Dynamic Investment, Capital Structure, and Debt Overhang*

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Abstract

We develop a dynamic contingent-claim framework to model the idea of Myers (1977) that a firm is a collection of growth options and assets in place. The firm’s composition between assets in place and growth options evolves endogenously with its investment opportunity set and its financing of growth options as well as its dynamic leverage and default decisions. The firm trades off tax benefits with the potential financial distress and endogenous debt overhang costs over its life cycle. Unlike the standard real options and dynamic capital structure models (McDonald and Siegel (1986) and Leland (1994)), our model shows that financing and anticipated endogenous default decisions have significant implications of firms’ growth option exercising decisions and leverage policies. The firm’s ability to use risky debt to borrow against its assets in place and growth options substantially influences its investment strategies and its value. Quantitatively, we find that the firm consistently chooses conservative leverage in line with empirical evidence documented in Graham (2000) in order to mitigate the debt overhang effect on the exercising decisions for future growth options.

Keywords: real options, endogenous default, leverage, growth options, capital structure, contingent claims analysis

JEL Classification: E2, G1, G3

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1 Introduction

Models of truly inter-temporal investments with irreversibility and models of dynamic financing with endogenous defaults have proceeded relatively independent of each other. The literature on inter-temporal investments with multiple rounds of investments often ignores the financing flexibility possessed by the firms. On the other hand, models of dynamic financing have tended to ignore the investment opportunity set to a point whereby proceeds from each round of new financing are paid out to equity holders. While considerable insights have emerged from each strand of literature, much remains to be done in integrating investment theory with dynamic financing. While much work has been done recently, our paper builds on the insights of both the real options and contingent claims/credit risk literatures with the objective of showing growth options exercising is very important for corporate leverage in a parsimonious and tractable way. By using our dynamic model of investment and financing, we show that a rational firm significantly lowers its leverage anticipating its future growth option exercising decisions. Our numerical exercise generates empirically observed low leverage once we incorporate multiple rounds of growth options indicating the important interaction of growth option exercising, leverage, and valuation.

Integration of multiple rounds of investments with multiple rounds of financing presents many modeling challenges: first, the firm must solve endogenously the upper threshold of its value when it must undertake new investments. In making this decision the firm must take into account the level of debt it must use to optimally trade-off the expected tax benefits with the possibility of premature termination of the firm when default occurs, taking into account all future investment and financing possibilities. The optimal default decision constitutes a lower threshold level of the value of the firm, which must also be decided endogenously. This paper provides an analytically tractable framework to analyze dynamic endogenous corporate
We provide a tractable model of real options in which the firm makes these endogenous lower (default) and upper (investment) decisions over time, while choosing its optimal debt level along the way. In so doing, we provide a methodological framework for assessing the how the life-cycle of the firm may influence the manner in which it makes inter-temporal investment, and financing decisions. Broadly speaking, we use the term "financing" to encompass both the level of debt as well as the optimal default decisions.

Several additional new insights emerge from our analysis. In thinking about inter-temporal investments and financing, we start with the intuitive premise that the firm starts its life as a collection of growth options, much as in Myers (1977). For simplicity, we assume that the collection of growth options possessed by the firm is known and does not change over time. Then, as the firm moves through time, it optimally decides when to exercise each growth option and how to finance each growth option, keeping in mind that several additional growth options may be available to the firm in future. When the firm has exercised all its growth options, it is left only with assets in place. On the other hand, it starts its life with no assets in place. At all other times, it has some future growth options and some assets in place. The composition of growth options and assets are endogenously determined in a dynamic optimizing framework. Thus, our model captures the life cycle of the firm in a natural way.

There is an important economic distinction between assets in place and growth option in terms of what fraction of each is available to residual claimants upon default. It is reasonable to argue that assets in place are “hard assets,” whose values are verifiable and hence may provide greater liquidation value upon default than growth options, which may

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have embedded human capital and hence may possess a different (and possibly much lower) liquidation value along the lines of Hart and Moore (1994). We explicitly incorporate the potential differences of recovery values for growth options and assets in place. The modeling of this difference is a new contribution to real options literature as well, as it requires the values of foregone growth options upon default (which are non-linear functions of primitive states) to bear on optimal exercise boundaries. Indeed, we provide explicit quantitative and qualitative characterization of the impact of embedded human capital in future growth options on optimal investment thresholds, default thresholds, and the level of debt used by the firm at each stage of its life cycle. We believe that this is a unique contribution of our paper.

Our paper provides a natural bridge between structural credit risk/capital structure models, and the dynamic irreversible investment theory. We find that even for firms with only one growth option, integrating investment and financing decisions generates new insights, not captured by either the standard real options models (e.g., McDonald and Siegel (1986)), or credit risk/capital structure models (e.g., Leland (1994)). For example, Leland (1994) shows that the default threshold decreases in volatility for the standard (put) option argument in a contingent claim framework based on the standard trade-off theory of Modigliani and Miller (1963). However, the default threshold in our model may either decrease or increase in volatility. The intuition is as follows: (i) a higher volatility raises the investment threshold in our model for the standard (call option) value of waiting argument; (ii) a higher investment threshold naturally leads to a greater amount of debt issuance. That is, the firm issues more liquidation value along the lines of Hart and Moore (1994). We explicitly incorporate the potential differences of recovery values for growth options and assets in place. The modeling of this difference is a new contribution to real options literature as well, as it requires the values of foregone growth options upon default (which are non-linear functions of primitive states) to bear on optimal exercise boundaries. Indeed, we provide explicit quantitative and qualitative characterization of the impact of embedded human capital in future growth options on optimal investment thresholds, default thresholds, and the level of debt used by the firm at each stage of its life cycle. We believe that this is a unique contribution of our paper.

debt (but at a later time), when volatility is higher. Larger debt issuance raises the default threshold, *ceteris paribus*. As a result, unlike Leland (1994), we have two opposing effects of volatility on the default threshold due to endogenous investment in our model.

In developing our analysis, we have made the analytically convenient assumption that the firm uses its financing flexibility only at times when it makes its optimal investments. At a first glance, the reader may think that this is a strong assumption. But it turns out that this assumption proves to be innocuous for the following reasons: first, when growth options are economically meaningful, investments occur over time at frequent (but stochastic) intervals. Hence the real cost of the assumption is rather slight. In addition, it is well known (Strebulaev (2007)) that even the introduction of low costs of financing leads to the well known result that firms will choose to adjust their capital structure at periodic intervals rather than continuously. Dudley (2008) shows that when there are fixed costs of adjustment, it is optimal for firms to synchronize capital structure adjustment with the financing of large investment projects. In our model, the primary reason for financing is investment, and investments require a lump sum cost. Hence, it is natural to model financing adjustments when investments occur. Moreover, the key focus of our paper is on the impact of financing on growth option exercising decisions.

In addition, our paper provides several additional insights on the valuation of equity and credit spreads at different stages in the life cycle of the firm. We have a natural benchmark to assess of our results here: after all the growth options are optimally exercised, the firm is left with only assets in place. At this final stage our results are exactly the same as either Leland (1994) (when no dynamic financing adjustments are allowed) or Goldstein, Ju and Leland (2001) (when dynamic financing adjustments are allowed). At all previous stages, the firm has a mixture of assets in place and growth options and they influence both equity valuation and credit spreads. The key insight is that the incremental financial flexibility at
times other than actual investments is less important when there are growth options.

**Related literature.** Recently, there is a growing body of literature that extends Leland (1994) to allow for strategic debt service\(^3\) and dynamic capital structure decisions. Fischer, Heinkel, and Zechner (1989), Goldstein, Ju, and Leland (2001), and Strebulaev (2007) formulate dynamic leverage decisions with exogenously specified investment policies.\(^4\) Leary and Roberts (2005) empirically find that firms rebalance their capital structure infrequently in the presence of adjustment costs. Following Leland (1994), most contingent claims models of credit risk/capital structure assume that the firm’s cash flows are exogenously given and focus on the firm’s financing and default decisions.\(^5\) Unlike these work, our model endogenizes growth option exercising decisions and induces dynamic leverage decisions via motives of financing investment.\(^6\) Titman and Tsyplakov (2007) also build a model that allows for dynamic adjustment of both investment and capital structure. Their model is based on continuous investment decisions, while our model focuses on the irreversibility of growth option exercising.\(^7\) We solve the model in closed form (up to coupled nonlinear equations), while Titman and Tsyplakov (2007) have three state variables and numerically solve the decision rules. Ju and Ou-Yang (2006) show that the firm’s incentive to increase firm risk *ex post* is mitigated if the firm wants to issue debt periodically. In the interest of parsimony, we

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\(^4\)Early important contributions towards building dynamic capital structure models include Kane, Marcus, and McDonald (1984, 1985).

\(^5\)Leland (1998) extends Leland (1994) by incorporating risk management with capital structure, and also allows the firm to engage in asset substitution by selecting volatility of the project.

\(^6\)Our model ties the investment and financing adjustments to occur at the same time. This assumption is made for analytical convenience. We leave extensions to allow for separate adjustments of investment and financing for future research.

\(^7\)Brennan and Schwartz (1984) is an early important contribution, which allows for the interaction between investment and financing.

2 Model

We first set up a dynamic formulation where the firm is a collection of growth options and assets in place. Assume that the firm behaves in the interests of existing equityholders at each point in time. At time 0, the firm starts with no assets in place, and knows that it has $N$ growth options. These growth options can only be exercised sequentially. One way to view these growth options is the discretized decisions for capital accumulation decisions.

The firm observes the demand shock $X$ for its product, where $X$ is given by the following geometric Brownian motion (GBM) process:

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t),$$

and $W$ is a standard Brownian motion. Equivalently, we may also interpret $X$ as the (stochastic) price process for the firm’s output. The risk-free interest rate $r$ is constant. For convergence, assume that the (risk-neutral) expected growth rate $\mu$ is lower than the

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9We ignore the conflicts of interests between managers and investors and leave for future research.

10One could certainly visualize growth options arriving with some intensity at random times in the future. In such an economy, the optimal investment decisions would reflect the arrival intensity in addition to the factors that we consider in our formulation. Extension of random arrivals of growth options is clearly an interesting topic for future research.

11Let $W$ be a standard Brownian motion in $\mathbb{R}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ and fix the standard filtration $\{\mathcal{F}_t : t \geq 0\}$ of $W$. Since all securities are traded here, we directly work under the risk-neutral probability measure $\mathbb{Q}$. Under the infinite horizon, additional technical conditions such as uniform integrability are assumed here. See Duffie (2001).

12In our model as in many other investment and capital structure models, the process $X$ captures both demand and productivity shocks.
interest rate, in that \( r > \mu \). Assume no production cost after the asset is in place.\(^{13}\) When the firm exercises its \( n \)-th growth option, it creates the \( n \)-th asset in place, which generates profit at the rate of \( m_n X \), where \( m_n > 0 \) is a constant. We may interpret \( m_n \) as the production capacity, or equivalently the constant rate of output produced by the firm’s \( n \)-th asset in place. Let the firm’s total profit rate from its first \( n \) assets in place be \( M_n X \), where 
\[
M_n = \sum_{j=1}^{n} m_j.
\]

Let \( T_n \) denote the endogenously chosen time at which the firm exercises its \( n \)-th growth option, where \( 1 \leq n \leq N \). Let \( I_n \) denote the fixed cost of exercising its \( n \)-th growth option. These exercising costs \( I_n \) are constant and known at time 0. At each endogenously chosen (stochastic) investment time \( T_n \), the firm issues a mixture of debt and equity to finance the exercising cost \( I_n \). As in standard trade-off models of capital structure, debt has a tax advantage. The firm faces a constant tax rate \( \tau > 0 \) on its income after servicing interest payments on debt. To balance the tax benefits, debt induces deadweight losses when the firm does poorly. The firm dynamically trades off the benefits and costs of issuing debt. For analytical convenience, assume that debt is perpetual and is issued at par. The assumption of perpetual debt simplifies the analysis substantially and has been widely adopted in the literature.\(^{14}\) Note that we have assumed that the firm can only issue debt at investment times \( \{T_n : 1 \leq n \leq N\} \). At a first glance, this may appear to be a strong assumption. In fact, our assumption is actually rather mild. Strebulaev (2007) has shown that in the presence of frictions firms adjust their capital structure only infrequently. Therefore, in a dynamic economy that we model, the leverage of the firm is likely to differ from the “optimum” leverage predicted by models that permit costless adjustment of leverage. Given this finding

\(^{13}\)Our model ignores operating leverage. We may extend our model to allow for operating leverage by specifying the firm’s profit from its \( n \)-th asset in place as \( m_n X - w_n \), where \( w_n \) is the operating cost for the \( n \)-th asset in place.

\(^{14}\)We may extend the model by allowing for a finite average maturity for debt as in Leland (1994b) at the cost of additional modeling complexity.
it is more natural to recapitalize when optimal investment decisions are warranted. In addition, models that permit re-leveraging in good times, implicitly or explicitly use the debt proceeds to pay dividends, which is at odds with the basic provision that senior claims (such as debt) may not be issued to finance junior claims (such as equity). Despite these strong grounds for not modeling “costless” dynamic financing, we treat this problem explicitly in a later section and show that our results are robust with respect to this extension.

Let $C_n$ and $F_n$ denote the coupon rate and the par value of the perpetual debt issued to finance the exercising of the $n$-th growth option at $T^i_n$. Let $T^d_n$ denote the endogenously chosen stochastic default time after the firm exercises $n$ growth options, but before exercising the $(n+1)$-th growth option, where $1 \leq n \leq N$. When exercising the new growth option at $T^i_{n+1}$, the firm calls back its outstanding debt with par $F_n$ and coupon $C_n$, and issues new debt with par $F_{n+1}$ and coupon $C_{n+1}$. That is, at each point in time, there is only one class of debt outstanding.\[15\]

Figure 1 describes the decision making process of the firm over its life cycle. The firm has $(N + 1)$ stages. In stage 0, the firm has no assets in place. We assume that the initial value of the demand shock is sufficiently low such that the firm always starts with waiting, the economically most interesting case. If the demand shock $\{X(t) : t \geq 0\}$ rises sufficiently high (i.e. greater or equal to an endogenous threshold $X^{i1}$ to be determined in Section 3) at the stochastic (endogenous) time $T^{i1}$, the firm exercises its first growth option by paying a one-time fixed cost $I_1$ at time $T^i_1$ as in McDonald and Siegel (1986). Note that since $X(0)$ is sufficiently low, we have $T^{i1} > 0$. Notation-wise, we use $X^{i1} = X(T^{i1})$. To finance the first growth option exercising cost $I_1$, the firm issues a mixture of equity and perpetual debt. This completes the description of the firm’s decision in its initial stage (stage 0). Next turn

\[15\]See Sundaresan and Wang (2006) and Hackbart and Mauer (2012) for analysis where more than one class of debt are outstanding. The design of priority structure of debt and its implications for real options exercise is a topic worthy of further research.
The firm’s decision-making process over its life cycle. The firm starts with $N$ sequentially ordered growth options. We divide its decision making over its life cycle into $(N + 1)$ stages. In stage 0, the firm exercises its first growth option when the stochastic process $X$ given in (1) rises sufficiently high (i.e. $X \geq X^i_1 = X(T^i_1)$). The firm waits otherwise. When exercising, the firm issues a mixture of equity and the first perpetual debt with coupon $C_1$ to finance the exercising cost $I_1$. This completes the description of the firm’s stage 0 decision. Now move to stage 1. Provided that $X^d_1 < X < X^i_2$, the firm generates cash flow $m_1X$ from its operation. If its cash flow drops sufficiently low, (i.e. $X \leq X^d_1$), the firm defaults. If the cash flow rises sufficiently high (i.e. $X \geq X^i_2$), the firm exercises its second growth option, and issues a mixture of equity and the second perpetual debt with coupon $C_2$ to finance the exercising cost $I_2$. After the second growth option is exercised, the firm generates stochastic cash flow $(m_1 + m_2)X$, provided that $X \geq X^d_2$. This process continues. If the firm reaches the final stage $N$, the firm has total $N$ assets in place and collects total cash flow $M_NX$, where $M_N = \sum_{n=1}^{N} m_n$. The decision variables include $N$ investment thresholds $X^i_n$, $N$ default thresholds $X^d_n$, and $N$ coupon policies $C_n$, where $n = 1, 2, \cdots, N$. Notation-wise, we define the $n$-th stage as $X$ such that $X^d_n \leq X < X^i_{n+1}$ where $X^i_{n+1} = X(T^i_{n+1})$ and $X^d_n = X(T^d_n)$.
to stage 1.

After the first asset is in place, the firm collects profit flow $m_1 X$ until it decides to either default on its debt or exercise its (second) growth option. If the firm defaults before exercising the second growth option ($T^d_1 < T^i_2$), it ceases to exist. All proceeds from liquidation go to creditors. However, liquidation is inefficient since it induces value losses from both the existing assets in place and foregone growth options. We will specify the liquidation payoff in the next paragraph when discussing the firm’s general stage $n$ problem. Intuitively, if the demand shock $X$ is sufficiently high, then it is optimal for the firm to exercise its second growth option. By incurring the fixed investment cost $I_2$, the firm exercises its second growth option at endogenously chosen time $T^i_2$. At the second investment time $T^i_2$, the firm calls back its outstanding debt with par $F_1$, and issues a mixture of equity and the new perpetual debt with par $F_2$ to finance the second growth option exercising cost $I_2$. This concludes the firm’s decision in stage 1.

It is straightforward to describe the firm’s stage-$n$ decision problem. After immediately exercising the $n$-th growth option, the firm operates its existing $n$ assets in place until the demand shock $X$ either rises sufficiently high, which triggers the firm to call back debt with par $F_n$, issue a mixture of new perpetual debt and equity to finance $I_{n+1}$ to exercise the $(n+1)$-th growth option, or the demand shock $X$ drops sufficiently low, which leads the firm to default on its outstanding debt with par $F_n$.

Let $A_n(X)$ denote the after-tax present value of all $n$ existing assets in place (under all equity financing), in that

$$A_n(X) = \left(\frac{1 - \tau}{r - \mu}\right) M_n X, \quad 1 \leq n \leq N,$$

(2)

where $M_n$ is the production capacity for all existing $n$ assets in place and is given by

$$M_n = \sum_{j=1}^{n} m_j, \quad 1 \leq n \leq N.$$

(3)
When equityholders default on debt, the firm is liquidated. Let $L_n(X)$ denote the proceeds from liquidation in stage $n$. Liquidation proceeds $L_n(X)$ has two components: one from the existing assets in place and the other from foregone growth options. Following Leland (1994), we assume that the firm uncovers $(1 - \gamma_A)$ fraction of the present value $A_n(X)$ from existing $n$ assets in place. Unlike Leland (1994), our model has growth options. While growth options may be less tangible, they still have scrap value upon liquidation. We calculate the liquidation value for unexercised growth options in an analogous way as we do for the liquidation value from existing assets in place. That is, we assume that the firm collects $(1 - \gamma_G)$ fraction of the present value of unexercised foregone growth options. We use the workhorse real option model to assess values for unexercised growth options, as if these options were stand-alone and financed solely by equity. Let $G_k(X)$ denote the present value of a stand-alone growth option (with exercise cost $I_k$ and cash flow multiplier $m_k > 0$) under all equity financing.

The following lemma summarizes the main results (McDonald and Siegel (1986)).

**Lemma 1** Consider an all-equity financed firm with a single growth option. The firm may exercise its stand-alone growth option by paying a one-time fixed cost $I_k$ and then generate a perpetual stream of after-tax stochastic cash flow $(1 - \tau) m_k X$, where $m_k > 0$ is a constant and the stochastic process $X$ is given by (4). The firm (option) value is given by

$$G_k(X) = \left( \frac{X}{X_k^{ae}} \right)^{\beta_1} \left[ \left( \frac{1 - \tau}{r - \mu} \right) m_k X_k^{ae} - I_k \right], \quad X < X_k^{ae},$$

(4)

where $X_k^{ae}$ is the optimal growth option exercising threshold and is given by

$$X_k^{ae} = \frac{r - \mu}{1 - \tau \beta_1 - 1 m_k} I_k,$$

(5)

and $\beta_1$ is the (positive) option parameter and is given by

$$\beta_1 = \frac{1}{\sigma^2} \left[ - \left( \mu - \frac{\sigma^2}{2} \right) + \sqrt{\left( \mu - \frac{\sigma^2}{2} \right)^2 + 2 r \sigma^2} \right] > 1.$$ 

(6)
The firm’s liquidation value in stage $n$, $L_n(X)$, is then given by

$$L_n(X) = (1 - \gamma_A) A_n(X) + (1 - \gamma_G) \sum_{k=n+1}^{N} G_k(X),$$

(7)

where $A_n(X)$ given in (2) is the after-tax present value of the existing $n$ assets in place, and $G_k(X)$ given in (4) is the after-tax present value of $k$-th unexercised growth options. The specification of our liquidation payoff is reasonably general and also intuitive. We allow for different loss rates $\gamma_A$ and $\gamma_G$ for assets in place and growth options, respectively. For example, the growth options may reflect the embedded human capital of current owners, which the new owners may not be able to replicate after liquidation. This may suggest that $\gamma_G$ is greater than $\gamma_A$, although our model specification does not require this condition. In addition to being realistic, the specification for liquidation payoffs is also quite tractable and we have closed-form solutions for both liquidation values of assets in place and of growth options, as shown above. Finally, we tie liquidation values for assets in place and growth options to their respective stand-alone values under all equity financing.

Having detailed the firm’s decision making in stage $n$, we introduce a few value functions, and leave the formal mathematical definition of these value functions to the appendix. Let $E_n(X)$ denote equity value in stage $n$, (i.e. the present discounted value of all future cash flows accruing to the existing equityholders after servicing debt and paying taxes). While equity value $E_n(X)$ does not internalize the benefits and costs of debt in stage $n$, it does include the tax benefits and distress costs of debt in future stages. Let $D_n(X)$ denote debt value in stage $n$. Recall that debt coupon $C_n$ is serviced until debt is either called back at par $F_n$ at investment time $T_{n+1}^i$, or is defaulted at $T_{n}^d$. At default, creditors collect $L_n(X(T_{n}^d))$ given in (7), which is less than $C_n/r$ in equilibrium.\footnote{Since default is endogenous, equity holders will never default when liquidation value exceeds the risk-free value of debt.} Let $V_n(X)$ denote stage-$n$ firm value, which is the sum of equity and debt values, in that $V_n(X) = E_n(X) + D_n(X)$. \footnote{Since default is endogenous, equity holders will never default when liquidation value exceeds the risk-free value of debt.}
The firm follows the sketched decision making process during each stage of its life cycle until it defaults. If the firm survives to exercise its last growth option (i.e. \( t \geq T_i^N \)), then the firm has exercised all its growth options. The firm then collects \( M_NX \) in profit flow from its \( N \) assets in place, servicing debt payment \( C_N \) and paying taxes, until profit drops sufficiently low, which triggers the firm to default on its outstanding debt with par \( F_N \) at time \( T_{N}^{d} \). The last stage default decision problem for the “mature” firm is the one analyzed in Leland (1994).

Having described the decision-making process over the life-cycle of the firm, we next solve the model using backward induction.

3 Solution

We solve our model in four steps. The first three steps take debt coupon levels \( \{C_n : 1 \leq n \leq N\} \) and liquidation payoff \( \{L_n(X) : 1 \leq n \leq N\} \) in each stage \( n \) as given, and analyze the firm’s growth option and default option exercising decisions. To be specific, first, we study the default decision in stage \( N \), (no investment decision in the last stage). This is effectively the classic capital structure/default problem treated in Leland (1994). Second, we characterize the firm’s optimal growth option and default option exercising decisions when the firm is in the intermediate stages of its life cycle, (stage 1 to stage \( (N - 1) \)). Third, we analyze the firm’s initial growth option exercising decision (no default decision in stage 0). After solving the investment and default decisions, we provide optimality conditions for the firm’s financing policies \( \{C_n : 1 \leq n \leq N\} \) over its life cycle (stage 0 to stage \( (N - 1) \)).

3.1 The final stage (stage \( N \)) in the firm’s life cycle

In the final stage, the firm has exercised all its growth options and hence operates \( N \) assets in place, which generate the profit flow at the rate of \( M_NX \) with \( M_N = \sum_{k=1}^{N} m_k \).
being the total production capacity. Using the standard valuation argument, we may value equity \( E_N(X) \) using the following ordinary differential equation (ODE):

\[
    rE_N(X) = (1 - \tau) (M_N X - C_N) + \mu X E'_N(X) + \frac{\sigma^2}{2} X^2 E''_N(X), \quad X \geq X^d_N, \tag{8}
\]

subject to the following conditions at the endogenously chosen default boundary \( X^d_N \):

\[
    E_N(X^d_N) = 0, \tag{9}
\]

\[
    E'_N(X^d_N) = 0. \tag{10}
\]

The value-matching (9) states that equity value is zero when equityholders default. The smooth-pasting (10) implies that equityholders optimally choose the default boundary \( X^d_N \). Moreover, the default option is completely out of money when \( X \) approaches \( \infty \).

Leland (1994) shows that equity value \( E_N(X) \) is given by

\[
    E_N(X) = \left( A_N(X) - \frac{(1 - \tau) C_N}{r} \right) + \left[ \frac{(1 - \tau) C_N}{r} - A_N(X^d_N) \right] \left( \frac{X}{X^d_N} \right)^{\beta_2}, \quad X \geq X^d_N, \tag{11}
\]

where \( A_N(X) \) given in (2) is the after-tax present value of \( N \) existing assets in place, and the optimal default threshold \( X^d_N \) for a given coupon \( C_N \) is given by

\[
    X^d_N = \frac{r - \mu}{M_N} \frac{\beta_2}{\beta_2 - 1} \frac{C_N}{r}, \tag{12}
\]

and \( \beta_2 \) is given by

\[
    \beta_2 = -\frac{1}{\sigma^2} \left[ \left( \frac{\mu - \sigma^2}{2} \right) + \sqrt{\left( \frac{\mu - \sigma^2}{2} \right)^2 + 2r\sigma^2} \right] < 0. \tag{13}
\]

Equity value \( E_N(X) \) is given by the after-tax present value of all \( N \) assets in place, \( A_N(X) \), minus the after-tax present value of the risk-free perpetual debt \((1 - \tau) C_N/r\), plus the option value of default, the last term in (11). The standard option value argument implies that the default threshold \( X^d_N \) decreases with volatility \( \sigma \), and equity value \( E_N(X) \) is convex in \( X \). Naturally, when \( X \leq X^d_N \), equity is worthless, i.e., \( E_N(X) = 0 \).
Similarly, using the standard valuation argument, we may value debt \( D_N(X) \) using the following ODE:

\[
r D_N(X) = C_N + \mu X D'_N(X) + \frac{\sigma^2}{2} X^2 D''_N(X), \quad X_N^d \leq X,
\]

subject to the following conditions:

\[
D_N(X_N^d) = L_N(X_N^d), \quad X \to \infty, \quad D_N(X) = \frac{C_N}{r},
\]

So we have the market value of debt \( D_N(X) \) is given by

\[
D_N(X) = \frac{C_N}{r} - \left[ \frac{C_N}{r} - L_N(X_N^d) \right] \left( \frac{X}{X_N^d} \right)^{\beta_2}, \quad X \geq X_N^d,
\]

where the second term captures the default risk. In stage \( N \), the firm only has assets in place, and therefore, \( L_N(X_N^d) = (1 - \gamma_A) A_N(X_N^d) \). Note that \( D_N(X) \) is concave in \( X \) because the creditor is short a default option. The second term in (17) measures the discount on debt due to the risk of default, which has two components: the loss given default \( (C_N/r - L_N(X_N^d)) \) for the creditor, and \( (X/X_N^d)^{\beta_2} \), the present discounted value for a unit payoff when the firm hits the default boundary \( X_N^d \). Firm value \( V_N(X) = E_N(X) + D_N(X) \) is given by

\[
V_N(X) = A_N(X) + \frac{\tau C_N}{r} - \left[ \gamma_A A_N(X_N^d) + \frac{\tau C_N}{r} \right] \left( \frac{X}{X_N^d} \right)^{\beta_2}, \quad X \geq X_N^d.
\]

Firm value \( V_N(X) \) is given by the after-tax value of the \( N \) assets in place \( A_N(X) \), plus the perpetuity value of the risk-free tax shield \( \tau C_N/r \), minus the cost of liquidation. Importantly, firm value \( V_N(X) \) is concave in \( X \), because the firm as a whole is short in a liquidation option. Intuitively, after \( T_N \), the firm is long in the \( N \) assets in place and the risk-free tax shield perpetuity \( \tau C_N/r \), and short in the liquidation option. Upon liquidation, the firm as a whole loses \( \gamma_A \) fraction of assets in place value \( A_N(X_N^d) \) and also the perpetual value of tax shields, the sum of the two terms in the square bracket in (18).
3.2 Intermediate stages (stage \((N-1)\) to stage 1)

Having analyzed the firm’s optimization problem in stage \(N\), we now use backward induction to analyze the firm’s decision problem in stage \((N-1)\). As Figure 1 indicates, the key decisions are \(i\) the \(N\)-th growth option exercising and \(ii\) the default decision on the existing debt with par \(F_{N-1}\). For generality, we solve the firm’s decision problem for its intermediate stage \(n\), including stage \((N-1)\) as a special case.

**Equityholders’ decisions and equity pricing.** For given investment threshold \(X_{n+1}^i\) and default threshold \(X_n^d\) in stage \(n\), equity value \(E_n(X)\) solves the following ODE:

\[
r E_n(X) = \left(1 - \tau\right) \left(M_n X - C_n\right) + \mu X E'_n(X) + \frac{\sigma^2}{2} X^2 E''_n(X), \quad X_n^d \leq X \leq X_{n+1}^i. \tag{19}\]

Now consider boundary conditions for investment. When exercising the \((n+1)\)-th growth option, equityholders are required to call back the old debt at the par value \(F_n\). Importantly, we will determine the value of \(F_n\) as part of the model solution that depends on the firm’s endogenous investment, default and coupon decisions.

Note that since the firm has to call back the debt at its par, the total cost of exercising the \((n+1)\)-th growth option is given by \((I_{n+1} + F_n)\), the sum of investment cost \(I_{n+1}\) and the face value of the debt \(F_n\). And part of this exercising cost is financed by new debt, which has market value \(D_{n+1}(X_{n+1}^i)\) at issuance time \(T_{n+1}^i\). The remaining part \((I_{n+1} + F_n - D_{n+1}(X_{n+1}^i))\) is financed by equity. Therefore, the net payoff to equityholders right after exercising is \(E_{n+1}(X_{n+1}^i) - (I_{n+1} + F_n - D_{n+1}(X_{n+1}^i))\). The value matching condition for the threshold \(X_{n+1}^i\) is then given by

\[
E_n(X_{n+1}^i) = V_{n+1}(X_{n+1}^i) - (I_{n+1} + F_n), \tag{20}\]

where \(V_{n+1}(X) = E_{n+1}(X) + D_{n+1}(X)\) is firm value in stage \((n+1)\). Since equityholders
optimally choose the threshold $X_{n+1}^i$, the following smooth pasting condition holds:

$$E'_n(X_{n+1}^i) = V'_{n+1}(X_{n+1}^i).$$

(21)

Now turn to the default boundary conditions. Using the same arguments as those for equity value $E_N(X)$ in the last stage, equityholders choose the default threshold $X_n^d$ to satisfy the value-matching condition $E_n(X_n^d) = 0$ and the smooth pasting condition $E'_n(X_n^d) = 0$.

Unlike the decision problem in the last stage, we now have a double (endogenous) barrier option exercising problem where the upper boundary is primarily about the real option exercising decision as in McDonald and Siegel (1986), and the lower boundary is effectively the financial default option decision as in Leland (1994). Of course, the upper (investment) and the lower (default) boundaries are interconnected. This is precisely how the investment and default decisions affect each other. Next, we formally characterize this interaction between investment and default decisions.

Let $\Phi^i_n(X)$ denote the present discounted value of receiving a unit payoff at $T_{n+1}^i$ if the firm invests at $T_{n+1}^i$, namely, $T_{n+1}^i < T_n^d$. Similarly, let $\Phi^d_n(X)$ denote the present discounted value of receiving a unit payoff at $T_n^d$ if the firm defaults at $T_n^d$, namely $T_n^d < T_{n+1}^i$. The closed-form expressions for $\Phi^i_n(X)$ and $\Phi^d_n(X)$ are given by

$$\Phi^i_n(X) = \mathbb{E}_t \left[ e^{-r(T_{n+1}^i-t)} 1_{T_n^d > T_{n+1}^i} \right] = \frac{1}{\Delta_n} \left[ (X_n^d)^{\beta_2} X^{\beta_1} - (X_n^d)^{\beta_1} X^{\beta_2} \right],$$

(22)

and

$$\Phi^d_n(X) = \mathbb{E}_t \left[ e^{-r(T_{n+1}^d-t)} 1_{T_n^d < T_{n+1}^i} \right] = \frac{1}{\Delta_n} \left[ (X_{n+1}^i)^{\beta_1} X^{\beta_2} - (X_{n+1}^i)^{\beta_2} X^{\beta_1} \right],$$

(23)

and

$$\Delta_n = (X_n^d)^{\beta_2}(X_{n+1}^i)^{\beta_1} - (X_n^d)^{\beta_1}(X_{n+1}^i)^{\beta_2} > 0.$$  

(24)

Using these formulae, we may write equity value $E_n(X)$ as follows:

$$E_n(X) = A_n(X) - \frac{(1 - \tau)}{r} C_n + c_n^i \Phi^i_n(X) + c_n^d \Phi^d_n(X), \quad X_n^d \leq X \leq X_{n+1}^i.$$  

(25)
where

\[ e_i^n = V_{n+1}(X_{n+1}^i) - (I_{n+1} + F_n) - \left( A_n(X_{n+1}^i) - \frac{(1 - \tau) C_n}{r} \right) > 0, \quad (26) \]

\[ e_d^n = - \left[ A_n(X_n^d) - \frac{(1 - \tau) C_n}{r} \right] > 0. \quad (27) \]

Equity value \( E_n(X) \) is given by the after-tax present value of assets in place \( A_n(X) \) minus the after-tax perpetuity value of risk-free debt with coupon \( C_n \), (i.e. \( (1 - \tau) C_n/r \)) plus two option values: the (real) growth option and the (financial) default option. The third term in (25) measures the present value of the growth option, which is given by the product of \( \Phi_i^n(X) \), and the net payoff \( e_i^n \) from exercising the growth option. The net payoff \( e_i^n \) is the difference between the payoff from growth option exercise \( V_{n+1}(X_{n+1}^i) - (I_{n+1} + F_n) \) and \( (A_n(X_{n+1}^i) - (1 - \tau) C_n/r) \), the forgone un-levered equity value when investing at the threshold \( X_{n+1}^i \). Note that the forgone “un-levered” equity value appears as an additional cost term in the net payoff \( e_i^n \) because the option payoff \( V_{n+1}(X_{n+1}^i) - (I_{n+1} + F_n) \) contains cash flows from the existing assets in place. Similarly, the fourth term in (25) is the present value of the (financial) default option, which is given by the product of \( \Phi_d^n(X) \) and the net payoff \( e_d^n \) upon default. Since equityholders receive nothing at default, the net payoff \( e_d^n \) is given by the savings, \( - (A_n(X_n^d) - (1 - \tau) C_n/r) > 0 \), from avoiding the loss of running the “un-levered” equity value at the default threshold \( X_n^d \).

**Debt pricing.** For given investment threshold \( X_{n+1}^i \) and default threshold \( X_n^d \) in stage \( n \), debt value \( D_n(X) \) solves the following ODE:

\[ rD_n(X) = C_n + \mu XD_n'(X) + \frac{\sigma^2}{2} X^2 D_n''(X), \quad X_n^d \leq X \leq X_{n+1}^i, \quad (28) \]

subject the following boundary conditions:

\[ D_n(X_n^d) = L_n(X_n^d), \quad (29) \]

\[ D_n(X_{n+1}^i) = F_n. \quad (30) \]
In the Appendix, we show that debt value $D_n(X)$ in stage $n$ where $X_n^d \leq X \leq X_{n+1}^i$ is given by:

$$D_n(X) = \frac{C_n}{r} - \left[ \frac{\Phi^d_n(X_{n+1}^i)}{1 - \Phi^i_n(X_n^i)} \Phi^i_n(X) + \Phi^d_n(X) \right] \left( \frac{C_n}{r} - L_n(X_n^d) \right), \quad (31)$$

where $\Phi^i_n(X)$ and $\Phi^d_n(X)$ are given in (22) and (23), respectively. Creditors incur losses when the firm default, i.e. $C_n/r > L_n(X_n^d)$. The second term in (31) gives the value discount on debt due to the risk of default. We may obtain the par value $F_n$ of this debt by evaluating $D_n(X)$ at the investment threshold $X_n^i$.

Since debt is priced at par $F_n$ at issuance time $T_n^i$, we have the following valuation equation for the par value $F_n$:

$$F_n = \frac{C_n}{r} - \frac{\Phi^d_n(X_{n+1}^i)}{1 - \Phi^i_n(X_n^i)} \left( \frac{C_n}{r} - L_n(X_n^d) \right). \quad (32)$$

Default is costly in that $C_n/r > L_n(X_n^d)$. The second term in (32) gives the value discount of debt at issuance due to default risk.

**Firm valuation.** Now, we may calculate firm value $V_n(X)$ as the sum of debt value $D_n(X)$ and equity value $E_n(X)$, in that

$$V_n(X) = A_n(X) + \frac{\tau C_n}{r} + v^i_n \Phi^i_n(X) + v^d_n \Phi^d_n(X), \quad X_n^d \leq X \leq X_{n+1}^i, \quad (33)$$

where

$$v^i_n = V_{n+1}(X_{n+1}^i) - I_{n+1} - \left( A_n(X_{n+1}^i) + \frac{\tau C_n}{r} \right), \quad (34)$$

$$v^d_n = L_n(X_n^d) - \left( A_n(X_n^d) + \frac{\tau C_n}{r} \right). \quad (35)$$

Having described the details to solve for the default threshold $X_n^d$ and the investment threshold $X_{n+1}^i$ for stage $n \geq 1$, we now turn to the investment decision for the initial stage. Unlike the intermediate stages, the initial stage (stage 0) has no default decision, and hence simplifies the analysis.
3.3 The initial stage (stage 0) in the firm’s life cycle

As in standard real option models, equity value \( E_0(X) \) in stage 0 solves the following ODE:

\[
rE_0(X) = \mu X E'_0(X) + \frac{\sigma^2}{2} X^2 E''_0(X), \quad X \leq X^*_i,
\]

subject to the following boundary conditions

\[
E_0(X^*_i) = V_1(X^*_i) - I_1, \quad (37)
\]

\[
E'_0(X^*_i) = V'_1(X^*_i). \quad (38)
\]

The intuition behind the value matching condition (37) builds and extends the one in McDonald and Siegel (1986). Without any assets and liability, the firm raises \( D_1(X^*_i) \) in debt to partially finance the exercising cost \( I_1 \). Immediately after exercising the first growth option at the threshold \( X^*_i \), equityholders collect \( E_1(X^*_i) - (I_1 - D_1(X^*_i)) \) giving rise to the value matching condition (37). The smooth pasting condition (38) states that the investment threshold \( X^*_i \) is chosen optimally. Finally, equity value \( E_0(X) \) also satisfies the standard absorbing barrier condition at the origin, in that \( E_0(X) \to 0 \), when \( X \to 0 \).

Equity value \( E_0(X) \), the solution to the above optimization problem, is given by

\[
E_0(X) = \left( \frac{X}{X^*_i} \right)^{\beta_1} \left( V_1(X^*_i) - I_1 \right), \quad X \leq X^*_i, \quad (39)
\]

where \( \beta_1 \) is given by (6), and the investment threshold \( X^*_i \) solves the following implicit equation:

\[
X^*_i = \frac{1}{1 - \frac{r - \mu}{m_1} \frac{\beta_1}{\beta_1 - 1}} \left[ \left( I_1 - \frac{\tau C_1}{r} \right) + \frac{\beta_1 - \beta_2}{\beta_1 \Delta_1} \left( (X^*_i)^{\beta_1} v_1^i - (X^*_2)^{\beta_1} v_2^i \right) \right], \quad (40)
\]

and \( \Delta_1 \) is a strictly positive constant given in (24) with \( n = 1 \). Unlike in the standard equity-based real options models (e.g. McDonald and Siegel (1986)), the payoff from investment in our model is total firm value \( V_1(X) \), which includes the present values of cash flows from
both operations and financing. Note that equity value $E_0(X)$ is convex in $X$, a standard result in the real options literature.

Having analyzed the firm’s investment and default thresholds, we now analyze the firm’s dynamic financial (debt) policies, and summarize the firm’s integrated dynamic decision making over its life cycle.

### 3.4 Dynamic debt policies and a summary of the firm’s life cycle decisions

First, review the decision problem in stage $N$. The firm chooses its last default threshold $X^d_N$ as a function of coupon $C_N$ by maximizing equity value $E_N(X; C_N)$. The solution for $X^d_N$ as a function of $C_N$ is given by (12), a well known problem treated in Leland (1994). Then, equityholders choose $C_N$ to maximize $V_N(X)$ and then evaluate the first-order condition (FOC) for $C_N$ at $X = X^i_N$. Intuitively, equityholders internalize all benefits and costs of debt issuance at $T^i_{n+1}$ and pay fair market value $D_N(X^i_N) = F_N$ when choosing coupon $C_N$. Because firm value $V_N(X)$ given by (18) is known in closed form, we obtain the following explicit solution for $C_N$ in terms of $X^i_N$:

$$C_N = \frac{r}{r - \mu} \frac{\beta_2 - 1}{\beta_2} M_N X^i_N,$$

where $h$ is given by

$$h = \left[ 1 - \beta_2 \left( 1 - \gamma_A + \frac{\gamma_A}{\tau} \right) \right]^{-1/\beta_2} > 1.$$

Using formula (12) for $X^d_N$ for a given coupon $C_N$, we obtain the relationship between the last default threshold $X^d_N$ and the last growth option exercising: $X^d_N = X^i_N / h$.

Now consider stage $(N - 1)$. Equityholders choose the thresholds $X^i_N$ and $X^{d}_{N-1}$ to maximize equity value $E_{N-1}(X; C_{N-1})$, taking the default threshold $X^d_N$ in (12) and optimal

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17The optimality for $C_N$ and $X^i_N$ and the envelope condition jointly imply that we do not need to consider the feedback effects between the investment threshold $X^i_N$ and the coupon policy $C_N$. 

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coupon $C_N$ in (11) in stage $N$ as given. Since equityholders internalize the tax benefits from issuing debt at $T_{N-1}^i$, equityholders choose coupon $C_{N-1}$ to maximize $V_{N-1}(X)$ and then evaluate at $X_{N-1}^i$.

Next turn to stage $n$, where $1 \leq n < (N - 1)$. As in stage $(N - 1)$, the firm chooses thresholds $X_{n+1}^i$ and $X_n^d$ to maximize equity value $E_n(X; C_n)$, taking into account the firm’s future optimality conditions described earlier. Then, the firm chooses the optimal coupon policy $C_n$ to maximize $V_n(X)$ and then evaluate at $X_n^i$.

Finally, stage 0 is a special case of stage $n$. The firm chooses the first investment threshold $X_1^i$ to maximize equity value $E_0(X)$. Note that $X_0^d = 0$ (no debt and no default). We have shown that equity value $E_0(X)$ is given by (39) and the investment threshold is given by the implicit nonlinear equation (40).

Our model thus have predictions for the dynamics of leverage choice by the firm, and how the leverage dynamics relate to the life-cycle of the firm. Unlike most existing dynamic financing models, which ignore investments, our model explicitly incorporates investment frictions, which are potentially important. The leverage dynamics under investment frictions will reflect the importance of remaining future growth options, and the potential for premature liquidation from excessive leverage. We explore this tension later in the paper.

Having outlined the solution methodology for the general model specification, we next summarize the setting where the firm is all equity financed, i.e. McDonald and Siegel (1986) with multiple growth options and taxes.

### 3.5 All-equity-financing model

We now consider an all-equity setting with multiple rounds of growth options. Recall that Lemma 1 gives the growth option value $G_k(X)$ and the exercise threshold $X_k^*$ for a firm with

\[\text{Our characterization of leverage dynamics only requires us to solve a system of non-linear equations for investment and default thresholds, and coupon policies. This substantially simplifies our analysis, in that we have solved out the endogenous default and investment thresholds up to a set of nonlinear equations.}\]
a stand-alone growth option. When the firm has \( N \) sequentially ordered growth options, the technology constraint requires that growth option \( k \) can only be exercised if and only if all previous \((k - 1)\) growth options have been exercised. Intuitively, this sequential exercising constraint binds when future growth options are worthy immediately exercising after growth options are exercised. We can show that simultaneously exercising growth options \( k \) and \((k - 1)\) is optimal if and only if \( m_k/I_k \geq m_{k-1}/I_{k-1} \). Then we may combine these two consecutive growth options into one, with an exercise cost \((I_k + I_{k-1})\) and a “new” cash flow multiplier \((m_k + m_{k-1})\) for the combined growth option. By redefining growth options, we can always focus on the setting where \( m_n/I_n \) strictly decreases in \( n \), in that

\[
\frac{m_1}{I_1} > \frac{m_2}{I_2} > \cdots > \frac{m_N}{I_N}.
\]

Under this condition, the option value of waiting is strictly positive between any two consecutive growth options in the first-best (all equity financing) benchmark. The following lemma summarizes the main results for the equity financing benchmark.

**Lemma 2** The firm’s investment decisions follow stopping time rules \( T_n^* = \inf\{t \geq 0 : X(t) = X_n^{ae}\} \) for \( n = 1, 2, \ldots, N \), where the investment threshold \( X_n^{ae} \) are given by (3), and the constant \( \beta_1 \) is given by (4). Firm (equity) value \( E_n(X) \) in stage \( n \) is given by the sum of assets in place \( A_n(X) \) and its unexercised growth options, in that

\[
E_n(X) = A_n(X) + \sum_{k=n+1}^{N} G_k(X), \quad X \leq X_n^{ae}, \quad 1 \leq n \leq N,
\]

where \( G_k(X) \) is the \( k \)-th growth option value and is given in (4).

For any stage \( n \), the investment threshold \( X_n^{ae} \) is the same as the one if the \( n \)-the growth option were stand-alone. Taxes reduce cash flows but do not provide benefits under all equity financing. This explains the factor \( 1/(1 - \tau) \) for the investment threshold \( X_n^{ae} \) given in (5). As in standard real options models (McDonald and Siegel (1986)), the investment threshold
$X^*_n$ increases in volatility. For the ease of future reference, let $X^*_n$ denote the $n$-th investment threshold without taxes ($\tau = 0$). We have

$$X^*_n = (r - \mu) \frac{\beta_1}{\beta_1 - 1} \frac{I_n}{m_n}, \quad 1 \leq n \leq N.$$  

(45)

Next, we analyze the investment and financing decisions for the one-growth-option setting ($N = 1$).

4 Benchmark: One-growth-option setting

When the firm has only one growth option, we obtain closed-form formulae for the joint investment, leverage, and default decisions. Our one-growth-option setting is essentially the case where McDonald and Siegel (1986) meet Leland (1994). The next proposition summarizes the main results for the one-growth-option setting.

**Proposition 1** The firm’s investment decision follows a stopping time rule $T^i = \inf\{t : X(t) \geq X^i\}$, where the investment threshold $X^i$ is given by

$$X^i = \left[1 + \frac{1}{h} \left(\frac{\tau}{1 - \tau}\right)\right]^{-1} X^ae,$$  

(46)

where the constant $h$ is given in (42) and $X^ae$ is the all-equity investment threshold given in (5). The default time $T^d$ is given by $T^d = \inf\{t > T^i : X(t) \leq X^d\}$, where the default threshold $X^d$ is given by $X^d = X^i / h < X^i$. The optimal coupon $C^*_1$ for debt issued at the investment time $T^i$ is given by

$$C^*_1 = \frac{r}{1 - \tau} \left(\frac{\beta_2 - 1}{\beta_2}\right) \left(\frac{\beta_1}{\beta_1 - 1}\right) \left(h + \frac{\tau}{1 - \tau}\right)^{-1} I_1.$$  

(47)

Mauer and Sarkar (2005) derive similar results for the one-growth-option setting. Their focus on the results and economic interpretations is very different. We derive explicit formulae and provide explicit link between investment and default thresholds, while they do not. They contain operating leverage (variable production costs) and we do not.
Firm value $V_1(X)$ (after investing at $T_1^i$) is given by

$$V_1(X) = A_1(X) + \frac{\tau C^*_i}{r} - \left( \gamma A_1(X^d_i) + \frac{\tau C^*_i}{r} \right) \left( \frac{X}{X^d_i} \right)^{\beta_2}, \quad X \geq X^d_i. \quad (48)$$

Firm (equity) value $E_0(X)$ (before investing at $T_1^i$) is

$$E_0(X) = \left( \frac{X}{X^d_i} \right)^{\beta_1} (V_1(X^d_i) - I_1) = \left( \frac{X}{X^d_i} \right)^{\beta_1} \left( \frac{m_1 X^*_i}{r - \mu} - I_1 \right), \quad X \leq X^d_i. \quad (49)$$

The investment threshold $X^d_i$, the default threshold $X^d_i$, and the optimal coupon $C^*_i$ are all proportional to the investment cost $I_1$. At investment time $T_1^i$, equity value $E_1(X^i_1)$, debt value $D_1(X^i_1)$, and firm value $V_1(X^i_1)$ are all proportional to the investment cost $I_1$. This implies that the market leverage at the moment of investment $T_1^i$, $D_1(X^i_1)/V_1(X^i_1)$ is independent of the size of the investment cost $I_1$.

**Ambiguous volatility effects on the value of growth options.** One of the fundamental insights of real options analysis is that option value increases with volatility (by drawing the analogy to the standard Black-Scholes-Merton option pricing insight). We show that debt financing invalids this widely quoted result of the real options literature, in that the value of growth option may decrease with volatility. The intuition is as follows. Debt financing generates an *ex post* default option. A more volatile environment makes the default more likely and hence inefficient liquidation more costly. This in turn lowers firm value. Intuitively, increasing volatility potentially lowers the *payoff* from growth option exercising, generating an opposing effect to the standard convexity payoff effect emphasized in the real options literature.\(^{20}\)

Figure 2 plots option values $E_0(X)$ for two levels of volatilities $\sigma = 10\%$ and $\sigma = 20\%$. For $X \leq \bar{X} = 0.073$, the standard real options argument holds, because the option value is deep in the money and hence the conventional convexity argument applies. More interestingly,\(^{21}\)Miao and Wang (2007) show the opposing effects of volatility on real options in an incomplete-markets setting where the entrepreneur cannot fully diversify the idiosyncratic risk of the underlying project.

\(^{20}\)Miao and Wang (2007) show the opposing effects of volatility on real options in an incomplete-markets setting where the entrepreneur cannot fully diversify the idiosyncratic risk of the underlying project.
Figure 2. Option value $E_0(X)$ and post-investment firm value $V_1(X)$. Panel A plots the option value $E_0(X)$ and the firm’s value $V_1(X)$ after the growth option exercising. Note that volatility $\sigma$ influences both the option exercising payoff $V_1(X)$ and the growth option value $E_0(X)$. As $V_1(X)$ is concave in $X$, the convex option value $E_0(X)$ may decrease with volatility when the growth option is closer to be in the money (as $X$ increases), as shown in Panel B. Note that for $X \geq \bar{X} = 0.073$, $E_0(X; \sigma = 20\%) < E_0(X; \sigma = 10\%)$.

when $\bar{X} \leq X \leq X^i = 0.1$, the growth option becomes sufficiently in the money. Then, the option value of waiting effect (the standard channel) becomes less important than the payoff effect (which decreases with volatility). Therefore, we see that $E_0(X; \sigma = 20\%) < E_0(X; \sigma = 10\%)$ for $X \geq \bar{X} = 0.073$.

While the option value $E_0(X)$ may decrease with volatility, it is worth noting that $E_0(X)$ is still convex in $X$. Therefore, the convexity of option value in $X$ does not imply a monotonic relation between option value and volatility, and vice versa. In the standard real options literature, these two predictions/properties of the model are often viewed as joint predictions of real option models.

We do not view our result that real option value does not necessarily increase with volatility simply as a theoretical possibility. Indeed, we think that our results have important practical implications for firm’s capital budgeting. Growth option/capital investments
are often financed by a mixture of debt and equity, but current real-options-based capital budgeting recommendations covered in standard MBA textbooks and practitioners’ journals hold only under all equity financed firms. When Modigliani-Miller theorem does not hold and there is room for debt financing, we need to be careful in providing real-option-style capital budgeting recommendations. It is no longer clear that we should emphasize the positive effect of volatility on the value of real options.

Having analyzed the two benchmarks, we next turn to the feedback effects between investment and financing when the firm has multiple rounds of growth options.

5 Analysis

To highlight the rich interactions between a firm’s investment and financing decisions over its life-cycle, we first analyze the two growth option setting \((N = 2)\), and then generalize our model to settings with multiple growth options, e.g., \(N = 3, 4, 5, 6\).

**Parameter choices.** For the baseline calculation, we use the following annualized parameter values summarized in Table 1. As in Leland (1994) and the follow-up dynamic capital structure literature, we choose the risk-free interest rate \(r = 0.05\), the expected growth rate \(\mu = 0.01\), annual volatility \(\sigma = 0.2\), and the effective tax rate \(\tau = 0.2\). The default cost parameters are \(\gamma_A = 0.25\) for assets in place and \(\gamma_G = 0.5\), implying that the recovery rate for assets in place is 75% of the unlevered asset value and 50% for the (unlevered) unexercised growth option value, respectively. Whenever applicable, all parameter values are annualized.

Without loss of generality, we normalize the cost of exercising the growth option to unity, in that \(I_n = 1\), for all stages \(1 \leq n \leq N\). Instead we capture the net value of each sequential growth option via the the production capacity (the rate at which output is generated from each asset in place) \(m_n\). We normalize the production capacity in the first stage to be unity, \(m_1 = 1\), and decrease the production capacity \(m_n\) at an exponential
rate, i.e. \( m_n = m_{n-1} \times (1 - g_n) \). We choose \( g_n = 0.2 \) for \( n = 2, 3, \ldots \), which implies that the profitability of each new asset in place decays at 20\%. Therefore, \( m_2 = 0.8 \) and \( m_3 = 0.8^2 = 0.64 \), \( m_4 = 0.8^3 \), and \( m_5 = 0.8^4 \).

Table 1: Parameter Values (annualized whenever applicable)

<table>
<thead>
<tr>
<th>( r )</th>
<th>( \tau )</th>
<th>( \mu )</th>
<th>( \sigma )</th>
<th>( I_n )</th>
<th>( m_n )</th>
<th>( \gamma_A )</th>
<th>( \gamma_G )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.2</td>
<td>0.01</td>
<td>0.2</td>
<td>1</td>
<td>( 0.8^{n-1} )</td>
<td>0.25</td>
<td>0.5</td>
</tr>
</tbody>
</table>

5.1 The settings with \( N = 1, 2, 3 \) growth options

Our framework is general enough that we can allow the firm to have multiple growth options. Earlier, we treated the case of a firm with two growth options. We now extend our analysis to the case of a firm which has three growth options. Panels A, B, and C in Table 2 report the firm’s decisions in three models (with 1, 2, and 3 growth options in total, respectively.)

One growth option \((N = 1)\). Panel A of Table 2 summarizes the closed-form solution for the one-growth-option setting \((N = 1 \text{ and } m_1 = 1, 0.8, 0.64)\). First, we analyzes the baseline case with \( m_1 = 1 \). The firm optimally chooses to invest when \( X \) exceeds \( X^i = 0.099 \) and defaults when its earning \( X \) falls below \( X^d = 0.038 \). When exercising its growth option at \( X^i = 0.099 \), the firm issues perpetual risky debt with coupon rate \( C = 0.082 \) at a credit spread of 108 basis points. The implied initial leverage is 62.1\%. For lower production capacity \( m_1 \), the firm chooses a higher investment threshold and also a higher default threshold. For the case with \( m_1 = 0.8 \), we have \( X^i = 0.124 \) and \( X^d = 0.047 \), and for the case with \( m_1 = 0.64 \), we have \( X^i = 0.155 \) and \( X^d = 0.059 \). Importantly, the production capacity has no effect on the optimal coupon \( C \) and leverage. This is due to the endogenous
Table 2: Models where the number of growth options $N = 1, 2, 3$.

This table reports results for three settings with $N = 1, 2, 3$. For the setting with $N = 1$, we consider three sub-cases with $m_1 = 1, 0.8, 0.64$. For the setting with $N = 2$, we set $m_1 = 1$ and $m_2 = 0.8$. Finally, for the case with $N = 3$, we set $m_1 = 1$, $m_2 = 0.8$, and $m_3 = 0.64$.

<table>
<thead>
<tr>
<th>stage</th>
<th>production capacity $m$</th>
<th>investment threshold $X^i$</th>
<th>default threshold $X^d$</th>
<th>coupon rate $C$</th>
<th>leverage $Lev$</th>
<th>credit spreads $cs$ (bps)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A. $N = 1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1st</td>
<td>1</td>
<td>0.099</td>
<td>0.038</td>
<td>0.082</td>
<td>62.1%</td>
<td>108</td>
</tr>
<tr>
<td>1st</td>
<td>0.8</td>
<td>0.124</td>
<td>0.047</td>
<td>0.082</td>
<td>62.1%</td>
<td>108</td>
</tr>
<tr>
<td>1st</td>
<td>0.64</td>
<td>0.155</td>
<td>0.059</td>
<td>0.082</td>
<td>62.1%</td>
<td>108</td>
</tr>
<tr>
<td>Panel B. $N = 2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1st</td>
<td>1</td>
<td>0.095</td>
<td>0.030</td>
<td>0.077</td>
<td>43.8%</td>
<td>129</td>
</tr>
<tr>
<td>2nd</td>
<td>0.8</td>
<td>0.126</td>
<td>0.048</td>
<td>0.188</td>
<td>62.1%</td>
<td>108</td>
</tr>
<tr>
<td>Panel C. $N = 3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1st</td>
<td>1</td>
<td>0.093</td>
<td>0.028</td>
<td>0.080</td>
<td>39.1%</td>
<td>127</td>
</tr>
<tr>
<td>2nd</td>
<td>0.8</td>
<td>0.117</td>
<td>0.036</td>
<td>0.156</td>
<td>46.6%</td>
<td>123</td>
</tr>
<tr>
<td>3rd</td>
<td>0.64</td>
<td>0.158</td>
<td>0.060</td>
<td>0.319</td>
<td>62.1%</td>
<td>108</td>
</tr>
</tbody>
</table>

adjustment of investment and default thresholds so that the firm achieves its optimal leverage ratio at 62.1% at the moment of debt issuance.

**Two growth options** ($N = 2$). Panel B of Table 2 reports the results for the two-growth-option setting. First consider the decision rules in the second (last) stage. After the first debt is in place, the firm exercises its second growth option when its earning exceeds $X^i_2 = 0.126$, larger than 0.124, the investment threshold for the setting with $N = 1$ and $m_1 = 0.8$ (see Panel A and B of Table 2). This reflects the impact of debt overhang in
that the firm exercises its investment option later than an otherwise identical firm with this stand-alone growth option (with $m_1 = 0.8$) does. The cost of exercising the second growth option is higher now because the equityholders need to call back the existing debt at par, which potentially involves the wealth transfer to creditors. However, conditioning on calling back the existing debt and exercising the growth option, equityholders optimally chosen its leverage and maximize the firm value going forward. This again gives rise to 62.1% leverage ratio, the same level as in the stand-alone one-growth-option setting, which is due to the scaling invariance property of the leverage ratio for assets in place as in Leland (1994).

Given that the equityholders are investing at a higher threshold and needs to call back the existing debt, the coupon is naturally much higher (i.e. $C_2 = 0.188$) than the coupon for stand-alone one-growth-option setting (i.e. $C = 0.082$). Despite a higher debt coupon, the credit risk of the second-stage debt is the same as the one in the stand-alone one-growth option setting. This is in the spirit of Leland (1994) and our one-growth-option benchmark result.

Now, we turn to the first-stage decision-making and see how the presence of the second growth option influences the optimal exercising and financing decisions of the first growth option. First, note the anticipation effect of the subsequent debt overhang as we have discussed in the previous paragraph. Equityholders anticipate future debt overhang and thus rationally lowers the leverage and take into account the future conflicts of interest between equityholders and debtholders. This is reflected via a lower leverage, 43.8%, in stage 1 compared with 62.1% in stage 2. Moreover, the presence of the second growth option raises the firm’s debt capacity. Therefore, benefits from exercising the first growth option are higher in the two-growth-option setting than in the stand-alone one-growth-option setting with $m_1 = 1$. Therefore, in the two-growth-option setting, the payoffs from investing in the

21There is another truncation effect which impacts the credit spread calculation, but that effect is dominated.
first round are greater, and hence the firm optimally invests earlier, i.e. \( X_1^i = 0.095 \) compared with \( X_1^i = 0.099 \) for the stand-alone one-growth-option setting with \( m_1 = 1 \). Additionally, the firm defaults later as we can see from \( X^d = 0.03 \), which is lower than 0.038 for the setting with \( N = 1 \) and \( m_1 = 1 \).

**Three growth options \( (N = 3) \).** Here, we show that the exercising timing decisions for early-rounds growth option are even earlier with more growth options in the future. One effect of more future growth options is that the firm can raise more debt against future cash flows, which effectively raises the firm’s ability to issue debt today and makes investment more attractive today. This explains the result that \( X_1^i = 0.093 \) in three-growth-option setting, which is lower than \( X_1^i = 0.095 \) in the two-growth-option setting. Additionally, the leverage chosen by the firm when it exercises its first growth option is now lowered to 39.1% from 43.8%. We see the pattern for eventual convergence as we increase the number of growth options.

By comparing our results for the two-growth option setting with those for the three-growth-option setting, we see that quantitatively the three-growth-option problem can be “somewhat approximately” decomposed into two two-growth-option optimization problems. The intuition is as follows. The additional impact of current growth option exercising and financing impact on any future growth option exercising and financing decisions beyond the immediate one is quantitatively less significant. Using this logic, we may simplify an \( N \)-stage growth option exercising/financing problem into \( (N - 1) \) two-growth-option exercising problem. Of course, our approximation based on the economic insight only holds for growth options that are sufficiently close to each other. For growth options that are somewhat different from each other, our insights for this approximation may work better if we decompose the \( N \)-growth option problem into a collection of 3-growth-option problem. Based on what we have shown and also what we will show in the following subsection, we conjecture that for
some sophisticated real-world corporate investment/financing decision problem with many growth options, we may obtain a good understanding at first pass (and potentially also in terms of quantitative analyses) by using a tractable and plausible setting with only a few growth options (perhaps as few as three or four.)

5.2 Multiple growth options

In principle, we can extend our analysis to treat a firm with any number, \( N \), of growth options. Such a treatment will allow us to see, how a firm optimally decide on its dynamic leverage strategy, while taking into its cognizance that it may have to issue additional debt to finance future growth options. Intuitively, we would expect such a firm to start with fairly low to moderate levels of debt in its early stages and slowly ramp up the debt level, By doing so, the firm can mitigate the debt overhang effects on its growth options in earlier stages, exploit its steadier cash flows from assets in place to service a higher level of debt in its later stages.

5.2.1 \( N \) growth options

In Table 3, we present the investment thresholds \( X_1^i \), default thresholds \( X_1^d \), the optimal coupon rate \( C_1 \) and optimal leverage \( Lev_1 \) when the firm is in its first stage and at the the moment of exercising its first growth option, in six setting where the firm faces one to as many as six growth options into the future. Note that the first growth option in all cases has the identical parameter values. So the differences in these models only arise from the future growth options and assets in places across them.

It is worth making the following observations. First, the investment threshold \( X_1^i \) monotonically decreases from \( X_1^i = 0.099 \) for the case with \( N = 1 \) to \( X_1^i = 0.092 \) for the case with \( N = 6 \). Intuitively, the additional benefit of not distorting future investment options in models with more growth options (a higher value of \( N \)) encourages the firm to exercise its
Table 3: The first-stage decisions in models with $N$ growth options

This table reports the first-stage decisions in a model with $N$ growth options. We increase $N$ from 1 to 6. Our parameter values are $m_1 = 1$, $m_2 = 0.8$, $m_3 = 0.64$, $m_4 = 0.512$, $m_5 = 0.410$ and $m_6 = 0.328$. Others are reported in Table 1.

<table>
<thead>
<tr>
<th>model with $N$ growth options</th>
<th>investment threshold $X_1$</th>
<th>default threshold $X_1^d$</th>
<th>coupon rate $C_1$</th>
<th>leverage ratio $Lev_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.099</td>
<td>0.038</td>
<td>0.082</td>
<td>62.1%</td>
</tr>
<tr>
<td>2</td>
<td>0.095</td>
<td>0.030</td>
<td>0.077</td>
<td>43.8%</td>
</tr>
<tr>
<td>3</td>
<td>0.093</td>
<td>0.028</td>
<td>0.080</td>
<td>39.1%</td>
</tr>
<tr>
<td>4</td>
<td>0.093</td>
<td>0.027</td>
<td>0.082</td>
<td>37.0%</td>
</tr>
<tr>
<td>5</td>
<td>0.092</td>
<td>0.026</td>
<td>0.083</td>
<td>35.3%</td>
</tr>
<tr>
<td>6</td>
<td>0.092</td>
<td>0.026</td>
<td>0.083</td>
<td>34.5%</td>
</tr>
</tbody>
</table>

growth options in earlier stages sooner. Second, the optimal leverage level $Lev_1$ decreases from 62.1% for the case with $N = 1$ to 34.5% for the case with $N = 6$. This partly reflects the debt conservatism as the firm worries about the debt overhang costs in the future if leverage is too high. Third, a firm with more growth options tends to default at a much later point than a firm with fewer growth options. For example, the default threshold $X_1^d$ decreases from $X_1^d = 0.038$ for the case with $N = 1$ to $X_1^d = 0.026$ for the case with $N = 6$. This makes intuitive sense: for the firm with many growth options, the cost of default should include the forgone opportunities associated with the loss of all future growth options and hence the firm chooses to default much later in the case with $N = 6$ than that with $N = 1$, in line with the predictions from the leverage pattern that we discussed earlier.

To reiterate, leverage, default, and investment thresholds in early stages all monotonically decrease as the firm has more growth options. This life-cycle pattern driven by the endogenous composition between growth options and assets in place is important. Quantitatively, in our model with $N = 3$ or more growth options, we obtain leverage in the empirically
plausible range of 1/3 for US corporations.

6 Debt Priority Structure

Our model can be generalized to investigate the effects of priority structure of debt on real investment decisions, leverage and credit spreads. We will just provide the economic intuition below. Readers are referred to Sundaresan and Wang (2006) for a detailed treatment of this issue. Hackbarth and Mauer (2012) also study debt priority structure.

First, fix the coupon and the amount of first debt issuance. Provided that the amount of pre-existing debt for the firm is not too high, the firm rationally delays exercising its future growth option. The intuition is as follows: After collecting the proceeds from (earlier) debt issuance, the firm no longer behaves in seasoned debtholders’ interests. Equityholders and new debtholders pay for the exercising cost, but the benefits from investment first go to seasoned debtholders under the absolute priority rule (APR). This ex post wealth transfer effect discourages the firm from investing. However, once the existing debt is sufficiently high, the firm starts to take excessive risks (risk shifting) by prematurely exercising growth options. This risk shifting incentive is another widely studied form of conflicts of interests between equityholders and debtholders (Jensen and Meckling (1976)). When the firm engages in risk shifting, the seasoned debtholders bear more default risks. Intuitively, when outstanding debt is too high, equityholders are better off by unloading some credit risks to senior debtholders. Thus our model implies that financial contracting such as the debt seniority structure plays a significant role for the degrees of debt overhang and risk shifting. We can show that the debt overhang distortions and risk shifting incentives are more severe under APR than under APR.


23By APR, we refer to the seniority structure that more seasoned debt has strict priority over the newer debt in default.
the *pari passu* structure, when debt is exogenously specified.\footnote{While we focus on the seniority of debt, there are studies which differentiate the priority structure between market debt and bank debt. For example, Hackbarth, Hennessy, and Leland (2005) study the optimal mixture and priority structure of bank and market debt using the tradeoff theory. They focus on the strategic debt service motives and do not model investment decisions.}

## 7 Conclusions

Our paper provides an integrated framework for thinking about multiple rounds of sequential investments simultaneously with dynamic financing. Our modeling approach is tractable and provides a coherent way to think about a complex dynamic problem which is at the heart of both investments theory and dynamic financing. Importantly, we show that the firm substantially lower its leverage in order to take advantage of its growth options going forward and indeed we obtain empirically plausible leverage (around 1/3) with as few as three growth options. Besides providing a natural way to model the life-cycle of the firm, our model also highlights the need to distinguish between the residual values of “assets in place” and that of remaining live growth options upon default. Extensions of our framework can help us to think about the manner in which senior and junior debt arise in a firm’s balance sheet. This endogenous development of the structure of debt seniority and its consequences for endogenous debt overhang is an exciting future avenue of research. Sundaresan and Wang (2006) and Hackbarth and Mauer (2012) provide an analysis in which the priority structure of debt is explicitly considered in the evolution of debt structure of the firm as it optimally exercises its growth options. Finally, we have side-stepped from informational asymmetry and moral hazard that may influence a firm’s financing decisions over its life-cycle. In a model with informational asymmetry and real options, Fulghieri, Garcia, and Hackbarth (2013) show that high-growth firms may prefer equity over debt, and then switch to debt financing as they mature. Their predictions under a different model formulation are in line with our model’s prediction in that leverage increases over the firm’s life-cycle in our model.
as the firm sequentially exercises its growth options and becomes one with more assets in place.
Appendices

A Appendix

We provide derivations and proofs for various results used in the main text.

A.1 Derivations of main results in Section 3

We solve the firm’s decision problem and its valuation equations for debt, equity, and firm using backward induction. First, note that the decision problem in the last stage is the standard model analyzed in Leland (1994). See Section 3.1 for the main results.

Now consider the firm’s intermediate stage \( n \), where \( 1 < n < N \). Given the default threshold \( X_d^n \) and the investment threshold \( X_i^{n+1} \), we may write down equity value as in (25), using \( \Phi_i^n(X) \) and \( \Phi_d^n(X) \), which are the present discounted value of receiving a unit payoff contingent on the \((n+1)\)-th growth option is exercised before the firm defaults at \( T_d^n \) or not, respectively. Formally,

\[
\Phi_i^n(X) = E_t \left[ e^{-r(T_{n+1}^i-t)}1_{T_d^n > T_{n+1}^i} \right], \quad \text{and} \quad \Phi_d^n(X) = E_t \left[ e^{-r(T_{n+1}^d-t)}1_{T_d^n < T_{n+1}^i} \right],
\]

where \( 1_{T_d^n > T_{n+1}^i} \) and \( 1_{T_d^n < T_{n+1}^i} \) are the indicator functions. If \( T_d^n > T_{n+1}^i \), we have \( 1_{T_d^n > T_{n+1}^i} = 1 \). Otherwise, \( 1_{T_d^n > T_{n+1}^i} = 0 \). It is immediate to see that \( \Phi_d^n(X_d^n) = \Phi_i^n(X_i^{n+1}) = 1 \), \( \Phi_d^n(X_i^n) = \Phi_i^n(X_d^n) = 0 \), and \( \Phi_d^n(X) > 0 \), \( \Phi_i^n(X) > 0 \), for \( X_d^n < X < X_i^{n+1} \).

Using the equity value formula (25) and the firm value formula (33), we have

\[
XE'_n(X) = A_n(X) + e_i^n \Phi_i^n(X)X + e_d^n \Phi_d^n(X)X,
\]
\[
XV'_{n+1}(X) = A_{n+1}(X) + v_i^{n+1} \Phi_i^{n+1}(X)X + e_{n+1}^d \Phi_d^{n+1}(X)X.
\]
Applying the smooth pasting condition $E_n'(X^i_{n+1}) = V'_{n+1}(X^i_{n+1})$ to (A.2) and (A.3) gives

$$
\left(\frac{1 - \tau}{r - \mu}\right)m_{n+1}X^i_{n+1} = \beta_1(X^i_{n+1})^{\beta_1} - \beta_2(X^i_{n+1})^{\beta_2} \left[ \frac{e_n^i(X^d_n)^{\beta_2} - e_n^d(X^i_{n+1})^{\beta_2}}{\Delta_n} - \frac{v_{n+1}^i(X^d_{n+1})^{\beta_2} - v_{n+1}^d(X^i_{n+1})^{\beta_2}}{\Delta_{n+1}} \right] - \beta_2(X^i_{n+1})^{\beta_2} \left[ \frac{e_n^i(X^d_n)^{\beta_1} - e_n^d(X^i_{n+1})^{\beta_1}}{\Delta_n} - \frac{v_{n+1}^i(X^d_{n+1})^{\beta_1} - v_{n+1}^d(X^i_{n+1})^{\beta_1}}{\Delta_{n+1}} \right].
$$

(A.4)

Similarly, the smooth pasting condition $E_n''(X^d_n) = 0$ gives

$$
0 = A_n(X^d_n) + \beta_1(X^d_n)^{\beta_1} \left( \frac{e_n^i(X^d_n)^{\beta_2} - e_n^d(X^i_{n+1})^{\beta_2}}{\Delta_n} \right) - \beta_2(X^d_n)^{\beta_2} \left( \frac{e_n^i(X^d_n)^{\beta_1} - e_n^d(X^i_{n+1})^{\beta_1}}{\Delta_n} \right).
$$

(A.5)

Using the ODE (28) for debt value $D_n(X)$ and the corresponding boundary conditions (29) and (30), we have the debt value is given by:

$$
D_n(X) = \frac{C_n}{r} - \left( \frac{C_n}{r} - F_n \right) \Phi^i_n(X) - \left( \frac{C_n}{r} - L_n(X^d_n) \right) \Phi^d_n(X), \quad X^d_n \leq X \leq X^i_{n+1}. \tag{A.6}
$$

Intuitively, debt value is given by the risk-free debt value $C_n/r$, minus the present value of the discount when debt is called back at $T^i_{n+1}$, and minus the present value of the loss when the firm defaults at $T^d_n$. Since debt is priced at par $F_n$ at issuance time $T^i_n$, using the above debt pricing formula, we have the par value is given by (32). And then using (32), debt value $D_n(X)$ is then given by (31).

Now turn to the firm’s decision making in stage 0. Substituting the conjectured equity value (39) into the ODE (36) and applying the endogenous default boundary conditions (37) and (38) give the following implicit equation for the first investment threshold $X^i_1$:

$$
A_1(X^i_1) = \frac{\beta_1}{\beta_1 - 1} \left[ I_1 - \frac{\tau C_1}{r} + \frac{\Phi^d_i(X^i_1)X^i_1 - \beta_1 \Phi^i_1(X^i_1)}{\beta_1} v^i_1 + \frac{\Phi^d_i(X^i_1)X^i_1 - \beta_1 \Phi^i_1(X^i_1)}{\beta_1} v^d_1 \right].
$$

(A.7)

Simplifying the above gives (40). Note that we may also obtain the same results directly using the general formulation (A.4) with the following properties: $X^d_0 = 0, e^d_0 = 0, e^i_0 = V_1(X^i_1) - I_1$, and $\Delta_0 = (X^d_0)^{-\beta_2}(X^i_1)^{\beta_1}$.
A.2 Proof of Lemmas 1 and 2

Following the standard real option analysis (e.g. McDonald and Siegel (1986), the value of the growth option, $G_k(X)$, for a stand-alone investment opportunity with one-time exercise cost $I_k$ which generates cash flow $(1 - \tau)m_k X$, solves the following ODE:

$$r G_k(X) = \mu X G_k'(X) + \frac{\sigma^2}{2} X^2 G_k''(X), \quad X \leq X_k^{ae},$$  \hspace{1cm} (A.8)

subject to the following boundary conditions

$$G_k(X_k^{ae}) = \frac{(1 - \tau)m_k X_k^{ae}}{r - \mu} - I_k,$$  \hspace{1cm} (A.9)

$$G_k'(X_k^{ae}) = \frac{(1 - \tau)m_k}{r - \mu}.$$  \hspace{1cm} (A.10)

In addition, we have the absorbing barrier condition $G_k(0) = 0$ because $X$ is a GBM process. Using the standard guess-and-verify procedure, we obtain the option value formula (4) for $G_k(X)$ for $X \leq X_k^{ae}$, and the growth option exercise threshold $X_k^{ae}$ given in (5).

When the firm is all equity financed and holds a sequence of decreasingly attractive growth option, the exercising decisions of each option is independent of the exercising decisions of other options. This is a robust result under all equity financing. Therefore, when $m_1/I_1 > m_2/I_2 > \cdots > m_N/I_N$ holds, all equity-financed firm value is given by the sum of assets in place and unexercised growth options. That is, in stage $n$, the firm has $n$ existing assets in place valued at $A_n(X)$ and $N - n$ unexercised growth options. Each growth option is valued at $G_k(X)$ with exercising cost $I_k$ and cash flow multiple $m_k$. Total firm value is then given by (44). Note that the growth options are ordered sequentially from the most attractive (1st growth option) to the least attractive is without loss of generality. See the main text for discussions on how we may redefine the growth options if the preceding growth option is more attractive. (For example, if $m_2/I_2 \geq m_1/I_1$, we can combine the first two growth options and relabel the option with exercising cost $I_1 + I_2$ and cash flow multiple $(m_1 + m_2)$.)
A.3 Proof of Proposition 1

When the firm has only one growth option, by definition, the exercise threshold for the second growth option is infinite, in that \( X_i^2 = \infty \). Therefore, (22) and (23) imply \( \Phi_i(X) = 0 \) and \( \Phi_i^d(X) = (X/X_i^d)^{\beta_2} \), for \( X \geq X_i^d \). Since the firm only has one growth option, we have

\[
L_1(X_i^d) - \left( A_1(X_i^d) + \frac{\tau C_1}{r} \right) = - \left( \gamma_A A_1(X_i^d) + \frac{\tau C_1}{r} \right) \tag{A.11}
\]

Equation (40) for the first investment threshold \( X_i^1 \) thus implies

\[
X_i^1 = \frac{1}{1 - \tau} \frac{r - \mu}{m_1 \beta_1 - 1} \left[ \left( 1 - \frac{\tau C_1}{r} \right) \beta_1 - \beta_2 \frac{\tau C_1}{r} \left( \gamma_A A_1(X_i^d) + \frac{\tau C_1}{r} \right) \left( \frac{X_i^1}{X_i^d} \right)^{\beta_2} \right] \tag{A.12}
\]

The optimal coupon policy is given by

\[
C_1 = \frac{r}{r - \mu} \frac{\beta_2 - 1}{\beta_2} h m_1 X_i^1 \tag{A.13}
\]

Re-arranging and simplifying (A.12) gives the following implicit equation for the investment threshold:

\[
(\beta_1 - 1) A_1(X_i^1) = \beta_1 I_1 - \beta_1 \frac{\tau C_1}{r} + (\beta_1 - \beta_2) \frac{C_1}{r} \left[ \gamma_A (1 - \tau) \left( \frac{\beta_2}{\beta_2 - 1} \right) + \frac{\tau}{r} h^{\beta_2} \right]
\]

where the first, second, and third line uses the explicit formulae for \( X_i^d \) as a function of \( C_1 \) given in (12) (when stage 1 is the last stage (i.e. \( N = 1 \)), \( h \) given in (42), and coupon \( C_1 \) given in (A.13), respectively. Finally, re-arranging the last expression gives \( X_i^1 \) in (46). Substituting (46) into (A.13) gives the coupon policy (47) and the default threshold \( X_i^d = X_i^1/h \).

If the initial value \( X_0 \) is below the investment threshold \( X_i^1 \) given in (46), the firm will wait before investing. Equity value before investment \( E_0(X) \) is given by (49).
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