) f'(p) \\
&\quad - \frac{\gamma r (1 - \rho^2)}{2r^2} \delta^2 p^2 (1 - p)^2 f'(p)^2.
\end{align*}
\]

Intuitively, if \( p = 0 \), the agent’s expected income growth is always equal to \( \alpha_2 \), and will never change. Substituting \( g(y; \alpha_2) \) given in (7) into (13) gives the following non-linear ODE for \( f(p) \):

\[
\begin{align*}
rf(p) &= \frac{\delta p}{r + \kappa} - \left( \frac{\gamma r}{r + \kappa} (1 - \rho^2) + \frac{\rho \sigma}{\sigma} \right) \delta p (1 - p) f'(p) + \frac{\delta^2}{2r^2} p^2 (1 - p)^2 f'(p)^2 \\
&\quad - \frac{\gamma r \delta^2 p^2 (1 - p)^2 (1 - \rho^2)}{2r^2} f'(p)^2, \quad 0 \leq p \leq 1.
\end{align*}
\]

Both \( p = 0 \) and \( p = 1 \) are absorbing states, in that there is no further belief updating once the belief reaches \( p = 0 \) or \( p = 1 \). Therefore, we simply use Proposition 1 to characterize the optimal consumption rule. Doing so, we obtain the following boundary conditions

\[
\begin{align*}
f(0) &= 0, \\
f(1) &= \frac{\delta}{r(r + \kappa)}.
\end{align*}
\]
Proof of Proposition 3. Consider the case where income growth parameter \( \{\alpha (t) : t \geq 0\} \) is stochastic but is known to the agent. The transition for the income growth \( \{\alpha (t) : t \geq 0\} \) follow a two-state Markov chain. The agent’s value function \( V(x, y, n) \) solves the following HJB equation:

\[
\beta V(x, y, 1) = \max_{c, \psi} u(c) + (rx + \psi \zeta + y - c)V_x(x, y, 1) + (\alpha_1 - \kappa y)V_y(x, y, 1) + \frac{\psi^2 \nu^2}{2}V_{xx}(x, y, 1)
\]

\[
+ \psi \rho \nu \sigma V_{xy}(x, y, 1) + \frac{\sigma^2}{2}V_{yy}(x, y, 1) + \lambda_1 \left( V(x, y, 2) - V(x, y, 1) \right).
\]

(17)

Conjecture that the value function \( V \) in regime \( n \) takes the following exponential form:

\[
V(x, y, n) = -\frac{1}{\gamma r} \exp \left[ -\gamma r (x + k_n(y)) \right],
\]

(18)

where \( k_n(y) \) is a regime-dependent function to be determined. Using the conjecture (18) for the value function, we have

\[
V_y(x, y, n) = \kappa' n(y) V_x(x, y, n),
\]

\[
V_{xx}(x, y, n) = -\gamma r V_x(x, y, n),
\]

\[
V_{xy}(x, y, n) = -\gamma r \kappa' n(y) V_x(x, y, n),
\]

\[
V_{yy}(x, y, n) = (\kappa'' n(y) - \gamma r \kappa' n(y)^2) V_x(x, y, n).
\]

The FOC with respect to \( c \) gives \( u'(c) = V_x(x, y, n) \), which implies

\[
c^* = r \left( x + k_n(y) \right).
\]

(19)

The FOC with respect to \( \psi \) for the HJB equation (17) gives

\[
\psi^* = \frac{\zeta}{\gamma r \nu^2} - \frac{\rho \sigma}{\nu} \kappa'_1(y).
\]

(20)

Substituting (19) and (20) into the HJB equation (17) for both regimes, we have the following coupled differential equations (24) and (25) for \( k_1(y) \) and \( k_2(y) \):

\[
rk_1(y) = \frac{\beta - r}{\gamma r} + y + \left( \frac{\zeta^2}{\gamma r \nu^2} - \frac{\rho \sigma}{\nu} k'_1(y) \right) + (\alpha_1 - \kappa y) k'_1(y) - \frac{\gamma r \nu^2}{2} \left( \frac{\zeta}{\gamma r \nu^2} - \frac{\rho \sigma}{\nu} k'_1(y) \right)^2
\]

\[
- \left( \frac{\zeta}{\gamma r \nu^2} - \frac{\rho \sigma}{\nu} k'_1(y) \right) \rho \nu \sigma \gamma r k'_1(y) + \frac{\sigma^2}{2} (k''_1(y) - \gamma r k'_1(y)^2) - \frac{\lambda_1}{\gamma r} \left[ e^{-\gamma r (k_2(y) - k_1(y))} - 1 \right].
\]

(21)

Using the risk-adjusted certainty equivalent wealth \( g(y; \alpha) \) in Model I (with constant and known income growth \( \alpha \)), I further conjecture that \( \{k_n(y) : n = 1, 2\} \) takes the following additively separable form:

\[
k_n(y) = g(y; \alpha_2) + \phi_n,
\]

(22)
where $\phi_1$ and $\phi_2$ are to be determined. Substituting (22) into (21), we may the following optimal allocation to the risky asset:

$$\psi^* = \frac{\zeta}{\gamma r \nu^2} - \frac{\rho \sigma}{\nu} \frac{1}{r + \kappa}. \quad (23)$$

Substituting (22) and (23) into (21) and simplifying gives

$$r (g(y; \alpha_2) + \phi_1) = \frac{\beta - r}{\gamma r} + \frac{\eta^2}{2 \gamma r} + y + \frac{\alpha_1 - \rho \sigma \eta - \kappa y}{r + \kappa} - \gamma r \frac{\sigma^2 (1 - \rho^2)}{2 (r + \kappa)^2} - \frac{\lambda_1}{\gamma r} \left[ e^{-\gamma r (\phi_2 - \phi_1)} - 1 \right], \quad (24)$$

Similarly, we obtain the following for $\phi_2$:

$$r (g(y; \alpha_2) + \phi_2) = \frac{\beta - r}{\gamma r} + \frac{\eta^2}{2 \gamma r} + y + \frac{\alpha_2 - \rho \sigma \eta - \kappa y}{r + \kappa} - \gamma r \frac{\sigma^2 (1 - \rho^2)}{2 (r + \kappa)^2} - \frac{\lambda_2}{\gamma r} \left[ e^{-\gamma r (\phi_1 - \phi_2)} - 1 \right]. \quad (25)$$

Applying (22) to we obtain the following coupled nonlinear equations for the regime-dependent constant coefficients $\phi_1$ and $\phi_2$:

$$r \phi_1 = \frac{\delta}{r + \kappa} - \frac{\lambda_1}{\gamma r} \left( e^{-\gamma r (\phi_2 - \phi_1)} - 1 \right), \quad (26)$$

$$r \phi_2 = -\frac{\lambda_2}{\gamma r} \left( e^{-\gamma r (\phi_1 - \phi_2)} - 1 \right). \quad (27)$$

I also verify that the transversality condition $\lim_{\tau \to \infty} E[e^{-r \tau} V(x_\tau, y_\tau, N_\tau)] = 0$ holds, where $N_\tau$ is the regime at time $\tau$. Model III includes Kimball and Mankiw (1989), Wang (2004), and Caballero (1991) as special cases.

For intuition, I now turn to the first-order approximate solutions for $\phi_1$ and $\phi_2$.

**Approximate Solutions of $\phi_1$ and $\phi_2$ for Model III.** Consider the following approximations of (26) and (27) in terms of $\delta$ up to the second order:

$$r \phi_1 \approx \frac{\delta}{r + \kappa} + \lambda_1 (\phi_2 - \phi_1), \quad (28)$$

$$r \phi_2 \approx \frac{\lambda_2}{r + \kappa} (\phi_1 - \phi_2). \quad (29)$$

First, keeping the first order terms and solving the above equations, we have

$$\phi_1 \approx \frac{r + \lambda_2}{r + \lambda_1 + \lambda_2} \frac{\delta}{r (r + \kappa)}, \quad (30)$$

$$\phi_2 \approx \frac{\lambda_2}{r + \lambda_1 + \lambda_2} \frac{\delta}{r (r + \kappa)}. \quad (31)$$

The above two equations correspond to the PIH analysis when income growth is stochastic.
Proof of Proposition 4. Consider the case where income growth \( \{ \alpha(t) : t \geq 0 \} \) is stochastic, but unknown. At time \( t \), \( \alpha(t) \) may take two possible values \( \alpha_1 \) and \( \alpha_2 \). Let \( p(t) \) denote the time-\( t \) conditional probability that \( \alpha(t) \) is equal to \( \alpha_1 \). By the principle of optimality, the agent’s value function \( J(x, y, p) \) solves the following HJB equation:

\[
\beta J(x, y, p) = \max_{c, \psi} \left[ u(c) + (r x + \psi \xi + y - c) J_x + \frac{\psi^2 \nu^2}{2} J_{xx} + (\alpha_2 + \delta p - \kappa y) J_y + \frac{1}{2} \sigma^2 J_{yy} \right. \\
+ \psi \rho \nu \sigma J_{xy} + (\lambda_2 - (\lambda_1 + \lambda_2) p) J_p + \frac{\delta^2}{2 \sigma^2} p^2 (1 - p)^2 J_{pp} \\
+ \psi \rho \nu \sigma^{-1} \delta p (1 - p) J_{xp} + \delta p (1 - p) J_{yp}. \tag{32}
\]

Conjecture that the agent’s value function takes the following exponential form:

\[
J(x, y, p) = -\frac{1}{\gamma r} \exp [-\gamma r (x + g(y; \alpha_2) + f(p))], \tag{33}
\]

where \( g(y; \alpha_2) \) and \( f(p(t)) \) are functions to be determined. Using (33), we have the following results: \( J_x = \exp [-\gamma r (x + g(y; \alpha_2) + f(p(t)))], J_y = g'(y; \alpha_2) J_x, J_{xx} = -\gamma r J_x, J_{xy} = -\gamma r g'(y; \alpha_2) J_x, J_{yy} = (g''(y; \alpha_2) - \gamma r g'(y; \alpha_2)^2) J_x, J_p = f'(p) J_x, J_{pp} = (f''(p) - \gamma r f'(p)^2) J_x, J_{xp} = -\gamma r f'(p) J_x, \) and \( J_{yp} = -\gamma r f'(p) g'(y; \alpha_2) J_x \). The first-order conditions (FOCs) with respect to consumption \( c \) and portfolio rules \( \psi \) are

\[
u' = J_x(x, y, p) = e^{-\gamma r (x + g(y; \alpha_2) + f(p))}, \tag{34}
\]

\[
\psi = -\frac{\zeta J_x + \rho \nu \sigma J_{xy} + \rho \nu \sigma^{-1} \delta p (1 - p) J_{xp}}{\nu^2 J_{xx}} \\
= -\frac{\zeta}{\gamma r \nu^2} - \xi(t), \tag{35}
\]

where \( \xi(t) \) is given by

\[
\xi(t) = \frac{\rho \sigma g'(y; \alpha_2) + \rho \sigma^{-1} \delta p (1 - p) f'(p)}{\nu}. \tag{36}
\]

The FOC (34) implies \( c = r (x + g(y; \alpha_2) + f(p)) \). Substituting \( g(y; \alpha_2) \) given by (7) into (35) gives the following portfolio rule:

\[
\psi = -\frac{\zeta}{\gamma r \nu^2} - \frac{\rho}{\nu} \left( \frac{\sigma}{r + \kappa} + \frac{\delta p (1 - p) f'(p)}{\sigma} \right). \tag{37}
\]

Substituting the implied consumption and portfolio allocation rules into the HJB equation
Unlike Model II, neither growth (lowest possible value). For a fixed small time interval $\Delta t$, instantaneous precautionary saving is zero. There is no contribution from the expected income growth change for exogenous reasons with transition intensities $\lambda_2$ and $\lambda_1$.

IV. Developing the explicit formulae for the PIH rule helps us understand the implications of precautionary saving.

Using the insights from Proposition 1, Substituting $g(y; \alpha_2)$ given in (7) into (38) gives the following non-linear ODE for $f(p)$:

$$0 = \frac{\beta - r}{\gamma r} + \left[(y - rg(y; \alpha_2)) + \frac{\zeta^2 - \gamma r \nu \sigma \zeta g'(y; \alpha_2)}{\gamma r \nu^2} \right] + (\alpha_2 - \kappa y) g'(y; \alpha_2) + \delta p g'(y; \alpha_2)$$

$$+ \frac{\sigma^2}{2} \left[\frac{\zeta - \gamma r \nu \sigma g'(y; \alpha_2)}{\gamma r \nu^2} \right]^2 - \frac{\zeta - \gamma r \nu \sigma g'(y; \alpha_2)}{\gamma r \nu^2} \rho \nu \sigma \gamma r g'(y; \alpha_2)$$

$$- \left(\frac{\rho \eta}{\sigma} + \frac{\gamma r}{\rho + \kappa} (1 - \rho^2) \right) \delta p (1 - p) f'(p) - \frac{\gamma r (1 - \rho^2)}{2\sigma^2} \delta^2 p^2 (1 - p)^2 f''(p).$$

(38)

Using the insights from Proposition 1, Substituting $g(y; \alpha_2)$ given in (7) into (38) gives the following non-linear ODE for $f(p)$:

$$rf(p) = \frac{\delta p}{r + \kappa} + \left[(\lambda_2 - (\lambda_1 + \lambda_2) p) - \left(\frac{\gamma r}{\rho + \kappa} (1 - \rho^2) + \frac{\rho \eta}{\sigma} \right) \delta p (1 - p) \right] f'(p)$$

$$+ \frac{\delta^2}{2\sigma^2} p^2 (1 - p)^2 f''(p) - \frac{\gamma r (1 - \rho^2)}{2\sigma^2} \delta^2 p^2 (1 - p)^2 f'(p)^2, \quad 0 < p < 1.$$

(39)

Unlike Model II, neither $p = 0$ nor $p = 1$ is an absorbing state, because the income growth may change for exogenous reasons with transition intensities $\lambda_2$ and $\lambda_1$.

Turn to the boundary conditions. First, consider the left boundary $p = 0$. Intuitively, the instantaneous precautionary saving is zero. There is no contribution from the expected income growth (lowest possible value). For a fixed small time interval $\Delta t$, the conditional probability of income growth change is $\lambda_2 \Delta t$. The value at which $f(p)$ changes is $f'(0)$ at $p = 0$. Therefore, we have $rf(0) = \lambda_2 f'(0)$. Similarly, when $p = 1$, the instantaneous precautionary saving is zero.

The contribution from the expected income growth to consumption is $\delta/(r + \kappa)$ in flow terms when $p = 1$ (i.e. $\alpha(t) = \alpha_1$). The expected change of $f(p)$ at $p = 1$ is $f'(1)$. The conditional probability is $\lambda_2 \Delta t$ for a given time interval $\Delta t$. Since the drift of $f(p)$ is negative at $p = 1$, we thus have $rf(1) = \delta/(r(r + \kappa)) - \lambda_1 f'(1)$. Summarizing the boundary conditions, we have

$$rf(0) = \lambda_2 f'(0),$$

$$rf(1) = \frac{\delta}{r + \kappa} - \lambda_1 f'(1).$$

(40)

(41)

2 Permanent Income Hypothesis for Models I-IV

In this section, I drive the corresponding permanent income hypothesis (PIH) for Models I-IV. Developing the explicit formulae for the PIH rule helps us understand the implications of precautionary saving.
The permanent income hypothesis (PIH) states that consumption is given by the annuity value of the sum of financial and human wealth (Friedman (1957)), in that

\[ c(t)^* = r (x(t) + h(t)), \quad (42) \]

where human wealth \( h \) is defined as the present value of future labor incomes discounted at the risk-free rate (Friedman (1957) and Hall (1978)). That is, we have

\[ h(t) = E \left( \int_t^{\infty} e^{-r(s-t)} y(s) \, ds \bigg| \mathcal{F}_t \right), \quad (43) \]

where \( \mathcal{F}_t \) is the agent’s information set at time \( t \). Using the definition of human wealth (43), this appendix calculates human wealth and derives consumption rules under the permanent income hypothesis for Models I-IV. For simplicity, we assume that the agent only invests in the risk-free asset following the tradition of the PIH.

**Model I: Known & constant growth parameter** \( \alpha \). The differential equation for human wealth is

\[ rh = y + (\alpha - \kappa y) \frac{dh}{dy} + \frac{\sigma^2 d^2h}{2 dy^2}. \quad (44) \]

Conjecture that human wealth is an affine function of income \( y \), in that \( h = a_y y + a_0 \). Substituting the conjectured (affine) form for human wealth \( h(y) \) into (44) gives

\[ h(t) = \frac{1}{r+\kappa} \left( y(t) + \frac{\alpha}{r} \right). \quad (45) \]

The rate of mean reversion \( \kappa \) lowers human wealth, \textit{ceteris paribus}. Recall \( r + \kappa > 0 \) is assumed for convergence. Note that human wealth is based on the first moment calculation, therefore, volatility \( \sigma \) has no impact on \( h \) as seen in (44).

**Model II: Unknown & constant growth parameter** \( \alpha \). We may solve for human wealth \( h \) in two ways. First, use the law of iterated expectation. Using the formula (45) for human wealth \( h \) for the case with known & constant growth parameter \( \alpha \) and applying \( E_t(\alpha) = \alpha_2 + \delta p(t) \), we obtain the following expression for human wealth:

\[ h(t) = \frac{1}{r+\kappa} \left( y(t) + \frac{\alpha_2 + \delta p(t)}{r} \right). \quad (46) \]
Alternatively, we may write down the following differential equation for human wealth $h$:

$$rh = y + (\alpha_1 p + \alpha_2 (1 - p) - \kappa y) \frac{dh}{dy} + \frac{\sigma^2 d^2 h}{2 dy^2} + \frac{\delta^2 p^2 (1 - p)^2 d^2 h}{2 \sigma^2 dp^2} + \delta p (1 - p) \frac{d^2 h}{dy dp}.$$  (47)

Conjecture that human wealth is affine in income $y$ and belief $p$, in that $h = h_0 + h_y y + h_p p$. Substituting the conjecture into (47) and matching conditions gives (46).

**Model III: Known & stochastic growth parameter $\alpha$.** Let $h_n$ denote human wealth when the current income growth parameter is $\alpha_n$, where $n = 1, 2$. Human wealth $h_1$ and $h_2$ then jointly solve the following coupled differential equations:

$$rh_1 = y + (\alpha_1 - \kappa y) \frac{dh_1}{dy} + \frac{\sigma^2 d^2 h_1}{2 dy^2} + \lambda_1 (h_2 - h_1),$$  (48)

$$rh_2 = y + (\alpha_2 - \kappa y) \frac{dh_2}{dy} + \frac{\sigma^2 d^2 h_2}{2 dy^2} + \lambda_2 (h_1 - h_2).$$  (49)

Conjecture that human wealth in each regime is affine in income $y$, in that $h_n = a_n y + b_n$, where $n = 1, 2$. Substituting the conjecture and solving the above two equations jointly gives

$$h_1(t) = \frac{1}{r + \kappa} \left[ (y(t) + \frac{\alpha_1}{r}) - \frac{\lambda_1}{r + \lambda_1 + \lambda_2} \frac{\delta}{r} \right],$$  (50)

$$h_2(t) = \frac{1}{r + \kappa} \left[ (y(t) + \frac{\alpha_2}{r}) + \frac{\lambda_2}{r + \lambda_1 + \lambda_2} \frac{\delta}{r} \right].$$  (51)

**Model IV: Unknown & stochastic growth parameter $\alpha$.** Using the standard valuation argument, human wealth satisfies the following differential equation:

$$rh = y + (\alpha_2 + \delta p - \kappa y) \frac{dh}{dy} + \frac{\sigma^2 d^2 h}{2 dy^2} + (\lambda_2 - (\lambda_1 + \lambda_2) p) \frac{dh}{dp} + \frac{\delta^2 p^2 (1 - p)^2 d^2 h}{2 \sigma^2 dp^2} + \delta p (1 - p) \frac{d^2 h}{dy dp}.$$  (52)

Conjecture that human wealth is linear in income $y$ and posterior belief $p$:

$$h = \frac{1}{r + \kappa} \left( y + \frac{\alpha_2}{r} \right) + h_0 + h_p p.$$  (53)

Substituting the above conjecture into the human wealth valuation equation (52) gives

$$r (h_0 + h_p p) + \frac{r}{r + \kappa} \left( y + \frac{\alpha_2}{r} \right) = y + \frac{\alpha_2 + \delta p - \kappa y}{r + \kappa} + (\lambda_2 - (\lambda_1 + \lambda_2) p) h_p.$$  (54)
Solving (54) gives

\[
    h_p = \frac{1}{r + \lambda_1 + \lambda_2} \frac{\delta}{r + \kappa} \tag{55}
\]

\[
    h_0 = \left( \frac{\lambda_1 + \lambda_2}{r} \right) \phi_1 h_p = \frac{\lambda_2}{r + \lambda_1 + \lambda_2} \frac{\delta}{r(\kappa + \kappa)} \tag{56}
\]

Substituting the above coefficients into the human wealth conjecture (53) gives the following form for human wealth:

\[
    h(t) = \frac{1}{r + \kappa} \left[ \left( y(t) + \frac{\alpha_2}{r} \right) + \frac{1}{r + \lambda_1 + \lambda_2} \delta \left( p(t) + \frac{\lambda_2}{r} \right) \right]. \tag{57}
\]

As expected, human wealth (57) includes \( h_1 \) given in (50) and \( h_2 \) given in (51) for Model III (with known but stochastic income growth) as special cases, when we set \( p(t) = 1 \) and \( p(t) = 0 \) respectively.

**References**


*Journal of Political Economy* 97, 863–79.

*Journal of Economic Theory* 3, 373–413.


