Dynamics of Entrepreneurship under Incomplete Markets *

Chong Wang†  Neng Wang‡  Jinqiang Yang§

July 18, 2011

Abstract
An entrepreneur faces significant non-diversifiable business risk and liquidity constraints, two frictions that lead to incomplete markets. We provide an integrated framework to study how these frictions influence a potential entrepreneur’s business start-up/entry, capital accumulation/asset sales, portfolio allocation, consumption/saving, and business exit decisions. With frictions, entrepreneurs invest substantially less in the business, consume less, and allocate less to the market portfolio. While the business liquidation option is completely out of the money under complete markets, with frictions, this endogenous exit option provides significant flexibility and is quite valuable for the entrepreneur to manage downside risk under incomplete markets. We further derive novel empirical predictions about the economic dynamics of entrepreneurship. When liquidation is sufficiently distant, business investment increases with financial wealth. However, when liquidation is sufficiently likely, the entrepreneur’s risk management considerations induce investment to decrease with financial wealth, generating non-monotonicity of investment in financial wealth. Furthermore, we show that the flexibility to accumulate wealth before entering entrepreneurship is highly valuable. The optimal entry decision critically depends on the outside option, the start-up cost, risk aversion, and wealth. Heterogeneity among entrepreneurs is thus important. Finally, we provide an operational framework to calculate the private equity idiosyncratic risk premium for an entrepreneurial firm and show that this premium critically depends on wealth, non-diversifiable risk, and risk aversion.

Keywords: idiosyncratic risk premium, the $q$ theory of investment, real options, precautionary saving, portfolio choice, investment, entry, exit, hedging, liquidity

JEL Classification: G11, G31, E2

*We thank Marco Bassetto, Patrick Bolton, Cristina De Nardi, Bob Hall, Jim Hamilton, Bob Hodrick, Boyan Jovanovic, Dirk Krueger, Nick Roussanov (WFA discussant), Tom Sargent, Anne Villamil, Laura Vincent, Annette Vissing-Jorgensen, Toni Whited, Noah Williams, James Withkin, and seminar participants at ASU, CIRPEE-IVEY Conference on Macroeconomics of Entrepreneurship, Columbia, UCSD, WFA 2011, and Wisconsin Madison for helpful comments.

†Naval Postgraduate School. Email: cwang@nps.edu.
‡Columbia Business School and NBER. Email: neng.wang@columbia.edu. Tel.: 212-854-3869.
§Columbia University and Shanghai University of Finance and Economics (SUFE).
1 Introduction

Agency conflicts and informational asymmetry induce entrepreneurs to hold a large fraction of wealth in their active businesses and make them liquidity constrained.\(^1\) Non-diversification and liquidity constraints, both of which we refer to as frictions, are important determinants for the economics of entrepreneurship. They lead to incomplete markets and cause business decisions (e.g. capital accumulation/size and entry/exit) and household decisions (inter-temporal consumption/saving and asset allocation) to be highly linked, invalidating the standard complete-markets profit-maximizing analysis for entrepreneurial firms.

We develop an intertemporal model of entrepreneurship to study the impact of frictions on interdependent household and business decision making from the pre-entry to the post-exit stage. We model entrepreneurship as an endogenous career choice followed by an optimal capital accumulation/business growth problem in an incomplete-markets consumption-saving and portfolio choice framework.

Being an entrepreneur often requires substantial start-up costs in terms of effort, time, attention, commitment, and resources. Additionally, being an entrepreneur often means giving up the outside option of being a worker elsewhere and earning wages, i.e. opportunity costs in terms of career/job choice. Moreover, entrepreneurship also involves a flexible entry timing and a project size choice. To capture these salient features of entrepreneurship, we endow the agent with a flexible timing option to become an entrepreneur and additionally the choice of the business project size. Engaging in entrepreneurship incurs both the fixed business start-up cost and the opportunity cost of giving up the outside option. Thus, becoming an entrepreneur is effectively optimally exercising a real option. However, unlike standard options, this career choice decision is rather different; it is a non-tradable and illiquid American-style real option which is dependent on the entrepreneur’s subsequent decisions and subject to important incomplete-markets frictions.

To understand the economics of entrepreneurship formation, by backward induction, we first study the entrepreneur’s post-entry decision making. Then, using the entrepreneur’s post-entry value function as the payoff from becoming an entrepreneur, we characterize the optimal entry into entrepreneurship as well as the optimal choice of the project size.

\(^1\)For example, see Evans and Jovanovic (1989), Gentry and Hubbard (2004), and Cagetti and De Nardi (2006). For a recent survey of research on entrepreneurship in macroeconomics, see Quadrini (2009), who discusses entrepreneurial career choice, entrepreneurial saving/investment, and economic development/growth.
From the pre-entry to the post-exit stage, whether being a worker or being an entrepreneur, our infinitely-lived agent additionally makes intertemporal consumption-saving and portfolio choice decisions between the risk-free asset and the risky (stock) market portfolio.

After setting up the business, the entrepreneur accumulates capital, incurs adjustment costs, and makes optimal investment decisions. The adjustment cost is critical because it makes entrepreneurial business illiquid and hence fundamentally differentiates business wealth from liquid financial wealth. Additionally, the entrepreneur’s liquid financial wealth mitigates the impact of frictions/financial constraints and hence influences investment.

The modern $q$ theory of investment studies optimal capital accumulation and the value of capital with costly capital adjustments. However, much of the $q$ theory was developed for risk-neutral firms and/or firms owned by and run in the interest of well diversified investors, where financial frictions do not matter and the Modigliani-Miller (MM) theorem holds.\(^2\) However, around the world, firms are often run by entrepreneurs, founders, families and/or controlling shareholders, even in publicly traded firms. La Porta, Lopez-de-Silanes, and Shleifer (1999) document ownership concentration by controlling shareholders for large publicly traded firms around the world.\(^3\)

One important contribution of this paper is to develop the counterpart of the modern $q$ theory of investment for private firms run by non-diversified entrepreneurs/controlling shareholders. We do so by incorporating the impact of incomplete-markets frictions into an otherwise standard $q$ theory of investment. With illiquidity and incomplete markets, a natural measure of value for private firms with concentrated entrepreneurial ownership is the entrepreneur’s certainty equivalent wealth of illiquid business. We further propose natural definitions of the average $q$ and the marginal $q$ for private firms. Average $q$ measures the entrepreneur’s certainty equivalent wealth of illiquid business per unit of the physical capital. Marginal $q$ measures the sensitivity (marginal changes) of the entrepreneur’s certainty equivalent wealth of illiquid business with respect to a (marginal) change in capital stock.

Under the complete-markets benchmark, business decisions are independent of household

---

\(^2\)Brainard and Tobin (1968) and Tobin (1969) define the ratio between the firms market value to the replacement cost of its capital stock, as $q$ and propose to use this ratio to measure the firms incentive to invest in capital. This ratio has become known as Tobins average $q$. Hayashi (1982) provides conditions under which average $q$ is equal to marginal $q$. Abel and Eberly (1994) develop a unified $q$ theory of investment in neoclassic settings. Lucas and Prescott (1971) and Abel (1983) are important early contributors.

\(^3\)See Burkart, Panunzi, and Shleifer (2003) for a model of a family firm. The law & finance/investor protection literature documents compelling evidence on concentrated ownership around the world.
consumption/portfolio decisions and the MM theorem holds. With the additional homogeneity assumption for production/capital accumulation, we uncover the seminal Hayashi (1982) result that the firm’s average \( q \) is equal to the marginal \( q \). However, with frictions, even with the homogeneity assumptions as in Hayashi (1982), we show that there is an important wedge between marginal \( q \) and average \( q \) for entrepreneurial firms.

Importantly, we show that the ratio \( w \) between liquid financial wealth and illiquid capital stock matters for entrepreneurial investment and household decisions because this ratio \( w \) directly impacts the entrepreneur’s ability to bear and manage business idiosyncratic risk. Intuitively, a larger business/firm requires more liquid wealth to achieve the same level of financial strength, \textit{ceteris paribus}. The ratio \( w \) is thus a natural scale-adjusted measure of financial constraints. The higher the \( w \), the less constrained the entrepreneurial firm. In the limit as \( w \) goes to infinity, entrepreneurs can perfectly self insure, frictions no longer matter, and markets are effectively complete.

With complete markets and/or as \( w \) goes to infinity, financing does not matter and debt capacity, the maximally allowed debt amount by the lender, has no effect on corporate investment. With frictions, for finite values of \( w \), the marginal value of liquid wealth is larger than unity because liquidity relaxes financial constraints and hence is more valuable than its nominal face/accounting value. Unlike the complete-markets setting, how much the entrepreneur can borrow against the collateral (physical capital) matters. The debt capacity depends on the liquidation value of the physical capital. The debt is fully collateralized and hence the lender faces no risk. While debt capacity matters, importantly we show that the entrepreneur may often voluntarily choose to liquidate physical capital before exhausting the debt capacity in order to limit the downside risk exposure. While this liquidation option is completely out of money under complete markets (because the technology is productive) as in the seminal Hayashi (1982) (risk-neutral) framework, this exit option provides an important downside risk protection against incomplete-markets frictions faced by entrepreneurs.

For sufficiently large \( w \), increasing liquid financial wealth relaxes financial constraints and mitigates underinvestment induced by frictions. In this “normal” region, average \( q \) for the entrepreneurial firm is concave in \( w \) and hence the marginal value of liquid wealth falls as liquid wealth becomes more abundant (i.e. higher \( w \)).

Recall that the entrepreneur may prefer liquidating over continuing the firm as a go-
ing concern before exhausting the debt capacity for risk management purposes. Therefore, by continuity, for sufficiently low $w$, the entrepreneur may scale back underinvestment (i.e. increase investment) as the entrepreneur’s liquidity $w$ decreases. Liquidation becomes increasingly attractive as the entrepreneur is sufficiently close to the optimal (lower) liquidation boundary. The convexity of average $q$ in liquidity $w$ and a monotonically decreasing relation of investment in $w$ for low levels of $w$ are sharp and distinctive predictions of our model for poorly diversified entrepreneurs.\footnote{For simplicity, we focus on the liquidation option as the exit option for downside risk protection. Without changing the analysis in any fundamental way, we can extend our model to allow the entrepreneur to have an exit option when doing well. For example, selling to diversified investors or going to an initial public offering (IPO) are two ways for the entrepreneur to exit when doing well. See Pastor, Taylor, and Veronesi (2009) and Chen, Miao, and Wang (2010) for models with IPO as an exit option in good times.}

Other models, such as the one based on moral hazard in a dynamic optimal contracting framework (e.g. Demarzo, Fishman, He, and Wang (2010)) and the one based on transaction/financing costs (e.g. Bolton, Chen, and Wang (2010)), generate neither this convexity for average $q$ nor a non-monotonicity relation between investment and liquidity. The entrepreneur’s diversification considerations fundamentally change the economics of capital accumulation and business entry/exit decisions.

By dynamically trading the market portfolio, the entrepreneur can hedge the systematic component of the business risk, but cannot hedge the idiosyncratic risk component.\footnote{The interesting and plausible case for entrepreneurs is the one where the market portfolio is not perfectly correlated with business risk, and hence markets are incomplete. Otherwise, the complete-market solution applies to the model. See the complete-markets solution for the benchmark model in Section 3.} The entrepreneur not only makes the consumption-saving and portfolio choice (between stocks and bonds) under incomplete markets, but also determines the relative mix of liquid assets (stocks and bonds) versus illiquid assets (entrepreneurial business) as a more broadly defined portfolio choice problem. Importantly, unlike for the complete-markets benchmark, both systematic and idiosyncratic risks matter for capital budgeting, portfolio choice and consumption-saving decisions.

Entrepreneurial Finance, as an academic field, so far offers no apparent theoretical guidance on how to calculate the cost of capital for entrepreneurial firms. Our model delivers an operational and analytically tractable framework for calculating the cost of capital for entrepreneurial firms. Naturally, idiosyncratic business risks as well as systematic ones play important roles on the private equity premium. We further link the model’s prediction to the empirically observed low private equity premium documented by Moskowitz and Vissing-
Jorgensen (2002).

Some predictions of our model have been empirically confirmed. For example, we find that the entrepreneur significantly underinvests in business, scales back consumption, and allocates less wealth to the market portfolio in order to mitigate frictions. Indeed, Heaton and Lucas (2000) find that entrepreneurs with high and variable business income hold less wealth in stocks than other similarly wealthy households. The private business valuation is also significantly lower than the complete-markets valuation. Additionally, we show that these frictions make wealth more valuable for both production and consumption purposes.

We have shown that the ability to liquidate capital makes the entrepreneur’s certainty equivalent wealth convex in $w$ for low values of $w$, generating optionality. Similarly, the ability to time the entry decision makes the option value of waiting to become an entrepreneur valuable. While these entry and exit options are subject to frictions and are not marked to the market, they are nonetheless valuable as risk management tools for the entrepreneur under incomplete markets. The entry and exit options make the certainty equivalent wealth convex for low values of $w$, and hence both investment and market portfolio allocation may decrease with $w$ for sufficiently low values of $w$.

While almost all existing work on the dynamics of entrepreneurship uses numerical programming, our model is analytically tractable. Recall that after becoming an entrepreneur, the key state variable is the ratio $w$ between the entrepreneur’s liquid and illiquid wealth. This ratio $w$ dictates the entrepreneur’s ability to absorb idiosyncratic risk due to incomplete markets frictions. The lower this ratio $w$, the more exposure to idiosyncratic risk.

Before becoming an entrepreneur, the key state variable is the liquid financial wealth. We solve for the optimal cutoff level for liquid wealth and initial project size for the to-be entrepreneur. Intuitively, this cutoff wealth level depends on the outside option, fixed start-up cost, risk aversion, and other important preference and technology parameters. The initial project size trades off the liquidity needs and the business profitability. Cross-sectional heterogeneity among entrepreneurs along preferences, business ideas/production technology, and outside options gives rise to different entrepreneurial entry, consumption-saving, portfolio choice, capital accumulation, and business exit decisions. As we will show, the tractability significantly facilitates our exposition of economic intuition and mechanism for our model.

Quantitatively, we also find that there are significant welfare costs for the entrepreneur
to bear non-diversifiable idiosyncratic risk. To give an example, for an expected-utility entrepreneur whose coefficient of relative risk aversion is equal to two and has no liquid wealth, in our baseline calculation, the subjective valuation of the entrepreneurial business is about 11% lower than the complete-markets benchmark.

**Related Literature.** Our paper links to several strands of literature in finance, macroeconomics, and entrepreneurship. The economics of entrepreneurship literature is fast growing. Hall and Woodward (2010) analyze the effects of non-diversifiable risk for venture-capital-backed entrepreneurial firms. Heaton and Lucas (2004) show that risky non-recourse debt helps the entrepreneur diversify business risk in a static framework with capital budgeting, capital structure, and portfolio choice decisions. Chen, Miao, and Wang (2010) study the effects of non-diversifiable risk on entrepreneurial finance by building on the workhorse contingent-claim capital structure model. They show that more risk averse entrepreneurs borrow more in order to lower their business risk exposure. Herranz, Krasa, and Villamil (2009) assess the impact of legal institutions on entrepreneurial firm dynamics. They also find that more risk averse entrepreneurs default more.

Evans and Jovanovic (1989) show the importance of wealth and liquidity constraints for entrepreneurship. Cagetti and De Nardi (2006) quantify the importance of liquidity constraints on aggregate capital accumulation and wealth distribution by constructing a model with entry, exit, and investment decisions. Hurst and Lusardi (2004) challenge the importance of liquidity constraints and provide evidence that the start-up sizes of entrepreneurial firms tend to be small. We provide a theory of entrepreneurship by accounting for endogenous entry/exit in a model with borrowing constraints and non-diversifiable risk.

Almost all models in the q theory literature are designed for firms held by diversified investors. We extend the q theory of investment to account for non-diversifiable risk and liquidity constraints. In addition, the entrepreneurial firm in our model has flexible entry and exit options. We show that non-diversifiable risk and liquidity constraints have first-order effects on capital accumulation and firm valuation. We develop the counterparts of marginal

---

6 Leland (1994) is an important paper which initiated this line of research. Morellec (2004) extends the framework to analyze managerial agency issues and leverage.

7 For tractability, Chen, Miao, and Wang (2010) adopt exponential utility, while this paper uses non-expected Epstein-Zin utility. Also, the economic issues addressed in these two papers are rather different.

Most models on portfolio choice with non-tradable income assume exogenous income. Our model endogenizes the non-marketable income from business via optimal entrepreneurial decisions. The endogenous business entry/exit and consumption/portfolio decisions are important margins for the entrepreneur to manage risk. The entry/exit options significantly alter the entrepreneur’s decision making. Some of our results are also related to the real options analysis under incomplete markets. Miao and Wang (2007) and Hugonnier and Morellec (2007) study the impact of non-diversifiable risk on real options exercising. These papers show that the non-diversifiable risk significantly alters option exercising strategies.

Finally, our model also relates to recent work on dynamic corporate finance. Bolton, Chen, and Wang (2011), henceforth BCW, analyze optimal investment, financing, and risk management decisions and valuation for a financially constrained risk-neutral firm. BCW focus on various transaction costs that a firm incurs when raising external funds. DeMarzo, Fishman, He, and Wang (2010), henceforth DFHW, integrate the risk-neutral dynamic agency framework of DeMarzo and Fishman (2007b) and DeMarzo and Sannikov (2006) with the neoclassic $q$ theory of investment. DFHW derive an optimal dynamic contract and provide financial implementation.

Unlike BCW and DFHW, our paper studies entrepreneurial finance. We show how non-diversification and liquidity constraint frictions influence a risk-averse entrepreneur’s consumption-saving, portfolio choice, and investment/entry/exit decisions. Additionally, we separate risk aversion from the elasticity of intertemporal substitution (EIS) by using non-expected homothetic recursive utility (Epstein and Zin (1989)). We further quantify idiosyncratic private equity risk premium for risk-averse entrepreneurs. Neither BCW nor DFHW addresses these issues given their risk-neutrality assumptions. While all three papers use the seminal model of Hayashi (1982) in the $q$ theory of investment literature as the MM benchmark, most importantly, the financing frictions and thus economic issues/mechanisms in these papers are fundamentally distinct.

\[9\] See Merton (1971) and Mayers (1974) for early contributions. Among others, Duffie, Fleming, Soner, and Zariphopoulou (1997), Koo (1998), and Viceira (2001) study the optimal consumption and portfolio rules for an investor with isoelastic utility and non-tradable labor income risk.

\[10\] This is a fast growing field. For studies on investment with financial constraints, for example, see Whited (1992), Gomes (2001), Hennessy and Whited (2005, 2007), Gamba and Triantis (2008), Riddick and Whited (2009), and Bolton, Chen, and Wang (2011).

\[11\] DeMarzo and Fishman (2007a) analyze the impact of agency on investment dynamics in discrete time.
2 The model

We first introduce the agent’s preferences and then set up the optimization problem.

Preferences. The agent has a preference featuring both constant relative risk aversion and constant EIS (Epstein and Zin (1989) and Weil (1990)). We use the continuous-time formulation of this non-expected utility introduced by Duffie and Epstein (1992a). That is, the agent has a recursive preference defined as follows

\[ J_t = \mathbb{E}_t \left[ \int_t^\infty f(C_s, J_s) ds \right], \]  

(1)

where \( f(C, J) \) is known as the normalized aggregator for consumption \( C \) and the agent’s utility \( J \). Duffie and Epstein (1992a, 1992b) show that \( f(C, J) \) for Epstein-Zin non-expected homothetic recursive utility is given by

\[ f(C, J) = \frac{\zeta}{1 - \psi^{-1}} C^{1-\psi^{-1}} - \left((1 - \gamma)J\right)^\chi, \]  

(2)

where

\[ \chi = \frac{1 - \psi^{-1}}{1 - \gamma}. \]  

(3)

The parameter \( \psi > 0 \) measures the EIS, and the parameter \( \gamma > 0 \) is the coefficient of relative risk aversion. The parameter \( \zeta > 0 \) is the agent’s subjective discount rate.

The widely used time-additive separable constant-relative-risk-averse (CRRA) utility is a special case of the Duffie-Epstein-Zin-Weil recursive utility specification where the coefficient of relative risk aversion is equal to the inverse of the EIS \( \psi \), i.e. \( \gamma = \psi^{-1} \) implying \( \chi = 1 \).\(^\text{12}\) In general, with \( \gamma \neq 1/\psi \), we can separately study the effects of risk aversion and the EIS.

Career choice and initial firm size. The agent is endowed with an entrepreneurial idea and initial wealth \( W_0 \). The entrepreneurial idea is defined by a productive capital accumulation/production function to be introduced soon. To implement the entrepreneurial idea, the agent chooses a start-up time \( T^0 \), pays a one-time fixed start-up cost \( \Phi \), and also

\(^{12}\)For this special case, we have \( f(C, J) = U(C) - \zeta J \), where \( U(C) \) is the expected CRRA utility with \( \gamma = \psi^{-1} \) and hence \( U(C) = \zeta C^{1-\psi^{-1}}/(1-\psi^{-1}) \). Note that for CRRA utility, \( f(C, J) \) is additively separable. By integrating (1) forward for this CRRA special case, we obtain \( J_t = \max_{C_t} \mathbb{E}_t \left[ \int_t^\infty e^{-\zeta(s-t)} U(C(s)) ds \right] \).
chooses the initial capital stock $K_{T^0}$. One example is being a taxi/limo driver. The agent can first start with a used car. After building up savings, the agent tolerates risk better and potentially upgrades the vehicle. With even more savings, the agent may further increase firm size by hiring drivers and running a limo service.

Before becoming an entrepreneur, the agent can take an alternative job (e.g. to be a worker) to build up financial wealth. Being an entrepreneur is a discrete career decision.\textsuperscript{13} We naturally assume that being an entrepreneur offers potentially a higher reward at a greater risk than being a worker. Hamilton (2000) finds that earnings of the self-employed are smaller on average and have higher variance than earnings of workers using data from Survey of Income and Program Participation. To contrast the earnings profile differences between an entrepreneur and a worker, we assume that the outside option (by being a worker) gives the agent a constant flow of income at the rate of $r\Pi$.

At the optimally chosen (stochastic) entry time $T^0$, the agent uses a combination of personal savings and collateralized borrowing to finance $(K_{T^0} + \Phi)$. Lenders make zero profit in competitive capital markets. If the entrepreneur reneges on debt, creditors can always liquidate the firm’s capital and recover fraction $l > 0$ per unit of capital. The borrower thus has no incentive to default on debt and can borrow up to $lK$ at the risk-free rate by using capital as the collateral.

We will show that initial wealth $W_0$ plays a role in how long it takes the agent to become an entrepreneur and the choice of the firm’s initial size. Borrowing constraints and non-diversifiable risk are conceptually and quantitatively important. Moreover, these two frictions interact and generate economically significant feedback effects on entrepreneurship.

**Entrepreneurial idea: capital investment and production technology.** The entrepreneurial idea is defined by a capital accumulation/production function. Let $I$ denote the gross investment. As is standard in capital accumulation models, the change of capital stock $K$ is given by the difference between gross investment and depreciation, in that

$$dK_t = (I_t - \delta K_t) dt, \quad t \geq 0,$$

\textsuperscript{13}We do not allow the agent to be a part-time entrepreneur and a part-time worker at the same time. This is a standard and reasonable assumption. For example, see Vereshchagina and Hopenhayn (2009) for a dynamic career choice model featuring the same assumption.
where $\delta \geq 0$ is the rate of depreciation. The firm’s productivity shock $dA_t$ is independently and identically distributed (i.i.d.), and is given by

$$dA_t = \mu_A dt + \sigma_A dZ_t,$$  \hspace{1cm} (5)

where $Z$ is a standard Brownian motion, $\mu_A > 0$ is the mean of the productivity shock, and $\sigma_A > 0$ is the volatility of the productivity shock. The firm’s operating revenue over time period $(t, t + dt)$ is proportional to its time-$t$ capital stock $K_t$, and is given by $K_t dA_t$. The firm’s operating profit $dY_t$ over the same period is given by

$$dY_t = K_t dA_t - I_t dt - G(I_t, K_t) dt,$$  \hspace{1cm} (6)

where the price of the investment good is set to unity and $G(I, K)$ is the adjustment cost.

Following Hayashi (1982), we assume that the firm’s adjustment cost $G(I, K)$ is homogeneous of degree one in $I$ and $K$, and write $G(I, K)$ in the following homogeneous form

$$G(I, K) = g(i)K,$$  \hspace{1cm} (7)

where $i = I/K$ is the firm’s investment-capital ratio and $g(i)$ is an increasing and convex function. With homogeneity, Tobin’s average $q$ is equal to marginal $q$ under perfect capital markets. However, as we will show, the non-diversifiable risk drives a wedge between Tobin’s average $q$ and marginal $q$ for the entrepreneur. For simplicity, we assume that

$$g(i) = \frac{\theta i^2}{2},$$  \hspace{1cm} (8)

where the parameter $\theta$ measures the degree of the adjustment cost. A higher value $\theta$ implies a more costly adjustment process.

The entrepreneur has an option to liquidate capital at any moment. Liquidation is irreversible and gives a terminal value $lK$, where $l > 0$ is a constant. Let $T_l$ denote the entrepreneur’s optimally chosen stochastic liquidation time. To focus on the interesting case, we assume capital is sufficiently productive. Thus, liquidating capital when capital markets are perfect is not optimal because doing so destroys going-concern value. However, when the entrepreneur is not well diversified, liquidation provides an important channel for the entrepreneur to manage the downside business risk exposure.
Our production specification features the widely used “AK” technology\textsuperscript{14} augmented with the adjustment cost technology. Our specification is a reasonable starting point and is also analytically tractable. Next, we turn to the agent’s financial investment opportunities.

**Financial investment opportunities.** The agent can invest in a risk-free asset which pays a constant rate of interest $r$ and the risky market portfolio (Merton (1971)). Assume that the incremental return $dR_t$ of the market portfolio over time period $dt$ is i.i.d., i.e.

$$dR_t = \mu_R dt + \sigma_R dB_t,$$

where $\mu_R$ and $\sigma_R$ are constant mean and volatility parameters of the market portfolio return process, and $B$ is a standard Brownian motion. Let

$$\eta = \frac{\mu_R - r}{\sigma_R}$$

(10)

denote the Sharpe ratio of the market portfolio. Let $\rho$ denote the correlation coefficient between the shock to the entrepreneur’s business and the shock to the market portfolio. With incomplete markets ($|\rho| < 1$), the entrepreneur cannot completely hedge business risk. Non-diversifiable risk will thus play a role in decision making and private valuation.

Let $W$ and $X$ denote the agent’s financial wealth and the amount invested in the risky asset, respectively. Then, $(W - X)$ is the remaining amount invested in the risk-free asset. Before becoming an entrepreneur ($t \leq T^0$), the wealth accumulation is given by

$$dW_t = r (W_t - X_t) dt + \mu_R X_t dt + \sigma_R X_t dB_t - C_t dt + r \Pi dt, \quad t < T^0.$$  

(11)

While being an entrepreneur, the liquid financial wealth $W$ evolves as follows:

$$dW_t = r (W_t - X_t) dt + \mu_R X_t dt + \sigma_R X_t dB_t - C_t dt + dY_t, \quad T^0 < t < T^l.$$  

(12)

Finally, after exiting from the business, the retired entrepreneur’s wealth evolves as follows:

$$dW_t = r (W_t - X_t) dt + \mu_R X_t dt + \sigma_R X_t dB_t - C_t dt, \quad t > T^l.$$  

(13)

The entrepreneur can borrow against capital $K$ at all times, and hence wealth $W$ can be negative. To ensure that entrepreneurial borrowing is risk-free, we require that the

\textsuperscript{14}Cox, Ingersoll, and Ross (1985) feature an equilibrium production economy with the “AK” technology. See Jones and Manuelli (2005) for a recent survey on endogenous growth models.
The liquidation value of capital $lK$ is greater than outstanding liability, in that

$$W_t \geq -lK_t, \quad T^0 \leq t \leq T^l. \quad (14)$$

Despite being able to borrow up to $lK_t$ at the risk-free rate $r$, the entrepreneur may rationally choose not to exhaust the debt capacity for precautionary reasons. Without capital as collateral, the agent cannot borrow: $W_t \geq 0$ for $t \leq T^0$ and $t \geq T^l$.

**The optimization problem.** The agent maximizes the utility defined in (1)-(2). The timeline can be described in five steps. First, before becoming an entrepreneur ($t \leq T^0$), the agent collects income as a worker and chooses consumption and portfolio allocations. Second, the agent chooses the optimal entry time $T^0$ to start up the firm and the initial firm size $K_{T^0}$ by incurring the fixed start-up cost $\Phi$, and financing the total costs ($K_{T^0} + \Phi$) with savings and/or potentially some collateralized borrowing. Third, the agent chooses consumption and portfolio choice while running the firm subject to the collateralized borrowing limit (14). Fourth, the agent optimally chooses the stochastic liquidation time $T^l$. Finally, after liquidating capital, the agent collects the liquidation proceeds, retires, allocates wealth between the risk-free and the risky market portfolio, and consumes.

### 3 Benchmark: Complete markets

With complete markets, the entrepreneur’s optimization problem can be decomposed into two separate ones: wealth maximization and utility maximization. We will show that our model has the homogeneity property. The lower case denotes the corresponding variable in the upper case scaled by $K$. For example, $w$ denotes the wealth-capital ratio $W/K$. The following proposition summarizes our main results under complete markets.

**Proposition 1** The entrepreneur’s value function $J^{FB}(K,W)$ is given by

$$J^{FB}(K,W) = \frac{(bP^{FB}(K,W))^{1-\gamma}}{1-\gamma}, \quad (15)$$

where the total wealth $P^{FB}(K,W)$ is given by the sum of $W$ and firm value $Q^{FB}(K)$

$$P^{FB}(K,W) = W + Q^{FB}(K) = W + q^{FB}K, \quad (16)$$

12
and
\[ b = \zeta \left[ 1 + \frac{1 - \psi}{\zeta} \left( r - \zeta + \frac{\eta^2}{2\gamma} \right) \right]^{\frac{1}{\psi}}. \]  \hspace{1cm} (17)

Firm value \( Q^{FB}(K) \) is equal to \( q^{FB}K \), where Tobin’s \( q \), \( q^{FB} \), is given by
\[ q^{FB} = 1 + \theta i^{FB}, \]  \hspace{1cm} (18)
where the first-best investment-capital ratio \( i^{FB} \) is given by
\[ i^{FB} = (r + \delta) - \sqrt{(r + \delta)^2 - \frac{2}{\theta} \left( \mu_A - \rho \sigma_A - (r + \delta) \right)}. \]  \hspace{1cm} (19)

The optimal consumption \( C \) is proportional to \( K \), i.e. \( C(K,W) = c^{FB}(w)K \), where
\[ c^{FB}(w) = m^{FB} (w + q^{FB}), \]  \hspace{1cm} (20)
and \( m^{FB} \) is the marginal propensity to consume (MPC) and is given by
\[ m^{FB} = \zeta + (1 - \psi) \left( r - \zeta + \frac{\eta^2}{2\gamma} \right). \]  \hspace{1cm} (21)

The market portfolio allocation \( X \) is also proportional to \( K \), \( X(K,W) = x^{FB}(w)K \), where
\[ x^{FB}(w) = \left( \frac{\mu_R - r}{\gamma \sigma_R^2} \right) (w + q^{FB}) - \frac{\rho \sigma_A}{\sigma_R}. \]  \hspace{1cm} (22)

The capital asset pricing model (CAPM) holds for the firm with its expected return given by
\[ \xi^{FB} = r + \beta^{FB} (\mu_R - r), \]  \hspace{1cm} (23)
where the firm’s beta, \( \beta^{FB} \), is constant and given by
\[ \beta^{FB} = \frac{\rho \sigma_A}{\sigma_R} \frac{1}{q^{FB}}. \]  \hspace{1cm} (24)

Equations (18) and (19) give Tobin’s \( q \) and the investment-capital ratio, respectively. The adjustment cost makes installed capital earn rents and, hence, Tobin’s \( q \) differs from unity. Note that the average \( q \) is equal to the marginal \( q \) as in Hayashi (1982). The entrepreneur’s total wealth is given by \( p^{FB}(w) = w + q^{FB} \), the sum of Tobin’s \( q \) and \( w \). Equation (20) gives consumption, effectively the permanent-income rule under complete markets. The entrepreneur’s MPC out of wealth \( m^{FB} \) generally depends on the risk-free rate \( r \), the EIS \( \psi \), the coefficient of risk aversion \( \gamma \), and the Sharpe ratio \( \eta = (\mu_R - r)/\sigma_R \). Equation (22) gives
The portfolio allocation to the market portfolio. The first term in (22) is the well-known mean-variance allocation, and the second term is the intertemporal hedging demand.

We explicitly account for the effects of risk on investment and Tobin’s \( q \). We decompose the total volatility of the productivity shock into systematic and idiosyncratic components. The systematic volatility is equal to \( \rho \sigma_A \) and the idiosyncratic component is given by

\[
\epsilon = \sqrt{1 - \rho^2} \sigma_A.
\]

The standard CAPM holds in our benchmark. The expected return is given in (23) and \( \beta \) is given by (24). As in the standard finance theory, the idiosyncratic volatility \( \epsilon \) carries no risk premium and plays no role under complete markets. However, importantly, the idiosyncratic volatility \( \epsilon \) will play a significant role in our incomplete-markets setting.

4 Incomplete-markets model solution after entry

Having characterized the complete-markets solution, we now turn to the incomplete-markets setting. We first consider the agent’s decision problem after liquidation, and then derive the entrepreneur’s interdependent decision making before exit.

The agent’s decision problem after exiting entrepreneurship. After exiting from entrepreneurship, the entrepreneur is no longer exposed to the business risk and faces a classic Merton consumption/portfolio allocation problem with non-expected recursive utility. The solution is effectively the same as the complete-markets results in Proposition 1 (without physical capital). We summarize the results as a corollary to Proposition 1.

Corollary 1 The entrepreneur’s value function takes the following homothetic form

\[
V(W) = \frac{(bW)^{1-\gamma}}{1-\gamma},
\]

where \( b \) is a constant given in (17). The optimal consumption \( C \) and allocation amount \( X \) in the risky market portfolio are respectively given by

\[
C = m^{FB} W, \quad \text{and} \quad X = \left( \frac{\mu_R - r}{\gamma \sigma_R^2} \right) W,
\]

where \( m^{FB} \) is the MPC out of wealth and is given in (21).

We will use the value function given in (26) when analyzing the agent’s pre-exit decisions.
The entrepreneur’s decision problem while running his business. Let \( J(K,W) \) denote the entrepreneur’s value function. The entrepreneur chooses consumption \( C \), real investment \( I \), and the allocation to the risky market portfolio \( X \) by solving the following Hamilton-Jacobi-Bellman (HJB) equation

\[
0 = \max_{C,I,X} f(C,J) + (I - \delta K)J_K + (rW + (\mu_R - r)X + \mu_A K - I - G(I,K) - C)J_W \\
+ \left( \frac{\sigma_A^2 K^2 + 2\rho\sigma_A \sigma_R K X + \sigma_R^2 X^2}{2} \right) J_W W.
\]

(29)

The entrepreneur’s first-order condition (FOC) for consumption \( C \) is given by

\[
f_C(C,J) = J_W(K,W).
\]

(30)

The above condition states that the marginal utility of consumption \( f_C \) is equal to the marginal utility of wealth \( J_W \). The FOC with respect to investment \( I \) gives

\[
(1 + G_I(I,K)) J_W(K,W) = J_K(K,W).
\]

(31)

To increase capital stock by one unit, the entrepreneur needs to forgo \((1 + G_I(I,K))\) units of consumption. The marginal utility of consumption is equal to the marginal utility of wealth \( J_W \). Therefore, the entrepreneur’s marginal cost of investing is given by the product of \((1 + G_I(I,K))\) and the marginal utility of wealth \( J_W \). The marginal benefit of increasing capital stock by a unit is \( J_K \). At optimality, the entrepreneur equates the two sides of (31).

The FOC with respect to portfolio choice \( X \) is given by

\[
X = -\frac{\mu_R - r}{\sigma_R^2} \frac{J_W}{J_W W} - \frac{\rho\sigma_A}{\sigma_R} K.
\]

(32)

The first term in (32) is the mean-variance demand, and the second term captures the hedging demand. Because the entrepreneur’s effective risk aversion depends on \( w \), the mean-variance demand is much more interesting under incomplete markets than it is under complete markets. Using the homogeneity property, we conjecture that the value function \( J(K,W) \) is given by

\[
J(K,W) = \frac{(bP(K,W))^{1-\gamma}}{1-\gamma},
\]

(33)

where \( b \) is given in (17). Comparing (33) with the value function without the business (26), we may intuitively refer to \( P(K,W) \) as the entrepreneur’s certainty equivalent wealth, the
minimal amount of wealth for which the agent is willing to permanently give up the business and liquid wealth $W$. Let $W$ denote the entrepreneur’s endogenous liquidation boundary and $w = W/K$. The following theorem summarizes the entrepreneur’s decision making and certainty equivalent wealth $p(w) = P(K, W)/K$.

**Theorem 1** The entrepreneur operates the business if and only if $w \geq \bar{w}$. The scaled certainty equivalent wealth $p(w)$ solves the following ordinary differential equation (ODE)

$$0 = \frac{m^{FB} p(w) (p'(w))^{1-\psi} - \psi p(w)}{\psi - 1} - \delta p(w) + (r + \delta) wp'(w) + (\mu_A - \rho \sigma_A) p'(w)$$

$$+ \frac{(p(w) - (w + 1)p'(w))^2}{2 \theta p'(w)} + \frac{\eta^2 p(w)p'(w)}{2 h(w)} - \frac{\epsilon^2 h(w)p'(w)}{2 p(w)}, \quad \text{if } w \geq \bar{w},$$

where $\epsilon$ is the idiosyncratic volatility given in (25) and $h(w)$ is given by

$$h(w) = \gamma p'(w) - \frac{p(w) p''(w)}{p'(w)}.$$  

When $w$ approaches $\infty$, $p(w)$ approaches complete-markets solution given by

$$\lim_{w \to \infty} p(w) = w + q^{FB}.$$  

Finally, the ODE (34) satisfies the following conditions at the endogenous boundary $\bar{w}$

$$p(w) = w + l,$$

$$p'(w) = 1.$$  

The optimal consumption $c = C/K$, investment $i = I/K$, and market portfolio allocation-capital ratio $x = X/K$ are given by

$$c(w) = m^{FB} p(w) (p'(w))^{-\psi},$$

$$i(w) = \frac{1}{\theta} \left( \frac{p(w)}{p'(w)} - w - 1 \right),$$

$$x(w) = -\frac{\rho \sigma_A}{\sigma_R} + \frac{\mu_R - r}{\sigma_R^2} \frac{p(w)}{h(w)},$$

where $h(w)$ is given in (35). The dynamics of the wealth-capital ratio $w$ are given by

$$dw_t = \mu_w(w_t) dt + \sigma_R x(w_t) dB_t + \sigma_A dZ_t,$$

where the drift $\mu_w(w)$ gives the expected change of $w$ and is given by

$$\mu_w(w) = (r + \delta - i(w))w + (\mu_R - r)x(w) + \mu_A - i(w) - g(i(w)) - c(w).$$  

However, if the conditions (37)-(38) do not admit an interior solution satisfying $w > -l$, the optimal liquidation boundary is then given by the maximal borrowing capacity, i.e. $w = -l$.  

16
Incomplete-markets model results after entry

We now explore the implications of the incomplete-markets model in Section 4. We choose parameter values as follows and, whenever applicable, all parameters are annualized. The risk-free interest rate is \( r = 4.6\% \) and the aggregate equity risk premium is \((\mu_R - r) = 6\%\). The annual volatility of the market portfolio return is \( \sigma_R = 20\% \) implying the Sharpe ratio for the aggregate stock market \( \eta = (\mu_R - r)/\sigma_R = 30\% \). The subjective discount rate is set to equal to the risk-free rate, \( \zeta = r = 4.6\% \).

On the real investment side, our model is a version of the \( q \) theory of investment (Hayashi (1982)). Using the sample of large firms in Compustat from 1981 to 2003, Eberly, Rebelo, and Vincent (2009) provide empirical evidence in support of Hayashi (1982). Using their work as a guideline, we set the expected productivity \( \mu_A = 20\% \) and the volatility of productivity shocks \( \sigma_A = 10\% \). Fitting the complete-markets \( q^{FB} \) and \( i^{FB} \) to the sample averages, we obtain the adjustment cost parameter \( \theta = 2 \) and the rate of depreciation for capital stock \( \delta = 12.5\% \). We choose the liquidation parameter \( l = 0.9 \) (Hennessy and Whited (2007)).

We set the correlation between the market portfolio return and the business risk \( \rho = 0 \), which implies that the idiosyncratic volatility of the productivity shock \( \epsilon = \sigma_A = 10\% \). We consider two widely used values for the coefficient of relative risk aversion, \( \gamma = 2 \) and \( \gamma = 4 \). We set the EIS to be \( \psi = 0.5 \), so that the first case corresponds to the expected utility with \( \gamma = 1/\psi = 2 \), and the second case maps to a non-expected utility with \( \gamma = 4 > 1/\psi = 2 \).

Table 1 summarizes the notations and if applicable, value choices for various parameters.

### 5.1 Decomposing the entrepreneur’s welfare

We measure the entrepreneur’s welfare via the value function given in (33). The value function \( J(K, W) \) is homogeneous of degree \((1 - \gamma)\) in the certainty equivalent wealth \( P(K, W) \). Therefore, we may equivalently quantify the agent’s welfare via \( P(K, W) \).

**Certainty equivalent wealth and private enterprise value.** In corporate finance, enterprise value is defined as firm value excluding liquid assets (e.g. cash and short-term marketable assets). For the entrepreneurial firm, we may similarly define the entrepreneur’s
A. Certainty equivalent wealth–capital ratio: \( p(w) \)

B. Private enterprise value–capital ratio: \( q(w) \)

C. Marginal value of liquid wealth: \( P_W(K,W) \)

D. Marginal value of physical capital: \( P_K(K,W) \)

Figure 1: **Certainty equivalent wealth** \( p(w) \), **private enterprise value ratio** \( q(w) \), **marginal value of wealth** \( P_W(K,W) \), and **marginal value of capital** \( P_K(K,W) \).

The private enterprise value \( Q(K,W) \) as follows

\[
Q(K,W) = P(K,W) - W. \tag{44}
\]

Dividing the private enterprise value \( Q(K,W) \) by illiquid physical capital \( K \), we have

\[
q(w) = \frac{Q(K,W)}{K} = p(w) - w. \tag{45}
\]

For a firm owned and managed by an entrepreneur, (45) captures the impact of non-diversifiable risk on the subjective valuation of capital. Importantly, the “private” average \( q \) defined in (45) depends on the entrepreneur’s preferences.

For Figures 1-5, we graph for two levels of risk aversion, \( \gamma = 2, 4 \). Panels A and B of Figure 1 plot \( p(w) \) and \( q(w) \), respectively. Note that \( q(w) = p(w) - w \). Thus, \( p(w) \) and
$q(w)$ effectively convey the same information. Graphically, it is easier to read Panel B for $q(w)$ than Panel A for $p(w)$, we thus discuss $q(w)$. Recall that the first-best Tobin’s $q$, $q^{FB}$, is independent of entrepreneurial preferences (complete-markets Arrow-Debreu separation results). For our baseline calculation, we have this complete-markets Tobin’s $q$, $q^{FB} = 1.31$. For finitely valued $w$, $q(w)$ increases with $w$. As $w \to \infty$, the entrepreneur effectively attaches no premium for the non-diversifiable risk and $q(w)$ approaches the complete-markets $q^{FB}$, $\lim_{w \to \infty} q(w) \to q^{FB} = 1.31$. However, quantitatively, the convergence requires a relatively high value of $w$. When $w = 3$, we have $q(3) = 1.23$ for $\gamma = 2$, and $q(3) = 1.21$ for $\gamma = 4$, both of which are significantly lower than $q^{FB} = 1.31$. The less risk-averse the entrepreneur, the higher Tobin’s $q$, $q(w)$.

More interestingly, $q(w)$ is not globally concave. Risk aversion does not necessarily imply that $q(w)$ is concave. Nonetheless, the risk-averse entrepreneur’s value function $J(K,W)$ is concave in $P(K,W)$ and is also concave in $W$. Figure 1 shows that $q(w)$ is concave in $w$ for sufficiently high $w$, i.e. $w \geq \tilde{w}$ where $\tilde{w}$ is the inflection point at which $p''(\tilde{w}) = q''(\tilde{w}) = 0$. For sufficiently low $w$, i.e. $w \leq \tilde{w}$, $q(w)$ is convex in $w$.

The entrepreneur has an option to eliminate the non-diversifiable business risk exposure by liquidating the firm. The option to exit from the business causes $q(w)$ to be convex in $w$ for sufficiently low $w$. Costly liquidation of capital provides a downside risk protection for the entrepreneur. Quantitatively, this exit option is quite valuable for low $w$. Recall that debt is fully collateralized and is risk-free. Thus, liquidation only provides an exit option which becomes in the money for the entrepreneur bearing significant non-diversifiable risks (i.e. being sufficiently low in $w$). In Zame (1993), Heaton and Lucas (2004), and Chen, Miao, and Wang (2010), the benefits of debt rely on the riskiness of debt, which creates state-contingent insurance. Both arguments rely on “put” options (a liquidation option in our model and a default option in risky debt models) providing downside protection. We now decompose the certainty equivalent wealth $P(K,W)$.

Decomposing the certainty equivalent (CE) wealth $P(K,W)$. Using the homogeneity property, we have

\begin{align}
P_W(K,W) & = p'(w), \quad (46) \\
P_K(K,W) & = p(w) - wp'(w). \quad (47)
\end{align}

19
For public firms owned by diversified investors, the marginal increase of firm value associated with a unit increase of capital is often referred to as marginal $q$. For a firm owned and managed by a non-diversified entrepreneur, the marginal increase of $P(K,W)$ associated with a unit increase of capital, $P_K(K,W)$, is the natural counterpart to the marginal $q$ for public firms. We refer to $P_K(K,W)$ as the private (i.e. subjective) marginal $q$ for the entrepreneurial firm. For public firms, the marginal increase of firm value associated with a unit increase of cash is referred to as the marginal value of cash (Bolton, Chen, and Wang (2011)). For entrepreneurial firms, the entrepreneur’s marginal value of wealth $P_W(K,W)$ is the natural counterpart to the marginal value of cash for public firms.

The marginal value of wealth $P_W(K,W)$. The lower left panel (Panel C) of Figure 1 plots $P_W(K,W) = p'(w)$. In perfect capital markets, $P_W^{FB}(K,W) = 1$. With incomplete markets, $P_W(K,W)$ is greater than unity because wealth has the additional benefit of mitigating the negative impact of financial frictions on investment and consumption. Panel C shows that $p'(w)$ is equal to unity at the liquidation boundary $w$, $p'(w) = 1$, because the agent is no longer exposed to non-diversifiable risk after exiting entrepreneurship. Then, $p'(w)$ increases with $w$ up to the endogenous inflection point $\tilde{w}$ (at which $p''(\tilde{w}) = 0$), decreases with $w$ for $w \geq \tilde{w}$, and finally approaches unity as $w \to \infty$ and non-diversifiable risk no longer matters.

Loose arguments may have led us to conclude that less constrained entrepreneurs (i.e. those with higher wealth) value their wealth less and $P_W(K,W)$ will globally decrease with wealth ($p''(w) < 0$). This is incorrect as we see from Panel C. The convexity of $p(w)$ (i.e. $p''(w) > 0$ for $w \leq \tilde{w}$) arises because the liquidation option provides a valuable exit option for non-diversified entrepreneurs under incomplete markets.

The marginal value of capital $P_K(K,W)$. The lower right panel (Panel D) of Figure 1 plots the private marginal $q$, $P_K(K,W)$. Perhaps surprisingly, the private marginal $q$ is not monotonic in $w$. One seemingly natural but loose intuition is that the (private) marginal $q$ increases with financial slack measured by $w$. Presumably, less financially constrained entrepreneurs face lower costs of investment and hence have higher marginal $q$. However, this intuition in general does not hold. Using the analytical formula (47) for private marginal
\[ \frac{dP_K(K, W)}{dw} = -wp''(w). \] (48)

Therefore, the sign of \( dP_K(K, W)/dw \) depends on both the sign of \( w \) and the concavity of \( p(w) \). When \( w > 0 \) and \( p(w) \) is concave, \( P_K(K, W) \) increases with \( w \) (see the right end of Panel D). When the entrepreneur is in debt \( (w < 0) \) and additionally \( p(w) \) is convex, \( P_K(K, W) \) also increases with \( w \) (see the left end of Panel D). In the intermediate region of \( w \), \( P(K, W) \) may decrease with \( w \) (for example, when \( w < 0 \) and \( p''(w) < 0 \)).

The private marginal \( q \), \( P_K(K, W) \), is linked to the average \( q \), \( q(w) \), as follows,
\[ P_K(K, W) = q(w) - w(p'(w) - 1). \] (49)

The wedge between private marginal \( q \) and private average \( q \), \( P_K(K, W) - q(w) \), can be either negative or positive depending on the sign of \( w \) because \( p'(w) \geq 1 \). For entrepreneurs with positive wealth \( (w > 0) \), increasing \( K \) mechanically lowers \( w = W/K \), which further lowers the marginal product of capital and gives rise to a negative wedge \( P_K(K, W) - q(w) \). However, for an entrepreneur in debt \( (W < 0) \), an increase \( K \) leads to an increase in \( w = W/K \) (by moving towards zero from the left of the origin) and hence implies a positive wedge \( P_K(K, W) - q(w) \).

### 5.2 Optimal capital accumulation

Using \( P(K, W) \), we rewrite the FOC (31) for investment as follows,
\[ (1 + \theta_i(w)) P_W(K, W) = P_K(K, W). \] (50)

To install a unit of capital, the entrepreneur needs to incur cost \( 1 + \theta_i(w) \) at the margin. The incremental cost \( \theta_i(w) \) is the marginal adjustment cost (e.g. over-time marginal labor costs and marginal installation costs) beyond the unit capital purchase cost. Moreover, the marginal cost of using a unit of wealth for the entrepreneur is \( P_W(K, W) = p'(w) \). Therefore, the marginal cost of installing a unit of capital is given by \( (1 + \theta_i(w)) P_W(K, W) \), the left side of (50). The right side is the marginal value of capital \( P_K \). The entrepreneur equates the two sides of (50) by optimally choosing investment. The FOC (50) states that the entrepreneur’s optimal investment decision depends on the ratio between the private marginal \( q \), \( P_K(K, W) \), and the private marginal value of wealth \( P_W(K, W) \). Both the private marginal \( q \) and \( P_W \) are endogenously determined. Moreover, they are highly correlated.
Figure 2: **Investment-capital ratio** $i(w)$ and investment-wealth sensitivity $i'(w)$.

Figure 2 plots $i(w)$ and $i'(w)$, the sensitivity of $i(w)$ with respect to $w$. Non-diversifiable business risk induces underinvestment, $i(w) < i^{FB} = 0.15$. The underinvestment result (relative to the first-best MM benchmark) is common in incomplete-markets models.

More interestingly and less intuitively, investment-capital ratio is not monotonic in $w$. That is, investment may decrease with wealth! This seemingly counter-intuitive result directly follows from the convexity of $p(w)$ in $w$. We may characterize $i'(w)$ as follows,

$$ i'(w) = -\frac{p(w)p''(w)}{\theta(p'(w))^2}. \tag{51} $$

Using the above result, we see that whenever $p(w)$ is concave, investment increases with wealth. However, whenever $p(w)$ is convex, investment decreases with $w$. Put differently, underinvestment is less of a concern when the entrepreneur is closer to liquidating the business because liquidation also has the benefit of leading the entrepreneur to exit incomplete markets. The entrepreneur has weaker incentives to cut investment if the distance to exiting incomplete markets is shorter. This explains why investment may decrease in $w$ when the exit option is sufficiently close to being in the money (i.e. when $w$ is sufficiently low).

The entrepreneur always has the option to exit the business by liquidating capital. The downside risk is thus capped by his exit option. After exiting, the entrepreneur is then only exposed to systematic shocks. This exit option induces a convexity effect of volatility on $p(w)$ for sufficiently low $w$. Note that the entrepreneur’s value function $J(K,W)$ is concave.
in $P(K, W)$ as well as concave in $W$. However, volatility increases $p(w)$ in the region $w \leq \tilde{w}$, where the inflection point $\tilde{w}$ is defined by $p''(\tilde{w}) = 0$.

### 5.3 Optimal liquidation decision

Now we turn to the entrepreneur’s liquidation decision. Because diversification benefits are more important for more risk-averse entrepreneurs, a more risk-averse entrepreneur liquidates capital earlier in order to avoid idiosyncratic risk exposure and achieve full diversification. For example, the optimal liquidation boundaries are $\underline{w} = -0.8$ and $\overline{w} = -0.65$ for $\gamma = 2$ and $\gamma = 4$, respectively. The (American) option to convert an illiquid risky business into liquid financial assets is more valuable for more risk-averse entrepreneurs. Note that the borrowing constraint does not bind even for a less risk-averse entrepreneur (e.g. $\gamma = 2$). The entrepreneur rationally liquidates capital before exhausting the debt capacity $w \geq -l = -0.9$ to ensure that wealth does not fall too low. While borrowing more to invest is desirable in terms of generating positive value for (diversified) investors, doing so may be too risky for non-diversified entrepreneurs. Moreover, anticipating that the liquidation option will soon be exercised, the entrepreneur has less incentive to distort investment when $w$ is close to the liquidation boundary. This option anticipation effect explains the non-monotonicity result for $i(w)$ in $w$. Next, we turn to the entrepreneur’s portfolio choice decisions.

### 5.4 Optimal portfolio allocation

The entrepreneur’s market portfolio allocation $x(w)$ has both a hedging demand term given by $-\rho \sigma_A / \sigma_R$ and a mean-variance demand term given by $\eta \sigma_R^{-1} p(w) / h(w)$. The constant hedging term $-\rho \sigma_A / \sigma_R$ is standard because of constant real and financial investment opportunities (Merton (1973)). We focus on the more interesting mean-variance demand term.

Unlike the standard portfolio allocation, the entrepreneur incorporates the impact of non-diversifiable risk by (1) replacing $w + q^{FB}$ with $p(w)$ in calculating “total” wealth and (2) adjusting risk aversion from $\gamma$ to the effective risk aversion $h(w)$ given in (35).

Figure 3 plots $h(w)$ for $\gamma = 2, 4$. In the limit as $w \to \infty$, idiosyncratic business risk has no role, markets are effectively complete for the entrepreneur, and $h(w)$ approaches the complete-markets value, $h(w) \to \gamma$. However, when the entrepreneur is close to liquidating the firm (i.e. $w \to \underline{w}$), the effective risk aversion $h(w)$ is lower than $\gamma$. This can be seen
from \( h(w) = \gamma - (w + l)p''(w) < \gamma \). The intuition is as follows. When the liquidation option is in the money, the entrepreneur behaves in a less risk averse manner than without the liquidation option under complete markets, because of the positive effect of volatility on the option value. This argument implies that at the moment of liquidation, the effective risk aversion \( h(w) < \gamma \). Figure 3 also shows that \( h(w) \) is not monotonic in \( w \). However, for most values of \( w \) (other than near the liquidation boundary), the effective risk aversion \( h(w) \) is higher than \( \gamma \) due to non-diversiable risk, consistent with our intuition.

Panel A of Figure 4 plots the demand for the market portfolio \( x(w) \) under incomplete markets and shows \( x(w) < x^{FB}(w) \) for both \( \gamma = 2 \) and \( \gamma = 4 \). The straight lines are for the complete-markets benchmark. The demand for the market portfolio for the non-diversified entrepreneur is lower than that under the complete-markets benchmark. This is consistent with empirical findings. For example, Heaton and Lucas (2000) find that entrepreneurs with high and variable business income hold less wealth in stocks than other similarly wealthy households. Note that \( x(w) \) is not monotonic in \( w \) due to the optionality of liquidation. Panel B of Figure 4 shows that \( x'(w) \) can be either positive or negative. On the left end, \( x'(w) < 0 \) implying that increasing \( w \), while making the entrepreneur less financially constrained, lowers the demand for risky assets due to the reduction of optionality (further away from the
5.5 Optimal consumption and the MPCs

Recall that the consumption-capital ratio is given by $c(w) = m^{FB}p(w)(p'(w))^{-\psi}$. Consumption is thus lower under incomplete markets than under complete markets, $c(w) < c^{FB}(w)$, for two reasons: $p(w) < p^{FB}(w)$ and $p'(w) > 1$. Panel A of Figure 5 plots $c(w)$ for $\gamma = 2, 4$, and confirm that $c(w) < c^{FB}(w)$ for all $w$. For both $\gamma = 2$ and $\gamma = 4$, the EIS $\psi$ is set at 0.5. The MPC $m^{FB} = 0.057$ for $\gamma = 2$, which is higher than $m^{FB} = 0.052$ for $\gamma = 4$. The less risk-averse entrepreneur consumes more (if $\psi < 1$). Panel B of Figure 5 plots the MPC out of wealth, $C_W(K,W) = c'(w)$. Note that the MPC $C_W = c'(w)$ is not monotonic in $w$. The MPC $C_W(K,W)$ first increases with $w$ and then decreases with $w$.

Let $m(w)$ denote the ratio between $C(K,W)$ and CE wealth $P(K,W)$. We have

$$m(w) = \frac{C(K,W)}{P(K,W)} = \frac{c(w)}{p(w)} = m^{FB}(p'(w))^{-\psi}. \quad (52)$$

Note that $m(w)$ is always lower than the MPC under complete markets, $m(w) \leq m^{FB}$ because $p'(w) \geq 1$. See Panel C of Figure 5. The entrepreneur has an additional motive to save, and consumption is more costly than under complete markets. Note that the sensitivity $m'(w)$ is not monotonic in $w$ (see Panel D of Figure 5). Indeed, $m'(w)$ has the same sign as that of $-p''(w)$. Since $p(w)$ is not globally concave, $m(w)$ is thus not monotonic in $w$. 

Figure 4: Market portfolio allocation-capital ratio $x(w)$ and its sensitivity $x'(w)$.

liquidation boundary). Next, we turn to the entrepreneur’s consumption decisions.
Figure 5: Consumption-capital ratio $c(w)$, the MPC out of wealth $c'(w)$, consumption-CE wealth ratio $m(w) = c(w)/p(w)$, and its sensitivity $m'(w)$.

## 6 Entrepreneurial entry: Career choice and firm size

We have studied the agent’s decision making and valuation after becoming an entrepreneur. However, what causes the agent to become an entrepreneur and when? The entrepreneurship choice is clearly an important decision. We analyze two cases: first, a time-0 binary career decision and then a richer model allowing for the choice of entry timing. For both cases, we will also study the determinants of the initial firm size.

### 6.1 When career choice is “now or never,” a binary decision

First consider the case where the agent has a static time-0 binary choice to be an entrepreneur or take the outside option. By taking the outside option, the agent collects a constant
perpetuity with payment \( r \Pi \), which has present value \( \Pi \). The agent’s optimal consumption and portfolio choice problem gives the value function \( V(W_0 + \Pi) \) where \( V(\cdot) \) is given in (26).

By being an entrepreneur, the agent incurs a fixed start-up cost \( \Phi \) and then chooses the initial project size \( K_0 \). Wealth immediately drops from \( W_0 \) to \( W_0 - (\Phi + K_0) \) at time 0. Note that the entrepreneur can borrow up to \( lK \), the liquidation value of capital, which implies

\[
W_0 \geq \Phi + (1 - l)K_0.
\]  

(53)

To rule out the uninteresting case where the entrepreneur makes instant profits by starting up the business and then immediately liquidating capital for profit, we require \( l < 1 \).

The value function is given by \( J(K_0, W_0 - (\Phi + K_0)) \), and the certainty equivalent wealth is \( P(K_0, W_0 - (\Phi + K_0)) = p(w_0 - 1 - \Phi/K_0)K_0 \). The agent chooses \( K_0 \) to maximize \( J(K_0, W_0 - (\Phi + K_0)) \), which is equivalent to maximizing \( P(K_0, W_0 - (\Phi + K_0)) \) by solving

\[
\max_{K_0} P(K_0, W_0 - (\Phi + K_0)),
\]  

(54)

subject to the borrowing constraint (53). Let \( K_0^* \) denote the optimal initial capital stock. Finally, the agent compares \( P(K_0^*, W_0 - (\Phi + K_0^*)) \) from being an entrepreneur with \( W_0 + \Pi \), and makes the career decision. The following theorem summarizes the main results.

**Theorem 2** At time 0, an agent chooses to be an entrepreneur if and only if the initial wealth \( W_0 \) is greater than the threshold wealth level \( \overline{W}_0 \), which is given by

\[
\overline{W}_0 = \frac{\Phi p'(w^*) + \Pi}{p'(w^*) - 1},
\]  

(55)

and \( w^* \) is the solution of the following equation

\[
p'(w^*) = \frac{p(w^*)}{1 + w^*}.
\]  

(56)

The entrepreneurial firm’s initial size \( K_0^* \) is given by

\[
K_0^* = \frac{W_0 - \Phi}{1 + w^*}.
\]  

(57)

The entrepreneur’s certainty equivalent wealth is then given by

\[
P(K_0^*, W_0 - \Phi - K_0^*) = p(w^*)K_0^* = p'(w^*) (W_0 - \Phi),
\]  

(58)

where \( w^* \) is given by (56). After starting up the firm, the agent chooses consumption, portfolio allocation, and firm investment/liquidation decisions as described by Theorem 1.
Figure 6: Entrepreneurial entry in a time-0 (now or never) binary setting: Initial firm size $K^*_0$ and initial certainty equivalent wealth $P(K^*_0, W_0 - \Phi - K^*_0)$. The outside option value is $\Pi = 0.5$ and the fixed start-up cost $\Phi = 0.05$.

Figure 6 plots the firm’s initial size $K^*_0$ and the initial certainty equivalent wealth $P(K^*_0, W_0 - \Phi - K^*_0)$ as functions of initial wealth $W_0$ for two levels of risk aversion, $\gamma = 2, 4$. First, risk aversion plays a significant role in determining entrepreneurship. The threshold for the initial wealth $W_0$ to become an entrepreneur increases significantly from 2.86 to 4.60 when risk aversion $\gamma$ increases from 2 to 4. Second, entrepreneurs are wealth constrained and the initial wealth $W_0$ has a significant impact on firm size $K^*_0$. Note that $K^*_0$ is a linear function of $W_0$ due to the homogeneity property. Additionally, a unit increase in initial wealth $W_0$ leads to an increase in $K^*_0$ by more than unity. Moreover, this effect is greater for less risk-averse entrepreneurs. For example, the slope of the linear function for $K^*_0(W_0)$ is 1.57 for $\gamma = 2$, which is significantly higher than 1.07, the slope of $K^*_0(W_0)$ for $\gamma = 4$. Finally, the marginal effect of initial wealth $W_0$ is also higher for less risk-averse entrepreneurs if we measure in the certainty equivalent wealth $P(K^*_0, W_0 - \Phi - K^*_0)$. Note that $P(K^*_0, W_0 - \Phi - K^*_0)$ is also linear in $W_0$. The slope of the initial certainty equivalent wealth in $W_0$ is 1.2 for $\gamma = 2$ and 1.12 for $\gamma = 4$ as seen in the right panel of Figure 6. The effects of initial wealth $W_0$ on the entry decision of entrepreneurship are not the reflection of liquidity constraints per se, but rather the interaction between the liquidity constraints and non-diversifiable risk.
6.2 When career choice is flexible: Optimal entry timing

We now allow the agent to choose the optimal entry time rather than restricting the decision to be binary at time 0. We will show that this additional flexibility allows the agent to build up financial strength before becoming an entrepreneur, which is highly valuable. For simplicity, we assume becoming an entrepreneur is irreversible.

Let $F(W)$ denote the agent’s value function before becoming an entrepreneur. Using an argument similar to our earlier analysis, we conjecture that $F(W)$ is given by

$$F(W) = \frac{(bE(W))^{1-\gamma}}{1 - \gamma},$$

where $b$ is the constant given by (17) and $E(W)$ is the agent’s certainty equivalent wealth.

We will show that the entrepreneurship decision is characterized by an endogenous cutoff threshold $\hat{W}$. When $W_t \geq \hat{W}$, the agent immediately enters entrepreneurship. Otherwise, the agent takes the outside option, builds up financial wealth, and becomes an entrepreneur when wealth reaches $\hat{W}$. We now summarize the results for career choice with flexible timing.

**Theorem 3** Provided that $W \leq \hat{W}$, the agent’s certainty equivalent wealth $E(W)$ solves

$$0 = \frac{mFBE(W)(E'(W))^{1-\psi} - \psi\zeta E(W)}{\psi - 1} + r(W + \Pi)E'(W) + \frac{\eta^2}{2 \gamma E'(W)^2 - E(W)E''(W)},$$

with the following boundary conditions

$$E(\hat{W}) = p'(w^*)(\hat{W} - \Phi),$$

$$E'(\hat{W}) = p'(w^*),$$

$$E(-\Pi) = 0,$$

and $w^*$ is given in Theorem 2. The agent’s consumption and portfolio rules are given by

$$C(W) = mFBE(W)E'(W)^{-\psi},$$

$$X(W) = \frac{\mu - r}{\sigma^2} \frac{E(W)E'(W)}{\gamma E'(W)^2 - E(W)E''(W)}.$$

The value-matching condition (61) states that the agent’s certainty equivalent wealth $E(W)$ is continuous at the endogenously determined cutoff level $\hat{W}$. The smooth-pasting condition (62) gives the agent’s optimal indifference condition between being an entrepreneur.
Figure 7: Certainty equivalent wealth \( E(W) \) and its sensitivity \( E'(W) \) before entrepreneurial entry. The outside option value is \( \Pi = 0.5 \) and the fixed start-up cost \( \Phi = 0.05 \).

or not with wealth \( \hat{W} \). Finally, being indebted with amount \( \Pi \) implies that the agent will never get out of the debt region and cannot pay back the fixed start-up cost. If this is the case, the certainty equivalent wealth is then zero as given by (63).

Figure 7 plots the agent’s certainty equivalent wealth \( E(W) \) before becoming an entrepreneur for two levels of risk aversion, \( \gamma = 2, 4 \). First, the less risk-averse agent is more entrepreneurial. For example, the threshold wealth \( \hat{W} \) is 5.66 for \( \gamma = 4 \), which is significantly higher than 4.3 for \( \gamma = 2 \). Unlike the standard real options problem, ours features incomplete markets. This means that the less risk-averse agent values the investment option more and hence exercises it earlier. Second, the less risk-averse agent values the future investment opportunity more by demanding a lower idiosyncratic risk premium as we will show in the next section. Finally, for all levels of \( W \), \( E'(W) \) is greater for less risk-averse agents.

Figure 8 quantifies the value of “entry timing flexibility” by comparing the time-0 binary entry with the flexible entry. For both \( \gamma = 2 \) and \( \gamma = 4 \), the convex curves in Figure 8 correspond to the the one with full timing flexibility (the “American” option), while the straight lines give the “now-or-never” time-0 binary version. First, it is immediate to see that timing flexibility is valuable. Second, the value of timing flexibility is highest when the agent’s wealth is in the intermediate range where building more financial strength substantially lowers the risk premium for the business project and hence enhances welfare. When the
Figure 8: The optimal entry decisions: Comparing the “optimal timing” and “time-0 binary decision” settings. The outside option value is $\Pi = 0.5$ and the fixed start-up cost $\Phi = 0.05$.

option value is sufficiently close to being in the money or deep out of the money, the wedge between the two versions of entrepreneurship entry is small. Finally, by allowing for flexible entry, the entrepreneur’s cutoff level of wealth significantly increases. For example, for $\gamma = 2$, the cutoff wealth increases from 2.86 in a time-0 binary setting (see Figure 6) to 4.3. To summarize, flexibility has significant value in entrepreneurship.

Figure 9 plots the optimal consumption $C(W)$ and the MPC out of wealth $C'(W)$ for four cases: worker with $\gamma = 2$ or $\gamma = 4$; and entrepreneur-to-be with $\gamma = 2$ or $\gamma = 4$. For workers receiving constant wage income at the rate of $r\Pi$, markets are effectively complete and the MPC out of wealth is constant: $m_{FB} = 0.0573$ for $\gamma = 2$ and $m_{FB} = 0.0516$ for $\gamma = 4$, respectively. Recall that the more risk-averse agent consumes less out of wealth when the EIS $\psi < 1$. More interestingly, the entrepreneur’s MPC out of wealth $C'(W)$ is lower than $m_{FB}$. Intuitively, frictions such as non-diversifiable risk and borrowing constraints make the agent consume less on the margin in order to build up wealth to become an entrepreneur sooner. This underconsumption effect is greater the closer the agent’s wealth $W$ is to the endogenous entry threshold $\hat{W}$, i.e. when the agent’s entry option is deeper in the money.

Figure 10 plots the optimal allocation to the market portfolio for a worker and for an
Figure 9: **Optimal consumption and the MPC before entry.** The outside option value is $\Pi = 0.5$ and the fixed start-up cost $\Phi = 0.05$.

entrepreneur-to-be with $\gamma = 2$. The left panel plots $X(W)/W$, the fraction of liquid financial wealth $W$ allocated to the risky market portfolio. The entrepreneur-to-be invests more in the risky market portfolio, i.e. $X(W)/W$ is higher for entrepreneurs-to-be than for workers. The right panel plots the portfolio allocation as a fraction of the agent’s certainty equivalent wealth $E(W)$, $X(W)/E(W)$. For workers, the certainty equivalent wealth is equal to the

Figure 10: **Optimal market portfolio allocation before entry.** The outside option value is $\Pi = 0.5$ and the fixed start-up cost $\Phi = 0.05$. 

entrepreneur-to-be with $\gamma = 2$. The left panel plots $X(W)/W$, the fraction of liquid financial wealth $W$ allocated to the risky market portfolio. The entrepreneur-to-be invests more in the risky market portfolio, i.e. $X(W)/W$ is higher for entrepreneurs-to-be than for workers. The right panel plots the portfolio allocation as a fraction of the agent’s certainty equivalent wealth $E(W)$, $X(W)/E(W)$. For workers, the certainty equivalent wealth is equal to the
sum of $W$ and $\Pi$, i.e. $E(W) = W + \Pi$. For entrepreneurs-to-be, $E(W)$ is convex and the option value makes $E(W) > W + \Pi$ for $W < \hat{W}$. The right panel shows that the entrepreneur-to-be allocates more to the market portfolio even after controlling for a higher level of $E(W)$ for workers than for entrepreneurs-to-be. However, quantitatively, the effects of the entry option on portfolio allocation are not significant. We have similar results for the comparison between the worker and the entrepreneur-to-be with $\gamma = 4$.

We next study the risk premium implications for entrepreneurial firms.

7 Idiosyncratic risk premium

A fundamental issue in entrepreneurial finance is to determine the cost of capital for private firms owned by non-diversified entrepreneurs. Intuitively, the entrepreneur demands both the systematic risk premium and an additional idiosyncratic risk premium for non-diversifiable risk. Compared to an otherwise identical public firm held by diversified investors, the cost of capital should be higher for the entrepreneurial firm. Using our model, we provide a procedure to calculate the cost of capital for the entrepreneurial firm.

Let $\xi(w_0)$ denote the constant yield (internal rate of return) for the entrepreneurial firm until liquidation. We have made explicit the functional dependence of $\xi$ on the initial wealth-capital ratio $w_0 = W_0/K_0$. By definition, $\xi(w_0)$ solves the following valuation equation

\[
Q(K_0, W_0) = \mathbb{E} \left[ \int_0^\tau e^{-\xi(w_0)t} dY_t + e^{-\xi(w_0)\tau} fK_\tau \right],
\]

where $\tau$ is the stochastic liquidation time. The right side of (66) is the present discounted value (PDV) of the firm’s operating cash flow plus the PDV of the liquidation value using the same discount rate $\xi(w_0)$. The left side is the “private” enterprise value $Q(K_0, W_0)$ that we have obtained earlier using the entrepreneur’s optimality.

Recall that the firm’s discount rate under complete markets, $\xi^{FB}$, is given in (23). We measure the idiosyncratic risk premium as the wedge between $\xi(w_0)$ and $\xi^{FB}$

\[
\alpha(w_0) = \xi(w_0) - \xi^{FB} = \xi(w_0) - r - \beta^{FB}(\mu_R - r).
\]

There is much debate in the empirical literature about the significance of this private equity risk premium. For example, Moskowitz and Vissing-Jorgensen (2002) document that the risk-adjusted returns to investing in a U.S. non-publicly traded equity are not higher than
the returns to private equity. Our model provides an analytical formula to calculate this private equity idiosyncratic risk premium.

Figure 11 plots the idiosyncratic risk premium for two levels of risk aversion, $\gamma = 2, 4$. For sufficiently high levels of wealth-capital ratio $w_0$, the idiosyncratic risk premium $\alpha(w_0)$ eventually disappears. Intuitively, this premium $\alpha(w_0)$ is higher for more risk-averse agents. Quantitatively, for entrepreneurs with positive wealth, we do not find a significant idiosyncratic risk premium. For both $\gamma = 2$ and $\gamma = 4$, the annual idiosyncratic risk premia are less than 1%. However, for entrepreneurs in debt, this premium $\alpha(w_0)$ is significant because the business carries significantly more weight in the entrepreneur’s portfolio, and non-diversifiable risk becomes much more important as Figure 11 shows.

8 Comparative analysis

There is significant heterogeneity among entrepreneurs in terms of preferences and production technology. In this section, we show how various structural parameters, including the EIS $\psi$, idiosyncratic volatility $\epsilon$, the adjustment cost parameter $\theta$, and the liquidation parameter $l$, on the entrepreneur’s decision making and business valuation. For all the figures,
we use parameter values given in the baseline model (Section 5) other than the parameter under study. In the preceding analysis, we have shown that risk aversion has substantial effects. First, we analyze the impact of EIS $\psi$.

**The EIS $\psi$.** In asset pricing, a high EIS is often used in the long-run risk literature (Bansal and Yaron (2004)). However, there is much disagreement about the empirical estimates of the EIS. Our previous calculations are based on $\psi = 0.5$. We now consider two commonly used but significantly different values for the EIS: $\psi = 0.25, 2$. Figure 12 shows that the effect of the EIS $\psi$ on consumption is quantitatively significant, while its effects on Tobin’s $q(w)$, investment $i(w)$, portfolio choice $x(w)$ and the idiosyncratic risk premium $\alpha(w)$ are much less significant. The large effect on consumption is similar to the intuition under complete markets. For example, the MPC $m^{FB}$ is only 0.014 when $\psi = 2$, which is substantially lower than the MPC $m^{FB} = 0.072$ when EIS is $\psi = 0.25$. Intuitively, an entrepreneur with a high EIS ($\psi = 2$) is willing to decrease consumption to build up wealth.

Insert Figure 12 here.

**Idiosyncratic volatility $\epsilon$.** In Figure 13, we plot for two values of the idiosyncratic volatility, $\epsilon = 0.1, 0.2$. We find that the idiosyncratic volatility $\epsilon$ has significant effects on investment $i(w)$ and Tobin’s $q(w)$. The entrepreneur invests significantly less in the firm (lower $i(w)$) and liquidates capital earlier when $\epsilon = 0.2$ than when $\epsilon = 0.1$. Firm value $q(w)$ increases significantly when the idiosyncratic volatility $\epsilon$ decreases from 0.2 to 0.1. The marginal value of financial wealth $q'(w)$ also strongly depends on the idiosyncratic volatility especially for low and intermediate values of $w$. Finally, intuitively, the effect of $\epsilon$ on the idiosyncratic risk premium $\alpha(w)$ is large. For example, when doubling the idiosyncratic volatility from 10% to 20%, the annual idiosyncratic risk premium for an entrepreneur with no liquid wealth ($w = 0$) increases from 0.5% to 2.3%!

Insert Figure 13 here.

**Adjustment cost parameter $\theta$.** In Figure 14, we plot for two values of the adjustment cost parameter: $\theta = 2$ and $\theta = 8$. Whited (1992) estimates this parameter to be around $\theta =$
Eberly, Rebelo and Vincent (2009) use an extended Hayashi (1982) model and provide a larger empirical estimate of this parameter value (close to seven) for large Compustat firms. Clearly, the adjustment cost has a first-order effects on investment $i(w)$ and Tobin’s $q(w)$ as in the first-best benchmark. However, due to frictions, investment and Tobin’s $q$ highly depends on $w$. Consumption $c(w)$ and portfolio allocation $x(w)$ depend very little on $\theta$. The effect of $\theta$ on the idiosyncratic risk premium $\alpha$ is also weak.

Liquidation parameter $l$. In Figure 15, we plot for two values of the liquidation parameter, $l = 0.6$ and $l = 0.9$. We show that liquidation value has a quantitatively significant impact on investment $i(w)$, Tobin’s $q(w)$, consumption $c(w)$, and the idiosyncratic volatility $\alpha(w)$ when the entrepreneur is in debt (i.e. the left sides of each panel). A higher value of $l$ provides a better downside protection for the entrepreneur and also allows the entrepreneur to borrow more (higher debt capacity). The entrepreneur thus operates the business longer with a higher $l$. Additionally, while running the business, the entrepreneur invests more, consumes more, and allocates more to the market portfolio with a higher value of $l$. A higher value of $l$ also lowers the idiosyncratic risk premium $\alpha(w)$ by providing a better downside risk protection and mitigating entrepreneurial underinvestment. When the liquidation option is sufficiently out of the money (i.e. when $w$ is sufficiently high), liquidation has almost no effect on entrepreneurial decision making and valuation, consistent with our intuition.

9 Conclusion

Non-diversifiable risk and liquidity constraints are important frictions in the real world. This paper provides an incomplete-markets framework with these two frictions to analyze the entrepreneur’s interdependent business entry, capital accumulation/growth, portfolio choice, consumption, and business exit decisions. Even when business is uncorrelated with the stock market, the entrepreneur rationally chooses to reduce business investment, lowers

---

16 Hall (2004) argues that the parameter $\theta$ is small using U.S. aggregate data.
consumption, and scales back portfolio investment in the stock market. Non-diversifiable risk and liquidity constraints also significantly influence the private business valuation. We provide an operational procedure to compute the private equity idiosyncratic risk premium and the cost of capital. Additionally, our framework helps us understand the empirical findings on the private equity premium (see Moskowitz and Vissing-Jorgensen (2002) and the follow-up research).

While being exposed to significant risk, the entrepreneur nonetheless has various options to manage risk. For example, the liquidation option substantially enhances the entrepreneur’s ability to manage downside risk. We show that the option value of building up financial strength before entering entrepreneurship is high. We show that wealth effects are significant for entrepreneurial entry in a dynamic setting with flexible entry timing. While entrepreneurs have important entry and exit options, these options are fundamentally different from the standard (real or financial) options analyzed in finance because the entrepreneurs’ entry and exit options are illiquid and not tradable. Additionally, the option exercising decisions interact with the agent’s consumption, saving, portfolio allocation and capital accumulation decisions in a fundamental way when incomplete-market frictions (non-diversification and liquidity constraints) are important. We provide a unified and analytically tractable dynamic framework to study the firm’s various decision margins in a life-cycle model of entrepreneurship.

Finally, our model is a single agent’s intertemporal decision problem. To study the impact of entrepreneurship on wealth distribution and economic growth, we need to construct a general equilibrium incomplete-markets model.17 Our model may provide one natural starting point for such a general equilibrium analysis.

17See Aiyagari (1994) and Huggett (1993) for foundational incomplete-markets equilibrium (Bewley) models and Cagetti and De Nardi (2006) for an application with entrepreneurship.
References


A. Private enterprise value: $q(w)$

$\psi = 0.25$

$\psi = 2$

B. Net marginal value of wealth: $q(w)$

C. Investment: $i(w)$

D. Consumption: $c(w)$

E. Market portfolio allocation: $x(w)$

F. Idiosyncratic risk premium: $\alpha(w)$

Figure 12: The effects of the EIS $\psi$. 
Figure 13: The effects of idiosyncratic volatility $\epsilon$. 

44
Figure 14: The effects of the adjustment cost parameter $\theta$. 
Figure 15: The effects of the liquidation parameter $l$. 
Table 1: Summary of Key Variables and Parameters

This table summarizes the symbols for the key variables used in the model and the parameter values for the baseline calculation of Section 5. For each upper-case variable in the left column (except $K$, $A$, $J$, $F$, $V$, $E$, $\bar{W}$, $\bar{W}$ and $K^*$), we use its lower case to denote the ratio of this variable to capital.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Symbol</th>
<th>Parameters</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Capital stock</td>
<td>$K$</td>
<td>Riskfree rate</td>
<td>$r$</td>
<td>4.6%</td>
</tr>
<tr>
<td>Cumulative Productivity Shock</td>
<td>$A$</td>
<td>Expected return of market portfolio</td>
<td>$\mu_R$</td>
<td>10.6%</td>
</tr>
<tr>
<td>Investment Adjustment Cost</td>
<td>$G$</td>
<td>Volatility of market portfolio</td>
<td>$\sigma_R$</td>
<td>20%</td>
</tr>
<tr>
<td>Cumulative Operating Profit</td>
<td>$Y$</td>
<td>Aggregate equity risk premium</td>
<td>$\mu_R - r$</td>
<td>6%</td>
</tr>
<tr>
<td>Financial wealth</td>
<td>$W$</td>
<td>Market Sharpe ratio</td>
<td>$\eta$</td>
<td>30%</td>
</tr>
<tr>
<td>Value function after entry</td>
<td>$J$</td>
<td>Subjective discount rate</td>
<td>$\zeta$</td>
<td>4.6%</td>
</tr>
<tr>
<td>Value function before entry</td>
<td>$F$</td>
<td>Adjustment cost parameter</td>
<td>$\theta$</td>
<td>2</td>
</tr>
<tr>
<td>Value function after exiting</td>
<td>$V$</td>
<td>Depreciation rate</td>
<td>$\delta$</td>
<td>12.5%</td>
</tr>
<tr>
<td>Certainty equivalent wealth after entry</td>
<td>$P$</td>
<td>Mean productivity shock</td>
<td>$\mu_A$</td>
<td>20%</td>
</tr>
<tr>
<td>Certainty equivalent wealth before entry</td>
<td>$E$</td>
<td>Volatility of productivity shock</td>
<td>$\sigma_A$</td>
<td>10%</td>
</tr>
<tr>
<td>Private enterprise value</td>
<td>$Q$</td>
<td>Correlation between market and firm</td>
<td>$\rho$</td>
<td>0</td>
</tr>
<tr>
<td>Effective risk aversion</td>
<td>$h$</td>
<td>Idiosyncratic volatility</td>
<td>$\epsilon$</td>
<td>10%</td>
</tr>
<tr>
<td>Market portfolio allocation</td>
<td>$X$</td>
<td>Relative Risk Aversion</td>
<td>$\gamma$</td>
<td>2.4</td>
</tr>
<tr>
<td>Consumption</td>
<td>$C$</td>
<td>Elasticity of Intertemporal Substitution</td>
<td>$\psi$</td>
<td>0.5</td>
</tr>
<tr>
<td>Business investment</td>
<td>$I$</td>
<td>Capital liquidation price</td>
<td>$l$</td>
<td>0.9</td>
</tr>
<tr>
<td>Liquidation Boundary</td>
<td>$W$</td>
<td>Outside option value</td>
<td>$\Pi$</td>
<td>0.5</td>
</tr>
<tr>
<td>Static entry threshold</td>
<td>$\bar{W}$</td>
<td>Fixed start-up cost</td>
<td>$\Phi$</td>
<td>0.05</td>
</tr>
<tr>
<td>Flexible entry threshold</td>
<td>$\hat{W}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Optimal initial capital stock</td>
<td>$K^*$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MPC out of wealth</td>
<td>$m$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Internal rate of return</td>
<td>$\xi$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Idiosyncratic risk premium</td>
<td>$\alpha$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Appendices

A Details for Theorem 1 and Proposition 1

We conjecture that the value function is given by (33). We then have

\[ J_K(K, W) = b^{1-\gamma}(p(w)K)^{-\gamma}(p(w) - wp'(w)), \tag{A.1} \]
\[ J_W(K, W) = b^{1-\gamma}(p(w)K)^{-\gamma}p'(w), \tag{A.2} \]
\[ J_{WW}(K, W) = b^{1-\gamma} \left( \frac{(p(w)K)^{-\gamma}p''(w)}{K} - \gamma(p(w)K)^{-\gamma-1}(p'(w))^2 \right). \tag{A.3} \]

The first-order conditions (FOCs) for \( C \) and \( X \) are

\[ f_C(C, J) = J_W(K, W), \tag{A.4} \]
\[ X = -\frac{\rho \sigma_A}{\sigma_R} K + \frac{(r - \mu_R)J_W(K, W)}{\sigma_R^2 J_{WW}(K, W)}. \tag{A.5} \]

Using the homogeneity property of \( J(K, W) \), we obtain the following for \( c(w) \) and \( x(w) \):

\[ c(w) = b^{1-\psi} \zeta^\psi p(w)(p'(w))^{-\psi}, \tag{A.6} \]
\[ x(w) = -\frac{\rho \sigma_A}{\sigma_R} + \frac{\mu_R - r}{\sigma_R^2 h(w)} p(w). \tag{A.7} \]

Substituting \( c(w) \) into (2), we have

\[ f(C, J) = \frac{\zeta}{1 - \psi^1} \left( \frac{(bp(w)K)^{1-\gamma}(bp'(w))^{1-\psi}}{\zeta^{1-\psi}} - (bp(w)K)^{1-\gamma} \right). \tag{A.8} \]

Substituting (A.1), (A.2), (A.3), (A.7) and (A.8) into (29) and simplifying, we obtain

\[ 0 = \max_i \left( \frac{\zeta^\psi (bp'(w))^{1-\psi}}{\psi - 1} - \frac{\psi \zeta}{\psi - 1} \right) p(w) + (i - \delta)(p(w) - wp'(w)) + (rw + \mu_A - \rho \eta \sigma_A - i - g(i)) p'(w) + \frac{\eta^2 p(w)p'(w)}{2h(w)} - \frac{\epsilon^2 h(w)p'(w)}{2p(w)}, \tag{A.9} \]

where \( h(w) \) is given in (35). Using the FOCs for investment-capital ratio \( i \), we obtain (40). Substituting it into (A.9), we obtain the ODE (34).

Using Ito’s formula, we obtain the following dynamics for the entrepreneur’s wealth-capital ratio \( w \),

\[ dw_t = d \left( \frac{W_t}{K_t} \right) = \frac{dW_t}{K_t^2} - \frac{W_t}{K_t^2} dK_t = \mu_w(w_t)dt + \sigma_R x(w_t)dB_t + \sigma_A dZ_t, \tag{A.10} \]
where $\mu_w(w)$ is given by (43).

Now consider the lower liquidation boundary $W$. When $W \leq W$, the entrepreneur liquidates the firm. Using the value-match condition at $W$, we have

$$J(K, W) = V(W + lK),$$

where $V(W)$ given by (26) is the agent’s value function after retirement and with no business. The entrepreneur’s optimal liquidation strategy implies the following smooth-pasting condition at the endogenously determined liquidation boundary $W$

$$J_W(K, W) = V_W(W + lK).$$

Using $W = wK$, (A.11), and (A.12), and simplifying, we obtain the scaled value-matching and smooth pasting conditions given in (37) and (38), respectively.

**Complete-markets benchmark solution.** When $w$ approaches infinity, markets are effectively complete. Non-diversifiable risk no longer matters for investment and consumption. Therefore, firm value approaches the complete-markets value and $\lim_{w \to \infty} J(K, W) = V(W + q^{FB}K)$, which implies (36). The certainty equivalent wealth $P(K, W)$ is equal to the sum of the financial wealth $W$ and complete-markets firm value $q^{FB}K$, in that

$$P^{FB}(K, W) = W + q^{FB}K.\tag{A.13}$$

Equivalently, we have $p^{FB}(w) = w + q^{FB}$. Substituting this linear relation into (34), taking the limit $w \to \infty$, we obtain the following equation:

$$0 = \left(\frac{\zeta \psi b^{1-\psi} - \psi \zeta}{\psi - 1} + \frac{\eta^2}{2\gamma}\right)(w + q^{FB}) + (i^{FB} - \delta)q^{FB} + rw + \mu_A - \rho \eta \sigma_A - i^{FB} - g(i^{FB}).\tag{A.14}$$

In order for the above to hold, we require that both the linear term coefficient and the constant term are equal to zero. This gives rise to

$$b = \zeta \left[1 + \frac{1-\psi}{\zeta} \left(r - \zeta + \frac{\eta^2}{2\gamma}\right)\right]^{\frac{1}{1-\psi}},\tag{A.15}$$

$$m^{FB} = b^{1-\psi} \zeta \psi = \zeta + (1-\psi) \left(r - \zeta + \frac{\eta^2}{2\gamma}\right).\tag{A.16}$$
We may now write (A.14) as follows

\[ rq^{FB} = \mu_A - \rho \eta \sigma_A - i^{FB} - g(i^{FB}) + (i^{FB} - \delta)q^{FB}. \] (A.17)

Using the FOC for investment, i.e. \( i^{FB} = \frac{(q^{FB} - 1)}{\theta} \), we obtain (19). Next, we calculate the rate of return for the firm. We have

\[
dR_t^{FB} = \frac{dY_t + dQ_t^{FB}}{Q_t^{FB}} = \frac{\mu_A dt + \sigma_A dZ_t - i^{FB} K_t dt - g(i^{FB}) K_t dt}{Q_t^{FB}} + \frac{q^{FB} dK_t}{Q_t^{FB}},
\]

Therefore, the expected return \( \mu^{FB} \) is given by

\[
\mu_r^{FB} = r + \beta^{FB} (\mu_R - r),
\]

where \( \beta^{FB} \) is given by

\[
\beta^{FB} = \frac{\rho \sigma_A}{\sigma_R q^{FB}}. \] (A.20)

### B Details for Theorem 2 and Theorem 3

**Theorem 2.** The entrepreneur chooses initial firm size \( K_0^* \) to maximize utility, which gives rise to the following FOC

\[
P_K(K_0^*, W_0 - \Phi - K_0^*) = P_W(K_0^*, W_0 - \Phi - K_0^*). \] (B.1)

The above condition (B.1) states that the marginal value of capital \( P_K(K_0^*, W_0 - \Phi - K_0^*) \) is equal to the marginal value of wealth \( P_W(K_0^*, W_0 - \Phi - K_0^*) \) at the optimally chosen \( K_0^* \). Simplifying (B.1) gives (56), which characterizes the optimal initial wealth-capital ratio \( w_0 \equiv (W_0 - \Phi)/K_0^* - 1 = w^* \). Note that (56) implies that \( w^* \) is independent of the fixed start-up cost \( \Phi \) and outside option value \( \Pi \).

Second, using the Euler’s theorem, we write \( P(K_0^*, W_0 - \Phi - K_0^*) \) as follows

\[
P(K_0^*, W_0 - \Phi - K_0^*) = P_K^* \times K_0^* + P_W^* \times (W_0 - \Phi - K_0^*) = p'(w^*) (W_0 - \Phi), \] (B.2)

where the second equality follows from (B.1). The entrepreneur’s certainty equivalent wealth \( P(K_0^*, W_0 - \Phi - K_0^*) \) is given by \( p'(w^*) \) multiplied by \( (W_0 - \Phi) \), the initial wealth after paying the fixed start-up cost \( \Phi \). If \( J(K_0^*, W_0 - (\Phi + K_0^*)) > V(W_0 + \Pi) \), the agent chooses to become an entrepreneur immediately. Otherwise, the agent will take the outside option. Therefore, the threshold level \( \overline{W} \) satisfies \( J(K_0^*, \overline{W} - (\Phi + K_0^*)) = V(\overline{W} + \Pi) \), which gives (55).
Theorem 3. Let $F(W)$ and $E(W)$ denote the agent’s value function and certainty equivalent wealth before entering entrepreneurship. Using the standard principle of optimality for recursive utility (Duffie and Epstein (1992b)), the following HJB equation holds

$$0 = \max_{C, X} \left[ f(C, F) + (rW + (\mu_R - r)X + r\Pi - C)F'(W) + \frac{\sigma_R^2X^2}{2}F''(W) \right].$$

(B.3)

The FOCs for $C$ and $X$ are given by

$$F'(W) = f_C(C, F),$$

(B.4)

$$X(W) = \frac{(r - \mu_R)F'(W)}{\sigma_R^2F''(W)}.$$  

(B.5)

Using the conjectured value function (59) and simplifying, we obtain

$$C(W) = b^1 - \psi E(W)(E'(W))^{-1},$$

(B.6)

$$X(W) = \frac{(\mu_R - r)E(W)}{\sigma_R^2} \frac{E'(W)}{\gamma E(W)^2 - E(W)E''(W)}.$$  

(B.7)

Substituting the above into $f(C, F)$, we obtain

$$f(C, F) = \frac{\zeta}{1 - \psi^{-1}} \left[ (bE(W))^{1-\gamma}(bE'(W))^{1-\psi} \right] - (bE(W))^{1-\gamma}.$$  

(B.8)

Substituting these results into (B.3), we obtain the following non-linear ODE

$$0 = \frac{m^{FB}E(W)(E'(W))^{1-\psi} - \psi bE(W)}{\psi - 1} + r(W + \Pi)E'(W) + \frac{\eta^2}{2} \frac{E(W)E'(W)^2}{\gamma E''(W)^2 - E(W)E''(W)}. $$  

(B.9)

The following value match and smooth-pasting conditions determine the threshold $\widehat{W}$

$$F(\widehat{W}) = J(K^*, \widehat{W} - \Phi - K^*),$$  

(B.10)

$$F'(\widehat{W}) = J_W(K^*, \widehat{W} - \Phi - K^*).$$  

(B.11)

Using (58) and (59), we obtain the following conditions for $E(W)$ at $\widehat{W}$

$$E(\widehat{W}) = p'(w^*) (\widehat{W} - \Phi),$$  

(B.12)

$$E'(\widehat{W}) = p'(w^*),$$  

(B.13)

where $w^*$ is defined by (56). Finally, we have the absorbing condition, $E(-\Pi) = 0$.  

51