A Simple Model for Friedman’s Conjecture on Consumption

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Abstract

Friedman’s contribution to the consumption literature goes well beyond the seminal permanent-income hypothesis. He conjectured that the marginal propensity to consume out of nonhuman wealth shall be larger than out of “human wealth,” the present discounted value of future labor income (Friedman (1957)). I present an explicitly-solved model to deliver this widely-noted consumption property by specifying that the conditional variance of income increases in its level. A larger realization of income not only implies a higher level of human wealth, but also signals a riskier stream of future labor income, inducing a higher propensity to save out of human wealth, and thus giving rise to Friedman’s conjecture. Appropriately adjusting human wealth for income risk, I show that Friedman’s conjecture may be formulated as a “generalized” permanent income hypothesis. I further show that Friedman’s conjecture captures the first-order effect of stochastic precautionary savings, and also propose a natural decomposition of the optimal saving rule to formalize various motives for holding wealth as emphasized in Friedman (1957).

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1 Introduction

The permanent-income hypothesis (PIH) of Friedman (1957) states that consumption is equal to the annuity value of total wealth given by the sum of nonhuman wealth (cumulative savings) and “human” wealth, the discounted expected value of future income, using the risk-free rate. This in turn implies that changes in consumption are not predictable, the well known martingale consumption result (Hall (1978)). Phrased in terms of saving, the PIH states that the agent only saves in anticipation of possible future declines in his labor income. Campbell (1987) dubbed the PIH-implied saving motive as saving for “rainy” days.

While Friedman is synonymous to the permanent-income hypothesis in the consumption literature, his contribution to this literature goes even beyond the PIH. For example, on page 16 of Friedman (1957), he wrote that “current consumption may be expected to depend not only on total permanent income and the interest rate, but also on the fraction of permanent income derived from nonhuman wealth, or—what is equivalent for a given interest rate, on the ratio of nonhuman wealth to permanent income. The higher this ratio, the less need there is for an additional reserve, and the higher current consumption may be expected to be. The crucial variable is the ratio for nonhuman wealth to permanent income, not the absolute amount of nonhuman wealth.” Friedman defined the permanent income as the annuity value of the sum of “human” and nonhuman wealth. Thus, he was suggesting a consumption rule with a lower marginal propensity to consume (MPC) out of human wealth than out of nonhuman wealth.

Friedman’s less known conjecture on a lower MPC out of human wealth than out of financial wealth has also found wide empirical supports in various papers after Hall (1978). For example, Flavin (1981) shows that empirically, changes in consumption are predicted by current income, which is inconsistent with the random-walk consumption prediction of the Friedman’s PIH (Hall (1978)). This empirical regularity on the predictability of changes in consumption by variables such as income is known as “excess-sensitivity” puzzle. I show that the excess sensitivity of consumption is exactly implied by a lower MPC out of human wealth than out of financial wealth. The “excess-smoothness” puzzle (Deaton (1987)) also provides support to this conjecture.

This paper formalizes Friedman’s conjecture on the consumption rule stated above by constructing an intertemporal precautionary savings model.\footnote{Leland (1968) is among earliest work on precautionary savings. Also see Sandmo (1970), Drèze and Modigliani (1972), Zeldes (1989), Caballero (1991), Deaton (1991), and Carroll (1997), among others. Deaton (1992) and Attanasio (1999) survey the literature.} I argue that a “lower” MPC out of human wealth than out of nonhuman wealth is among the most important feature of a sensible consumption rule, because this feature captures the first-order effect of precautionary
saving. A precautionary agent rationally values a unit of human wealth less than a unit of nonhuman wealth.\(^2\) This translates into a lower MPC out of human wealth than out of nonhuman wealth, in terms of the consumption rule.

An essential ingredient of the model that gives rise to a lower MPC out of human wealth than out of nonhuman wealth is that the conditional variance of income changes in levels increases in the level of income. Simply put, I assume that the conditional variance of income changes for an agent whose annual income is 50\(K\) is larger than that for an agent whose annual income is less than 50\(K\). This assumption does not restrict on the structure of the conditional variance of percentage changes of income. There is much empirical evidence in support of a conditionally heteroskedastic income process in levels. A common specification of the income process in the empirical literature is a conditionally homoskedastic income process in logarithm.\(^3\) A conditionally homoskedastic process in logarithm states that the conditional variance of percentage changes of income is equal at all income levels. This implies that the conditional variance of income changes in levels must increase with the level of labor income. This paper proposes to use a large class of stochastic processes, known as affine processes,\(^4\) to model the dynamics of the income process. A key feature of affine processes is that the conditional variance of percentage changes of income is an increasing linear function of the level of current income. The intuition of using affine processes to model income is as follows. With an increase in income, human wealth has increased, but its volatility has also increased, so the associated increase in optimal consumption is less than proportional, because of a precautionary savings motive. This in turn implies an optimal consumption rule with a lower MPC out of human wealth than out of nonhuman wealth. In addition to capturing these empirical features, the affine process is also analytically tractable.

My work is closely related to Zeldes (1989) and Caballero (1990, 1991). Zeldes (1989) derived an incomplete-markets consumption model with constant-relative-risk-averse (CRRA) utility, by using numerical dynamic programming.\(^5\) Based on his numerical consumption policy rule, Zeldes concluded that “one possible remedy to this problem”\(^6\) would be to put a weight of

\(^2\)Recall that the definition of human wealth adopted in the literature does not adjust for the risk of the labor income process.

\(^3\)See MaCurdy (1982), for example. Also see Attanasio (1999) for a survey.

\(^4\)Affine models are widely used in finance literature. See Vasicek (1977), Cox, Ingersoll, Jr., and Ross (1985), and Duffie and Kan (1996), among others.

\(^5\)Hayashi (1982) tested a generalized version of the permanent-income model, using a higher discount rate for human wealth than the risk-free rate. However, his model is not based on optimality. Also see Nagatani (1972) for some early attempts to capture a lower MPC out of human wealth than out of nonhuman wealth in an “ad hoc” way.

\(^6\)This problem refers to the inconsistency between the CRRA-utility-based optimal consumption that he solved numerically, and martingale consumption result implied by the PIH.
less than one on human wealth before adding it to nonhuman wealth, or to discount expected future income at a higher discount rate.” Zeldes’ work provides a justification for Friedman’s conjecture on a lower MPC out of human wealth than out of nonhuman wealth.

Caballero (1991) proposed an analytically tractable optimal consumption rule, based on additively-separable constant-absolute-risk-averse (CARA) utility, and an autoregressive moving average income process, including the unit-root process as a special case. He showed that the optimal consumption level is lower than that under certainty equivalence by a term that captures a precautionary premium. Because of conditionally homoskedastic income shocks, his model predicts constant precautionary saving demand, which in turn implies that the MPC out of human wealth is equal to that out of nonhuman wealth. In a related paper, Caballero (1990) mentioned the potential importance of conditional heteroskedasticity of income shocks in explaining consumption puzzles.

Following Merton (1971), Kimball and Mankiw (1989), and Caballero (1991), I assume CARA utility for technical convenience. This is not surprising, as Zeldes (1989) noted in his abstract that “no one has derived closed-form solutions for consumption with stochastic labor income and constant relative risk aversion utility.” Both Zeldes (1989) and this paper conclude that a lower MPC out of human wealth is crucial for a realistic consumption rule. Unlike Caballero (1991), this paper delivers a lower MPC out of human wealth than out of nonhuman wealth. In this model, with stochastic precautionary saving, changes in consumption are naturally predicted by current income, consistent with the empirical regularity, known as “excess-sensitivity” (Flavin (1981)).

If we take the PIH rule as the first-order approximation of a realistic optimal consumption rule, then a linear consumption rule (in nonhuman wealth and human wealth) with a lower MPC out of human wealth is the natural second-order approximation of a realistic consumption rule. What the second-order linear approximation captures and the PIH rule does not capture is the “stochastic” precautionary saving. However, the second-order approximation does not capture the obvious non-linearity feature of the consumption rule. In order to check the robustness of my model’s prediction, I compare the model’s prediction on precautionary savings with that of a related CRRA-utility-based model. The key result that the MPC out of human wealth is lower than out of nonhuman wealth remains valid, consistent with the conclusion in Zeldes (1989). Finally, I use the explicitly derived consumption rule to decompose saving and formalize Friedman’s insights on various motives of holding wealth.

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7Merton (1971) derived an explicitly-solved consumption rule using Poisson processes to model the income dynamics. Kimball and Mankiw (1989) used a continuous-time Markov chain to model the income process and solved the optimal consumption rule in closed form.
The remainder of the paper is organized as follows. Section 2 describes the setup of the model and solves the optimal consumption rule. In Section 3, I interpret the consumption rule, decompose the saving rate, and check the robustness of the result. Section 4 concludes. Appendix A derives and proves the optimal consumption rule. Appendix B proposes alternative risk-adjusted measures of wealth for labor income.

2 The Model

I model an agent who lives forever and maximizes his expected utility for consumption. The only source of uncertainty is his labor-income process. I sidestep the agent’s labor-supply decision, as is conventionally done in the consumption literature. The agent can borrow or lend at a constant positive risk-free interest rate $r$. There exist no other financial assets, hence markets are incomplete with respect to labor-income uncertainty. For technical convenience, the model is cast in continuous time. I first introduce a natural parametric model for the labor-income process, one leading to a lower MPC out of human wealth than out of nonhuman wealth. Then, I characterize the optimal consumption rule.

The macroeconomic consumption literature often postulates a Gaussian autoregressive income process,\(^8\) which has a few drawbacks. It is unbounded from below and symmetrically distributed, with no excess kurtosis. Empirically, labor income is positively skewed, fat-tailed, and bounded from the below. Moreover, the conditional variance of changes in labor income, in a Gaussian setting, is deterministic, and thus cannot depend on income outcomes. Another frequently adopted model assumes that the logarithm of income, rather than its level, is a conditionally homoskedastic Markov process,\(^9\) which implies that the conditional variance of income increases in the level of income.

The affine income process introduced here is also conditionally heteroskedastic, in that a higher level of income implies a higher conditional volatility of changes in income, signaling a riskier stream of future labor income. Furthermore, the “affine” process is shown to be more tractable. Specifically, I propose a conditionally-heteroskedastic affine income process, allowing for positive skewness, excess kurtosis and boundedness (from below). Mathematically, I fix a probability space $(\Omega, \mathcal{F}, P)$ and an information filtration \(\{\mathcal{F}_t\}\), and suppose that the agent receives labor income at time $t$ at the rate $y_t$, solving a stochastic differential equation (SDE),

\[
d y_t = \mu(y_t) \, dt + \sigma(y_t) \, dW_t, \quad t \geq 0,
\]

\(^8\)See Deaton (1992) and Attanasio (1999) for reviews.
with an initial income level $y_0$, and where $W$ is a standard Brownian motion. The drift and volatility functions $\mu(\cdot)$ and $\sigma(\cdot)$ in (1) are defined by

$\mu(y) = \theta - \kappa y$, \hspace{1cm} (2)

$\sigma(y) = \sqrt{l_0 + l_1 y}$, \hspace{1cm} (3)

respectively, where $\theta, \kappa, l_0$ and $l_1$ are constant parameters. This is an example of an affine diffusion, because the drift $\mu(y)$ and the conditional variance function $\sigma^2(y)$ are affine in the income level $y$. A positive coefficient $l_1$ captures a monotonically increasing relationship between the conditional variance of and the level of labor income. I split this class of processes into two groups.

One group has conditionally homoskedastic shocks ($l_1 = 0$), wherein the coefficient $\sigma_0 \equiv \sqrt{l_0}$ is the volatility of the income process. The process need not be stationary. The conventional autoregressive Gaussian process, possibly unit root ($\kappa = 0$), with or without drift, is a special case. The focus of this paper is the second group, for which innovations are conditionally heteroskedastic ($l_1 > 0$), and for which a higher income signals a more volatile future labor income.

Given the labor-income process $y$ and an adapted consumption process $c$ from the admissible set $L(x_0, y_0)$ defined by transversality condition (A.8) in Appendix A, the agent’s financial wealth process $x$ evolves according to

$$dx_t = (rx_t + y_t - c_t) \, dt, \hspace{1cm} t \geq 0,$$

with an initial wealth endowment $x_0$.

To facilitate later discussions, I follow Friedman (1957) and Hall (1978) to define human wealth as follows:

**Definition 1** Human wealth $h_t$ at time $t$ is the expected present value of future labor income, discounted at the risk-free interest rate $r$, conditioning on $\mathcal{F}_t$, the agent’s information set at time $t$. That is,

$$h_t = \mathbb{E}_t \left( \int_t^\infty e^{-r(s-t)} y_s \, ds \right),$$

where $\mathbb{E}_t$ denotes $\mathcal{F}_t$-conditional expectation. I assume that $r + \kappa > 0$, ensuring that human wealth is finite.

With (1) and (2), human wealth is affine in current labor income, in that

$$h_t = \frac{1}{r + \kappa} \left( y_t + \frac{\theta}{r} \right).$$
This definition of human wealth ignores risk and therefore overstates wealth associated with stochastic uninsurable labor income. Appendix B provides an alternative risk-adjusted measure of wealth for income.

For tractability reasons, I assume CARA utility, following Merton (1971), Kimball and Mankiw (1989), and Caballero (1991). That is, the agent has an expected intertemporal utility,

\[ U(c) = \mathbb{E} \left( \int_0^\infty e^{-\beta s} u(c_s) \, ds \right) = \mathbb{E} \left( -\frac{1}{\gamma} \int_0^\infty e^{-\beta s} e^{-\gamma c_s} \, ds \right), \]

with \( \beta, \gamma > 0 \), for any consumption process \( c \) from the admissible set \( L(x_0, y_0) \), defined by the transversality condition (A.8). The agent’s objective is to solve

\[ J(x_0, y_0) = \sup_{c \in L(x_0, y_0)} U(c). \]

The optimal consumption rule \( c^* \) is affine in financial wealth \( x \) and current income \( y \), in that, for all \( t \),

\[ c^*_t = r(x_t + a_y y_t + a_0), \]

where

\[ a_y = \frac{a_h}{r + \kappa}, \]

\[ a_h = \begin{cases} \frac{1}{\Delta_1} \left( \sqrt{1 + 2\Delta_1} - 1 \right) < 1, & \text{if } l_1 > 0, \\ 1, & \text{if } l_1 = 0, \end{cases} \]

\[ a_0 = \frac{1}{r} \left( \frac{\beta - r}{\gamma r} + \frac{\theta}{r + \kappa} a_h - \frac{1}{2} \Delta_0 a_h^2 \right), \]

\[ \Delta_1 = \frac{\gamma l_1}{(r + \kappa)^2} \geq 0, \]

\[ \Delta_0 = \frac{\gamma l_0}{(r + \kappa)^2}. \]

Appendix A contains both a derivation and a proof for the optimality results stated above.

Although the CARA-utility-based model lacks wealth effect, the next section shows that rich and desirable implications for optimal consumption (9), such as a lower MPC out of human wealth than out of financial wealth, may still be obtained, provided that income shocks are conditionally heteroskedastic.

3 Model Implications

This section analyzes the implications of the consumption rule (9). Subsection 3.1 provides a theoretical justification of the Friedman’s conjecture on a lower MPC out of “human” wealth.
than out of financial wealth. Subsection 3.2 proposes a saving decomposition and uses it to analyze the three saving motives, stated in Friedman (1957). Subsection 3.3 perform robustness checks against CRRA-utility-based models.

3.1 The Marginal Propensities to Consume

The decision rule (9) relates the optimal consumption to his wealth, a stock variable, and current income, a flow variable. In order to shed light on how consumption responds to income shocks, it is helpful to convert current income in the optimal consumption rule (9) to a stock measure for income, for example, “human” wealth, defined in (5). Expressing the optimal consumption rule (9) in terms of stock variables (financial and human wealth) gives

\[ c_t^* = r (x_t + a_h h_t - b_0), \]

where

\[ b_0 = \frac{1}{r} \left( \frac{1}{2} \Delta_0 a^2_h - \frac{\beta - r}{\gamma r} \right). \]  

The MPC out of human wealth \( \omega_h = ra_h \) is always less than that out of financial wealth. This inequality holds strictly when the conditional variance of labor income depends directly on its level \( (l_1 > 0) \). The intuition is as follows. A larger realization of current income not only implies a higher level of human wealth, but also signals a more volatile stream of future labor income. An agent with a precautionary savings motive will therefore consume less out of his human wealth than out of his financial wealth, not only in terms of the absolute level, but also at the margin.

In order to quantify the agent’s precautionary savings motive, I first introduce a natural benchmark: certainty-equivalence consumption level \( c^p \), that is obtained by setting \( l_0 = l_1 = 0 \) in (15). This gives

\[ c_t^p = r \left( x_t + h_t + \frac{\beta - r}{\gamma r} \right) = w_t^p + \frac{\beta - r}{\gamma r}, \]

where \( w_t^p = r(x_t + h_t) \) is “permanent income,” as defined in Friedman (1957). The precautionary saving premium\(^{10}\) \( \pi \) is measured as the difference between the certainty-equivalence consumption and optimal consumption \( c^* \), in that \( \pi \equiv c^p - c^* \). From (15) and (17), the precautionary saving premium is given by

\[ \pi_t = \frac{1}{2} a^2_h (\Delta_0 + r \Delta_1 h_t) = \frac{\gamma r}{2} a^2_y (l_0 + l_1 y_t). \]

If shocks are conditionally heteroskedastic, then \( \pi_t \) increases in labor income \( y \) and naturally in human wealth \( h \).

\(^{10}\)Kimball (1990) introduced the concept of prudence and measures precautionary premium, based on the convexity of marginal utility.
A common notion that consumption responds one to one to permanent shocks is based on the PIH and is actually misleading. This can be seen from (15) and (18). The precautionary savings demand \( \pi_t \) increases in the level of income, because a higher level of income signals a more volatile stream of future income, even when shocks are permanent. This implies that consumption must respond less to a unit increase in human wealth than a unit increase in financial wealth, even when shocks are permanent. For a unit-root process \((\kappa = 0)\), the MPC \( a_y \) out of the current income is equal to the MPC out of human wealth \( \omega_h = ra_h = 2/(1 + \sqrt{1 + 2\gamma l_1/r}) < 1 \). That is, the agent will save some portion out of income for precautionary reasons, even if income shocks is permanent.

A special case of (18) is a continuous-time version of Caballero (1991), obtained by setting \( l_1 = 0 \). The precautionary savings premium is \( \pi = \Delta_0/2 = 0.5\gamma r l_0/(r + \kappa)^2 \), a constant, independent of the agent’s financial and human wealth. All agents, regardless of differences in their current labor incomes, have the same precautionary saving premium. Wang (2003) shows that Caballero-type agents\(^ {11}\) behave effectively as Friedman-type permanent-income consumers, in a Bewley-style incomplete-markets equilibrium model\(^ {12}\) with a continuum of *ex ante* identical, but *ex post* heterogeneous agents. The market-clearing condition in the risk-free asset leads to an equilibrium interest rate, that is lower than the subjective discount rate. As a result, the constant dis-saving due to impatience is *exactly* offset with constant precautionary saving premium in equilibrium, leaving the consumer effectively only saves in anticipation of possible future changes in income (Campbell (1987)), and thus behaves as a permanent-income consumer. With conditionally heteroskedastic income innovations \((l_1 > 0)\) and CARA utility, the agent’s precautionary savings demand increases linearly in the level of income. Agents accumulate wealth at different rates for precautionary reasons, depending upon their income levels. They do not behave in accordance with the PIH in equilibrium Bewley models with heteroskedastic income shocks.

The traditional definition of human wealth as given in (5) ignores risk, and consequently overstates the value that a precautionary agent attaches to his future stochastic labor income. In Appendix B, I propose a natural risk-adjusted measure (B.1) of wealth for income, based on weighting future income by the agent’s marginal rate of substitution across time.

A conventional wisdom in the consumption literature is that if risk associated with stochastic income is “appropriately” incorporated, then the certainty-equivalence consumption rule will hold. Friedman (1957) wrote “the rate of interest at which an individual can borrow on

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\(^{11}\) A Caballero-type agent is defined with time-additive separable CARA utility and conditionally homoskedastic uninsurable income process.

the basis of his future earnings may be different from the rate at which he can borrow on
the basis of nonhuman capital.” More recently, Zeldes (1989) suggested that a remedy to the
failure of the certainty-equivalence-based permanent-income model is “to discount expected
future income at a higher discount rate” than the interest rate. In order to highlight the
insights of Friedman and Zeldes, I re-write the optimal consumption rule (9) as follows:13
\[ c_t^* = r \left( x_t + h_t^R \right) = r \left[ x_t + \frac{1}{r + \kappa + u t_1/2} \left( y_t + \frac{\theta - u t_0/2}{r} \right) \right], \] (19)
where \( h_t^R \) is given in (B.19), and \( u = \gamma r a_y \). That is, the “permanent-income” hypothesis would
hold at an elevated discount rate of \( r + u t_1/2 \) for the income process.

This section has so far focused on precautionary savings. However, saving for precaution
is only one of the motives. Discussing different motives for saving has a long tradition in the
consumption literature, at least dating back to Friedman (1957). The next section formalizes
Friedman’s insights on various motives for holding wealth, by providing a saving decomposition.
I then use the decomposition to compare different consumption models and to evaluate my
model along each saving motive.

3.2 Motives for Saving: A Decomposition Analysis

I offer a natural and explicitly-solved decomposition of the agent’s saving. The decomposition
analysis allows us to relate this model to the large consumption literature in an intuitive
way. It therefore helps us to judge the performance of the newly proposed model along each
saving motives. Gourinchas and Parker (2001) offered a decomposition analysis of savings
motive. Their decomposition is based on the second-order Taylor expansion of the Euler
equation, not on the consumption-saving rule. This is primarily due to the numerical nature
of the consumption rule in their model.14 Unlike their work, the decomposition proposed here
relates to the Friedman’s original insights on the three different saving motives to be discussed
below in detail.

The optimal saving rate may be obtained by plugging the optimal consumption rule (9)
into the wealth accumulation equation (4). This gives
\[ s_t = \frac{dx_t}{dt} = (1 - r a_y) y_t - r a_0, \] (20)
where \( a_y \) and \( a_0 \) are given in (10) and (12), respectively. A simple saving rule (20) has rich
implications for different saving motives. This paper provides a natural decomposition of
the total saving rate into three motives stated in Friedman (1957): (i) to straighten out of

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13To simplify the formula, let \( \beta = r \). For a more general formula, see Appendix B.
14Recall that there is no closed-form solutions for the consumption rule using CRRA utility.
consumption stream; (ii) to earn interests on assets; and (iii) to save for unexpected low incomes. The saving rate $s_t$ may be written as:

$$s_t = d_t + \pi_t - \phi_t,$$

(21)

where

$$d_t = \frac{\kappa y_t - \theta}{r + \kappa},$$

(22)

$$\phi_t = \frac{\beta - r}{\gamma r},$$

(23)

and $\pi_t$ is given by (18). The first term $d_t$ formalizes the agent’s motive to “straighten out of consumption stream,” as stated in Friedman (1957). It measures the portion of the saving that is due to “expected” future declines in labor income. Campbell (1987) dubbed this “saving for a rainy day,” an equivalent way of phrasing Friedman’s PIH in terms of saving. If $\kappa \leq 0$, the agent’s income grows over time in expectation and, hence, he borrows against future income ($d_t < 0$). If $\kappa > 0$, income is stationary and “saving for a rainy day” is

$$d_t = \frac{\kappa}{r + \kappa} (y_t - \bar{y}) = y_t - rh_t,$$

(24)

where $\bar{y} = \theta/\kappa$ is the long-run mean of income. When $y_t > \bar{y}$, the agent expects that his income falls in the long run (due to mean reversion), and thus saves a portion of current income in excess of its long-run mean $\bar{y}$ in anticipation of future “rainy” days. A higher rate $\kappa$ of mean reversion or a larger difference between current income and its long-run mean induces a higher saving rate, ceteris paribus. The saving $d_t$ is positive if current income is above the annuity value of human wealth (Friedman (1957)).

If $d_t$ is the only saving component, the implied consumption is a martingale (Hall (1978)). This confirms Friedman’s intuition of “straightening out of consumption stream,” in a stochastic sense, in that changes in consumption are not predictable. Note that “saving for a rainy day” is independent of the agent’s utility function. Any forward-looking consumption model contains this PIH-implied component of saving for “rainy” days (Campbell (1987)). Empirically, changes in consumption are predicted by variables such as labor income. This is known as the excess-sensitivity puzzle (Flavin (1981)). My model shows that a direct dependence of the conditional variance of labor income on its own level ($l_1 > 0$) is consistent with the “excess sensitivity” of consumption. In order to highlight the mechanism, consider the implications of
Equation (B.7) implies that
\[
E_t (\Delta c_{t+1}^s) = -\frac{\beta - r}{\gamma} + \frac{1}{2} r^2 a_y^2 \int_t^{t+1} (l_0 + l_1 y_s) \, ds,
\]
where \( \Delta c_{t+1}^s \equiv c_{t+1}^s - c_t^s \), and \( \tilde{\theta} = \theta + \kappa l_0 / l_1 \), for \( l_1 > 0 \). When the labor-income innovation is conditionally homoskedastic, as in Caballero (1991), (25) implies that current income does not predict future movements in consumption. That is, the precautionary savings motive and stochastic labor income together are not sufficient for the excess sensitivity of consumption. With \( l_1 > 0 \), however, the model predicts a strictly positive regression coefficient \( r(1 - a_h)(1 - e^{-\kappa})/\kappa \) of \( \Delta c_{t+1}^s \) on current income \( y_t \), supporting the empirical finding of the excess-sensitivity. Equation (25) implies that consumption is a submartingale, expected to grow over time, consistent with the empirical findings of excess-growth (Deaton (1987)).

An optimization-based consumption model differs from the PIH in the other two components of saving: (i) dissaving \( \phi_t \) because of being relatively impatient \((\beta > r)\), and (ii) precautionary savings demand \( \pi_t \) due to the uncertain nature of income. The dissaving due to relative impatience corresponds to the motive of paying interests, the second motive pointed out in Friedman (1957). Dissaving \( \phi_t \) is driven by \((\beta - r)\), the difference between the market discount rate and individual’s discount rate for future consumption. If the subjective discount rate \( \beta \) is larger than the interest rate \( r \), then the agent dissaves because he is relatively impatient. In this model, the dissaving \( \phi_t \) is constant, because CARA utility lacks wealth effect. I compare this component with the prediction in a CRRA utility model in Subsection 3.3. The precautionary saving term \( \pi_t \) captures Friedman’s insight that the agent may want to save in order to build a reserve for emergency. Subsection 3.1 contains a detailed treatment about the precautionary saving term \( \pi_t \), and shows the implication of stochastic precautionary saving on a lower MPC out of human wealth than out of financial wealth.

Next, I check the robustness of the optimal consumption rule (9), by comparing these individual saving demands with CRRA-utility-based models. I show that the proposed model captures the first-order precautionary effect.

3.3 Comparisons with CRRA-Utility-Based Models

I compare the implications of the optimal consumption rule (9) with a CRRA-utility-based consumption model in two steps, first under certainty and then with uncertainty, following

\[\text{This is under an additional assumption } \beta = r. \text{ Otherwise, } \{z_t \equiv c_t^s + (\beta - r) t / r\}_{t \geq 0} \text{ is a submartingale, as shown in the Appendix B.}\]
Friedman (1957). Doing so allows us to single out and concentrate on each of the three saving motives.

With deterministic income and CRRA utility, the optimal consumption is given by \( c_t = \omega (x_t + h_t) = [r + \psi (\beta - r)] (x_t + h_t) \), where \( h \) is human wealth, and parameters \( \psi \) and \( \beta \) are the elasticity of intertemporal substitution and the subjective discount rate, respectively. The coefficient \( \omega \) is both the MPC out of financial wealth and that out of human wealth. Applying the saving decomposition (21) gives \( s_t = d_t - \phi_t \), where \( d_t \) is the PIH-implied saving, given by \( d_t = y_t - rh_t \), and \( \phi_t \) is the dissaving due to relative impatience. I assume that the drift of the income process is the same as the one in Section 2, in that \( dy_t = (\theta - \kappa y_t) dt \) for \( \kappa > 0 \). Then, the PIH-implied saving is given by \( d_t = (\kappa y_t - \theta) / (r + \kappa) = \kappa (y_t - \bar{y}) / (r + \kappa) \), same as in (24). This is indeed expected, because the PIH-implied saving \( d_t \) is independent of utility specification, as shown in Section 3. The dissaving for being relatively impatient \( (\beta > r) \) in a CRRA-utility-based model is given by

\[
\phi_t = \psi (\beta - r) (x_t + h_t).
\]

A higher elasticity \( \psi \) of intertemporal substitution, or a larger degree of impatience, measured by \( (\beta - r) \), implies a larger dissaving \( \phi_t \), \textit{ceteris paribus}. The dissaving \( \phi_t \) is increasing in both wealth and “human” wealth. This is fundamentally different from the constant dissaving result given in (23). CARA utility’s lacking wealth effect accounts for the implication of this constant dissaving \( \phi_t \).

Because both the PIH-implied saving “for rainy days” \( d_t \) and dissaving \( \phi_t \) for being relatively impatient are independent of income shocks, therefore, previous analyses and comparisons about \( d_t \) and \( \phi_t \) also apply in a model with stochastic income. From now on, we need only focus on the precautionary savings demand \( \pi_t \) due to uninsurable income shocks.

I construct a CRRA-utility-based comparison model, and numerically solve for the optimal consumption rule. I choose the CRRA coefficient to be 2, a commonly used and plausible value. Based upon the decomposition analysis, I set \( r = \beta \). (The dissaving \( \phi_t \) due to impatience has been analyzed.) Let the (annual) subjective discount rate \( \beta \) be 5%. I calibrate a non-negative income process of the following type,

\[
dy_t = \kappa (\bar{y} - y_t) dt + \sigma \sqrt{y_t} dW_t,
\]

an example of the affine process (1). Empirically, labor income is highly persistent.\(^{17} \) I choose the mean-reversion rate \( \kappa \) by setting an annual autoregressive coefficient \( \rho = .98 \), in that

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\(^{16}\)I have assumed that the income stream is such that non-negative consumption constraint is never binding. This is a technical point that can be easily checked for any given income process.

\(^{17}\)See MaCurdy (1982), Abowd and Card (1989), and Attanasio and Davis (1996), for example. Storesletten,
\[ \kappa \equiv -\log \rho = -\log .98 = .02. \]

I normalize the long-run mean \( \bar{y} \) of income to unity, without loss of generality. This gives \( \theta = \kappa \bar{y} = 0.02 \). I calibrate the volatility parameter \( \sigma = 0.18 \) in (27) by setting the unconditional variance of income to 0.80. This chosen value (0.80) of the unconditional variance of income is based on the estimates for the autoregressive logarithmic income process in Storesletten, Telmer, and Yaron (2002). I set the initial wealth of the agent to be half of his average human wealth \( \bar{h} = \bar{y}/r = 20 \). This gives \( \bar{x} = \bar{h}/2 = 10 \). I use the Chebyshev polynomial method\(^{18} \) to approximate the value function and solve for the optimal consumption rule numerically in this CRRA-utility model.

Figure 1 plots the optimal consumption, and precautionary savings demands in the left and right diagrams, for the above calibrated CRRA-utility model, the CARA-utility model,\(^{19} \) and the PIH. First, consumption levels in both CARA-utility and CRRA-utility models are lower than the PIH-implied consumption level, because of positive precautionary savings. Second, the consumption levels in both CARA-utility and CRRA-utility models are increasing in income and to the first-order approximation, close to each other. Third, the precautionary saving demand increases with income in both models – consistent with a lower MPC out of “human” wealth than out of financial wealth. Fourth, the precautionary saving demand in the CRRA-utility model is higher than that in the CARA-utility model, for relatively low income levels; and lower than that in the CARA-utility model, for relatively high income levels. The intuition is that, a lower (higher) income leads to a lower (higher) level of consumption,\(^{19} \) \( \text{ceteris paribus} \). For the CRRA utility function, its marginal utility has a bigger (smaller) curvature at the lower (higher) levels of consumption than the marginal utility associated with the CARA utility function does, therefore suggesting a higher (lower) precautionary premium than the CARA utility does, \( \text{ceteris paribus} \). Kimball (1990) introduced the coefficients of absolute and relative prudence, measures for the convexity of the marginal utility, to quantify the precautionary saving demand.\(^{20} \) For CRRA utility, the coefficient of absolute prudence, \(-u'''(c)/u''(c)\), is stochastic and is equal to \((\alpha + 1)/c\), where \( \alpha \) is the coefficient of relative risk

---

\(^{18}\)See Judd (1998) for a textbook description of this computational method.

\(^{19}\)As a starting point, I choose the CARA coefficient \( \gamma \) by setting \( \gamma \bar{c} = \alpha \), where the CRRA coefficient \( \alpha \) is chosen to be 2. The agent’s average consumption is approximately equal to the permanent income, in that \( \bar{c} = r(\bar{x} + \bar{h}) = r\bar{x} + \bar{y} = 1.5 \). An equilibrium model with a continuum of agents such as that of Aiyagari (1994) may be constructed to support an average consumption level that is equal to permanent income. This procedure gives \( \gamma = 1.33 \).

\(^{20}\)Kimball (1990) showed that the theory of precautionary saving is isomorphic to the theory of risk aversion (Pratt (1964)), after making the observation that precautionary savings to the convexity of marginal utility is as risk aversion to the concavity of the utility function.
aversion. Therefore, it decreases in consumption level \( c \). Unlike CRRA utility, CARA utility has a constant coefficient of absolute prudence \( \gamma \).

Zeldes (1989) was among the first to note that the optimal consumption rule for a CRRA-utility-based model with stochastic labor income is conceptually and quantitatively different from the optimal consumption rule with deterministic income. Based on his numerical solutions, he stated that possible remedies over the PIH rule would be to (i) put a weight less than one on human wealth before adding it to nonhuman wealth, or (ii) to discount expected future income at a higher discount rate. This paper shows that both his suggestions (i) and (ii) are consistent with optimality, using CARA utility and a conditionally heteroskedastic affine income process. Although CARA utility lacks wealth effect, it offers a first-order precautionary effect, provided that an empirically plausible conditionally heteroskedastic income process is used.

4 Concluding Remarks

This paper provides the first explicitly-solved optimal consumption model with a lower marginal propensity to consume out of human wealth than out of financial wealth, a widely noted desirable property of the consumption rule (Friedman (1957) and Zeldes (1989)). The main ingredient of the model is to have the conditional variance of changes in income as an increasing function of current income. Empirically, there is well documented evidence in support of conditional heteroskedasticity of income shocks in levels. Specifically, I model the income
process, using a conditionally heteroskedastic process, known as an affine process. A higher level of current income not only implies a higher level of human wealth, but also signals a riskier stream of future income. As a result, a precautionary agent therefore saves more than the PIH-implied saving “for rainy days” (Campbell (1987)). I also compare the proposed analytical model with a numerically solved CRRA-utility-based model, and show that my model captures the first-order precautionary effect. A saving decomposition analysis is used to help understand various motives for holding wealth pointed out by Friedman (1957).
Appendices

A Proof of the Optimal Consumption Rule (9)

The proof proceeds in two steps. First, I derive the consumption rule using dynamic programming. Then, I verify the optimality of the policy function.

A.1 Derivation of Optimal Consumption

I conjecture that the value function $J$ of (8) solves the Hamilton-Jacobi-Bellman (HJB) equation:

$$0 = \sup_{\bar{c}} \{ u(\bar{c}) - \beta J(x, y) + \mathcal{D}^c J(x, y) \}, \tag{A.1}$$

where

$$\mathcal{D}^c J(x, y) = (rx + y - \bar{c})J_x(x, y) + (\theta - \kappa y)J_y(x, y) + \frac{1}{2}(l_0 + l_1 y)J_{yy}(x, y). \tag{A.2}$$

I further conjecture that the value function takes an exponential-affine form:

$$J(x, y) = -\frac{1}{\gamma r} \exp \left[ -\beta r(y + a_y y + a_0) \right]. \tag{A.3}$$

The first-order condition for the HJB equation is $u'(\bar{c}) = J_x(x, y)$. Plugging the implied values for $\bar{c}$, $J_x$, $J_y$, and $J_{yy}$ into (A.1) leaves

$$0 = -\frac{1}{\gamma} + \frac{\beta}{\gamma r} + (1 - ra_y) y - ra_0 + (\theta - \kappa y) a_y - \frac{1}{2}(l_0 + l_1 y) \gamma r a_y^2, \tag{A.4}$$

Because the above equality holds for any $y$, therefore

$$0 = 1 - ra_y - \kappa a_y - \frac{1}{2} \gamma r l_1 a_y^2, \tag{A.5}$$

$$a_0 = \frac{r}{\gamma} \left( \frac{\beta - r}{\gamma r} + \theta a_y - \frac{1}{2} \gamma r l_0 a_y^2 \right). \tag{A.6}$$

Equation (A.5) may be written as

$$\frac{1}{2} \Delta_1 a_h^2 + a_h - 1 = 0. \tag{A.7}$$

where $a_h$ and $\Delta_1$ are given in (10) and (13), respectively. If innovations are conditionally homoskedastic in that $l_1 = 0$, then $a_h = 1$. Two candidate roots for $a_h$ arise from the quadratic equation, if shocks are conditionally heteroskedastic $l_1 > 0$. I discard the negative root, since it implies a negative MPC out of current income. The positive root, between zero and one, is given in (11).
A.2 Verification of Optimality and Transversality

Fixing initial wealth $x_0$ and initial income $y_0$, on the given probability space $(\Omega, \mathcal{F}, P)$, with information filtration $\{\mathcal{F}_t\}_{t=0}^\infty$, an adapted consumption process $c$ is defined to be in the set $L(x_0, y_0)$ if the following “transversality condition” is satisfied:

$$\lim_{t \to \infty} E \left[ e^{-\beta t} |J(x_t, y_t)| \right] = 0,$$

where $x$ is the wealth process associated with the consumption process $c$. Condition (A.8) restricts the rate at which the debt is allowed to grow.

First, I compute the total differential of a discounted conjectured value function as follows,

$$d \left[ e^{-\beta t} J(x_t, y_t) \right] = e^{-\beta t} \left[ D^{(c)} J(x_t, y_t) - \beta J(x_t, y_t) \right] dt + \delta_t dB_t,$$

(A.9)

$$\leq -e^{-\beta t} u(c_t) dt + \delta_t dB_t,$$

(A.10)

where

$$\delta_t = e^{-\beta t} J_y(x_t, y_t) \sigma(y_t),$$

(A.11)

and the inequality follows from the HJB equation. The integral form of the above inequality is then given by

$$e^{-\beta T} J(x_T, y_T) + \int_0^T e^{-\beta t} u(c_t) dt \leq J(x, y) + \int_0^T \delta_t dB_t.$$

(A.12)

Without loss of generality, I require that any candidate consumption rule $c$ in $L(x_0, y_0)$ satisfies $U(c) \geq U(y)$:

$$E \left[ \int_0^\infty e^{-\beta t} u(c_t) dt \right] \geq E \left[ \int_0^\infty e^{-\beta t} u(y_t) dt \right] > -\infty.$$

(A.13)

The intuition behind (A.13) is that for any candidate consumption process $c$ to be optimal, it should at least yield a higher utility level $U(c)$ than $U(y)$ attainable under autarky. Also, recall that $U(c) \leq 0$, since $u(c) = -e^{-\gamma c}/\gamma < 0$. Therefore,

$$E \left[ \int_0^\infty e^{-\beta t} u(c_t) dt \right] \leq E \left[ \int_0^\infty e^{-\beta t} u(y_t) dt \right] < \infty.$$

(A.14)

With (A.14), the dominated convergence theorem implies that, for any feasible consumption process $c$ in $L(x_0, y_0),

$$\lim_{T \to \infty} E \left( \int_0^T e^{-\beta t} u(c_t) dt \right) = E \left( \int_0^\infty e^{-\beta t} u(c_t) dt \right).$$

(A.15)

Assuming that for any candidate optimal consumption rule $c$, we have

$$E \left[ \left( \int_0^T \delta_t^2 dt \right)^{1/2} \right] < \infty, \quad T > 0.$$
Therefore, \( \int \delta dB \) is martingale. As a result, we have

\[
J(x_0, y_0) \geq \mathbb{E} \left( \int_0^\infty e^{-\beta t} u(c_t) \, dt \right) = U(c),
\]

(A.16)
after taking expectation on both sides of (A.12) and limits, and using (A.15).

Next, I check the optimality of the candidate consumption rule, given by (9), whose associated candidate value function is given in (A.3). The HJB equation and the same calculations leave

\[
e^{-\beta T} J(x^*_T, y_T) + \int_0^T e^{-\beta t} u(c^*_t) \, dt = J(x_0, y_0) + \int_0^T \delta_t dB_t,
\]

(A.17)
where the equality appears because the proposed optimal control achieves the supremum in the HJB equation.

The rest of the proof is to show (i) the transversality condition (A.8) is met for candidate consumption \( c^* \) and (ii) \( \int \delta^* dB \) is a martingale, for \( \delta^* \) associated with candidate optimal consumption rule \( c^* \). With (i) and (ii), equations (A.17) and (A.12) imply that \( J(x_0, y_0) = U(c^*) \). That is, \( J \) is the value function and \( c^* \) of (9) is optimal.

Equation (9) implies that the associated endogenous wealth \( x^*_\tau \) is given by

\[
x^*_\tau = x_0 + \int_0^\tau (rx^*_t + y_t - c^*_t) \, dt = x_0 + \int_0^\tau [(1 - ra_y)y_t - ra_0] \, dt.
\]

(A.18)
Equation (A.18) implies that

\[
e^{-\beta \tau} J(x^*_\tau, y_\tau) = -\frac{1}{\gamma r} e^{-\gamma r(x_0 + a_0)} \mathbb{E} \left[ \exp \left( -\int_0^\tau (\rho_0 + \rho_1 y_s) \, ds \right) e^{uy_\tau} \right],
\]

(A.19)
where

\[
u = -\gamma r a_y = -\frac{\gamma r a_h}{r + \kappa} < 0,
\]

(A.20)
\[
\rho_0 = \beta - \gamma r a_0 = r - \gamma r \left( \frac{\theta}{r + \kappa} a_h - \frac{1}{2} \Delta_0 a_h^2 \right),
\]

(A.21)
\[
\rho_1 = \gamma r (1 - ra_y) = \gamma r \left( \frac{(1 - a_h)r + \kappa}{r + \kappa} \right) > 0.
\]

(A.22)

Duffie, Pan, and Singleton (2000) showed that under technical regularity conditions,

\[
\mathbb{E} \left[ \exp \left( -\int_0^\tau (\rho_0 + \rho_1 y_s) \, ds \right) e^{uy_\tau} \right] = \exp \left[ A(\tau) + B(\tau)y_0 \right],
\]

where coefficients \( A(\tau) \) and \( B(\tau) \) solve Riccati equations,

\[
\dot{B}(\tau) = -\kappa B(\tau) + \frac{1}{2} l_1 B^2(\tau) - \rho_1,
\]

(A.23)
\[
\dot{A}(\tau) = \theta B(\tau) + \frac{1}{2} l_0 B^2(\tau) - \rho_0,
\]

(A.24)
with initial conditions \( B(0) = u \) and \( A(0) = 0 \).

Note that
\[-\kappa u + \frac{l_1}{2} u^2 - \rho_1 = \gamma r \left( \frac{\gamma r l_1}{2} a_y^2 + (\kappa + r)a_y - 1 \right) = 0,\]
using (A.20), (A.22) and (11). Therefore, (A.23) and the above identity imply that \( B(\tau) = u \) for all \( \tau \geq 0 \). Therefore, (A.24) implies that
\[ A(\tau) = \left( \theta u + \frac{l_0}{2} u^2 - \rho_0 \right) \tau = -\tau, \quad (A.25) \]
using (A.20), (A.21), and (14). Therefore, the transversality condition (A.8) is satisfied if the interest rate is positive \((r > 0)\). Next is to verify that \( \int \delta^* dB \) is a martingale, where \( \delta^* \) is associated with candidate optimal consumption \( c^* \) and given by (A.11). Equation (A.18) implies that
\[ \delta^*_t = a_y \sigma(y_t) \exp \left( -\beta t - \gamma r (x^*_t + a_y y_t + a_0) \right), \quad (A.26) \]
\[ = a_y \sigma(y_t) e^{-\gamma r (x_0 + a_0)} \exp \left( -\int_0^t (\rho_0 + \rho_1 y_s) ds + uy_t \right). \quad (A.27) \]
It is sufficient to check
\[ \mathbb{E} \left( \int_0^T \delta^2_t dt \right) < \infty \quad (A.28) \]
in order to verify that \( \int \delta^* dB \) is a martingale.

If innovations are conditionally homoskedastic \((l_1 = 0)\), then \((x^*_t, y_t)\) is jointly normal. Therefore,
\[ \mathbb{E} \left( \int_0^T \delta^2_t dt \right) = K_0 \mathbb{E} \left[ \int_0^T \exp \left( -2\beta t - 2\gamma r \left( 1 + a_y \right) \left( x^*_t \right) y_t \right) dt \right] < \infty, \quad (A.29) \]
where \( K_0 = l_0 a_y^2 e^{-2\gamma a_0} \).

If innovations are conditionally heteroskedastic \((l_1 > 0)\), earnings \( y \) is bounded from below \((y \geq -l_0/l_1)\). Let \( w = y + l_0/l_1 \). Fubini’s theorem implies that
\[ \mathbb{E} \left( \int_0^T \delta^2_t dt \right) = K_1 \mathbb{E} \left[ \int_0^T w_t \exp \left( -2 \int_0^t (\bar{\rho}_0 + \rho_1 w_s) ds \right) e^{2w_t} dt \right], \]
\[ \leq K_1 \mathbb{E} \left[ \int_0^T w_t e^{-\bar{\rho}_0 t} dt \right] = K_1 \int_0^T \left[ \left( n(t; \kappa) \theta + e^{-\kappa t} y_0 + \frac{l_0}{l_1} \right) e^{-\bar{\rho}_0 t} \right] dt, \]
\[ = K_1 \left[ \frac{\theta}{\kappa} (n(T; \bar{\rho}_0) - n(T; \bar{\rho}_0 + \kappa)) + \frac{l_0}{l_1} n(T; \bar{\rho}_0) + y_0 n(T; \bar{\rho}_0 + \kappa) \right] < \infty, \]
using \( \rho_1 > 0, u < 0, \) non-negative \( w, \) and where \( \bar{\rho}_0 = \rho_0 - l_0 \rho_1/l_1, \) function \( n(v; \delta) \) is given in (B.8), and \( K_1 = l_1 a_y^2 e^{-2\gamma r (x_0 + a_0)} e^{-2u a_0/l_1}. \)
B Risk-adjusted Measures of Wealth for Income

This appendix proposes a risk-adjusted measure of “human wealth,” incorporating precautionary savings motives. I propose that the agent’s future labor income is discounted by weighting it with the agent’s marginal rate of substitution, in that

$$S_t \equiv \mathbb{E}_t \left[ \int_t^{\infty} \left( \frac{M_u}{M_t} y_u \right) du \right]. \quad \text{(B.1)}$$

The stochastic discount factor $M$ associated with $c^*$ given in (9) is

$$M_t = e^{-\beta t} u'(c^*_t) = e^{-\beta t - \gamma c^*_t}, \quad \text{(B.2)}$$

a function of optimal consumption $c^*$ and time $t$. In order to derive the dynamics for this stochastic discount factor, I need to first derive the dynamics of consumption process.

Equations (9) and (4) together imply that

$$dc^*_t = \frac{1}{2} \left[ \frac{1}{\gamma} \eta_t^2 - (\beta - r) \right] dt + \frac{1}{\gamma} \eta_t dW_t, \quad \text{(B.3)}$$

where

$$\eta_t = u a(y_t) \geq 0, \quad \text{(B.4)}$$

$$u = \gamma r a_y. \quad \text{(B.5)}$$

I let $z_t \equiv c^*_t + \left( \frac{\beta - r}{\gamma} \right) t$. From (B.3),

$$\mathbb{E}_t (z_s - z_t) = \mathbb{E}_t \left( \int_t^s \frac{\eta_v^2}{2\gamma} dv \right), \quad \text{(B.6)}$$

$$= \begin{cases} r \left( 1 - a_h \right) \left[ y_t + l_0/l_1 \right] n(s - t; \kappa) + \tilde{\theta} m(s - t; \kappa), & l_1 > 0, \\ r{\Delta}_0 (s - t)/2, & l_1 = 0, \end{cases} \quad \text{(B.7)}$$

for $s > t$, where $\tilde{\theta} = \theta + \kappa l_0/l_1$, for $l_1 > 0$, and

$$n(v; \delta) = \begin{cases} (1 - e^{-\delta v}) / \delta, & \delta \neq 0, \\ v, & \delta = 0, \end{cases} \quad \text{(B.8)}$$

$$m(v; \delta) = \begin{cases} (v - n(v; \delta)) / \delta, & \delta \neq 0, \\ v^2/2, & \delta = 0. \end{cases} \quad \text{(B.9)}$$

I note that both $n(v; \delta) > 0$ and $m(v; \delta) > 0$ for $v > 0$. Equation (B.7) implies that $z$ is a submartingale.

Using (B.3), (13), (14), and (A.7) and applying Ito’s formula give

$$dM_t = - r M_t dt - \eta_t M_t dW_t, \quad \text{(B.10)}$$
where $\eta_t$ is given in (B.4). Integrating (B.10) gives

$$M_t = M_0 e^{-rt} \xi_t,$$

(B.11)

where

$$\xi_t = \exp \left( -\frac{1}{2} \int_0^t \eta_s^2 ds - \int_0^t \eta_s dW_s \right)$$

(B.12)

is a stochastic exponential, also known as the Doléans-Dade exponential associated with $\eta$ given in (B.4) (Protter (1990)). I assume that $\xi$, which is a local martingale, is actually a martingale, a technical regularity condition (Duffie et al. (2000)).

By Girsanov’s Theorem, (B.1) implies that $S_t$, a risk-adjusted measure of wealth, can be computed in the same way as human wealth, but rather under a risk-adjusted probability measure $Q$, whose density process $\xi$ is given by (B.12) (Duffie (2001)). That is,

$$S_t = E^Q_t \left( \int_t^\infty e^{-r(u-t)} y_u du \right).$$

(B.13)

By Girsanov’s Theorem (Karatzas and Shreve (1991)), the income process (1) under $Q$ is given by

$$dy_t = (\theta^Q - \kappa^Q y_t) dt + \sigma(y_t) dW^Q_t,$$

(B.14)

where $W^Q$ is a standard Brownian motion under $Q$, and

$$\theta^Q = \theta - ul_0,$$

(B.15)

$$\kappa^Q = \kappa + ul_1 \geq \kappa,$$

(B.16)

and where $u$ is given in (B.5). From (B.14) and (B.13),

$$S_t = \frac{1}{r + \kappa^Q} \left( y_t + \frac{\theta^Q}{r} \right).$$

(B.17)

It is an algebraic manipulation to show that the optimal consumption is given by

$$c^*_t = r \left[ x_t + \left( 1 + \frac{1}{2} \Delta_1 a_h^2 \right) S_t + \frac{1}{r} \left( \frac{1}{2} \Delta_0 a_h^2 + \beta - r \right) \right].$$

(B.18)

Alternatively, by using the risk-adjusted index of income\(^{21}\)

$$h^R_t = \frac{1}{r + \kappa^R} \left( y_t + \frac{\theta^R}{r} \right),$$

(B.19)

\(^{21}\)This particular form of the income index arises from another application of Girsanov’s Theorem. Details are available upon request.
where
\[
\begin{align*}
\theta^R &= \theta - \frac{1}{2} ul_0, \\
\kappa^R &= \kappa + \frac{1}{2} ul_1 \geq \kappa,
\end{align*}
\] (B.20)

we can re-express the optimal consumption rate (9) in a certainty-equivalent form
\[
c_t^* = r \left( x_t + h_t^R + \frac{\beta - r}{\gamma \tau^2} \right). 
\] (B.21)

We note that \( h_t > h_t^R > S_t \) for all \( t \).
References


