Investment under Uncertainty and the Value of Real and Financial Flexibility*

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Abstract

We develop a model of investment under uncertainty for a financially constrained firm. Facing external financing costs, the firm prefers to fund its investment through internal funds, so that the firm’s optimal investment policy and value now depend on both its earnings fundamentals and liquidity holdings. We show that financial constraints significantly alter the standard real options results, with the financial flexibility conferred by internal funds acting as a complement, and at times as a substitute, to the real flexibility given by the optimal timing of investment. We show that: 1) the investment hurdle (whose deviation from the first-best Modigliani-Miller benchmark measures investment distortions) is highly nonlinear and non-monotonic in the firm’s internal funds, as the firm may prefer accumulating internal funds rather than accessing external capital markets to finance investment when internal funds are sufficiently high; 2) with multiple rounds of growth options, a value-maximizing financially constrained firm may choose to over-invest via accelerated investment timing in earlier stages in order to mitigate under-investment problems in later stages.

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1 Introduction

In their influential textbook Dixit and Pindyck (1994) condense the essence of investment decisions to three key attributes: i) the degree of irreversibility; ii) the risk over future revenue; and, iii) the flexibility in the timing of the decision. In this paper we add a fourth attribute: the funding cost of the investment. Much of the real-options-based theory of investment under uncertainty following McDonald and Siegel (1986) assumes that firms operate in frictionless capital markets. This is for good reason, as the firm’s investment decision can then be formulated as a simple real option problem involving the optimal exercise and valuation of an American option. The option pricing tools developed by Black and Scholes (1973) and Merton (1973) can then be deployed to analyze the real options problem.

A major limitation of this approach, however, is that firms in practice do not operate in frictionless capital markets. They face significant external financing costs and as a result may often rely on internal funds and internally generated (current and future) cash flows to finance their investments. The recent financial crisis is an important reminder of how severe external financing costs can be in extreme situations and how much they can affect corporate investment and the macroeconomy. The extensive surveys of chief financial officers (CFOs) by Graham and Harvey (2001, 2002) have also revealed the great importance CFOs attach to maintaining financial flexibility. Thus, while determining the value of real flexibility obtained from the optimal timing of irreversible investments one cannot abstract from the requirement of financial flexibility to be able to fully take advantage of real options. Accordingly, the important theoretical question we address in this paper is how corporate investment decisions and the valuation of investments under uncertainty are affected by external financing costs and the firm’s efforts to relax these costs by dynamically managing its liquidity holdings. In effect, we are integrating two classical strands of literature, one started by Miller and Orr (1966) on corporate cash balances and the other started by McDonald and Siegel (1986) on real options.

Although the classical tools of option pricing theory can no longer be directly applied (and although the analysis of the investment problem and its financing involves solving a considerably more involved two-dimensional partial differential equation), the results we obtain are intuitive and striking. First, both the growth options in the start-up phase and the abandonment options in the mature phase of the firm’s life-cycle are worth less when the firm faces external financing constraints. Second, the firm is more likely to abandon an asset.
(captured by a higher abandonment hurdle) when the firm faces external financing costs: The firm facing external financing costs is capitalizing expected future external financing costs and setting these against the present value of the asset. However, should the financially constrained firm decide to continue operating the asset when it runs out of cash by raising external funds, the firm will tend to raise more external funds the higher are external financing costs, as the firm seeks to limit the risk of having to return to capital markets again in the future. Third, with multiple rounds of growth options, a financially constrained firm may surprisingly lower the hurdle for early investments (a form of over-investment) in order to gain access to higher operating cash-flows and thus mitigate under-investment problems in later stages.

Fourth, the hurdle for investment in the start-up phase is a non-monotonic function of the firm’s stock of liquid assets: When the firm’s internal funds are sufficient to entirely cover the costs of the investment then the firm’s hurdle for investment is lower the higher the firm’s internal funds. In contrast, when the firm’s internal funds cannot cover the entire cost of the investment then the hurdle may be sharply increasing with the firm’s internal funds. The reason is that when the firm is approaching the point where it may be able to entirely fund its investment with retained earnings it has a strong incentive to delay investment until it has sufficient funds to be able to entirely avoid tapping costly external funds. This is in our view the most striking effect of the presence of external financing costs. An important implication of this result is that investment is not necessarily more likely when the firm has more cash. Investment could well be delayed further, as the firm’s priority becomes avoiding reliance on external funds. In sum, for firms facing external financing costs the value of an investment opportunity is not just tied to timing optionality but also to flexibility, which depends on both the financial ability to seize an investment opportunity and on timing optionality.

Our analysis contributes to the literature on financial frictions and corporate investment in the vein of Fazzari, Hubbard, and Petersen (1988), Froot, Scharfstein, and Stein (1993), and Kaplan and Zingales (1997). These models are purely static and thus not are set up to study the value of financial and real flexibility, which arises from the optionality of investment and financing decisions.

While all static financial-constraint-based models unambiguously predict that firms under-invest due to costly external financing, we show that a financially constrained firm may over-invest as doing so allows the firm to accumulate more internal funds sooner which in turn mitigate the the firm’s under-investment distortions in the future. This dynamic mechanism
is important as it highlights that ignoring the firm’s ability to adjust its financing margin (accumulating internal funds versus issuing external equity) significantly changes the impact of financial constraints on investment decisions. Put differently, in a dynamic setting, a firm has multiple margins to adjust its decisions. Hence an empirical hypothesis (e.g., that a firm underinvests due to financial constraints based on static models) can be mis-leading. Also our dynamic model predicts that the firm’s flexibility with regard to its real investment timing decision (summarized by the optimal investment threshold policy) is non-monotonic in the firm’s internal funds, again due to the firm’s intertemporal financial flexibility (by accumulating internal funds rather than having to access external equity). Third, the amount of equity financing is also non-monotonic in its internal funds in our model, also due to the firm’s real and financial flexibilities. The firm’s option to access external financing is highest when its real option value is moderate. The intuition is as follows. For a firm whose investment option is sufficiently out of the money, financial flexibility has little value to the firm. On the other hand, a firm whose investment option is deep in the money, the option value of accumulating internal funds is of little value. Therefore, the multiple options of financing (internal versus external as well as the flexible timing decision of external financing) are most valuable for a firm whose real option value is in the medium range. The positive synergies between the real and financial option values are the greatest in the medium range.

models with managerial agency. Miao and Wang (2007) develop an incomplete-markets real-options problem and show that the standard positive effect of volatility on option value is balanced by the risk-averse entrepreneur’s precautionary savings motive.

Boyle and Guthrie (2003) is the first study of real options in the presence of financial constraints. They model financial constraints by assuming that the firm can only pledge a fraction of its value, which constrains its ability to fund investments. Unfortunately, however, their framework is flawed and contains two inconsistencies, the main one being that the firm is allowed to continue operations even when it accumulates arbitrarily large negative earnings, although it is not able to raise funds to finance capital expenditures.¹ An earlier study by Mauer and Triantis (1994) considers a real options problem for a levered firm, which may face recapitalization costs when its operating performance is poor. However, as in Leland (1994) the firm does not otherwise incur any external financing costs. In their model the levered firm has a lower hurdle as it seeks to bring tax-shield benefits of debt financing forward in time.

Closer to our setup, Décamps and Villeneuve (2007) consider a financially constrained firm with an asset in place generating cash-flows that are subject to i.i.d shocks and a growth option, which raises the drift of the cash-flow process and can only be financed with internal funds. They characterize the firm’s optimal investment and dividend policy. Asvanunt, Broadie and Sundaresan (2007) also consider a real options problem for a levered firm. Unlike Mauer and Triantis (1994) and Leland (1994), they do introduce external equity financing costs in the form of dilution costs and allow the firm to accumulate internal funds. However, neither of these studies provides a systematic analysis of the firm’s optimal cash management problem and the interplay of real and financial flexibility. More recently, four independent studies have analyzed a similar problem to ours: First, Copeland and Lyasoff (2013) who consider a somewhat narrower framework to ours, which does not allow for either abandonment or sequential growth options. Second, Boot and Vladimirov (2014) consider a financially constrained entrepreneurial firm with an asset in place that generates random cash flows following a geometric Brownian motion and a new investment opportunity. Their focus is on comparing private firms with less liquid assets but better aligned objectives

¹The other inconsistency is that the firm’s investment project value is assumed to follow a geometric Brownian motion, while cumulative operating cash-flows are assumed to follow an arithmetic Brownian motion. Basically, Boyle and Guthrie (2003) assume that by exercising its growth option the firm swaps out an asset in place with iid returns for another asset with value $V$ which follows a geometric Brownian motion. Moreover, while swapping out these assets the firm looses all its accumulated cash $X$. 

4
between the entrepreneur and other investors and public firms with more liquid assets but greater potential disagreements between the manager and other investors. Third, Babenko and Tserlukevich (2013) consider the optimal hedging policy (in a dynamic model with cash accumulation) for a financially constrained firm with a decreasing returns to scale technology and lumpy investment opportunities. And fourth, Hugonnier, Malamud, and Morellec (2014) study the impact of financial constraints on growth options under capital supply uncertainty, assuming, however, that the cash-flow process (both before and after the option is exercised) follows an arithmetic Brownian motion (that is, the cash flow shocks are $iid$).

Our paper is also related to Décamps, Mariotti, Rochet, and Villeneuve (2011) and Bolton, Chen, and Wang (2011), who consider optimal dynamic corporate policies for firms facing external financing costs. Décamps, Mariotti, Rochet, and Villeneuve (2011) consider a financially constrained firm’s optimal dynamic payout and equity issuance policy and Bolton, Chen, and Wang (2011) extend the $q$-theory of investment to firms facing external financing costs (Anderson and Carverhill (2012) also study a related liquidity management problem).

The major advance of this paper is that it allows for permanent earnings shocks, fixed investment and operating costs, and irreversibility, which together generate a demand for optionality and flexibility. In contrast, neither Décamps, Mariotti, Rochet, and Villeneuve (2011) nor Bolton, Chen, and Wang (2011) allow for permanent shocks, a key simplification which allows them to stay with a one-dimensional dynamic optimization problem. This paper provides a significant generalization of a financially constrained firm’s dynamic optimization problem to two dimensions—earnings fundamentals and liquidity buffers—in contrast to the earlier contributions with the (scaled) liquidity buffer as the single state variable. The economic analysis in the two-dimensional problem is significantly enriched, as the firm now takes account not only of its current stock of internal funds but also of the information about its future cash flow prospects contained in current persistent earnings shocks. Despite the significantly more complex formulation of the two-dimensional problem, we are still able to provide an intuitive and remarkably tractable analysis of this problem.

Finally, our work is also distantly related to dynamic contracting models of corporate financing and investment. Thus, DeMarzo, Fishman, He, and Wang (2012) develop a dynamic contracting model of corporate investment and financing with managerial agency in a $q$-theoretic framework with capital adjustment costs, by building on Bolton and Scharfstein (1990) and using the dynamic contracting framework of DeMarzo and Sannikov (2006) and DeMarzo and Fishman (2007). Rampini and Viswanathan (2010, 2013) develop dynamic
models of investment and risk management with endogenous financing constraints, in which the firm is subject to endogenous collateral constraints induced by limited enforcement.

2 Model

Operating revenues and profits. We consider a firm with an investment opportunity modeled as in McDonald and Siegel (1986). At any point in time $t \geq 0$, the firm can exercise an investment opportunity by paying a fixed investment outlay $I > 0$. Upon exercising, the firm then immediately obtains a perpetual stream of non-negative stochastic revenue $Y_t$. We assume that $Y$ follows a geometric Brownian motion (GBM) process:

$$dY_t = \mu Y_t dt + \sigma Y_t dB_t,$$

where $\mu$ is the drift parameter, $\sigma$ the volatility parameter, and $B$ is a standard Brownian motion. Once it has exercised its growth option, the firm also incurs a constant flow operating cost $Z > 0$ to operate the asset. The operating profit is then given by

$$Y_t - Z$$

per unit of time for as long as it continues operating the project. Should the firm deem that it is no longer worth continuing the operation, it can stop the project and there will be no scrap value. The firm has an American-style liquidation option where the timing of the option exercising decision is endogenously chosen.

Before undertaking the investment, the firm does not incur any costs, and while the firm is waiting to invest, the revenue process $Y_t$ continues to evolve according to (1). In sum, the simplest formulation of the life-cycle of the firm in our model allows for three phases: a start-up phase, a mature phase, and a liquidation (a strong form of scale-down) phase.

We assume that investors are risk neutral, so that all cash flows are discounted at the risk-free rate, $r$. Equivalently, we may interpret that the revenue generating process (1) has already captured the risk adjustment, i.e., under the risk-neutral measure and hence we may use the risk-free rate to discount the firm’s profits. One may pursue the risk-neutral measure interpretation when analyzing a firm’s risk-return tradeoff.

Liquidity hoarding. For a financially constrained firm, the key is the cash accumulation dynamics. We next discuss cash accumulation in both phases.
The startup phase. At the beginning of the start-up phase \((t = 0)\) the firm is endowed with a stock of cash (or, more generally, a liquidity hoard comprising both cash and marketable securities) of \(W_0\). As long as the firm does not spend this cash it simply accumulates liquid wealth at the risk-free rate \(r\) as follows:

\[
dW_t = rW_t dt, \quad t \geq 0.
\]  

(3)

Note that since the firm earns the risk-free rate \(r\) on its cash it does not need to pay out any cash to its shareholders, and shareholders weakly prefer hoarding cash inside the firm. If the firm were to earn less than the risk-free rate on its cash, it would also face an optimal payout decision. For simplicity we do not consider this generalization of the model.\(^2\) As the firm incurs no cost in carrying cash, without loss of generality, the firm never pays out its cash as long as it operates the asset.

When the firm’s liquidity \(W\) is insufficient to cover investment costs, i.e. \(W < I\), obviously the firm will have to raise external financing or continue to accumulate internal funds in order to finance the cost of exercising the growth option. Note that the firm will also need funds to finance potential operating losses after the growth option is exercised.

We introduce the standard specification for the external financing costs as follows: if the firm needs external funds \(F\) net of fees, it incurs an external financing cost \(\Phi(F)\). Hence, the firm must raise a gross amount \(F + \Phi(F)\). For simplicity, we assume that the equity issuance cost function for external financing takes the following form:

\[
\Phi(F) = \phi_0 + \phi_1 F,
\]  

(4)

where \(\phi_0 \geq 0\) is the fixed cost parameter and \(\phi_1 \geq 0\) is the marginal cost of external financing. Intuitively, when the fixed equity issuance cost is sufficiently high, the firm prefers liquidation over equity issuance. While in theory, we may allow for different equity issuance cost functions in the start-up and mature phases to capture different degrees of financing frictions (e.g., agency costs and informational frictions) in the start-up and mature phases, we keep the functional forms to be the same in both phases for simplicity.

The mature phase. During the mature phase (after exercising its growth option) the firm’s liquidity \(W_t\) accumulates as follows:

\[
dW_t = (rW_t + Y_t - Z)dt + dC_t, \quad W_t \geq 0.
\]  

(5)

\(^2\)For dynamic models with cash-carrying costs, see Bolton, Chen, and Wang (2011) and DéCamps, Mariotti, Rochet, and Villeneuve (2011) for example.
The first term in (5) denotes the firm’s internal funds $W_t$ (which earn the interest rate $r$) plus the operating revenue $Y_t$ minus the operating cost $Z$. The second term in (5), $dC_t$, denotes the net external funds that the firm chooses to raise through an external equity issue. We assume that the equity issuance cost function in this phase is also given by (4).

Note that the firm weakly never pays out and hence $dC_t \geq 0$. Total profits include interest income $rW$ and operating profits $Y_t - Z$. Without external financing (i.e., $dC_t = 0$), profits increase liquidity hoard $W$, i.e. when $rW_t + Y_t - Z > 0$, and liquidity hoard finances losses when $rW_t + Y_t - Z < 0$. Finally, after the firm has chosen to liquidate/scale-down its investment, it simply pays out its remaining cash $W_t$ to shareholders and closes down.

The firm’s dynamic optimization problem thus involves a sequence of two optimal stopping decisions: an investment timing decision followed by an abandonment timing decision. Importantly, liquidity plays a critical role in both phases. Before providing the solution for a financially constrained firm’s optimization problem, we first summarize the main results for a financially unconstrained firm.

3 The First-Best Benchmark

In the perfect capital markets world, where the MM holds, a firm is financially unconstrained and solves its value maximization problem. We summarize the first-best solution for the value-maximizing firm. First, consider the firm’s value in the mature phase.

3.1 The Mature Phase

Let $P^*(W,Y)$ denote the firm’s value in the mature phase in the MM world. With perfect capital markets, the firm’s value is simply given by the sum of its cash $W$ and the value of its asset in place:

$$P^*(W,Y) = Q^*(Y) + W.$$  \hspace{1cm} (6)

Here, $Q^*(Y)$ is the value of the firm’s asset in place, which equals the present discounted value of its future operating profits $Y_t - Z$. Importantly, the calculation accounts for the firm’s optimal exercising of its abandonment option. For sufficiently low values of $Y$, i.e. $Y \leq Y_a^*$ where $Y_a^*$ is the optimal abandonment hurdle to be determined soon, the asset is abandoned and hence $Q^*(Y) = 0$.

For the more interesting case where $Y > Y_a^*$, we may write $Q^*(Y)$ as the solution of the
following ODE:

\[ rQ(Y) = Y - Z + \mu Y Q'(Y) + \frac{\sigma^2 Y^2}{2} Q''(Y), \quad Y > Y_a^*, \]  

(7)

with the standard value-matching and smooth-pasting conditions:

\[ Q(Y_a^*) = 0, \quad (8) \]

\[ Q'(Y_a^*) = 0. \quad (9) \]

Importantly, the abandonment hurdle \( Y_a^* \) is endogenous and part of the model solution. The value \( Q^*(Y) \) admits the following unique closed-form solution:

\[ Q^*(Y) = \left( \frac{Y}{r - \mu} - \frac{Z}{r} \right) + \left( \frac{Y}{Y_a^*} \right) \gamma \left( \frac{Z}{r} - \frac{Y_a^*}{r - \mu} \right), \quad \text{for} \quad Y \geq Y_a^*. \quad (10) \]

The first term in (10) is the present discounted value of its operating profits if the firm were to remain in operation forever (which would be suboptimal for sufficiently low \( Y \)). The second term gives the additional value created if the firm were to optimally exercise its abandonment option. Without capital market frictions, the firm optimally operates its physical asset if and only if \( Y \geq Y_a^* \), where

\[ Y_a^* = \frac{\gamma}{\gamma - 1} \frac{r - \mu}{r} Z, \quad (11) \]

and the constant \( \gamma \) is given by

\[ \gamma = \frac{1}{\sigma^2} \left[ -\left( \mu - \frac{\sigma^2}{2} \right) - \sqrt{\left( \mu - \frac{\sigma^2}{2} \right)^2 + 2r\sigma^2} \right] < 0. \quad (12) \]

In the mature phase, the firm’s value is convex in earnings \( Y \) due to its abandonment option. Due to the option value of keeping the asset as a going concern, at the optimal abandonment hurdle \( Y_a^* \), the present discounted value of future revenues is less than the perpetual value of the operating cost \( Z/r \), in that

\[ \frac{Y_a^*}{r - \mu} = \frac{\gamma Z}{\gamma - 1} \frac{Z}{r} < \frac{Z}{r}, \quad (13) \]

as \( \gamma < 0 \). Next, we turn to the firm’s value in the start-up phase.
3.2 The Start-up Phase

We denote the \textit{first-best} value of a financially unconstrained firm by $G^*(W,Y)$. As for the firm’s value $P^*(W,Y)$ in the mature phase, the first-best value $G^*(W,Y)$ takes the following simple additive form:

$$G^*(W,Y) = H^*(Y) + W. \quad (14)$$

Here, $H^*(Y)$ is the value of the firm’s growth option, which includes the present discounted value of its future operating profits $Y_t - Z$, the value of optimal growth option exercising, and the value of abandonment. Intuitively, with perfect capital markets, again, firm value is given by the sum of its cash holding $W$ and the value of its growth option, $H^*(Y)$, which can be valued independently from its liquidity holding.

For sufficiently high values of $Y$, i.e. $Y \geq Y_i^*$ where $Y_i^*$ is the optimal investment hurdle to be determined, the growth option is immediately exercised and $H^*(Y) = Q^*(Y) - I$, where $Q^*(Y)$ is the value of assets in place in the mature phase, given in (10). For the more interesting case where $Y < Y_i^*$, we may write $H^*(Y)$ as the solution of the following ODE:

$$rH(Y) = \mu Y H'(Y) + \frac{\sigma^2 Y^2}{2} H''(Y), \quad (15)$$

subject to the value-matching and smooth-pasting boundary conditions:

$$H(Y_i^*) = Q^*(Y_i^*) - I, \quad (16)$$
$$H'(Y_i^*) = Q'(Y_i^*). \quad (17)$$

Additionally, the growth option is worthless at the origin $Y = 0$ as it is an absorbing state for a GBM process, i.e. $H(0) = 0$.

The optimal investment hurdle $Y_i^*$ is the solution to the following equation:

$$(\beta - \gamma) \left( \frac{Y_i^*}{Y_a^*} \right)^\gamma \left( \frac{Z}{r} - \frac{Y_a^*}{r - \mu} \right) + (\beta - 1) \frac{Y_i^*}{r - \mu} - \beta \left( \frac{Z + rI}{r} \right) = 0. \quad (18)$$

The option value $H^*(Y)$ has the following closed-form solution:

$$H^*(Y) = \left( \frac{Y}{Y_i^*} \right)^\beta (Q^*(Y_i^*) - I), \quad \text{for} \quad Y \leq Y_i^*, \quad (19)$$

where $\beta$ is a constant given by

$$\beta = \frac{1}{\sigma^2} \left[ - \left( \mu - \frac{\sigma^2}{2} \right) + \sqrt{\left( \mu - \frac{\sigma^2}{2} \right)^2 + 2r\sigma^2} \right] > 1. \quad (20)$$

Again, here the firm’s value is convex in $Y$. Next, we turn to the analysis for a financially constrained firm.
4 Abandonment and Financing in the Mature Phase

In the mature phase, the firm manages its asset in place and has a liquidity hoard \( W \). With external financing costs, liquidity hoard \( W \) influences its decision and its valuation. First we note that the firm will never voluntarily issue external equity provided that it has liquidity to keep the firm solvent, as the firm is better off by postponing its equity issue. Intuitively, the firm’s financing term does not change and delaying equity issue saves the forgone interest income on the financing cost. Therefore, the firm can be in one of the three regions depending on its liquidity \( W \) and earning \( Y \):

1. the “financially unconstrained” region where more \( W \) does not influence the firm’s decision in any way and the firm is permanently financially unconstrained and hence behaves in the same way as it does in an MM world;

2. the “interior” financially constrained region where the firm does not raise any external financing nor pays anything out but simply hoards and accumulates its liquidity \( W \) and continues its operation;

3. the “equity issuance” region where the firm will only issue equity when it incurs an operating loss, i.e. \( Y - Z < 0 \) and moreover only when it exhausts its liquidity capacity, i.e., \( W = 0 \), as the firm can always defer the option of financing to save the time value of financing costs.

Note that in all three regions, the firm may liquidate its operations when \( Y \) is sufficiently below its operating cost \( Z \), which reflects the firm’s dynamic tradeoff between covering operating losses \( Y - Z \) using its internal funds and the option value of capturing the upside when the earning eventually turns out to be high enough.

Denote the firm’s value in the mature phase by \( P(W,Y) \). We may write it as

\[
P(W_t,Y_t) = \max_{\tau_L, dC \geq 0} \mathbb{E}_t \left[ -\int_{t}^{\tau_U \wedge \tau_L} e^{-r(s-t)} I_{dC_s > 0} [dC_s + \Phi(dC_s)] \\ + e^{-r(\tau_U - t)} P^\ast(W_{\tau_U},Y_{\tau_U}) I_{\tau_U < \tau_L} + e^{-r(\tau_L - t)} W_{\tau_L} I_{\tau_U > \tau_L} \right], \tag{21}
\]

where \( P^\ast(W_t,Y_t) \) is the first-best firm value for a financially unconstrained firm given by (6), \( dC \) is the net equity issuance and \( \Phi(dC) \) is the corresponding equity issuance costs, and \( I_{dC_s > 0} \) is an indicator function which takes the value of one when \( dC_s > 0 \) and zero otherwise. Note that we have two stopping times: \( \tau_U \) is the stopping time that the firm
accumulates sufficient liquidity such that it permanently becomes unconstrained and hence attains the first-best firm value \( P^*(W_t, Y_t) \), and \( \tau_L \) is the endogenous stochastic liquidation time at which the firm is liquidated and releases its cash \( W_{\tau_L} \) to shareholders. \( I_{\tau_U < \tau_L} \) is an indicator function which takes the value of one when the firm becomes financially unconstrained \( \tau_U < \tau_L \) and zero otherwise.

If liquidation is suboptimal, the firm must raise costly external financing to be able to continue operating the project should it run out of cash. That is, at any time \( s \) when the firm incurs operating losses, \( Y_s < Z \), and also is out of cash, \( W_s = 0 \), it has to raise funds \( dC_s \) at least sufficient to cover operating losses, if it were to continue operations.

4.1 The Financially Unconstrained Region

Unlike in a static setting, a firm is financially unconstrained in a dynamic setting if and only if it faces no financial constraint with probability one at the current and all future times. There are two ways that the firm can be financially unconstrained: (1) it internally generates sufficient liquidity at all times; or (2) the firm already has a sufficient liquidity hoard: \( W \geq \Lambda \), where \( \Lambda \) denotes the lowest level of liquidity needed for a mature firm to be permanently financially unconstrained. We will provide an explicit formula for \( \Lambda \).

**Type-1 financially unconstrained firm:** \( Y \to \infty \).

If the firm’s internally generated cash flow \( Y \) is very high, the firm will be fully liquid even without any liquidity hoard \( W \), as internally generated revenue \( Y \) is fully sufficient to cover its flow operate cost \( Z \) per period without ever having to raise external funds. In the limit where \( Y \to \infty \), the following boundary condition must hold:

\[
\lim_{Y \to \infty} P(W, Y) = \lim_{Y \to \infty} P^*(W, Y),
\]

(22)

where \( P^*(W, Y) \) is the value for a financially unconstrained firm and is given by (6).

**Type-2 financially unconstrained firm:** \( W \geq \Lambda \).

The firm is able to implement the first-best abandonment option policy and avoid costly external financing permanently with probability one provided that the firm has sufficiently high liquidity hoarding. The question is how high the firm’s liquidity hoard \( W \) has to be for a firm to be permanently unconstrained. As long as the firm does not abandon its asset in place prematurely and involuntarily, that is, when \( Y > Y^*_a \), the firm achieves its first-best policy and thus it is financially unconstrained. That is, as long as the corporate saving rate \((rW + Y - Z)\) is weakly positive at \( Y^*_a \), then the firm will never be forced to liquidate the
firm sub-optimally. That is, we need $rW + Y_a^* - Z \geq 0$, which implies

$$W \geq \frac{Z - Y_a^*}{r} = \Lambda. \tag{23}$$

Using the explicit formula (23) for the abandonment hurdle $Y_a^*$, we write $\Lambda$ as

$$\Lambda = \frac{r - \gamma \mu}{r^2(1 - \gamma)} Z. \tag{24}$$

In summary, as long as condition (23) holds, the firm is permanently financially unconstrained, and hence the firm’s value $P(W,Y)$ equals the first-best firm’s value:

$$P(W,Y) = P^*(W,Y), \quad \text{for} \quad W \geq \Lambda. \tag{25}$$

Note that the firm can finance its efficient continuation entirely out of its cash hoard in the first-best continuation region, $Y \geq Y_a^*$. And for $Y < Y_a^*$, the firm voluntarily and efficiently abandons its asset and distributes $W$ to shareholders.

In macro savings literature, e.g. Aiyagari (1994), a core concept is the natural borrowing limit, which refers to the maximal amount of risk-free credit that a consumer can tap with no probability of ever defaulting. Hence, the consumer can borrow at the risk-free rate up to that limit, but any additional amount of borrowing will give rise to default risk. Here is the analogy. For a firm, $\Lambda$ given in (24) is the the minimum of the liquidity hoard that it needs in order to implement its first-best abandonment strategy. Any liquidity hoard lower than $\Lambda$ may induce underinvestment via inefficient liquidation in the future with positive probability.

Next, we turn to the “interior” region where the firm hoards liquidity and operates its asset in place. In this region, the firm is financially constrained but does not raise any external financing as the firm strictly prefers deferring equity issuance decision to the future.

### 4.2 The Interior Liquidity Hoarding Region

Using the standard principle of optimality, we characterize the firm’s value $P(W,Y)$ as the solution to the following Hamilton-Jacobi-Bellman (HJB) equation:

$$rP(W,Y) = (rW + Y - Z)P_W(W,Y) + \mu Y P_Y(W,Y) + \frac{\sigma^2 Y^2 P_{YY}(W,Y)}{2}, \tag{26}$$

subject to various boundary conditions to be discussed later. The first term on the right side of (26), given by the product of the firm’s marginal value of cash $P_W(W,Y)$ and the firm’s
saving rate \((rW + Y - Z)\), represents the effect of the firm’s savings on its value. The second term (the \(P_Y\) term) represents the marginal effect of expected earnings change \(\mu Y\) on firm value, and the last term (the \(P_{YY}\) term) encapsulates the effects of the volatility of changes in earnings \(Y\) on firm value. Intuitively, the expected change of firm value \(P(W, Y)\), given by the right side of (26), equals \(rP(W, Y)\), as the firm’s expected return is \(r\).

With financial frictions, liquidity generally is more valuable than its pure monetary value. Typically, the firm’s decision of whether to abandon the project or not is influenced by its financial considerations and the prospect of having to incur external financing costs in the future. All else equal, the costs of external financing ought to be an additional inducement to abandon a project yielding low revenues. Therefore, one would expect that the prospect of having to incur external financing costs would lower the firm’s valuation for its asset in place and result in a higher abandonment hurdle.

Now consider the situation where a financially constrained firm is just indifferent between abandoning the firm or not. At the moment of indifference, firm value must be continuous,

\[
P(W, \underline{Y}(W)) = W,
\]

which states the the firm’s value equals to its liquidity hoard \(W\) at the moment of abandoning its asset. Equation (27) implicitly defines the abandonment hurdle \(\underline{Y}(W)\). Moreover, since the abandonment decision is optimally made, the marginal values along both \(Y\) and \(W\) dimensions shall also be matched before and after the abandonment of the asset in place,

\[
P_Y(W, \underline{Y}(W)) = 0.
\]

The smooth-pasting condition (28) states that at the optimal abandonment hurdle \(\underline{Y}(W)\), \(P_Y\) equals zero.\(^3\) Next, we turn to the equity issuance region.

### 4.3 The Equity Issuance Region

As Bolton, Chen, and Wang (2011) show, in a world with constant financing opportunities where financing terms do not change over time, the firm has no need to issue equity unless it absolutely has to. By delaying equity issuance, the firm saves the time value of money for the financing costs. In the mature phase, with positive liquidity hoarding \(W\), the firm has

\(^3\)As a result, the firm’s marginal value of liquidity \(P_W(W, \underline{Y}(W))\) evaluated at the optimal abandonment hurdle \(\underline{Y}(W)\) equals one.
sufficient slack to cover any operating losses over a given small time interval. Therefore, the firm never issues equity before it exhausts its cash.

When the firm runs out of its cash \((W = 0)\), it finds itself in one of the three regions. First, when \(Y > Z\), the firm is solvent even without savings as its internally generated cash flow \(Y\) covers its operating cost \(Z\), and hence the firm needs no external financing. When the firm’s internally generated revenue cannot fully cover its operating cost \((Y < Z)\) and with no savings \((W = 0)\), the firm has an option to either issue equity or to simply liquidate its asset in place altogether, whichever is in the interest of its shareholders. Intuitively, whether the firm issues equity or liquidates itself depends on how valuable the firm’s going concern value is, i.e. how high the revenue \(Y\) is.

With a fixed equity issuance cost \(\phi_1 > 0\), the firm optimally chooses the amount of financing \(F\) and the endogenous hurdle \(Y(0)\) to satisfy the following value-matching boundary conditions at \(W = 0\):

\[
P(0, Y) = \begin{cases} 
P(F, Y) - F - \Phi(F), & Y(0) < Y < Z, \\ 0, & Y \leq Y(0). \end{cases}
\]  

(29)

The endogenous hurdle \(Y(0)\) given in condition (29) provides the boundary between the equity-issuance region and the liquidation region, \(Y \leq Y(0)\). The first case in (29) corresponds to the equity issuance region, \(Y(0) < Y < Z\). Let \(F_a(Y)\) denote the external financing amount \(F\) as a function of earning \(Y\) (recall that financing only occurs when \(W = 0\)). And the second case in (29) corresponds to the liquidation region, \(Y \leq Y(0)\). By the firm’s revealed preference, the dividing boundary \(Y(0)\) shall be higher than the first-best abandonment hurdle \(Y^*_a\), a form of underinvestment.

Should the firm seek to raise new funds, its optimal external financing amount \(F\) is given by the first-order condition (FOC)\(^4\) for the case with \(\phi_0 > 0\):

\[
P_W(F, Y) = 1 + \Phi'(F) = 1 + \phi_1, \quad Y(0) < Y < Z.
\]  

(30)

Intuitively, conditional on the firm’s issuance decision at \(W = 0\), the marginal value of cash \(P_W(F, Y)\) equals the marginal cost of financing \(1 + \Phi'(F)\). Note that the firm’s marginal value of cash \(P_W(W, Y)\) depends on its revenue \(Y\), and \(P_W(W, Y)\) is greater than 1 at the moment of financing.

\(^4\)We will verify the second-order condition (SOC) to ensure that the FOC solution yields the maximal value.
Summary. In a dynamic environment, the condition for a firm to be financially unconstrained is much tighter than in a static setting. The reason is as follows. For a firm to be financially unconstrained in a dynamic setting, the firm cannot have demand for funds at any moment. That is, with probability one, the firm has no demand for funding. Only under this condition, can the firm be assured that its marginal value of cash is one. In our model, the firm can be financially unconstrained in one of the two ways; it either internally generates enough amount of funds or it has sufficient liquidity to cover all the potential needs.

We next turn to the firm’s problem in the start-up phase. Obviously, a rational forward-looking firm fully anticipates its financial constraints in the mature phase and acts accordingly in the start-up phase.

5 Investment and Financing in the Start-up Phase

In the start-up phase, the firm maximizes its present value by solving the optimal investment timing problem. Let $G(W_t, Y_t)$ denote this value. Specifically, the firm chooses the optimal investment timing $\tau_i$ to maximize the value of the growth option by solving

$$G(W_t, Y_t) = \max_{\tau_i, F \geq 0} \mathbb{E}_t \left[ e^{-r(\tau_i-t)} (P(W_{\tau_i} + F - I, Y_{\tau_i}) - (F + \Phi(F)) I_{F>0}) \right] ,$$

where $\tau_i$ is the endogenous investment timing, and $I_{F>0}$ is an indicator function which takes the value of one when $F > 0$ and zero otherwise. Recall that $P(W, Y)$ is the firm’s value in the mature phase. To be able to invest, the firm must have total available funds $W + F$ that cover at least the investment outlay $I$, i.e., $W + F \geq I$.

When choosing its optimal investment timing, the firm incorporates both the one-time lumpy investment cost $I$ and also its future operating (flow) cost $Z$. Before analyzing the effect of financial constraints on the firm’s investment option exercising and financing decisions, we first reason how much liquidity the firm needs in order to be financially unconstrained.

5.1 The Financially Unconstrained Region: $W \geq I + \Lambda$

For a firm to be dynamically financially unconstrained, it should have the first-best investment and abandonment decisions under all circumstances. Intuitively, with liquidity hoard greater than $\Lambda + I$, then with probability one, it can cover both its investment cost $I$ and its future liquidity shortfall to continue an efficient operation of its asset with liquidity amount $\Lambda$. Therefore, the firm in its start-up phase is financially unconstrained if and only
if
\[ W \geq I + \Lambda. \] (31)

In summary, as long as (31) holds, the firm is permanently financially unconstrained, and hence in the startup phase, its value \( G(W, Y) \) achieves the first-best value given by
\[ G(W, Y) = G^*(W, Y) = H^*(Y) + W, \quad \text{for} \quad W \geq I + \Lambda, \] (32)
where \( H^*(Y) \) is given by (19) and the first-best investment hurdle \( Y_i^* \) is given by (18). Recall that \( H^*(Y) \) and \( Y_i^* \) are independent of liquidity \( W \). Note that in the mature phase, the firm can also finance its efficient continuation entirely out of its cash hoard with an optimal abandons its asset and distributes \( W \) to shareholders.

When \( W < I + \Lambda \), the firm is financially constrained. There are two sub-cases:

- When \( I \leq W < I + \Lambda \), the firm has a sufficient liquidity hoard to fund the investment outlay \( I \) entirely out of its internal funds, but may not have sufficient funds to avoid the liquidity shortage in the mature phase and hence equity issuance or involuntary liquidation may occur with positive probability.

- When \( W < I \), the firm cannot cover its investment cost \( I \), and hence the firm may require external financing to cover both the investment cost and the liquidity need in the mature phase.

Note that a financially constrained firm has a option value of building up financial slack internally. The tradeoff between internal and external financing, the timing decision, and the consideration of future liquidity needs in the mature phase make the financially constrained firm’s decision a complex but very important one.

5.2 The Medium Cash-holding Region: \( I \leq W < I + \Lambda \)

Consider now the situation of a firm with moderate financial slack. This firm has sufficient internal funds \( W \) to cover the investment cost \( I \), but not quite enough cash to ensure that it will never run out of internal funds in the mature phase: \( I \leq W < I + \Lambda \). In the forward-looking sense, the firm is still financially constrained and the marginal value of cash is greater than one. For such a firm it is optimal not to raise any external funds when it chooses to exercise its growth option, so that \( F = 0 \). Note that the firm may not choose to exercise the growth option. Importantly, the firm realizes that exercising the investment option drains its
cash holding by $I$ and hence the firm may be led to raise external funds in the future to cover operating losses in the mature phase. Therefore, the firm is still financially constrained, as it may have potential liquidity demand in the mature phase.

In the waiting region, the firm’s value $G(W, Y)$ solves the following HJB equation:

$$rG(W, Y) = rWG_W(W, Y) + \mu Y G_Y(W, Y) + \frac{\sigma^2 Y^2}{2} G_{YY}(W, Y),$$

subject to various boundary conditions to be discussed later. Note that the first term on the right side of (33) reflects the firm’s savings effect on firm value. The remaining two terms are the standard earnings drift and volatility effects on the option value.

As in the standard real options literature, at the endogenously chosen moment of investment, firm value $G(W, Y)$ is continuous and hence we have an implicit function $\bar{Y}(W)$ defined by

$$G(W, \bar{Y}(W)) = P(W - I, \bar{Y}(W)).$$

(34)

The value-matching condition (34) characterizes the investment hurdle as an implicit function of liquidity $W, \bar{Y}(W)$. In this region, the investment cost $I$ is entirely financed out of internal funds, and hence liquidity $W$ decreases by $I$, as seen on the right side of (34). Additionally, because the investment hurdle $\bar{Y}(W)$ is optimally chosen, we have the following smooth-pasting condition along the earning $Y$ dimension:

$$G_Y(W, \bar{Y}(W)) = P_Y(W - I, \bar{Y}(W)).$$

(35)

Finally, when the absorbing state $Y = 0$ is reached, there is no investment opportunity and the only valuable asset of the firm is its cash $W$, and hence

$$G(W, 0) = W.$$

(36)

Next, we turn to the low cash-holding region where investment cannot be financially solely with internal funds, $W < I$.

5.3 The Low Cash-holding Region, $0 \leq W < I$

In the region where internal funds are insufficient to cover the investment cost $I$, the firm has to raise external financing should it decide to invest. Intuitively, no matter how large its current earning $Y$ is, the firm has to access external capital markets if choosing to immediately invest, as the investment cost $I$ is lumpy while the earning $Y$ is a flow. At the
moment of investing, the firm’s value must be continuous, and hence we have an implicit function $\bar{Y}(W)$ defined by

$$G(W, \bar{Y}(W)) = P(W + F - I, \bar{Y}(W)) - F - \Phi(F).$$

(37)

The right side of the value-matching condition (37) gives the firm’s value after it issues net amount $F$ and incurs a cost $\Phi(F)$. The left side of (37) is the firm’s value before investing. Note that the post-financing/investment liquidity is $W + F - I$.

Of course, it is quite plausible that the firm may want to wait. In this waiting region, the firm’s value $G(W,Y)$ also solves the HJB equation (33) for the same argument as the one used the previous subsection (in the medium cash-holding region). In addition to the investment hurdle $\bar{Y}(W)$, the firm also needs to choose the net equity issue amount $F$ to at least cover the needed financing for investment $I - W$.

We denote by $F_g(W)$ the amount of external financing by the firm as a function of $W$ in the region $0 \leq W < I$. The minimal amount of issuance required so that the post-issuance liquidity is non-negative is $I - W$. Thus, under optimal external financing the following inequalities must hold:

$$P_W(W + F_g(W) - I, \bar{Y}(W)) \leq 1 + \Phi'(F_g(W)) \text{ and } F_g(W) \geq I - W.$$  

(38)

That is, the firm will issue equity such that the marginal value of liquidity is weakly lower than the marginal cost of issuance. We write the optimality conditions in this way because it is possible that the constraint $F_g(W) \geq I - W$ may bind, in which case the firm chooses to rely solely on its ability to generate sufficient liquidity from operating earnings after it has invested in the productive asset.

The firm’s optimality implies that the marginal value of earning $Y$ is continuous before and after the investment option is exercised. Therefore, we have the following two smooth-pasting conditions:

$$G_Y(W, \bar{Y}(W)) = P_Y(W + F_g(W) - I, \bar{Y}(W)).$$

(39)

Finally, $G(W, 0) = W$ as $Y = 0$ is an absorbing state.

6 Analysis

As is standard in the literature, we set the risk-free interest rate $r = 5\%$, the expected earnings growth rate $\mu = 0$, and the earnings growth volatility $\sigma = 15\%$. The investment
cost is set at $I = 2$ and the operating cost is $Z = 1$. When applicable, the parameter values are annualized.

The first-best liquidation hurdle is $Y_a^* = 0.625$ which implies that a financially unconstrained firm will continue as a going concern even when it incurs a loss of $Z - Y_a^* = 0.375$, 37.5% of the (flow) operating cost $Z = 1$. This indicates a significant option value for a firm to continue as a going concern under MM. In the startup phase, the firm exercises its growth option when its earnings $Y$ reaches the first-best investment hurdle $Y_i^* = 1.544$. At the moment of exercising, the value of the assets in place (including the option value of abandonment) is $Q^*(Y_i^*) = 12.54$. As the investment cost $I = 2$, at the moment of investment, the firm’s value (netting the investment cost) is $H^*(Y_i^*) = 10.54$.

With sufficiently high cash holding $W$, the firm is financially unconstrained at all times with probability one. In our example, the minimal amount of liquidity needed for a firm in the mature phase to be permanently financially unconstrained is

$$
\Lambda = \frac{r - \gamma \mu}{r^2(1 - \gamma)} Z = 7.50,
$$

which is 7.5 times the operating cost $Z = 1$. In the startup phase, the minimal amount of liquidity needed for the firm to be permanently financially unconstrained is thus $\Lambda + I = 9.50$ which covers both the investment cost $I$ and the liquidity needs in the mature phase.

For the external financing cost, we choose the marginal issuance cost $\phi_1 = 0.01$ motivated by the empirical analysis in Altinkilic and Hansen (2000). The fixed equity issuance cost induces lumpy issuance, which is empirically important. We thus focus on the parameter $\phi_0$ in our comparative static analysis. For our baseline case, we choose $\phi_0 = 0.4$ which implies that the fixed equity issuance cost is about 4% of $H^*(Y_i^*) = 10.54$, the first-best (net) firm value at the moment of investment and also this value is broadly in line with the empirical estimate reported in Altinkilic and Hansen (2000). To highlight the impact of the fixed financing cost $\phi_0$, we thus consider three values: $\phi_0 = 0.01$, 0.4, 2. By varying $\phi_0$ we see how the financing optionality interacts with the real optionality.

### 6.1 The Mature Phase

We define the firm’s *enterprise value* as its total value in excess of cash,

$$
Q(W, Y) = P(W, Y) - W.
$$

(41)
Figure 1: **The optimal liquidation hurdle \( Y(W) \) for a financially constrained firm in the mature phase.** The endogenous liquidation hurdle \( Y(W) \) is monotonically decreasing with liquidity holding \( W \) and approaches the first-best level \( Y^*_a = 0.625 \) independent of the financing cost \( \phi_0 \). For a given value of \( W \), the larger the fixed equity issuance cost \( \phi_0 \), the higher the liquidation hurdle \( Y(W) \) indicating a higher degree of under-investment. At \( W = 0 \), \( Y(0) \) equals 0.645, 0.735, 0.895 for \( \phi_0 = 0.01, 0.4, 2 \), respectively.

Because cash is valuable beyond its face value for a financially constrained firm, the enterprise value also depends on \( W \). Under the MM condition, the enterprise value is independent of the firm’s cash holding, and we have \( Q^*(Y) = P^*(W,Y) - W \), as given by (6).

**The Liquidation Decision.** Figure 1 plots the optimal liquidation hurdle \( Y(W) \) for a financially constrained firm. First note that the firm becomes permanently financially unconstrained when its cash holding reaches \( \Lambda = 7.50 \), which is 7.5 times the (annual) operating cost \( Z = 1 \). For a financially unconstrained firm, it is optimal to liquidate its asset when its earning \( Y \) falls below the first-best liquidation hurdle \( Y^*_a = 0.625 \). And importantly, the firm will never be forced into sub-optimally abandoning its asset due to the shortage of its liquidity \( W \) as long as \( W \geq \Lambda = 7.50 \). Quantitatively, Figure 1 shows that the firm with liquidity \( W \) larger than 3 is effectively dynamically financially unconstrained.

When the firm exhausts its internal funds \( W_r = 0 \) and its cash flow is larger than its operating cost \( Z \) (i.e., \( Y > Z = 1 \)), the firm remains solvent purely relying on its internally generated cash flow. However, when its earning \( Y \) falls below its operating cost (i.e., \( Y < 1 \)), it has to either raise external funds or will be liquidated otherwise. In our baseline case
where $\phi_0 = 0.4$, if a constrained firm’s earning $Y$ lies in the region $1 > Y > Y(0) = 0.735$, it is optimal to issue equity keeping the firm alive. Only when its earnings $Y$ falls below 0.735, the firm will abandon its asset rather than issue equity to finance its liquidity shortfall. Recall that the first-best liquidation boundary is $Y^*_a = 0.625$. Hence, a firm with $W = 0$ will be inefficiently liquidated due to lack of internal funds in the region $0.625 < Y \leq 0.735$.

Figure 1 also illustrates that the liquidation hurdle $Y(W)$ decreases with the firm’s liquidity $W$ in the region $[0, \Lambda] = [0, 7.50]$ for all three levels of $\phi_0$. Hence, inefficiently liquidation occurs in the region $0 \leq W < \Lambda$, as $P_W > 1$ in this region and liquidity is valuable. For example, when $\phi_0 = 2$, the abandonment hurdle decreases from $Y(0) = 0.895$ to $Y^*_a = 0.625$ as $W$ increases from the origin to $\Lambda = 7.50$. Intuitively, the higher the liquidity holding $W$, the less inefficient the firm’s liquidation decision. For the case with a small fixed equity issuance cost, $\phi_0 = 0.01$, the impact of financial constraints is negligible; the cashless firm will only be abandoned inefficiently if its earning $Y$ falls inside the tight region $0.625 < Y \leq Y(0) = 0.645$. How important is the impact of financing costs on liquidation? At the origin $W = 0$, as we increase the fixed issuance cost $\phi_0$ from 0.01 to 0.4 and then from 0.4 to 2, the abandonment hurdle $Y(0)$ increases from 0.645 to 0.735 and then from 0.735 to 0.895, respectively. The implied real inefficiencies are significant. Finally, we note that quantitatively the effect of financial constraints essentially disappears as the firm’s liquidity hoard $W$ reaches 1.6, even when the fixed equity issuance cost is relatively high, $\phi_0 = 2$.

The Equity Issuance Decision. The firm will consider the possibility of issuing equity when it runs out of cash ($W = 0$) and its earning cannot cover its operating cost ($Y < Z$). Intuitively, it is always preferable for the firm to postpone raising external funds whenever feasible. Additionally, the firm will not issue equity if its earning falls below its first-best abandonment hurdle $Y^*_a$, as it must be optimal for a financially constrained firm to abandon its assets in place if it is optimal for a financially unconstrained firm to do so. Hence, in Figure 2, we only need to plot the optimal equity issuance amount $F_a(Y)$ as a function of its earning $Y$ in the region $[Y^*_a, Z] = [0.625, 1]$ for the three cases, $\phi_0 = 0.01, 0.4, 2$.

Importantly, the amount of equity financing $F_a(Y)$ may be non-monotonic in $Y$. For the baseline case with $\phi_0 = 0.4$, $F_a(Y)$ first increases with $Y$ in the region $Y(0) < Y < 0.80$ and then decreases with $Y$ in the region $0.80 < Y < 1$. The net issuance amount $F$ peaks at $Y = 0.80$ with a value of $F_a(0.80) = 1.80$. In the region $0.735 \leq Y < 0.80$, the firm’s future prospects are not sufficiently encouraging for it to raise much external funding as
Figure 2: The optimal external financing $F_a(Y)$ for a financially constrained firm in the mature phase. Firms will choose to raise external funds only when it runs out of its liquidity and when its internally generated cash flow cannot cover the operating cost $Z$ but is sufficiently high (i.e., only when $W = 0$ and $Y(0) < Y < Z = 1$), where $Y(0)$ is the optimal abandonment hurdle for a financially constrained firm. For $\phi_0 = 0.01, 0.4, 2$, we have shown that $Y(0) = 0.645, 0.735, 0.895$, respectively. Interestingly, the firm’s net financing amount $F_a(Y)$ is non-monotonic in its earning $Y$ over the region $(Y(0), 1)$. For example, for the case with $\phi_0 = 0.4$, the external financing $F_a(Y)$ first increases in the region $Y \in (0.735, 0.80)$, peaks at $Y = 0.80$ with a value of $F_a(0.80) = 1.80$, and then decreases with $Y$ in the region $Y \in (0.80, 1)$.
the liquidation threat (in the future) is not low. This is the dominant consideration in this region, so that when \( Y \) increases, it is marginally worth raising more funds (paying the marginal cost \( \phi_1 = 0.01 \)) given that the firm’s survival likelihood is improving. In the region where \( Y \in (0.80, 1) \) on the other hand, the dominant consideration when \( Y \) increases is the greater likelihood that the firm will be able to generate funds internally via its earning \( Y \) and interest income \( rW \) in the near future so that the firm does not need much external funds. As a result the firm optimally chooses to rely less on current external financing in the expectation of larger future internally generated funds. Intuitively, the firm’s expectation about its own future ability to generate internal funds (e.g., from operations and interest incomes) significantly influences the firm’s current financing policy.

Additionally, conditional on choosing to raise funds, the firm raises more if the the fixed costs of external funding \( \phi_0 \) are higher. For example, at \( Y = 0.9 \), as we increase the fixed issuance cost \( \phi_0 \) from 0.01 to 0.4 and then from 0.4 to 2, the firm’s external financing \( F_a(0.9) \) increases from 0.81 to 1.71, and then from 1.71 to 1.88. Intuitively, a firm that faces a larger fixed issuance cost \( \phi_0 \) has a stronger incentive to issue more and hence capitalize on the fixed cost \( \phi_0 \) as the cost of going back to the capital markets is greater in the future \textit{ceteris paribus}. Importantly, this prediction is the opposite to that based on the intuition from static models such as Froot, Scharfstein, and Stein (1993) and Kaplan and Zingales (1997). In these static models, the higher the financing cost, the more constrained the firm, and the lower the amount of equity financing.

Figure 3 plots the firm’s enterprise value \( Q(W,Y) \) and its \textit{marginal enterprise value of cash} \( Q_W(W,Y) \) against liquidity \( W \) for two different levels of earning, \( Y = 0.65 \) and \( Y = 1.5 \), and for three fixed equity issuance costs, \( \phi_0 = 0.01, 0.4, 2 \). Intuitively, the higher the financing cost \( \phi_0 \), the lower the firm’s enterprise value \( Q(W,Y) \). Also note that the \textit{net marginal value of cash} \( Q_W(W,Y) \) is always positive implying that the firm is financially constrained and hence liquidity is valuable in the constrained region, i.e., \( W < \Lambda = 7.50 \).

The firm’s value and the marginal value of liquidity. A central observation emerging from Figure 3 is that the firm’s marginal enterprise value of cash \( Q_W(W,Y) \) can vary nonmonotonically with its liquidity \( W \). Panel B (with \( Y = 0.65 \)) highlights the non-concavity of \( Q(W,Y) \) in \( W \). Specifically, \( Q(W,Y) \) can be either concave or convex in liquidity \( W \). First, consider the case with a small fixed equity issuance cost, \( \phi_0 = 0.01 \). Even with a low earning, e.g., \( Y = 0.65 \), the firm will not abandon its asset as the firm’s abandonment
hurdle \( Y(0) = 0.645 \) and hence the firm will never liquidate. In this case, liquidity provides value by mitigating the firm’s external funding needs. Hence, the firm’s liquidity is valuable and firm value is concave in \( W \) with the marginal value \( Q_W(W, Y) \) monotonically decreasing from 0.280 to zero as \( W \) increases from zero to \( \Lambda = 7.50 \). However, as we increase the fixed cost \( \phi_0 \) from 0.01 to 0.4, the marginal enterprise value of liquidity \( Q_W(W, Y) \) is no longer concave. Indeed, \( Q_W(W, Y) \) first equals zero for \( W \leq 0.29 \) and then increases for \( 0.29 < W < 0.6 \) and finally decreases with \( W \) in the region with sufficiently high \( W > 0.6 \). The intuition is as follows. A firm with a low earning \( Y = 0.65 \) optimally chooses to exercise its abandonment option provided that its liquidity \( W \leq 0.29 \), as the abandonment hurdle
\( Y(0.29) = 0.65 \). Hence, in the low liquidity region \( 0.29 < W < 0.6 \), increasing \( W \) lowers the firm’s likelihood of tapping costly external financing, but the firm still has a low probability to survive. In this case, the firm may be endogenously risk-loving with respect to exercising its equity issuance option. This can only be the case if the firm faces strictly positive fixed costs of equity issuance (\( \phi_0 > 0 \)). And hence \( Q_W \) increases with \( W \) in this low-liquidity region implying that firm value is locally convex in \( W \) in this region. Finally, when \( W > 0.6 \), the firm has sufficiently high liquidity and equity issuance becomes much less likely causing the marginal enterprise value of liquidity \( Q_W \) to decrease as the firm becomes less financially constrained. We also see that with a higher \( \phi_0 \) (an increase of \( \phi_0 \) from 0.4 to 2) shifts \( Q_W(W,Y) \) as a function of \( W \) to the right while retaining the general non-monotonic shape. This makes sense as a firm facing a larger external equity financing cost \( \phi_0 \) chooses a higher abandonment hurdle \( Y(W) \).

Now, we consider the case with a relative high earning, e.g., \( Y = 1.5 \), the firm’s enterprise value \( Q(W,Y) \) is now concave in \( W \), as its earning is significantly larger than the abandonment hurdle \( Y(W) \) and the financing option is sufficiently deeply in the money. In these cases, the firm behaves as it were risk averse \( (P_{WW} < 0) \) with respect to liquidity \( W \).

### 6.2 The Start-up Phase

As for the mature phase, we again define the firm’s enterprise value in the start-up phase as

\[
H(W,Y) = G(W,Y) - W. \quad (42)
\]

For a financially unconstrained firm, its enterprise value is independent of its cash holding, i.e., \( H^*(Y) = G^*(W,Y) - W \), and the closed-form solution for \( H^*(Y) \) is given by (19). In general, a financially constrained firm’s enterprise value depends on its cash holding \( W \) and is lower than the first-best value, \( H(W,Y) \leq H^*(Y) \) due to investment timing distortions.

**The Investment Decision.** Figure 4 plots a financially constrained firm’s optimal investment hurdle \( Y(W) \) for three values of the fixed cost, \( \phi_0 = 0.01, 0.4, 2 \). Recall that in the start-up phase, a firm is financially unconstrained if its liquidity holding is greater than \( I + \Lambda = 9.50 \). The firm pays the cost of \( I = 2 \) to exercise the investment option if and only if the earning \( Y \) exceeds the endogenously chosen first-best time-invariant investment hurdle \( Y_1^* = 1.544 \).
First, consider the region where \( 2 \leq W < 9.5 \). In this medium cash-holding region, the firm’s internal funds \( W \) are sufficient to cover its investment cost \( I = 2 \) and hence there will be no equity issuance decision. The higher the liquidity \( W \), the less distorted investment timing decision. Panel B of Figure 4 shows the decreasing function of \( Y(W) \) against liquidity \( W \). For the case with \( \phi = 2 \), the investment hurdle decreases from \( Y(2) = 1.555 \) to the \( Y^*_a = 1.544 \) as \( W \) increases from 2 to 3. Note that the quantitative effect of the financing friction in this medium cash-holding region is small.

In contrast, in the low cash-holding region \( W \in [0, I) \), the quantitative effect of financing costs on investment is much greater. Panel A of Figure 4 plots the investment hurdle \( Y(W) \) as a function of \( W \) in the region \([0, 2)\). Consider first the case with \( \phi_0 = 0.01 \). In this case, \( Y(W) \) is non-monotonic in \( W \); it first decreases slightly from 1.547 to 1.545 as \( W \) increases from \( W = 0 \) to \( W = 1.85 \). This is intuitive: as \( W \) increases the firm can rely on a greater and greater portion of internal funds, thus lowering its overall cost of investment, so that it is willing to take on investments with lower and lower earning \( Y \) mitigating distortions due to delayed investment timing decision. But eventually \( W \) rises with \( W \), and asymptotes to \( \infty \), as \( W \to I = 2 \). Thus, far from behaving more and more like an unconstrained firm as \( W \to I \), it behaves like a more and more constrained firm. What is the logic behind this behavior? In essence, as \( W \to I \) the opportunity cost to the firm of entirely avoiding the fixed external financing cost \( \phi_0 = 2 \) by waiting until it has accumulated sufficient internal funds to pay for the investment outlay \( I \) becomes smaller and smaller. It therefore takes a larger and larger earning shock \( Y \) to get the firm to invest right away and incur the cost \( \phi_0 = 2 \) rather than wait a little, accumulate \( W \) at the interest rate \( r \) until \( W \geq I \), and avoid this cost entirely.

The Financing Decision. Figure 5 plots the post-investment liquidity \( W_{\tau_i+} = W_{\tau_i} + F_g(W_{\tau_i}) - I \) implied by the financing amount \( F_g(W) \) at the moment of exercising the growth option in the region \( W < I \). First, consider the case with \( \phi_0 = 0.4 \). In the region \( W \geq 1.6 \), as \( W \) increases, the firm becomes increasingly less willing to issue equity to finance investment as the alternative of simply waiting for its internal funds to build up to cover the investment cost \( I \) is getting cheaper. As a result, the firm’s investment hurdle increases significantly, approaching \( \infty \) as \( W \to I \), as we discussed earlier. As a result, the amount of post-investment liquidity needed ever to cover operating losses is becoming increasingly small as internally generated cash flows (now and in the future) can easily finance the firm’s potential liquidity
Figure 4: The optimal investment hurdle $\overline{Y}(W)$ as a function of pre-investment liquidity $W$.

need in the future. For example, post-investment liquidity $W_{\tau_i+} = W_{\tau_i} + F_9(W_{\tau_i}) - I$ decreases from 0.202 at $W_{\tau_i} = 0$ to 0 at $W_{\tau_i} = 1.89$. For $W < 1.89$, should the firm decided to issue equity, (e.g., at $\overline{Y} = 1.592$ when $W = 1$), it issues some more funds than needed to purely finance investment so that its post-investment liquidity is positive, $W + F - I = 0.196$. For lower values of $\phi_0$, e.g., $\phi_0 = 0.01$, the post-investment liquidity is essentially zero for all levels of $W$, as the firm essentially has no need to hoard more liquidity than needed to cover the financing gap $I - W$.

Panel B shows that the marginal enterprise value of cash at the moment of investing, $Q_W(W_{\tau_i+}, \overline{Y}(W_{\tau_i}))$, is weakly lower than the proportional financing cost $\phi_1 = 0.01$. When $Q_W(W_{\tau_i+}, \overline{Y}(W_{\tau_i})) = \phi_1 = 0.01$, the firm’s post-investment liquidity $W_{\tau_i+}$ is positive, as it wants to hoard liquidity to cover the operating cost in the future. When $Q_W(W_{\tau_i+}, \overline{Y}(W_{\tau_i})) < 0.01$, $F_9(W_{\tau_i}) = I - W_{\tau_i}$ and the post-investment liquidity $W_{\tau_i+} = 0$.

The Value of the Start-up Firm. Figure 6 plots the firm’s enterprise value $H(W,Y)$ in the start-up phase and the corresponding marginal value $H_W(W,Y)$ for $Y = 0.65, 1.5$ and the same three fixed cost of issuing equity ($\phi_0 = 0.01, 0.4, 2$).

With a small fixed financing cost, $\phi_0 = 0.01$, $H(W,Y)$ is increasing and concave in $W$ as the firm is financially constrained and needs cash in order to invest. In contrast, with fixed financing costs, for both $\phi_0 = 0.4$ and $\phi_0 = 2$ cases, the (net) marginal value of cash
H_w(W,Y) first increases with W and then decreases with W as W becomes sufficiently large. For example, consider the case with \( \phi_0 = 0.4 \) and \( Y = 0.65 \), \( H_w(W,0.65) \) is upward sloping in \( W \) and reaches its highest value 0.035 around \( W = 1.05 \), and then declines as \( W \) decreases, effectively approaching 0 as \( W \) exceeds 1.85.

Why is the firm’s growth option value \( H(W,Y) \) convex in \( W \) in the region \( W < 1.05 \) for \( Y = 0.65 \)? Intuitively, if the firm was able to take a mean-preserving spread with \( W \) in that region, it would be better off, as either it accumulates internal funds faster or upon incurring losses, the firm gets closer to issue external equity to finance the exercising of its investment option. This additional benefit ameliorates the costly delay of the investment exercising decision.

In contrast, in the region where \( W > 1.05 \), the financially constrained firm has sufficiently high liquidity and hence has much weaker incentives to tap external financing and prefers waiting until the time when it has accumulated sufficient internal funds to finance investment. Therefore, the marginal value of cash \( H_w(W,Y) \) decreases with \( W \) in that region.

7 Simulation Illustration

To provide further insight into the dynamics of cash balances, financing, investment, and abandonment decisions in our model, we simulate one sample path with two different initial values for the firm’s cash holdings under our baseline parameter values. The implied time-
Figure 6: The firm’s enterprise value in the startup phase, $H(W,Y)$, and the marginal value of liquidity $H_W(W,Y)$ for three levels of fixed costs $\phi_0 = 0.01, 0.4, 2$. 
series of corporate decisions are illustrated in Figures 7 and 8 for the two cases, the first with $W_0 = 1.8$ and the second with $W_0 = 0.5$. Table 1 also summarizes the implied equity ownership dynamics that follow from the firm’s decisions over the simulated path used in Figures 7 and 8.

In the Modigiani-Miller scenario where there are no external financing costs, the optimal investment time for this sample path is $t = 2.6$, just when the firm’s earnings $Y_t$ reach 1.544, and the abandonment time is $t = 29.4$, just as the firm’s earnings $Y_t$ hit the low level of 0.625. Thus, the firm is willing to fund losses of up to 37.5% of operating costs $Z = 1$ under this scenario to maximize the value of its abandonment option.

Recall that when the firm incurs an external financing cost $\Phi(F_t)$ as it raises equity $F_t$ at time $t$, the firm’s post-issuance value is $P(W_{t+}, Y_{t+})$. If we then denote the equilibrium fraction of the newly issued equity held by outside investors by $\omega_{t+}$ we have:

$$\omega_{t+} = \frac{F_t + \Phi(F_t)}{P(W_{t+}, Y_{t+})} = 1 - \frac{P(W_t, Y_t)}{P(W_{t+}, Y_{t+})},$$

(43)
given that the new investors just break even under perfectly competitive capital markets. Under the simulated sample-path we can highlight the dynamics of equity dilution by keeping track of the equity ownership of the original investors who have stayed with the firm since its inception. We denote by $\alpha_t$ the ownership share of the original equity holders at time $t$, with $\alpha_0 = 1$ by construction. As the firm issues equity to finance investment and/or replenish liquidity over time, the original equity investors’ ownership then evolves as follows:

$$\alpha_{t+} = \alpha_t (1 - \omega_{t+}).$$

(44)

In other words, with no issuance $\omega_{t+} = 0$ and $\alpha_{t+} = \alpha_t$, so that $\alpha$ does not change. But when new equity is issued at time $t$, with a strictly positive ownership stake for new investors of $\omega_{t+} > 0$, the original equity investors’ equity is diluted to $\alpha_{t+}$ from $\alpha_t$ according to (44).

Figure 7 plots the scenario where the firm starts with a high cash stock of $W_0 = 1.80$. Panel A plots the path of earnings $Y_t$ starting with $Y_0 = 1$. Note first that when the firm faces external financing costs it only exercises its investment option at time $t = 2.9$ when $Y_t$ reaches 1.546 (compared with $t = 2.6$ when the firm faces no external financing costs). It then pays the investment cost $I = 2$ solely out of internal funds, thus depleting its stock of cash down to 0.08, as illustrated in Panel B. Given that the firm already starts out with a relatively high cash stock of $W_0 = 1.8$ it is optimal for the firm to wait until its internal funds have accumulated sufficiently to be able to finance its investment cost entirely out of internal funds, thus deferring costly external financing.
Next, at \( t = 7.5 \), when the firm’s earnings fall to 0.746 and it has burned through its internal funds, the firm issues net equity of \( F(0.746) = 1.79 \) to replenish its liquidity \( W_t \) from zero to 1.79, by selling 94.1% of its equity. At that point the firm’s original owners are nearly wiped out and only retain a stake of 5.9%.

Finally, at \( t = 19.8 \), when the firm runs out cash for the second time and \( Y_t \) reaches 0.656, it simply abandons its asset and distributes the remaining cash of 0.07 to its shareholders. At this low point the cost of new external financing simply outweighs the benefit of keeping the firm as a going concern.

Figure 8 plots the scenario where the firm starts with a low cash stock of \( W_0 = 0.5 \). Panel A again plots the identical simulated path of earning \( Y_t \) starting with \( Y_0 = 1 \). When at \( t = 3.7 \) expected earnings \( Y_t \) reach 1.592, the firm decides to raise net equity \( F(1.592) = 1.60 \) (see Panel C) to finance its investment cost \( I = 2 \) via a combination of external equity (1.60) and internal funds (0.40), leaving the firm with a stock of cash of 0.20 post-investment (as shown in Panel B). As a result, the original owners are diluted down to an ownership stake of only 85.3% (as shown in Panel D).

At \( t = 6.3 \), when earnings \( Y_t \) have collapsed to 0.789 and the firm’s liquidity has been drained, the firm returns to the capital markets and raises a net amount of \( F = 1.80 \), thus further diluting the firm’s original owners’ down to 13.8% (see Panel D). Next, at \( t = 18.6 \), as the firm’s liquidity \( W_t \) is again drained out, the firm yet again issues equity, raising a total net amount of \( F = 1.73 \) and diluting the original owners down to a small stake of 5.7%. Finally, at \( t = 28.6 \) the firm almost runs out of cash again and abandons its asset given that expected operating earnings hit the low level of \( Y_t = 0.692 \); it then distributes the remaining cash 0.03 to its shareholders.

Comparing the two scenarios, we can make the following observations: First, in all cases where the firm issues equity for purposes of replenishing its liquidity it chooses different financing levels because each time it faces different expected operating earnings when it exhausts its liquidity. Second, the firm with \( W_0 = 1.8 \) turns out to abandon its asset sooner (at \( t = 19.8 \)) than the firm with \( W_0 = 0.5 \) (at \( t = 28.6 \)). This may at first appear surprising, as one may think that a firm with more liquidity should have less distorted abandonment decisions. The intuition for this seemingly counter-intuitive result is as follows. As it is optimal for the firm with \( W_0 = 1.8 \) to finance its investment cost \( I = 2 \) purely via internal funds, this firm ends up with lower post-investment liquidity and hence an earlier inefficient liquidation decision. Put differently, a less financially constrained firm could turn out to have
Table 1: Equity ownership dynamics and (investment, equity issue, and abandonment) timing decisions for a simulated path. The parameter values are $r = 5\%$, $\mu = 0$, $\sigma = 15\%$, $I = 2$ and $Z = 1$. For the given simulated path with $Y_0 = 1$, we consider two cases with $W_0 = 1.8$ and $W_0 = 0.5$. Under the first-best case, the optimal investment time is $t = 2.6$ and the abandonment time is $t = 29.4$.

<table>
<thead>
<tr>
<th>time</th>
<th>firm’s earning</th>
<th>net equity issue $F_t$</th>
<th>gross equity issue $F_t + \Phi(F_t)$</th>
<th>% of equity issue $\omega_{t+}$</th>
<th>original investors’ ownership $\alpha_{t+}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A: $W_0 = 1.8$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>100%</td>
</tr>
<tr>
<td>2.9</td>
<td>1.546</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>100%</td>
</tr>
<tr>
<td>7.5</td>
<td>0.746</td>
<td>1.79</td>
<td>2.21</td>
<td>94.1%</td>
<td>5.9%</td>
</tr>
<tr>
<td>19.8</td>
<td>0.656</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>liquidation</td>
</tr>
<tr>
<td>Panel B: $W_0 = 0.5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>100%</td>
</tr>
<tr>
<td>3.7</td>
<td>1.592</td>
<td>1.60</td>
<td>2.02</td>
<td>14.7%</td>
<td>85.3%</td>
</tr>
<tr>
<td>6.3</td>
<td>0.789</td>
<td>1.80</td>
<td>2.22</td>
<td>83.8%</td>
<td>13.8%</td>
</tr>
<tr>
<td>18.6</td>
<td>0.887</td>
<td>1.73</td>
<td>2.14</td>
<td>58.9%</td>
<td>5.7%</td>
</tr>
<tr>
<td>28.6</td>
<td>0.692</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>liquidation</td>
</tr>
</tbody>
</table>

more severe distortions ex post due to the endogenous adjustments along other margins (in this case, the financing decision for the investment option). This is yet another example illustrating how dynamic models can provide new insights for the design of empirical tests on the implications of financial constraints on firm behavior, which are unavailable in static models such as Froot, Scharfstein, and Stein (1993) and Kaplan and Zingales (1997), and which consequently may yield imprecise empirical hypotheses.

8 Sequential Investment Opportunities

We now generalize our model to allow for two rounds of investment options. Accordingly, we refer to stage 0 as the start-up phase when the firm has not acquired any productive asset, stage 1 as the growth phase when the firm operates its first asset in place but still holds the second growth option, and stage 2 as the mature phase that starts after the second investment option has been exercised. As before, due to operating leverage, the firm may exercise its abandonment option in either the growth phase or the mature phase.
Figure 7: **Growth option, abandonment option and equity issues for a simulated path.** The firm’s initial cash holding is $W_0 = 1.80$. Panel A plots a simulated path of earning $Y$ with $Y_0 = 1$. At $t = 2.9$, as $Y$ reaches 1.546, the firm exercises its investment option by paying the investment cost $I = 2$ solely with internal funds leaving the post-investment liquidity 0.08 in the firm (Panel B). At $t = 7.5$, as the firm’s earning falls to 0.746 and its liquidity drains out, the firm issues equity 1.80 in net to replenish liquidity $W$ from zero to 1.80 by diluting the firm’s original equity investors’ ownership to 5.9%. Finally, at $t = 19.8$, as the firm runs out cash again and $Y$ reaches 0.656, it simply abandons its asset and distributes the remaining cash to shareholders as the cost of using costly external financing outweighs the benefit of keeping the firm as a going concern.
Figure 8: **Growth option, abandonment option and equity issues for a simulated path.** The firm’s initial cash holding is $W_0 = 0.5$. Panel A plots the identical simulated path of earning $Y$ with $Y_0 = 1$ as the one in Figure 7. At $t = 3.7$, as $Y$ reaches 1.592, the firm raises equity $F = 1.60$ in net (Panel C), finances its investment cost $I = 2$ via a combination of external equity (1.60) and internal funds (0.40) leaving the firm with 0.20 liquidity post-investment (Panel B), and consequently dilutes the original equity investors’ ownership to 85.3% (Panel D). At $t = 6.3$, as $Y$ falls to 0.789 and liquidity $W$ drains out, the firm raises $F = 1.80$ in net increasing $W$ from zero to 1.80, and consequently further dilutes the firm’s original equity investors’ ownership to 13.8% (Panel D). At $t = 18.6$, as $Y$ again reaches 0.887 and its liquidity $W$ drains out, the firm issues equity, again, $F = 1.73$ in net, further diluting the original equity investors’ ownership to 5.7%. Finally, at $t = 28.6$, as the firm runs out cash and $Y = 0.692$, it abandons its asset and distributes the remaining cash to shareholders.
This extension allows us to analyze two novel effects. The first is the change in incentives to exercise its first investment option caused by the prospect of acquiring valuable operating cash-flows that help relax the internal funding constraint to finance the second growth option. The second is how incentives to exercise the later investment option and other corporate decisions are modified by operating cash-flow risk from the first productive asset.

A striking new result we obtain is that when the firm faces multiple rounds of growth options it may choose to invest in the earlier opportunities sooner than under the first-best benchmark. In other words, the firm optimally engages in a form of over-investment relative to the first-best as a way of building up its liquidity which helps mitigate the incentives to delay the second growth option. Accelerated timing of investment in the start-up phase may be in the firm’s best interest if the cost of early investment in the initial productive asset outweighs the costs of under-investment in the later expansion phase.

We denote by $Z_n$ the firm’s total operating cost (per unit of time) after exercising the $n$th growth option, where $n \in \{1, 2\}$. Correspondingly, we denote the stochastic revenue process generated by all existing productive assets by $x_nY_t$, where $Y_t$ is given in (1) and where $x_1 < x_2$ denotes the production capacity after exercising respectively the first and second investment option. That is, upon exercising the second growth option, the firm’s operating revenue increases by $\Delta_x Y$ where $\Delta_x = x_2 - x_1$ and the operating cost increases by $\Delta_Z = Z_2 - Z_1$. Intuitively, we may interpret $\Delta_x Y - \Delta_Z$ as the additional net revenue generated by the second asset in place.

We also denote by $\tau^*_n$ the abandonment time in phase $n$. In the growth phase, when the firm exercises its abandonment option it releases its cash to shareholders and loses both its first asset in place and its second growth option. In the mature phase the firm distributes its cash to shareholders and liquidates all assets in place.

By backward induction, in the mature phase we essentially have the same firm value, the same dynamics for liquidity, and the same abandonment decision as in the baseline model when we replace the baseline cash flow with $x_2Y$ and the baseline operating cost with $Z_2$. The main changes to the analysis are thus in stages 0 and 1, to which we now turn. The liquidity dynamics are the same as in the baseline model for the start-up phase and are given in (3). As for stage 1, the growth phase, the firm’s liquidity $W_t$ then accumulates as follows:

$$dW_t = (rW_t + x_1Y_t - Z_1)dt + dC_t, \quad W_t \geq 0,$$

where $C_t$ denotes as before the undercounted cumulative equity issue and $dC$ is the net
equity issued. Note that the revenue is now $x_1 Y_t$ and the operating profit is $x_1 Y_t - Z_1$.

**Results.** For the first growth option we set $x_1 = 1$, $Z_1 = 1$ and $I_1 = 2$, (the same as in the baseline model), and for the expansion option we set: $x_2 = 3$, $Z_2 = 3$ and $I_2 = 7$. For these parameter values the expansion asset is less valuable than the first asset, a form of decreasing returns to scale. For all other parameters, we use the same values as those in the baseline model. With these parameter values, we obtain that the first-best abandonment and investment thresholds in stage 1 are given by $Y_{1,a}^* = 0.534$ and $Y_{1,i}^* = 1.586$ respectively, and the first-best abandonment and investment thresholds in stage 2 are $Y_{2,a}^* = 0.625$ and $Y_{2,i}^* = 1.678$, respectively.

For these parameter values we also obtain the following cut-offs for when the firm is forever financially unconstrained in respectively the growth and mature phases: $\Lambda_1 = 9.29$ and $\Lambda_2 = 22.5$. That is, when $W \geq \Lambda_2$ the firm is forever financially unconstrained in

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**Figure 9:** The optimal investment threshold $\overline{Y}_1(W)$ and $\overline{Y}_2(W)$ for a financially constrained firm in stage 0 and 1, respectively.
phase 2 (the mature phase). And in phase 1 (the growth phase) it is permanently financially unconstrained if \( W \geq \max\{\Lambda_1, I_2 + \Lambda_2\} = 29.5 \). Finally, in the stage 0 (the start-up phase) the firm is permanently financially unconstrained if and only if \( W \geq I_1 + \max\{\Lambda_1, I_2 + \Lambda_2\} = 31.5 \).

Figure 9 plots the optimal investment thresholds \( \overline{Y}_1(W) \) and \( \overline{Y}_2(W) \) in stages 0 and 1 against liquidity \( W \). Panel A shows that \( \overline{Y}_1(W) \) is non-monotonic in \( W \) in the region \( W < I_1 = 2 \), consistent with our earlier findings for the baseline model with a single growth option. Panel B shows that the investment threshold \( \overline{Y}_1(W) \) is also non-monotonic in the region \( W \geq 2 \) where the firm does not need external funds to finance the investment cost \( I_1 \) for the first growth option. Importantly, the firm may over-invest by exercising its second growth option before the first-best level \( Y_1^{*} \). Intuitively, the firm accelerates the timing of its first growth option in order to generate internal cash flows from the first asset in place. By using the internally generated cash flows from the first asset in place, the firm may finance the investment cost for the second growth option and also provide additional liquidity to cover operating costs. This is an inter-temporal tradeoff in the interest of shareholders. By over-investing in the first growth option, the firm gains overall because the benefit of relaxing the financial constraint for the second growth option outweighs the cost of accelerating the first growth option.

Panels C and D of Figure 9 report similar patterns to those we see in our baseline model. That is, the investment threshold \( \overline{Y}_2(W) \) is higher than the first-best level and it is non-
monotonic in the firm’s liquidity $W$.

Finally, we turn to Figure 10, which plots the firm’s abandonment thresholds $Y_1(W)$ and $Y_2(W)$. Note that both abandonment thresholds decrease with liquidity and asymptotically approach the first-best levels $Y_{1,a}^*$ and $Y_{2,a}^*$, respectively. These results are consistent with our earlier findings for the baseline model.

9 Conclusion

We develop a theory of real options for a financially constrained firm facing costly external financing. We model the rich dynamic interactions between how a corporation manages its real and financial flexibilities. Real options well capture the corporate flexibility on the “real” side of corporate decision making (e.g., investment) and liquidity management is at the core of corporate financial flexibility. Unlike in the standard MM framework, internal funds significantly influence a financially constrained firm’s real investment and abandonment option exercising decisions. Our analysis also brings out rich and subtle interactions between the optimal investment and abandonment timing decisions and the firm’s sources of funds, whether external, internal, or from current and future retained earnings. When the firm is able to accumulate more internal funds it may well be induced to slow down its investment even further as the prospect of being able to entirely avoid incurring external funding costs is closer. Our framework is also sufficiently parsimonious to be able to provide quantitatively plausible estimates of the value of the firm’s real options and its marginal value of cash, as well as its optimal investment and external funding policy. Importantly, we find that with multiple rounds of growth options, a financially constrained firm may over-invest in earlier stages with the hope of replenishing liquidity in order to mitigate under-investment problems in the future. We have, however, left other important features out of our analysis. We plan to extend the framework by incorporating a credit line, risk management opportunities, and capital structure margins as alternative sources of liquidity and risk management instruments into our analysis.
Appendices

Before solving for the financially constrained firm’s value and optimal investment decisions in the setting with multiple growth options, it is helpful to characterize the first-best benchmark where the firm faces no external funding costs.

A The First-Best Benchmark

Let $H^*(Y)$ denote the solution for the first-best enterprise value in stage 0, $Q_n^*(Y)$ the solutions for the enterprise values in respectively stages $n = 1, 2$. Also, let $Y_{n,a}^*$ and $Y_{n,i}^*$ denote the optimal abandonment boundaries and exercising thresholds of the $n$th growth options in stages $n = 1, 2$, respectively.

We can then write $Q_1^*(Y)$ as the solution to the following ODE:

$$r Q_1(Y) = (x_1 Y - Z_1) + \mu Y Q'_1(Y) + \frac{\sigma^2 Y^2}{2} Q''_1(Y), \quad (A.1)$$

subject to the following value-matching and smooth-pasting boundary conditions for the investment and abandonment options:

$$Q_1(Y_{2,i}^*) = Q_2^*(Y_{2,i}^*) - I_2, \quad (A.2)$$
$$Q'_1(Y_{2,i}^*) = Q''_2(Y_{2,i}^*), \quad (A.3)$$
$$Q_1(Y_{1,a}^*) = 0, \quad (A.4)$$
$$Q'_1(Y_{1,a}^*) = 0. \quad (A.5)$$

Note that equation (A.1) for $Q_1(Y)$ is identical to equation (7) for $Q(Y)$ except for the first term, which reflects the fact that the firm is not operating at full scale and only has one productive asset with capacity $x_1$ and operating costs $Z_1$. The upper boundary $Y_{2,i}^*$ at which the firm decides to scale up to full capacity is such that the growth-phase enterprise value $Q_1(Y_{2,i}^*)$ is just equal to the mature-phase enterprise value $Q_2^*(Y_{2,i}^*)$ net of investment costs $I_2$.

As we have established in Section 3.1, in the mature-phase, the firm’s value is given by $Q_2^*(Y; Z_2) = Q^*(x_2 Y; Z = Z_2)$ and hence

$$Q_2^*(Y) = \left( \frac{x_2 Y}{r - \mu} - \frac{Z_2}{r} \right) + \left( \frac{Y}{Y_{2,a}^*} \right)^\gamma \left( \frac{Z_2}{r} - \frac{x_2 Y_{2,a}^*}{r - \mu} \right), \quad \text{for} \; Y \geq Y_{2,a}^*, \quad (A.6)$$
where
\[ Y_{2,a}^* = \frac{\gamma}{\gamma - 1} \frac{r - \mu Z_2}{x_2}. \] (A.7)

After solving for \( Q_2^*(Y) \) we move backwards to the growth phase and solve for \( Q_1^*(Y) \). As can be readily verified, the firm’s value \( Q_1^*(Y) \) in the region \( Y_{2,i}^* \leq Y \leq Y_{1,a}^* \) takes the following form:
\[ Q_1^*(Y) = \left( \frac{x_1 Y}{r - \mu} - \frac{Z_1}{r} \right) + \Psi_i(Y) \left( Q_2^*(Y_{2,i}^*) - I_2 + \frac{Z_1}{r} - \frac{x_1 Y_{2,i}^*}{r - \mu} \right) + \Psi_a(Y) \left( \frac{Z_1}{r} - \frac{x_1 Y_{1,a}^*}{r - \mu} \right), \] (A.8)

where
\[ \Psi_i(Y) = \left( \frac{Y_{1,a}^* Y_{2,i}^* Y - Y_{1,a}^* Y_{2,i}^* Y}{Y_{1,a}^* Y_{2,i}^* Y - Y_{1,a}^* Y_{2,i}^* Y} \right) \] and \[ \Psi_a(Y) = \left( \frac{Y_{2,i}^* Y_{2,i}^* Y - Y_{2,i}^* Y_{2,i}^* Y}{Y_{1,a}^* Y_{2,i}^* Y - Y_{1,a}^* Y_{2,i}^* Y} \right), \] (A.9)

where \( \beta \) and \( \gamma \) are given by (20) and (12), the positive and negative roots of the fundamental quadratic equation, and the first abandonment threshold \( Y_{1,a}^* \) and the second investment threshold \( Y_{2,i}^* \) satisfy the boundary conditions (A.2)-(A.5).

Finally, we can move backwards to the start-up phase and solve for \( H^*(Y) \), which satisfies the ODE (15) and the boundary conditions:
\[ H(Y_{1,i}^*) = Q_1^*(Y_{1,i}^*) - I_1, \] (A.10)
\[ H'(Y_{1,i}^*) = Q_1''(Y_{1,i}^*). \] (A.11)

Again, it is straightforward to verify that \( H^*(Y) \) takes the following form:
\[ H^*(Y) = \left( \frac{Y}{Y_{1,i}^*} \right)^\beta \left( Q_1^*(Y_{1,i}^*) - I_1 \right) \text{ for } Y \leq Y_{1,i}^*, \] (A.12)

and \( Y_{1,i}^* \) solves the following implicit equation:
\[ \beta \left[ Q_1^*(Y_{1,i}^*) - I_1 \right] = Q_1''(Y_{1,i}^*) Y_{1,i}^*. \] (A.13)

B General Solution under External Financing Costs

In Section 4, we showed that a firm can be in one of three liquidity regions in the mature phase: \( i \) a financially unconstrained region (when \( W \) is sufficiently large or \( Y \rightarrow \infty \)); \( ii \) an interior financially constrained region, and; \( iii \) an equity issuance region. This naturally continues to be the case for the mature phase in our more general setting.
B.1 The Mature Phase

The solution for the firm’s value in the mature phase, \( P^{(2)}(W,Y; Z_2) \), is identical to \( P(W, x_2 Y; Z = Z_2) \) given in Section 4. Note that the firm is permanently financially unconstrained in this region when its liquidity \( W \geq \Lambda_2 \) where \( \Lambda_2 \) is given by

\[
\Lambda_2 \equiv \frac{Z_2 - x_2 Y_{2,a}^*}{r} \equiv \frac{r - \gamma \mu}{r^2(1 - \gamma)} Z_2.
\] (B.1)

The analysis is a bit more complicated but remains intuitive in the growth phase as the firm has both a liquidation option and also the second growth option to exercise.

B.2 The Growth Phase

Again, the firm may be in one of these three regions in the growth phase (stage 1). Let \( \Lambda_1 \) denote the lowest level of liquidity such that the firm will not be inefficiently liquidated in the growth phase. As long as the firm’s saving rate is positive at the first-best liquidation threshold \( Y_{1,a}^* \), which may be equivalently written as \( W \geq \Lambda_1 \), where

\[
\Lambda_1 \equiv \frac{Z_1 - x_1 Y_{1,a}^*}{r},
\] (B.2)

the firm will not be inefficiently liquidated in phase 1. However, for a firm to be permanently financially unconstrained, it shall distort neither its liquidation decision nor its (second) growth option exercising decision. For a firm to choose the first-best investment strategy for the second growth option and to incur no further distortions, it needs at least an amount of liquidity \( \Lambda_2 + I_2 \). This gives the following analytical characterization for the financially unconstrained region.

**The Unconstrained Region:** When the firm neither satisfies

\[
W \geq \max\{\Lambda_1, I_2 + \Lambda_2\}
\] (B.3)

or its revenues approaches infinity, i.e., \( Y \to \infty \), the firm will be permanently financially unconstrained with probability one, and the value of the firm will be equal to the first-best value: \( P^{(1)}(W,Y) = Q_1^*(Y) + W \).

**The Financially Constrained Region:** When \( W < \max\{\Lambda_1, \Lambda_2 + I_2\} \), the firm’s value \( P^{(1)}(W,Y) \) satisfies the following HJB equation:

\[
r P^{(1)}(W,Y) = (rW + x_1 Y_1 - Z_1)P^{(1)}_W(W,Y) + \mu Y P^{(1)}_Y(W,Y) + \frac{\sigma^2 Y^2 P^{(1)}_{YY}(W,Y)}{2},
\] (B.4)
subject to the following conditions:

\[ P^{(1)}(W, Y_1(W)) = W, \quad (B.5) \]

and

\[ P_Y^{(1)}(W, Y_1(W)) = 0. \quad (B.6) \]

Intuitively, (B.5) states that the value of abandonment is simply equal to the firm’s cash released to shareholders upon liquidation and (B.6) gives the smooth pasting condition for the optimality of the endogenously chosen abandonment threshold \( Y_1(W) \). Additionally, the firm’s value also satisfies the financing boundary condition at \( W = 0 \),

\[
P^{(1)}(0, Y) = \begin{cases} 
P^{(1)}(F, Y) - F - \Phi(F), & Y_1(0) < Y < Z_1, \\ 0, & \frac{Y}{Y} \leq Y_1(0), \end{cases} \quad (B.7)
\]

where the optimal amount of funds raised \( F \) (net of issuing costs) is given by:

\[
P_W^{(1)}(F, Y) = 1 + \Phi'(F) = 1 + \phi_1, \quad Y_1(0) < Y < Z_1. \quad (B.8)
\]

Finally, \( P^{(1)}(W, Y) \) also satisfies a set of boundary conditions that characterize the optimal growth option exercising decisions. Let \( F_g^{(2)}(W) \) denote the amount of external funds that the firm raises. Clearly, \( F_g^{(2)}(W) \geq I_2 - W \) as \( W < I_2 \). The additional conditions needed to characterize the investment decisions are:

1. In the region \( I_2 \leq W < \max\{\Lambda_1, I_2 + \Lambda_2\} \), the firm requires no external funds and indeed will not choose to tap external funds to finance its second growth option exercising cost. Hence, we have the standard value-matching condition,

\[
P^{(1)}(W, Y_2(W)) = P^{(2)}(W - I_2, Y_2(W)) \quad (B.9)
\]

and the standard smooth-pasting condition,

\[
P_Y^{(1)}(W, Y_2(W)) = P_Y^{(2)}(W - I_2, Y_2(W)). \quad (B.10)
\]

2. In the region \( W < I_2 \), at the growth option exercising boundary \( Y_2(W) \), we have

\[
P^{(1)}(W, Y_2(W)) = P^{(2)}(W + F_g^{(2)}(W) - I_2, Y_2(W)) - F_g^{(2)}(W) - \Phi(F_g^{(2)}(W)) \quad (B.11)
\]

which is the standard value-matching condition, and

\[
P_Y^{(1)}(W, Y_2(W)) = P_Y^{(2)}(W + F_g^{(2)}(W) - I_2, Y_2(W)), \quad (B.12)
\]
which is the standard smooth-pasting condition. Importantly, we pin down the amount of financing $F_g^{(2)}(W)$ via the following conditions:

$$P_{W}^{(2)}(W + F_g^{(2)}(W) - I_2, Y_2(W)) \leq 1 + \Phi'(F_g^{(2)}(W)) \quad \text{and} \quad F_g^{(2)}(W) \geq I_2 - W. \quad (B.13)$$

As in the baseline case, the constraint $F_g(W) \geq I - W$ may bind as it may be optimal for the firm to rely solely on its ability to generate sufficient liquidity from operating earnings after it has exercised its second growth option. Note that the firm’s marginal value of liquidity $P_W^{(2)}(W + F_g^{(2)}(W) - I_2, Y_2(W))$ cannot exceed the marginal cost of financing at the moment of issuing equity to finance investment cost. Otherwise, the firm should issue more and keep the proceeds inside the firm as retained earnings.

### B.3 The Start-up Phase

We now characterize the solution for the start-up phase for the corresponding three regions depending on the firm’s liquidity $W$. In the financially unconstrained region, where

$$W \geq I_1 + \max\{\Lambda_1, I_2 + \Lambda_2\}, \quad (B.14)$$

we have the first-best solution for the firm’s option value:

$$G(W, Y) = H^*(Y) + W. \quad (B.15)$$

In the financially constrained region where $W < I_1 + \max\{\Lambda_1, I_2 + \Lambda_2\}$ the firm’s value $G(W,Y)$ solves the HJB equation:

$$rG(W,Y) = rWG_W(W,Y) + \muYG_Y(W,Y) + \frac{\sigma^2 Y^2}{2}G_{YY}(W,Y), \quad (B.16)$$

subject to the following various boundary conditions. Let $F_g^{(1)}(W)$ denote the amount of external funds that the firm raises. Clearly, $F_g^{(1)}(W) \geq I_1 - W$ as $W < I_1$. The additional conditions needed to characterize the investment decisions are:

1. In the region $I_1 \leq W < I_1 + \max\{\Lambda_1, I_2 + \Lambda_2\}$, the firm issues no equity to finance the exercising cost of the first growth option. We have the standard value-matching and smooth-pasting conditions:

$$G(W, \bar{Y}_1(W)) = P^{(1)}(W - I_1, \bar{Y}_1(W)), \quad (B.17)$$

$$G_Y(W, \bar{Y}_1(W)) = P^{(1)}_Y(W - I_1, \bar{Y}_1(W)). \quad (B.18)$$
2. In the region \(0 \leq W < I_1\), the firm has to raise external financing, denoted by \(F_g^{(1)}(W)\), if it wants to exercise its first growth option. We thus have the following value-matching and smooth-pasting conditions:

\[
G(W, \overline{Y}_1(W)) = P^{(1)}(W + F_g^{(1)}(W) - I_1, \overline{Y}_1(W)) - F_g^{(1)}(W) - \Phi(F_g^{(1)}(W)), \quad (B.19)
\]

\[
G_Y(W, \overline{Y}_1(W)) = P_Y^{(1)}(W + F_g^{(1)}(W) - I_1, \overline{Y}_1(W)), \quad (B.20)
\]

To pin down the external financing amount, \(F_g^{(1)}(W)\), we use the following conditions:

\[
P_W^{(1)}(W + F_g^{(1)}(W) - I_1, \overline{Y}_1(W)) \leq 1 + \Phi'(F_g^{(1)}(W)) \quad \text{and} \quad F_g^{(1)}(W) \geq I_1 - W. \quad (B.21)
\]

The argument for the optimality condition (B.21) for \(F_g^{(1)}(W)\) is essentially the same as the optimality condition (B.13) for \(F_g^{(2)}(W)\). Finally, \(G(W, Y)\) also satisfies the absorbing barrier condition \(G(W, 0) = W\).

We have now completely characterized the HJB equations and various boundary conditions in the three phases of the firm. We next numerically solve the model and provide intuitive explanations for our model’s main results.
References


