The Economics of Hedge Funds:  
Alpha, Fees, Leverage, and Valuation*  

Yingcong Lan† Neng Wang‡ Jinqiang Yang§  
January 17, 2011  

Abstract  
Hedge fund managers are compensated via management fees on the assets under management (AUM) and incentive fees indexed to the high-water mark (HWM). We study the effects of managerial skills (alpha) and compensation on dynamic leverage choices and the valuation of fees and investors' payoffs. Increasing the investment allocation to the alpha-generating strategy typically lowers the fund's risk-adjusted excess return due to frictions such as price pressure. Without agency conflicts, the manager optimally chooses time-invariant leverage to balance the size of allocation to the alpha-generating strategy against the negative impact of increasing size on the fund's alpha. With agency, we show that (i) the high-powered incentive fees encourage excessive risk taking, while management fees have the opposite effect; (ii) conflicts of interest between the manager and investors have significant effects on dynamically changing leverage choices and the valuation of fees and investors' payoffs; (iii) the manager optimally increases leverage following strong fund performances; (iv) investors' options to liquidate the fund following sufficiently poor fund performances substantially curtail managerial risk-taking, provide strong incentives to de-leverage, and sometime even give rise to strong precautionary motives to hoard cash (in long positions); and (v) managerial ownership concentration has incentive alignment effects.  

Keywords: assets under management (AUM), high-water mark, alpha, management fees, incentive fees, conflicts of interest, liquidation option, managerial ownership  

JEL Classification: G2, G32  

*First Draft: May 2010. We thank Patrick Bolton, Markus Brunnermeier, Kent Daniel, Pierre Collin-Dufresne, Will Goetzmann, Bob Hodrick, Suresh Sundaresan, and seminar participants at Brock, Columbia, PREA for helpful comments.  
†Cornerstone Research. Email: ylan@cornerstone.com. Tel.: 212-605-5017.  
‡Columbia Business School and NBER. Email: neng.wang@columbia.edu. Tel.: 212-854-3869.  
§Hunan University and School of Finance, Shanghai University of Finance and Economics. Email: huda518@gmail.com.
1 Introduction

Hedge funds’ management compensation contracts typically feature both management fees and performance/incentive fees. The management fee is charged periodically as a fraction, e.g. 2%, of assets under management (AUM). The incentive fee, a key characteristic that differentiates hedge funds from mutual funds, is calculated as a fraction, e.g. 20%, of the fund’s profits. The cost base for the profit calculation is the fund’s high-water mark (HWM), which effectively keeps track of the maximum value of the invested capital and critically depends on the fund manager’s dynamic investment strategies. Presumably, incentive fees are intended to reward talented managers and to align the interests of the manager and investors more closely than flat management fees do. However, incentive fees may also have unintended consequences because they tend to encourage managerial excessive risk taking and do not lead to fund value maximizing leverage choices.\footnote{Private equity funds also charge both management and incentive fees. While the compensation structure is similar in essence, institutional details such as fees and profits calculations differ significantly for hedge funds and private equity funds. Metrick and Yasuda (2010) provide an economic analysis of private equity funds.}

Sophisticated use of leverage is an important feature of hedge funds (Ang, Gorovyy, and van Inwegen (2010)). Hedge funds may borrow through the repo markets or from their prime brokers. Hedge funds can also use various implicit leverage, often via options and other derivatives. Skilled managers, by leveraging on strategies with positive alphas, can potentially create significant value for investors. That is, with a limited amount of capital, leverage can be rewarding for both skilled managers and the funds’ investors. However, the fund’s marginal return may decrease with leverage because increasing investment exposure to the alpha generating technology potentially lowers alpha for reasons such as negative price pressure. The manager takes into account the negative impact of leverage on the portfolio’s alpha when choosing dynamic investment strategies.

We develop an analytically tractable framework of leverage and valuation for hedge funds with the following features: (i) an alpha generating strategy; (ii) the management fee specified as a fraction of AUM; (iii) the high-powered incentive fee linked to the HWM, and (iv) the fund’s alpha decreasing with its leverage. The manager dynamically chooses leverage to maximize the present value (PV) of total managerial rents, the sum of PVs of both management and of incentive fees.
For both conceptual and quantitative reasons, it is important to incorporate both management and incentive fees into our analysis. We will show that management and incentive fees have different and often opposite implications on managerial risk taking. With management fees only, the manager effectively behaves in the investors’ interest because management fees make the manager a de facto equity investor in the fund and there is no agency issue between the manager and investors. However, when the manager is paid via both fees, conflicts of interest between the manager and investors arise. The optionality embedded in incentive fees encourage the manager to choose leverage more than the desired level from the investors’ perspective. The closer the fund’s AUM is to its HWM, the more leverage the manager chooses because the incentive fee is closer to being realized (i.e. the embedded call option is deeper in the money). This standard optionality argument suggests that the PV of incentive fees is increasing and convex in AUM.

On the other hand, paying incentive fees lowers the AUM and hence crowds out current management fees and also potentially reduces future management fees. There is thus a tension between the PV of management fees and the PV of incentive fees. Importantly, the PV of management fees is increasing but concave in the AUM because (i) management fees effectively give the manager an “equity-like” stake in the fund, and (ii) equity investors in the fund are effectively short in a call option (embedded in incentive fees) to the manager. As a result, management fees encourage the manager to behave prudently and hence reduce leverage, ceteris paribus.

When the fund’s performance improves (i.e. its AUM rises and gets closer to its HWM), the manager is more likely to collecting the incentive fees, and thus behaves in a more risk-seeking way by increasing leverage. When the fund’s performance deteriorates, incentive fees are out of the money and the manager thus puts more weights on management fees, which encourages a more prudent investment strategy. The incentive to lower leverage is stronger if the downside risk is greater. For example, as we show, the manager is even willing to go long in the risk-free asset to lower the fund’s return volatility when facing serious performance-based liquidation threats from investors.

To study the impact of performance-triggered liquidation risk by investors, we extend our baseline model to allow investors to liquidate the fund when the fund loses a significant frac-

\footnote{See Duffie, Wang, and Wang (2009) for a similar result in a different setting with management fees only.}
tion (e.g. 50%) of its AUM from its HWM (i.e. effectively its maximum AUM ever achieved historically). The investors’ liquidation/withdrawal option allows investors to better manage their downside risk exposure. As we noted earlier, the fund manager is now not just long in the incentive call option but also short in investors’ liquidation option. The manager rationally manages the fund’s leverage in order to maximize the PV of total fees. Quantitatively, we show that liquidation options significantly reduce managerial risk-taking and sometimes cause the manager to be too conservative even from the perspective of investors. As a result, investors’ liquidation options significantly curtail the manager’s total rents. Moreover, management fees become much more important than incentive fees in contributing to managerial total rents when we allow investors to liquidate the fund based on performances. In our numerical example, without the liquidation option, the total managerial rents is about 60 cents for each dollar of the AUM out of which two-thirds of total rents, i.e. 40 cents, come from incentive fees. On the other hand, when investors can liquidate the fund, say after the manager loses 25% of its AUM from its HWM, the PV of total fees falls about half to 30 cents for each dollar of the AUM, out of which only one-third of total rents, i.e. 10 cents, are due to incentive fees. This calculation illustrates the importance of performance-based liquidation option on total managerial rents and also on the relative weights of incentive fees and management fees in contributing to managerial rents. We also find that concentrated managerial ownership helps to align the incentives between the manager and investors.

Our model provides the first elements of an analytically tractable operational framework to study the effects of managerial compensation on managerial leverage and dynamic valuation for the managerial and investors’ stakes. In addition to deriving the manager’s dynamic leverage decisions, we explicitly link the PV of managerial compensation to managerial talent/alpha. Our model thus provides a valuation toolbox for investors to evaluate different managerial compensation contracts and their investments given their beliefs about managerial skills. Once investors choose their priors about the manager’s unlevered alpha and the Sharpe ratio, we can use our dynamic valuation framework to determine managerial leverage and also to quantify PVs of various fees and investors’ payoffs.

Related literature. There are only a few theoretical papers studying the hedge fund’s valuation and leverage decisions. Goetzman, Ingersoll, and Ross (2003), henceforth abbreviated GIR, provides the first quantitative inter-temporal valuation framework for management and
incentive fees in the presence of the HWM. Despite the sophisticated HWM dynamics induced by embedded optionality among other features, they derive closed-form solutions for various value functions for a fund with a constant alpha and Sharpe ratio. CIR focuses solely on valuation and thus takes AUM dynamics as given without allowing for leverage decisions. Building on GIR, we introduce investment-dependent alpha-generating technology, derive time-varying optimal endogenous leverage choice, and study the implied dynamic valuation due to time-varying incentives to manage leverage and the fund’s performances.

Panageas and Westerfield (2009) study the manager’s investment decisions when the manager receives incentive fees without management fees. They obtain explicit time-invariant investment strategies for a risk-neutral patient manager.\(^3\) In terms of the model setup, our paper differs from theirs by incorporating management fees and also the assumption that the alpha for the levered position may not necessarily scale up with the position. We show that optimal leverage chosen by the manager critically depends on the ratio of the AUM and the HWM due to the interaction between both incentive and management fees. Management and incentive fees play important but different roles on leverage and valuation. In terms of results, the value of incentive fees in our paper is convex in AUM, while it is concave in their model with no liquidation boundary.\(^4\)

Duffie, Wang, and Wang (2009) study leverage management when the manager is compensated via management fees but not incentive fees. Compensation contracts with only management fees effectively make the manager a co-investor in the fund whose cash flows are management fees proportional to AUM, and thus effectively behaves in the investors’ interest by optimally balancing the trading costs (e.g. bid/ask spread) with the benefits of leverage on alpha. Unlike their paper, our paper focuses on agency issues induced by incentive fees and being indexed to the high-water mark and other frictions such as liquidation boundaries imposed by investors. Dai and Sundaresan (2010) point out that the hedge fund is short in two important options: investors’ redemption option and funding options (from prime brokers and short-term debt markets). These two short positions have significant effects on optimal leverage and risk management policies (e.g. the use of unencumbered cash).

\(^3\)While their portfolio strategy solution looks similar to the one in the classic portfolio choice problem for investors with constant relative risk aversion as in Merton (1971), the economic mechanism is different as they pointed out in their paper.

\(^4\)Other differences between the two papers include payouts to investors and the allowance for a lower liquidation boundary. For more detailed comparisons with Panageas and Westerfield (2009), see Section 5.2.
They do not model the effects of incentive fees and the HWM on leverage and valuation. Like their paper, we also model the investors’ liquidation option, but in the context of managerial compensation. Additionally, our paper also studies the importance of managerial ownership on mitigating agency conflicts.

There has been much recent and continuing interest in empirical research on hedge funds. Fung and Hsieh (1997), Ackermann, McEnally, and Ravenscraft (1999), Agarwal and Naik (2004), and Getmansky, Lo, and Makarov (2004) among others, study the nonlinear feature of hedge fund risk and return.\(^5\) Lo (2008) provides a detailed treatment of hedge funds for their potential contribution to systemic risk in the economy.

2 The baseline model

First, we introduce the fund manager’s alpha-generating investment opportunity. Second, we describe the fund’s managerial compensation contracts including both management and incentive fees. Then, we discuss the dynamics for the fund’s AUM, and define various value functions for the manager and investors. Finally, we state the manager’s intertemporal optimization problem subject to investors’ voluntary participation.

The fund’s investment technology. The fund has a trading strategy which generates expected excess returns after risk adjustments. The fund’s expected excess returns may be attributed to managerial talent and may not be traded. In reality, fund managers are sometimes secretive about their trading strategies and sometimes take measures to make replications of their strategies difficult.

In addition to investing in the risky strategy, the manager can also invest in the risk-free asset, which offers a constant rate of return \(r\). Let the amount of investment in the risk-free asset be \(D\). Let \(W\) denote the fund’s AUM. The amount invested in the risky alpha-generating technology is then \(W - D\). When \(D < 0\), the fund takes on leverage, often via short-term debt. Hedge funds can obtain leverage from the fund’s prime brokers, repo markets and the use of derivatives.\(^6\)

---

\(^5\)For the presence of survivorship bias, selection bias, and back-filling bias in hedge funds databases, see Brown, Goetzmann, Ibbotson, and Ross (1992) for example.

\(^6\)Few hedge funds are able to directly issue long-term debt or secure long-term borrowing.
Let \( \pi \) denote the asset-capital ratio \( \pi = (W - D)/W \). Given investment strategy \( \pi_t \), the instantaneous incremental return \( dR_t \) is given by

\[
dR_t = \mu(\pi_t)dt + \sigma dB_t,
\]

where \( B_t \) is a standard Brownian motion. For expositional simplicity, we assume that the volatility parameter \( \sigma \) is constant for the strategy and focus on the expected return \( \mu(\pi) \). We may interpret that the expected return \( \mu(\pi) \) is already after the adjustment of the systematic risk. Intuitively, investment performance potentially depends on the amount of investment the manager deploys to the alpha-generating strategy. The larger the fund’s position per unit of capital \( \pi \), the lower the expected excess return on the marginal unit of investment. This may be due to the price pressure, decreasing returns to scale for the fund manager’s ability to scale up its alpha-generating skill/position and/or liquidity issues with the fund’s increasingly large position. Motivated by these considerations, we thus assume that \( \mu(\pi) \) is decreasing in \( \pi \), i.e. \( \mu'(\pi) < 0 \). Berk and Green (2004) make the same assumption in their study of mutual funds industry equilibrium.

We define the expected excess return beyond the risk-free rate, \( \alpha(\pi) \), as follows:

\[
\alpha(\pi) = \mu(\pi) - r.
\]

Positive alpha measures managerial talent in our model (Note that we have already accounted for the systematic risk in the return generating process (1)). Scarce managerial skills earn rents in equilibrium. The manager will optimally choose \( \pi \) taking into account the effects of the investment \( \pi \) on the alpha of the fund’s portfolio. Economically sensible solution requires convergence. By increasing \( \pi \), the firm earns alpha on a larger amount of total assets, however, each unit of asset earns a lower level of \( \alpha \). The fund optimally trades off between these two offsetting effects in a dynamic optimizing framework as we will show.

**Managerial compensation contracts.** The typical hedge fund management compensation contract has two main components: the management fees and the incentive fees. The management fee is specified as a constant fraction \( c \) of the AUM \( W \): \( \{cW_t : t \geq 0\} \). The incentive fee links compensation to the fund’s performance.

To describe the incentive fees, we need to understand the fund’s high-water mark (HWM) process \( \{H_t : t \geq 0\} \). For the purpose of exposition, first consider the simplest example where
the HWM $H_t$ is the highest level that the AUM $W$ has attained up to time $t$, i.e. $H$ is the running maximum of $W$: $H_t = \max_{s \leq t} W_s$.

More generally, the HWM may also change due to indexed growth or investors’ withdrawal.\(^7\) Let $g$ denote the (normal) rate at which $H$ grows. As in GIR, investors in our model are paid continuously at a rate $\delta W_t$ where $\delta$ is a constant. Naturally, the fund’s high-water mark is adjusted downward due to investors’ withdrawal at the rate $\delta$.

Provided that the fund is in operation and the fund’s AUM $W$ is below its HWM ($W < H$), the evolution of $H$ is locally deterministic and is given by

$$dH_t = (g - \delta)H_t dt, \quad \text{when} \quad W_t < H_t. \quad (3)$$

For example, $g$ may be set to zero, or the interest rate $r$ or other benchmark levels. This growth of $H$ to some extent may capture the time value of money. Whenever the AUM $W$ exceeds its HWM, the manager collects a fraction $k$ of the fund’s performance exceeding its HWM and then the HWM resets. We provide a detailed analysis for the dynamics of the HWM $H$ on the boundary $W = H$ in Section 4. We often see that $c$ and $k$ are around 2% and 20% for typical hedge funds. Of course, managers with different track records charge differently. Our analysis provides a link between the managerial compensation contract and managerial skills.

The dynamic of the AUM $W$. In our baseline model and in GIR, investors stochastically liquidate the fund at a constant rate $\lambda$ per unit of time. This assumption implies that the fund has a finite average duration and also keeps the model stationary and analytically tractable. Upon liquidation at exogenous stochastic time $\tau$, the manager receives nothing and investors collect AUM $W$.

While the manager runs the fund (i.e. before stochastic liquidation time $\tau$, i.e. $t < \tau$), the AUM $W_t$ evolves as follows:

$$dW_t = \pi_t W_t (\mu(\pi_t)dt + \sigma dB_t) + (1 - \pi_t)r W_t dt - \delta W_t dt - cW_t dt - k[dH_t - (g - \delta)H_t dt] - dJ_t, \quad t < \tau. \quad (4)$$

\(^7\)Sometimes, the HWM is also negotiated downward if the fund has performed poorly. Sometimes, it is argued that resetting the high-water mark helps to re-align the incentives between the manager and the investors. To offset the potential increase in incentive fees collected, the manager in exchange may reduce management fees by lowering $c$. 

7
The first and second terms in (4) describe the change of AUM $W$ from the manager’s investment strategies in its alpha-generating technology and the risk-free asset as in standard portfolio choice problem (Merton (1971)). The third term $-\delta Wdt$ gives the continuous payout rate to investors. The sum of these three terms gives the change of $W$ in the absence of fees. The fourth term represents the management fees to the manager in flow terms (e.g. $c = 2\%$). The fifth term gives the incentive/performance fees when the AUM exceeds the HWM (e.g. $k = 20\%$). The process $J$ in the last (sixth) term is a pure jump process which describes this liquidation risk: The AUM $W$ is set to zero with probability $\lambda$ per unit of time when investors exogenously liquidate the fund.

**Various value functions for investors and the manager.** We next define present values for various streams of cash flows. For a given dynamic investment strategy $\pi$, we use $M(W, H; \pi)$ and $N(W, H; \pi)$ to denote the present values (PVs) of the management fees and that of the incentive fees, respectively. Recall that the alpha-generating technology (1)-(2) is already after the systematic risk adjustment. For example, when the market portfolio is the only source of systematic risk (i.e. CAPM holds for investments with no alpha), (1) should be interpreted as market-neutral excess return after factoring out the market risk premium. We may thus discount cash flows using the risk-free rate as follows:

$$M(W, H; \pi) = \mathbb{E}_t \left[ \int_t^\tau e^{-r(s-t)}cW_s ds \right], \quad (5)$$

$$N(W, H; \pi) = \mathbb{E}_t \left[ \int_t^\tau e^{-r(s-t)}k [dH_s - (g - \delta)H_s ds] \right], \quad (6)$$

where the manager collects neither management nor incentive fees after stochastic liquidation time $\tau$.\(^8\) Let $F(W, H; \pi)$ denote the PV of total fees, which is given by

$$F(W, H; \pi) = M(W, H; \pi) + N(W, H; \pi). \quad (7)$$

Similarly, we define investors’ value $E(W, H)$ as follows:

$$E(W, H; \pi) = \mathbb{E}_t \left[ \int_t^\tau e^{-r(s-t)}\delta W_s ds + e^{-r(\tau-t)}W_\tau \right]. \quad (8)$$

\(^8\)The manager collects the incentive fees if and only if $dH_t > (g - \delta)H_t dt$, which can only happen on the boundary ($W_t = H_t$). However, after reaching the new HWM, the manager may not collect incentive fees if the fund does not make profits.

\(^9\)We may allow the manager to start a new fund after paying certain start-up costs and extend the model and analysis accordingly. In those extensions, the manager will maximize the PV of fees from the current fund and the “continuation” value from managing future funds. This important extension will significantly complicate the analysis and lengthen the paper. We leave it for future research.
Finally, the total PV of the fund $V(W, H)$ is given by the sum of $F(W, H)$ and $E(W, H)$:

$$V(W, H; \pi) = F(W, H; \pi) + E(W, H; \pi).$$  \hspace{1cm} (9)

The manager’s optimization problem and investors’ voluntary participation. The manager chooses the optimal dynamic investment policy $\pi$ to maximize the PV of total fees $F(W, H; \pi)$ as follows:

$$\max_{\pi} F(W, H; \pi) = \max_{\pi} \{ M(W, H; \pi) + N(W, H; \pi) \},$$  \hspace{1cm} (10)

subject to the investors’ participation condition to which we now turn.

Anticipating that the manager behaves in own interest, investors rationally demand that the PV of their payoffs is at least higher than their time-0 investment $W_0$ in the fund in order to break even, risk-adjusted and time-value adjusted. At time 0, by definition, the fund’s HWM is set at $H_0 = W_0$. Given the manager’s optimal dynamic investment strategy $\pi$, we thus require the investors’ value $E(W_0, W_0; \pi)$ at time 0 to satisfy the following condition:

$$E(W_0, W_0; \pi) \geq W_0.$$  \hspace{1cm} (11)

Intuitively, how much surplus investors collect depend on their relative bargaining power against the manager. In perfectly competitive markets where we assume that the skilled manager collects all the surplus, the above participation constraint (11) holds with equality. In moments when investors may have certain bargaining power (such as in the financial crisis periods where capital supply is potentially limited), their participation constraint may hold with slack, i.e. earning some rents by providing valuable capital to the manager.

Before analyzing the general case where the manager receives both management and incentive fees, we first study the case where the manager only receives management fees.

3 First-best benchmark: management fees only

With only management fees and no incentive fees ($k = 0$), we obtain closed-form solutions for the manager’s investment strategy and various value functions for the manager and investors. As we will show, without high-powered incentive schemes ($k = 0$), there are no conflicts of interest between the manager and investors. The management fees only compensation
effectively gives the manager an equity stake in the fund and hence managerial incentives are perfectly aligned with investors’ incentives. The manager therefore simple maximizes the fund’s total surplus, which is equivalent to maximizing his own surplus and investors’ interest. Therefore, in our model, the first-best leverage is chosen by the manager. Leverage is valuable in the model because the manager has alpha and trades off earnings alpha on a larger pool of capital and lower alpha per unit of capital due to price pressure (i.e. $\alpha'(\pi) < 0$) as in a monopoly pricing problem. Using a different model setup, Duffie, Wang, and Wang (2009) also obtain this result when the manager is only paid via management fees. We thus view our results as complementary and reinforcing theirs.\textsuperscript{10} The following proposition summarizes the results for the special case with management fees only.

\textbf{Proposition 1} With no incentive fees ($k = 0$), the PV of total fees $F^*(W)$ is given by

$$F^*(W) = M^*(W) = \frac{c}{c + \delta + \lambda - \alpha'^*} W,$$

(12)

where $\alpha'$ is a constant given by the following equation:

$$\alpha'^* = \pi'^* \alpha(\pi^*),$$

(13)

and the unique optimal time-invariant investment strategy is given by:

$$\alpha(\pi^*) + \pi'^* \alpha(\pi^*) = 0.$$

(14)

With optimal investment strategy $\pi^*$, the PV of investors’ payoff is given by

$$E^*(W) = \frac{\delta + \lambda}{c + \delta + \lambda - \alpha'^*} W.$$

(15)

The PV of the total fund, $V^*(W)$, is given by:

$$V^*(W) = M^*(W) + E^*(W) = \frac{c + \delta + \lambda}{c + \delta + \lambda - \alpha'^*} W.$$  

(16)

Note that $\alpha'$ can be interpreted as the level of the fund’s optimally levered alpha. For convergence, we need to ensure that levered alpha $\alpha'^*$ cannot be too high, i.e.

$$\alpha'^* < c + \delta + \lambda,$$

(17)

\textsuperscript{10}Duffie, Wang, and Wang (2009) use the proportional transaction cost setting as in the well known portfolio choice model of Davis and Norman (1990). Our results reported in the benchmark model of Section 3 share the same focus and similar results to theirs despite different detailed settings.
where $\alpha^*$ is given by (13)-(14). Intuitively, the above inequality ensures that investment strategy is not too good to be true so that the value of the alpha-generating technology (1) even after leverage is still finite. Otherwise, convergence conditions (similar to the one in typical Gordon growth models) are violated.

Our model features managerial skills ($\alpha > 0$ and hence $\alpha^* > 0$). If the skilled manager has all the bargaining power as in perfectly competitive markets, we expect that investors break even, i.e. $E^*(W) = W$. Using (15), we obtain the equilibrium compensation contract which features management fee $c = \alpha^*$. Intuitively, the manager charges $c$ at the managerial optimally levered skill premium $\alpha^*$. Consequently, using (12), we obtain that the equilibrium value of management fees to be $F^*(W) = \alpha^*W/(\delta + \lambda)$. Note that management fees are collected each period as a fraction of AUM, the sum of the fees can be quite significant as the formula suggests and our calculation in Section 5 shows. In markets where capital is scarce, more even distributions of bargaining powers between the manager and investors may exist. In that case, we naturally expect potentially more balanced distributions of surpluses between investors and the manager.

As we will show, the model features homogeneity. Anticipating this feature, we use the lower case to denote various value functions in the corresponding upper case, per unit of the HWM $H$. For example, $f^*(w) = F^*(W)/H = \alpha^*w/(\delta + \lambda)$ and $e^*(w) = E^*(W)/H = w$.

While our benchmark model is set up for the case with $k = 0$, it also applies to settings where the high-water mark $H$ approaches infinity ($H \to \infty$). For that case, the incentive fees are effectively completely out of the money, and hence the manager de facto only collects the management fees, and therefore, the solution is the same as the one given here.

We next turn to the general setting where the manager collects both management and incentive fees. The high-water mark plays a fundamental role in leverage choices and valuation, resulting in imperfect alignment of incentives between the manager and investors.

4 Model solution: The general case with the HWM

We solve the manager’s optimization problem using dynamic programming. First, we study the manager’s behavior and implied valuation when the AUM is below the HWM (i.e. in the interior region $W < H$). Second, we analyze the manager’s behavior when the evolution of the AUM leads to setting the new HWM. Third, we use the homogeneity property of
our model to solve for and then discuss the economics of the manager’s optimal investment strategy and various value functions for the manager and investors.

**The manager’s intertemporal decision problem when** $W < H$. When the fund’s AUM is below its HWM ($W < H$), instantaneously the manager will not receive the incentive fees. The fund is in the “interior” region where the option component (incentive fees) is inactive over the short horizon, and the manager only collects management fees. However, being forward looking, the manager chooses dynamic investment strategy $\pi$ to maximize $F(W, H)$, the PV of total fees. Using the principle of optimality, we have the following Hamilton-Jacobi-Bellman (HJB) equation in the interior region:

\[
(r + \lambda)F(W, H) = \max_\pi \left[ cW + \left[ \pi \alpha(\pi) + (r - \delta - c) \right] WF_W(W, H) 
+ \frac{1}{2} \pi^2 \sigma^2 W^2 F_{WW}(W, H) + (g - \delta) HF_H(W, H) \right].
\] (18)

The left side of (18) elevates the discount rate from the interest rate $r$ to $(r + \lambda)$ to reflect the stochastic liquidation of the fund. The first term on the right side of (18) gives the management fees in flow terms, $cW$. The second and third terms give the drift (expected change) and the volatility effects of the AUM $W$ on the value of total fees $F(W, H)$, respectively. Finally, the last term on the right side of (18) describes the effect of the HWM $H$ change on $F(W, H)$ in the interior region ($W < H$). The manager optimally chooses the dynamic investment strategy $\pi$ to equate the two sides of (18). Next, we analyze the properties of $F(W, H)$ when the AUM equals the HWM and moves along the boundary $W = H$.

**The manager’s intertemporal decision problem when** $W = H$. Our reasoning for the boundary behavior ($W = H$) essentially follows GIR and Panageas and Westerfield (2009). Start with $W = H$. A positive return shock increases the AUM from $W = H$ to $H + \Delta H$. The PV of total fees for the manager is then given by $F(H + \Delta H, H)$ before the HWM adjusts. Immediately after the positive shock, the HMW adjusts to $H + \Delta H$. The manager then collects the incentive fees in flow terms $k\Delta H$, and consequently the AUM is lowered from $H + \Delta H$ to $H + \Delta H - k\Delta H$. The PV of total managerial fees is then equal to $F(H + \Delta H - k\Delta H, H + \Delta H)$. Using the continuity of $F(\cdot, \cdot)$ before and after the adjustment of the HWM, we have

\[
F(H + \Delta H, H) = k\Delta H + F(H + \Delta H - k\Delta H, H + \Delta H).
\] (19)
By taking the limit $\Delta H$ to zero and using the Taylor’s rule, we obtain the following result:

$$kF_W(H, H) = k + F_H(H, H).$$  \hspace{1cm} (20)

The above condition can be viewed as a value-matching condition for the manager when moving on the boundary $W = H$. By using essentially the same logic, we obtain the following boundary conditions for the PV of management fees $M(W, H)$ and the PV of incentive fees $N(W, H)$ at the boundary $W = H$: $kM_w(H, H) = M_H(H, H)$ and $kN_w(H, H) = k + N_H(H, H)$.

Finally, we turn to the left boundary condition. When the fund runs out of assets ($W = 0$), there is no more AUM in the future and hence no fees: $F(0, H) = 0$. In Section 6, we extend our model to allow investors to have more control rights. For example, investors may prevent the AUM from falling below a certain fraction of the HWM by being able to contractually liquidating the fund. We show that this lower liquidation boundary has significant effects on the manager’s dynamic investment strategies and valuation.

As we will show, our model captures dynamic leverage decisions in a tractable framework. The tractability of our model solution critically rests on the homogeneity property. That is, if we simultaneously double the AUM $W$ and the HWM $H$, the PV of total fees $F(W, H)$ will accordingly double. The effective state variable is thus the ratio between the AUM $W$ and the HWM $H$: $w = W/H$. As we mentioned earlier, we will use the lower case to denote the corresponding variable in the upper case scaled by the contemporaneous HWM $H$. For example, $f(w) = F(W, H)/H$. The following theorem summarizes the main results on $\pi(w)$ and the PV of total fees $F(W, H)$.

**Theorem 1** The scaled PV of total fees $f(w)$ solves the following ODE:

$$(r + \lambda - g + \delta)f(w) = cw + [\pi(w)\alpha(\pi(w)) + r - g - c]wf'(w) + \frac{1}{2}\pi(w)^2\sigma^2w^2f''(w),$$  \hspace{1cm} (21)

where the optimal investment strategy $\pi(w)$ solves the following equation:

$$\pi(w) = \frac{\alpha(\pi(w))}{\sigma^2\psi(w) - \alpha'(\pi(w))},$$  \hspace{1cm} (22)

and $\psi(w)$ is given by

$$\psi(w) = -\frac{wf''(w)}{f'(w)}.$$  \hspace{1cm} (23)
The ODE (21) for $f(w)$ is solved subject to the following boundary conditions:

\begin{align}
  f(0) &= 0, \\
  f(1) &= (k + 1)f'(1) - k.
\end{align}

Equation (22) gives the manager’s optimal dynamic investment strategy. While the manager is risk neutral, incentive issues and other frictions generate curvatures for the managerial value function $f(w)$. The numerator in (22) gives the fund’s optimally chosen levered alpha $\alpha(\pi(w))$. The denominator has two components. First, the curvature of the manager’s value function $f(w)$, i.e. the manager’s effective risk attitude $\psi(w) = -wf''(w)/f'(w)$, matters for the portfolio choice $\pi(w)$ due to an “effective” risk attitude for the manager, $\psi(w)$, which works in an interactive way with the variance of the return process, $\sigma^2$. Second, with market pressure (i.e. $\alpha'(\pi) < 0$), the manager chooses investment $\pi$ by anticipating its diminishing effect on total expected excess return $\alpha(\pi)$. Provided that the sum of the two terms is positive, i.e. $\sigma^2\psi(w) - \alpha'(w) > 0$, the optimal portfolio rule $\pi(w)$ has an interior solution.

The formula (23) for $\psi(w)$ measures the curvature of the manager’s value function $f(w)$, and may thus intuitively refer to $\psi(w)$ as the manager’s effective risk attitude. When $f(w)$ is convex (say due to optionality induced by the HWM), the risk-neutral manager behaves effectively in a risk-loving manner. When $f(w)$ is concave, the risk-neutral manager behaves effectively in a risk-averse manner (e.g. potentially for precautionary reasons to avoid investors’ liquidation). Whether the manager behaves effectively as a risk-averse or risk-seeking agent depends on various constraints imposed by investors and also managerial incentives. We will discuss the properties of $\psi(w)$ in Sections 5 and 6 under different settings.

Finally, the ODE (21) is a Black-Merton-Scholes type of valuation equation adapted for the manager’s dynamic optimal leverage and management/incentive fees. Equation (24) states that $w = 0$ is an absorbing boundary with $f(0) = 0$. Equation (25) gives the condition at the upper boundary $w = 1$ when the manager is close to collecting incentive fees.

We now turn to the implied dynamic valuation for various components of the fund. We summarize the main results in the following proposition.

**Proposition 2** Given investment strategy $\pi(w)$, the dynamics of $w = W/H$ is given by:

\[ dw_t = \mu_w(w_t)dt + \sigma\pi(w_t)w_tdB_t - dJ_t, \]
where $J$ is a pure jump process which sets $w = 0$ upon the stochastic arrival of the jump with intensity $\lambda$, and the drift function $\mu_w(w)$ is given by

$$
\mu_w(w) = \left[ \pi(w)\alpha(\pi(w)) + r - g - c \right]w .
$$

(27)

The value functions $M(W,H)$, $N(W,H)$, and $E(W,H)$ are all homogeneous with degree one in $AUM W$ and $HWM H$, i.e. $M(W,H) = m(w)H$, $N(W,H) = n(w)H$, $E(W,H) = e(w)H$, where $m(w)$, $n(w)$, $e(w)$ solve the following ODEs respectively:

$$
(r + \lambda - g + \delta)m(w) = cw + \mu_w(w)m'(w) + \frac{1}{2}\pi(w)^2\sigma^2 w^2m''(w) ,
$$

(28)

$$
(r + \lambda - g + \delta)n(w) = \mu_w(w)n'(w) + \frac{1}{2}\pi(w)^2\sigma^2 w^2n''(w) ,
$$

(29)

$$
(r + \lambda - g + \delta)e(w) = (\delta + \lambda)w + \mu_w(w)e'(w) + \frac{1}{2}\pi(w)^2\sigma^2 w^2e''(w) ,
$$

(30)

with the following boundary conditions:

$$
m(0) = n(0) = e(0) = 0 ,
$$

(31)

$$
m(1) = (k + 1)m'(1) ,
$$

(32)

$$
n(1) = (k + 1)n'(1) - k ,
$$

(33)

$$
e(1) = (k + 1)e'(1) .
$$

(34)

The total fund’s value is $V(W,H) = v(w)H$, where $v(w) = m(w)+n(w)+e(w) = f(w)+e(w)$.

**Investors’ voluntary participation.** In order for investors to voluntarily participate, we need to ensure that investors do not lose money in present value by investing with the manager, i.e. $E(W_0,W_0) \geq W_0$ as given in (11). It is equivalent to require $e(1) \geq 1$ due to the homogeneity property of $E(W,H)$. In our baseline model with perfectly competitive capital markets, the manager collects all the rents from their managerial skills, and therefore, we have the equilibrium outcome for the investors’ value:

$$
e(1) = 1 .
$$

(35)

In general, depending on the relative bargaining powers between the manager and investors, investors may collect some surpluses especially when capital supply is limited as in financial crisis periods. We now turn to analyze the implications for this general case with the HWM.
5 Results: Alpha, fees, leverage, and valuation

We first specify a simple alpha-generating technology and then choose the parameter values for the baseline model.

Specifying an alpha-generating technology. For expositional simplicity, in our numerical analysis, we use the following linear alpha-generating function:

\[ \alpha(\pi) = \theta_0 - \theta_1 \pi, \]  

(36)

where \( \theta_0 > 0 \) and \( \theta_1 \geq 0 \) are constant. Intuitively, the fund’s alpha decreases with its position \( \pi \). In equilibrium, the manager’s optimality implies that \( \alpha(\pi) \) is positive at the optimally chosen \( \pi \). The larger \( \theta_1 \), the more the manager is concerned about taking a bigger position due to the reduction of \( \alpha \) from the size of the position.

Applying the specification (36) to the benchmark with management fees only as in Section 3, we obtain the first-best investment level: \( \pi^* = \theta_0/(2\theta_1) \), and the optimally levered alpha for the portfolio is: \( \alpha^* = \theta_0^2/(4\theta_1) \). Recall that the manager has no conflicts of interest with investors in the benchmark without incentive fees and volatility has no influence on leverage because the manager is risk neutral. The manager optimally chooses leverage to trade off more investments via leverage yielding excess returns against lowering alpha on each unit of investments due to higher investment (i.e. price pressure).

With both incentive fees and management fees, using the alpha-generating technology (36), we obtain the following optimal investment strategy \( \pi(w) \):

\[ \pi(w) = \frac{\pi^*}{1 + \sigma^2 \psi(w)/(2\theta_1)}, \]  

(37)

where \( \psi(w) = -wf''(w)/f'(w) \) is the manager’s effective risk attitude defined in (23) and \( \pi^* \) is the first-best leverage and is given by \( \pi^* = \theta_0/(2\theta_1) \). Importantly, (37) implies that volatility \( \sigma \), the manager’s effective risk attitude \( \psi(w) \), and the price pressure parameter \( \theta_1 \) all influence the manager’s dynamic leverage choice \( \pi(w) \). For example, if \( f(w) \) is convex (due to incentive fees), the manager behaves in a risk-seeking manner (\( \psi(w) < 0 \)) and we thus have excessive managerial risk taking compared with the first-best: \( \pi(w) > \pi^* \).

Parameter choices. All rates are annualized and continuously compounded, whenever applicable. For the baseline calculation, we set the interest rate \( r = 5\% \), the payout rate
to investors $\delta = 5\%$, the annual liquidation probability $\lambda = 5\%$, and the target growth rate of the HWM $\gamma = 5\%$ so that the net growth rate of the HWM $\gamma$ in the interior region is zero in the absence of stochastic liquidation, i.e. $\gamma - \delta = 0$. That is, the HWM $\gamma$ does not change unless the AUM $W$ reaches a new record, i.e. $\gamma$ is the running maximum of the AUM $W$: $H_t = \max\{W_s : s \in [0, t]\}$. We specify the alpha technology by choosing $\theta_0 = 2.5\%$, $\theta_1 = 0.25\%$, and the volatility of risky investment strategy $\sigma = 9\%$. We choose the commonly seen 2-20 management contract (i.e. $c = 2\%$ and $k = 20\%$). As we will show later, these parameter values jointly imply $e(1) = 1$ so that investors simply break even in PV and the manager collects all the surplus from managerial skills.

5.1 The HWM $\gamma$, leverage $\pi$, and managerial rents $f(w)$

The benchmark with no incentive fees ($k = 0$). With the baseline parameter values, the optimal leverage is time-invariant and is equal to $\pi^* = 5$. That is, for each dollar of the AUM, the fund borrows four dollars and invests total five dollars in the risky positive-alpha generating technology delivering an optimally levered alpha for the fund: $\alpha^* = 6.25\%$. To ensure that investors break even under perfectly competitive capital markets ($e(1) = 1$), we need to set the equilibrium management fee to the value of the portfolio’s levered alpha: $c = \alpha^* = 6.25\%$ using (15) so that the manager collects all the surplus. The managerial rents are significant in this example: 62.5% of the AUM, i.e. $F^*(W) = 0.625W$. Investors always value his investments at AUM, i.e. $E^*(W) = W$. The dashed line in Figure 1 plots the time-invariant optimal leverage $\pi^* = \theta_0/(2\theta_1) = 5$.

The effect of incentive fees ($k > 0$). The solid line in Figure 1 plots $\pi(w)$ as a function of $w$ for a typical 2-20 compensation contract ($c = 2\%$ and $k = 20\%$). Reducing management fees $c$ from 6.25% to 2% to accompany the introduction of incentive fees from $k = 0$ to $k = 20\%$ ensures that investors break even at the moment of participation, i.e. $e(1) = 1$.

Compared with the $k = 0$ benchmark, the manager invests more in its positive-alpha technology due to incentive fees: $\pi(w) > \pi^* = 5$ for $w > 0$. Intuitively, the manager now maximizes a sum of an “equity” like stake (management fees) and an “option” like stake (incentive fees). The manager’s overall position is thus more levered than investors’ pure equity stake. The call optionality embedded in incentive fees ($k > 0$) encourages
excessive managerial risk taking, the degree of which increases with \( w \) because the moneyness of the incentive fees increases with \( w \). In the limit as \( w \to 1 \), we obtain the maximal leverage: \( \pi(1) = 6.33 \), which is about 27% higher than the no-incentive-fee benchmark leverage: \( \pi^* = 5 \). At the other end as \( w \to 0 \), excessive risk taking disappears and optimal leverage \( \pi(0) \) converges to \( \pi^* = 5 \) because the call option is completely out of money.

\[
\pi(1) = 6.33, \quad \text{which is about 27\% higher than the no-incentive-fee benchmark leverage:} \quad \pi^* = 5.
\]

\[
\pi(0) \text{ converges to } \pi^* = 5 \text{ because the call option is completely out of money.}
\]

Figure 1: Dynamic investment strategy \( \pi(w) \). The solid line corresponds to the 2-20 compensation contract: \( c = 2\% \), \( k = 20\% \); and the dashed line corresponds to the benchmark contract with no incentive fees: \( k = 0, c = 6.25\% \). Both contracts satisfy \( e(1) = 1 \). The other parameter values are: \( r = 5\%, \sigma = 9\%, \lambda = 5\%, g = 5\%, \delta = 5\%, \theta_0 = 2.5\%, \) and \( \theta_1 = 0.25\% \).

The left panel of Figure 2 plots the PV of total fees \( f(w) \) under the 2-20 contract and compares with \( f^*(w) \) for the \( k = 0 \) benchmark. Greater risk taking induced by incentive fees lowers total managerial rents: \( f(w) < f^*(w) \). Note that this result follows from the following two results: (i) the no-incentive fee (\( k = 0 \)) benchmark yields the highest total value of the fund \( v(1) \) and (ii) the management compensation contracts adjust so that \( e(1) = 1 \) under both settings and hence the manager collects all the surpluses: \( f(1) = v(1) - 1 \). In our example, under the 2-20 contract, \( f(1) = 0.589 \), which is about 5.76\% smaller than the first-best benchmark level \( f^*(1) = 0.625 \).

The right panel of Figure 2 plots the manager’s risk attitude \( \psi(w) = -wf''(w)/f'(w) \), which is defined in (23). In the benchmark with \( k = 0 \), we have \( \psi^*(w) = 0 \) for all \( w \), i.e.
Figure 2: The manager’s PV of total fees $f(w)$ and the “effective” risk attitude: $\psi(w) = -wf''(w)/f'(w)$. The manager’s value $f(w)$ is convex, indicating that excessive managerial risk seeking ($\psi(w) < 0$) with incentive fees ($k > 0$) than without ($k = 0$).

the manager is risk neutral. When $k > 0$, the risk-neutral manager takes on excessive risk. Using the optimal leverage formula (37), we see that $\pi(w) > \pi^*$ because the manager is effectively risk seeking ($f(w)$ is convex and $\psi(w) < 0$). Moreover, as $w$ increases, the call option embedded in incentive fees becomes deeper in the money, the manager becomes more risk seeking ($\psi(w) < 0$ and $|\psi(w)|$ increases with $w$), and therefore, leverage $\pi(w)$ increases with $w$ as we see in Figure 1.

The marginal values of the AUM $W$ and of the HWM $H$: $F_W(W, H)$ and $F_H(W, H)$. Using the homogeneity property of the PV of total fees $F(W, H)$, we have the following results for the marginal values of the AUM $W$ and of the HWM $H$ on managerial rents $F(W, H)$:

$$F_W(W, H) = f'(w),$$

$$F_H(W, H) = f(w) - wf'(w).$$

The solid line in the left panel of Figure 3 plots $F_W(W, H) = f'(w)$ for the case with incentive fees ($k > 0$). For comparison purposes, the dashed horizontal line in the left panel of Figure 3 depicts the corresponding constant marginal value of the AUM for the $k = 0$ benchmark.
at $F^*_W(W) = f'_w(w) = 0.625$ for all $w$, which implies that the managerial rents are equal to 62.5% of the AUM, a significant amount. Unlike the $k = 0$ benchmark, the marginal value of the AUM $F_W(W,H) = f'(w)$ is increasing in $w$, which implies that $F(W,H)$ is convex in the AUM $W$ (i.e. $f''(w) > 0$) due to the call option feature embedded in incentive fees.

The marginal value of the AUM $F_W(W,H)$ varies significantly with $w$. If $w$ is sufficiently high, the marginal value of the AUM $F_W(W,H)$ is high because of the high powered incentives from the incentive fee structure. For example, $F_W(W,H)$ reaches $0.658$ at $w = 1$: $f'(1) = 0.658$, which is above the horizontal line for the $k = 0$ benchmark. For sufficiently low $w$, the marginal value of AUM $F_W$ can be quite low. For example, when $w = 0$ (say due to $H \to \infty$), the incentive fee is completely out of the money and $F_W(0,H) = f'(0) = c/(\delta + \lambda + c - \alpha^*) = 0.02/(0.05 + 0.05 + 0.02 - 0.0625) = 0.348$, significantly below the horizontal line (note that $c = 2\%$ in the 2-20 contract compared with $c = 6.25\%$ in the $k = 0$ benchmark).

The right panel of Figure 3 plots $F_H(W,H)$, the marginal effect of the HWM $H$ on $F(W,H)$. The HWM $H$ is effectively the strike price of $F(W,H)$, however, it is stochastic and endogenous. Increasing the HWM $H$ reduces the optionality and hence lowers the PV of total fees $F(W,H)$ giving rise to the intuitive result: $F_H(W,H) < 0$. See the negatively valued $F_H(W,H)$ in the right panel of Figure 3. Because $F_{HH}(W,H) = w^2 f''(w)/H > 0$
and $f''(w) > 0$ with $k > 0$, $F(W, H)$ is thus convex in the HWM $H$, which is similar to the result that option value is convex in the strike price.

5.2 Valuing various components for the manager and investors

Figure 4: Decomposing managerial rents $f(w)$ into the PV of incentive fees $n(w)$ and the PV of management fees $m(w)$: $f(w) = n(w) + m(w)$.

PVs of incentive fees $N(W, H)$ and of management fees $M(W, H)$. Figure 4 plots the scaled PVs of incentive fees $n(w)$ and of management fees $m(w)$. Not surprisingly, both $n(w)$ and $m(w)$ increase with $w$. For this numerical example, the PV of incentive fees at $w = 1$ is $n(1) = 0.39$, which is approximately twice as much as the PV of management fees: $m(1) = 0.20$. Incentive fees matter much more than management fees in this example. The total surplus $F(W, W)$ is about 59% of the AUM $W$, ie. $f(1) = 0.59$, which is significant. We will show that both absolute and relative quantitative significances of incentive and management fees will fundamentally change when we allow investors to contractually liquidate the fund conditional on managerial performances.
Figure 5: *Sensitivities of the PV of management fees* $M(W,H)$ and the PV of incentive fees $N(W,H)$ with respect to the AUM $W$ and the HWM $H$: $N_W(W,H)$, $N_H(W,H)$, $M_W(W,H)$, and $M_H(W,H)$.\[11\]

The upper and lower left panels of Figure 5 plot $M_W(W,H) = m'(w)$ and $N_W(W,H) = n'(w)$, respectively. First, $n'(w)$ is increasing in $w$. That is, the value of incentive fees $N(W,H)$ is convex in the AUM $W$ due to the embedded optionality of incentive fees. Second, $m'(w)$ is decreasing in $w$. That is, the value of management fees $M(W,H)$ is concave in the AUM $W$. Incentive fees crowd out management fees, and the payment of incentive fees lowers the AUM. The substitution of incentive fees for management fees effectively makes the value of management fees $M(W,H)$ concave in $W$.\[11\]

\[11\] Of course, there is an overall risk-taking effect induced by the incentive fees, i.e. the value of total fees $F(W,H)$ is convex in $W$. The crowding-out effect is stronger than the overall risk-taking effect which makes $F(W,H)$ and $N(W,H)$ convex in $W$ but $M(W,H)$ concave in $W$.\[22\]
We now turn to the effects of the HWM $H$ on $M(W,H)$ and $N(W,H)$. Intuitively, the higher the HWM $H$, the less likely the manager collects the incentive fees, which suggests that the PV of incentive fees $N(W,H)$ decreases with the HWM $H$: $N_H(W,H) < 0$. Indeed, the upper right panel in Figure 5 plots $N_H(W,H)$, which is negative for all $w$.

The higher the HWM $H$, the less likely the manager gets paid via the incentive fees, which implies a higher AUM for a given dynamic investment strategy and hence a larger PV of management fees $M(W,H)$, i.e. $M_H(W,H) > 0$. The lower right panel in Figure 5 plots $M_H(W,H)$, which is indeed positive for all $w$.

It is worth pointing out the differences between our model and Panageas and Westerfield (2009). In their model, the manager is only compensated with incentive fees and is more patient than investors.\textsuperscript{12} The main differences are as follows. First, our model predicts that the PV of incentive fees $N(W,H)$ is convex in the AUM $W$ while $N(W,H)$ is concave in the AUM $W$ in Panageas and Westerfield (2009). Second, our model predicts that the PV of incentive fees $N(W,H)$ decreases with the HWM $H$. Third, our model predicts that ceteris paribus $N(W,H)$ increases with the incentive fee parameter $k$. These predictions are opposite of those given by Panageas and Westerfield (2009). In their model, the manager prefers keeping money inside the fund due to the fund’s alpha-generating technology and managerial patience. Increasing the HWM keeps the manager’s future fees inside the fund longer, which allows the manager to generate more expected excess returns by building up the AUM. As a result, the PV of incentive fees $N(W,H)$ increases with the HWM $H$. The mechanism in our model fundamentally differs from theirs.

**Investors’ value $E(W,H)$ and the total value of the fund $V(W,H)$**. The upper row of Figure 6 plots scaled investors’ value $e(w)$, $E_W(W,H)$ and $E_H(W,H)$. The scaled investors’ value $e(w)$ is increasing and concave in $w$ because investors are short in the incentive fee induced call option, whose value is convex in the AUM $W$. The mid-panel shows that $E_W(W,H)$ is decreasing and hence implies a concave $E(W,H)$ in $W$. The higher the HWM $H$, the less valuable the incentive fees and as a result, the higher investors’ value: $E_H(W,H) > 0$ given the managerial alpha-generating skill. The lower row of Figure 6 plots the scaled total fund value $v(w)$, $V_W(W,H)$, and $V_H(W,H)$. While investors’ value $e(w)$ is

\textsuperscript{12}There is no payout from the fund to investors other than exogenous liquidation of the fund which stochastically arrives at a constant rate.
concave and the manager’s value $f(w)$ is convex, the sum of the two, $v(w) = f(w) + e(w)$, is concave. The net effect of paying the manager incentive fees encourages excessive risk taking which in turn lowers the total fund value. Therefore, the total fund is short of a risk-seeking option ex post and hence its ex ante value is concave in the AUM $W$. Finally, the total fund value $V(W, H)$ increases with the HWM $H$ because the gain to investors from a higher $H$ (an increase in $E(W, H)$) outweighs the corresponding loss to the manager (decreases in $N(W, H)$ and $F(W, H)$).

![Graphs showing value of investors' payoff, total value of the fund, and sensitivities](image)

Figure 6: The scaled investors’ value $e(w)$, the scaled total fund value $v(w)$, and sensitivities of investors’ value $E(W, H)$ and the total fund value $V(W, H)$ with respect to the AUM $W$ and the HWM $H$.

6 Extensions: Liquidation and managerial ownership

In order to focus on economic intuition, we have so far intentionally kept our baseline model parsimonious. However, there are two important features that we often observe in the real
world hedge fund contracts but are missing in the baseline model. One key simplification in the baseline model is that the fund has no performance-triggered withdrawal. In our baseline model, the manager still manages the fund even after losing 99% of the AUM from the peak performance. In reality, when the fund performance deteriorates sufficiently (for example, if the fund loses 50% or more of its AUM from its running maximum, i.e. the HWM), investors are likely to withdraw and sometimes even liquidate the fund.

We will show that investors’ option to liquidate/withdraw the fund significantly changes the economics of managerial leverage and also has important quantitative implications on valuations of fees. We model liquidation by allowing investors to liquidate the fund if the manager performs sufficiently poorly. Specifically, we assume that the fund is liquidated whenever the AUM $W$ falls by a fixed fraction $(1-b)$ from the fund’s HWM $H$, i.e. the fund’s running maximum/best record in the simple example. GIR makes the same assumption on the liquidation boundary but the AUM evolution in their paper is exogenously given.

The manager is now potentially averse to performance-linked withdrawal/liquidation because the manager loses both management and incentive fees if the fund is liquidated. The manager rationally manages the fund’s leverage to influence the AUM evolution and hence the liquidation likelihood. As we will show, the manager will significantly delever the fund when it is sufficiently close to the liquidation boundary in order to preserve the fund as a going-concern and collect future management and incentive fees.

One way to mitigate managerial agency is to require managers to have their own skins in the game. In equilibrium, rational investors take into account managerial ownership in the fund when making investment decisions. We next extend our baseline model of Section 2 to incorporate both performance-based liquidation and managerial ownership.

### 6.1 A generalized setup and solution

Let $\phi$ denote the managerial ownership in the fund. For simplicity, we assume that $\phi$ is constant over time. Let $Q(W, H)$ denote the manager’s total value, which includes both the value of total fees $F(W, H)$ and $\phi E(W, H)$, the manager’s share as an investor:

$$Q(W, H) = F(W, H) + \phi E(W, H).$$

The manager dynamically chooses the investment policy \( \pi(w) \) to maximize (40). The following theorem gives the optimal \( \pi(w) \) and the scaled manager’s total value \( q(w) = f(w) + \phi e(w) \).

**Theorem 2** The manager’s scaled total value \( q(w) \) solves the following ODE:

\[
(r + \lambda - g + \delta)q(w) = [c + \phi(\delta + \lambda)]w + \mu_w(w)q'(w) + \frac{1}{2}\pi(w)^2 \sigma^2 w^2 q''(w),
\]

(41)

where \( \mu_w(w) \) is given by (27). The ODE (41) is solved subject to the boundary conditions:

\[
q(b) = \phi b ,
\]

(42)

\[
q(1) = (k + 1)q'(1) - k .
\]

(43)

With the alpha-generating technology (36), the optimal investment strategy \( \pi(w) \) is given by:

\[
\pi(w) = \frac{\pi^*}{1 + \sigma^2 \psi_q(w) / (2\theta_1)},
\]

(44)

where \( \pi^* = \theta_0 / (2\theta_1) \) is the first-best leverage ratio and \( \psi_q(w) \) is the manager’s effective risk aversion defined by:

\[
\psi_q(w) = -\frac{w q''(w)}{q'(w)}.
\]

(45)

There are two key differences between Theorem 2 and Theorem 1: concentrated managerial ownership \( \phi \) in the fund and the lower liquidation boundary \( b \). The cash flow to the manager (prior to liquidation) in the region \( b \leq w < 1 \) is given by \( (c + \phi \delta)w \). With probability \( \lambda \) per unit of time, the fund is exogenously liquidated. Therefore, the expected cash flow per unit of time (including exogenous liquidation risk) is \( (c + \phi(\delta + \lambda))w \). When the fund’s performance is sufficiently poor reaching the lower liquidation boundary \( b \), the manager loses all fees but collects pro rata share of the liquidating AUM: \( q(b) = \phi b \).

Requiring the risk-neutral manager to have skin in the game via equity ownership improves the alignment between managerial interest and investors’. As in the baseline model, the manager’s optimal investment strategy \( \pi(w) \) depends on the manager’s effective risk attitude. When calculating the manager’s effective risk aversion \( \psi_q(w) \), we need to use the manager’s total value \( q(w) \), which includes both managerial pro rata share of the fund \( \phi e(w) \) and the PV of total fees \( f(w) \). Finally, we show that the liquidation boundary \( q(b) = \phi b \) serves as an important discipline to the manager’s leverage choice. The next proposition summarizes the results on dynamic valuation.
Proposition 3 The scaled value functions \( m(w), n(w), \) and \( e(w) \) solve (28), (29) and (30), subject to the following boundary conditions:

\[
m(b) = n(b) = 0, \quad e(b) = b.
\]

The upper boundary conditions at \( w = 1 \) remain the same as in Proposition 2, i.e. conditions (32), (33), and (34), respectively. The total fund’s value is \( V(W, H) = v(w)H \), where \( v(w) = m(w) + n(w) + e(w) = f(w) + e(w) \). The manager’s scale value is \( q(w) = f(w) + \phi e(w) \).

6.2 Performance-triggered liquidation

Dynamic investment strategy \( \pi(w) \) under a liquidation boundary \( b > 0 \). Figure 7 plots the manager’s optimal leverage \( \pi(w) \) when the liquidation boundary is set at \( b = 0.5 \) and no managerial ownership \( (\phi = 0) \). The new equilibrium compensation contract (with \( c = 2.48\% \) and \( k = 20\% \)) ensures \( e(1) = 1 \), i.e. investors break even. All other parameter values remain unchanged.

Figure 7: Dynamic investment strategy \( \pi(w) \): The case with liquidation boundary \( b = 0.5 \) and no managerial ownership \( (\phi = 0) \). The managerial compensation contract \( (c, k) \) is set at \( (2.48\%, 20\%) \) to ensure that investors break even: \( e(1) = 1 \). Other parameter values are: \( r = 5\%, \sigma = 9\%, \lambda = 5\%, g = 5\%, \delta = 5\%, \theta_0 = 2.5\%, \) and \( \theta_1 = 0.25\% \).

First, we note that the overall leverage is much lower when \( b = 0.5 \) than when \( b = 0 \).
This is due to the manager’s aversion to liquidation. For sufficiently high \( w \) (i.e. \( w \geq 0.6 \)), the manager takes on leverage \((\pi > 1)\). However, the leverage level is much lower with liquidation boundary \( b = 0.5 \) than without as in the baseline case (i.e. \( b = 0 \)). For example, even at \( w = 1 \), the leverage is only \( \pi(1) = 2.91 \) with \( b = 0.5 \), which is substantially lower than the level in the baseline model with \( b = 0 \): \( \pi(1) = 6.33 \).

When \( w \) is sufficiently close to the liquidation boundary \( b = 0.5 \) (i.e. \( 0.5 < w < 0.6 \)), fighting for survival is the manager’s primary concern. The manager rationally allocates some of the fund’s AUM to the risk-free asset \((0 < \pi(w) < 1)\) in order to lower the volatility of the AUM return even though the risky investment strategy has a positive alpha. That is, unlike the case without liquidation \((b = 0)\), the optimal investment strategy \( \pi(w) \) can sometimes be less than unity. The manager is not only long an incentive-fees-induced call option but also short a liquidation option. This liquidation option is closer to being in the money following sufficiently poor performances, i.e. \( w \) drifts leftward toward the liquidation boundary \( b = 0.5 \). The short position in the liquidation option makes the manager to behave in an effectively risk averse manner and hence de-leveraging is often optimal. Casual observations seem consistent with this prediction. Fund managers tend to reduce leverage in bad times following poor performances.

Liquidation is costly because the manager loses future rents from the alpha generating technology.\(^{14}\) By reducing exposure to the risky positive alpha technology, the manager gives up some current and near-term fees but in exchange increases longer-term survival probability. Dai and Sundaresan (2010) also show that hedge funds may use conservative levels of leverage if they properly recognize the short option positions due to contractual arrangements with investors and prime brokers.

**Managerial rents \( f(w) \) and risk attitude \( \psi(w) \).** The left panel of Figure 8 plots \( f(w) \) when the liquidation boundary is set at \( b = 0.5 \). Compared with the case of \( b = 0 \), the PV of total fees \( f(w) \) is substantially lower. Liquidation shortens the duration of the fund’s life and also reduces the manager’s incentive to take on risk, both of which lower \( f(w) \). With \( b = 0.5 \), the manager makes a rent of 29.6 cents for each dollar in AUM: \( f(1) = 0.296 \), which is substantially lower than a rent of 58.9 cents per dollar of AUM: \( f(1) = 0.589 \) when

\(^{14}\)We do not consider managerial option to start a new fund, which may be costly but potentially valuable strategy.
$b = 0$. Note that this comparison has already accounted for the adjustment of managerial compensation from $c = 2\%$ for $b = 0$ to $c = 2.48\%$ for $b = 0.5$ while keeping $k = 20\%$ so that investors break even under perfectly competitive capital markets: $e(1) = 1$.

The right panel of Figure 8 plots the manager’s effective risk aversion $\psi(w)$ for the case with $b = 0.5$. The lower liquidation boundary turns the risk-neutral manager from behaving in a risk-seeking way ($\psi(w) < 0$ for $b = 0$) to behaving in an effectively risk-averse way ($\psi(w) > 0$) when $b = 0.5$). The manager’s “effective” risk aversion at the liquidation boundary $b = 0.5$ is $\psi(0.5) = 4.91$, which is quantitatively significant. Even when incentive fees are close to being in the money (when $w = 1$), the manager continues to behave in an effectively risk-averse manner with an effective risk aversion: $\psi(1) = 0.45 > 0$.

![Figure 8](image_url)

Figure 8: The PV of total fees $f(w)$ and the manager’s effective risk aversion: $\psi(w) = -wf''(w)/f'(w)$: The case with liquidation boundary $b = 0.5$ and no managerial ownership ($\phi = 0$). Increasing the liquidation boundary from $b = 0$ to $b = 0.5$ substantially lowers the PV of total fees $f(w)$ and makes the manager effectively risk averse ($\psi(w) > 0$).

Our results challenge the conventional wisdom that high-powered incentive fees often encourage excessive risk taking. While in isolation, the high-powered incentives encourage excessive risk taking as we have shown in the baseline model with $b = 0$, other frictions such as the liquidation option held by investors fundamentally changes the managerial incentive
to take on leverage. Indeed, the manager sometimes behaves too conservatively from the investors’ perspective. Compared with the first-best leverage $\pi^* = 5$, the fund is underlevered for all $w$ when $b = 0.5$. This provides one explanation why sometimes we see that investors require a lower bound for the fund’s leverage. Managers may sometimes choose excessively low leverage primarily for survival and fee collection. Next, we characterize various value functions and provide some quantitative results from this numerical example.

Figure 9: The value of management fees $m(w)$, the value of incentive fees $n(w)$, investors’ value $e(w)$ and the total fund value $v(w)$: The case with lower liquidation boundary $b = 0.5$.

**Various value functions:** $n(w)$, $m(w)$, $e(w)$, and $v(w)$. Figure 9 plots the PV of incentive fees $n(w)$, the PV of management fees $m(w)$, investors’ value $e(w)$, and the total fund value $v(w)$ for the case with $b = 0.5$ and no managerial ownership ($\phi = 0$). Not surprisingly,
we also find the PV of management fees $m(w)$ is increasing and concave, the PV of incentive fees $n(w)$ is increasing and convex, and the value of total fund $v(w)$ is increasing and concave in $w$. These results are consistent with those in Section 5 for the baseline model with $b = 0$.

Quantitatively, the liquidation option significantly changes the economics of managerial fees and valuation. For the baseline model with $b = 0$, in our numerical example, the PV of incentive fees $n(1) = 0.39$, which is approximately twice as much as the PV of management fees $m(1) = 0.2$. Increasing $b = 0$ to $b = 0.5$, the value of incentive fees drops to $n(1) = 0.1$ which is about half of the value of management fees $m(1) = 0.2$. Therefore, increasing the liquidation boundary from $b = 0$ to $b = 0.5$ not only lowers the PV of total fees from $f(1) = 0.39 + 0.2 = 0.59$ to $f(1) = 0.1 + 0.2 = 0.3$ by about half, it also changes the relative weights of incentive fees to management fees from 2:1 to 1:2. With liquidation boundary $b = 0.5$, the manager acts much more conservatively with leverage and management fees become the larger portion of the total fees.

Unlike in the baseline model with $b = 0$, the PV of investors’ value $e(w)$ is no longer globally concave. It is convex in $w$ for sufficiently low $w$ ($0.5 \leq w \leq 0.617$) and concave in $w$ for sufficiently high $w$ ($0.617 \leq w \leq 1$). The intuition is as follows. For any given $w$, the investors are simultaneously short in the incentive call option but long the liquidation option, in addition to the unlevered “equity” claim in the fund’s after-fees cash flow process. For lower values of $w$, the long position of the liquidation option is quantitatively more important than the short position of the incentive option. The net effect is that $e(w)$ is convex in $w$ for low values of $w$. For higher values of $w$, the opposite holds and hence the net effect is that $e(w)$ is concave in $w$ at the right end of $w$. The high non-linearity of $e(w)$ reflects various option features embedded in the hedge fund management compensation contracts. The lower left panel of Figure 9 depicts $e(w)$ which is convex in $w$ for $0.5 \leq w \leq 0.617$ and concave in $w$ for $0.617 \leq w \leq 1$.

To summarize, the liquidation boundary makes the manager’s investment strategy much more conservative because preserving the fund as a going-concern is valuable for the manager. Management fees carry more weights when the liquidation option becomes more valuable for investors. Liquidation boundary can significantly lower the value of total fees by truncating the fund’s horizon and also substantially lowering the managerial incentive to take on leverage. We next analyze the effects of managerial ownership.
Table 1: The effects of managerial ownership and the liquidation boundary on fees, leverage, and valuation.

This table reports the investment strategy $\pi(1)$, the PV of the management fees $m(1)$, the PV of incentive fees $n(1)$, the PV of total fees $f(1)$, the investor’s payoff $e(1)$ and the total value of the fund $v(1)$ under various managerial ownership $\phi$ and the liquidation boundary $b$. The compensation contract is fixed at $c = 2.61\%$ and $k = 20\%$. Other parameter values are: $r = 5\%$, $\sigma = 9\%$, $\lambda = 5\%$, $g = 5\%$, $\delta = 5\%$, $\theta_0 = 2.5\%$, and $\theta_1 = 0.25\%$.

<table>
<thead>
<tr>
<th>Managerial ownership $\phi$</th>
<th>Panel A. Liquidation boundary $b = 0.5$</th>
<th>[\pi(1)]</th>
<th>[m(1)]</th>
<th>[n(1)]</th>
<th>[f(1)]</th>
<th>[e(1)]</th>
<th>[v(1)]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.8381</td>
<td>0.2055</td>
<td>0.0947</td>
<td>0.3002</td>
<td>0.9906</td>
<td>1.2908</td>
<td></td>
</tr>
<tr>
<td>0.05</td>
<td>2.7646</td>
<td>0.2046</td>
<td>0.0955</td>
<td>0.3001</td>
<td>0.9958</td>
<td>1.2958</td>
<td></td>
</tr>
<tr>
<td><strong>0.10</strong></td>
<td><strong>2.7069</strong></td>
<td><strong>0.2035</strong></td>
<td><strong>0.0963</strong></td>
<td><strong>0.2998</strong></td>
<td><strong>1.0000</strong></td>
<td><strong>1.2998</strong></td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>2.5921</td>
<td>0.2002</td>
<td>0.0981</td>
<td>0.2983</td>
<td>1.0090</td>
<td>1.3072</td>
<td></td>
</tr>
<tr>
<td>0.50</td>
<td>2.4971</td>
<td>0.1953</td>
<td>0.1002</td>
<td>0.2956</td>
<td>1.0165</td>
<td>1.3121</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Liquidation boundary $b$</th>
<th>Panel B. Managerial ownership $\phi = 0.1$</th>
<th>[\pi(1)]</th>
<th>[m(1)]</th>
<th>[n(1)]</th>
<th>[f(1)]</th>
<th>[e(1)]</th>
<th>[v(1)]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5.7732</td>
<td>0.2642</td>
<td>0.3452</td>
<td>0.6095</td>
<td>1.0116</td>
<td>1.6210</td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>4.7106</td>
<td>0.2407</td>
<td>0.2262</td>
<td>0.4669</td>
<td>1.0504</td>
<td>1.5173</td>
<td></td>
</tr>
<tr>
<td><strong>0.50</strong></td>
<td><strong>2.7069</strong></td>
<td><strong>0.2035</strong></td>
<td><strong>0.0963</strong></td>
<td><strong>0.2998</strong></td>
<td><strong>1.0000</strong></td>
<td><strong>1.2998</strong></td>
<td></td>
</tr>
<tr>
<td>0.75</td>
<td>0.3044</td>
<td>0.1742</td>
<td>0.0083</td>
<td>0.1826</td>
<td>0.9071</td>
<td>1.0897</td>
<td></td>
</tr>
</tbody>
</table>

6.3 Managerial ownership and liquidation threshold

Table 1 reports the investment strategy $\pi(1)$, the PV of management fees $m(1)$, the PV of incentive fees $n(1)$, the PV of total fees $f(1)$, the investor’s payoff $e(1)$ and the total value of the fund $v(1)$ under various managerial ownership $\phi$ and the liquidation boundary $b$. The managerial compensation contract is fixed at $c = 2.61\%$ and $k = 20\%$ for all cases.\(^{15}\) Other parameters are the same as in Section 5.

Panel A of Table 1 reports the results at $w = 1$ (e.g. when the fund is started) for various levels of managerial ownership $\phi$ with $b = 0.5$. Recall that when $\phi = 10\%$, investors break even (i.e. $e(1) = 1$) under the given compensation contract $c = 2.61\%$ and $k = 20\%$. The third row highlighted in bold summarizes the results for this base case: the optimal

\(^{15}\)We use the condition that capital markets are perfectly competitive and investors break even (i.e. $e(1) = 1$) for the case with liquidation boundary $b = 0.5$ and managerial ownership $\phi = 0.1$ to pin down the management fee at $c = 2.61\%$. We set $k = 20\%$ for this calculation.
leverage \( \pi(1) = 2.7 \), the PV of total fees is about 30% of the AUM, i.e. \( f(1) = 0.3 \), out of which management fees and incentive fees account for two-thirds and one-third, respectively: \( m(1) = 0.2 \) and \( n(1) = 0.1 \).

As we increase managerial ownership \( \phi \) from 0 to 50%, leverage \( \pi(w) \) falls from 2.8 to 2.5.\(^\text{16}\) The more equity ownership \( \phi \) that the manager has in the fund, the less the fund will be levered. The value of incentive fees \( n(1) \) thus also falls with ownership \( \phi \), but the value of management fees \( m(1) \), the value of investors’ payoff \( e(1) \), and the total value of the fund \( v(1) \) all increase.

Next, we turn to the impact of liquidation boundary \( b \). Panel B of Table 1 reports the results for various levels of \( b \) with \( \phi = 0.1 \). As we increase the liquidation boundary \( b \) from 0 to 0.5, the manager significantly decreases leverage \( \pi(1) \) from 5.77 to 2.71. The PV of total fees \( f(1) \), the PV of incentive fees \( n(1) \) and the PV of management fees \( m(1) \) all decrease with the liquidation boundary \( b \) because liquidation shortens the fund’s horizon and discourages leverage. Comparing the case of \( b = 0 \) with that of \( b = 0.5 \), the PV of total fees \( f(1) \) drops by about half from 0.61 to 0.30, while the PV of incentive fees \( n(1) \) drop from 0.35 to 0.1 by more than 70% and the PV of management fees \( m(1) \) decrease from 0.26 to 0.20 by 23%. The relative weights of management and incentive fees also significantly change with the liquidation boundary \( b \). For example, while incentive fees account for 57% of \( f(1) \) with \( b = 0 \), incentive fees only account for less than 33% of \( f(1) \) with \( b = 0.5 \).

With sufficiently high liquidation risk (e.g. \( b = 0.75 \)), the manager optimally places a significant fraction of the AUM, about 70%, in cash to earn the risk-free rate: \( \pi(1) = 0.3 \). Managerial strong desire for the fund’s survival dominates the manager’s dynamic investment decision. As a result, the total fund’s value \( v(1) \) is significantly lower, only about 1.09, which implies that the fund is about 9 percent more valuable than its AUM \( W \). For the manager, the PV of total fees \( f(1) = 0.18 \), out of which the vast majority is from management fees: \( m(1) = 0.17 \) leaving only the value of incentive fees being \( n(1) = 0.01 \).

The preceding analyses fix the managerial compensation contract at \( c = 2.61\% \) and \( k = 20\% \), keep all the parameter values unchanged, and focus on the comparative effects of managerial ownership \( \phi \) and liquidation boundary \( b \) on leverage and valuation. However, it

\(^{16}\)Note that in this exercise, as we vary managerial ownership \( \phi \), we no longer impose the restriction that \( e(1) = 1 \). We are focusing on the impact of managerial ownership \( \phi \) and later liquidation boundary \( b \) on leverage and valuation for a given fixed compensation contract.
is important to note that the surplus created from more concentrated managerial ownership and/or better incentive alignments is not going to be captured by investors but rather by the manager under perfectly competitive capital markets \((e(1) = 1)\). The compensation contract adjusts to a new equilibrium to reflect the equilibrium division of the surplus.\(^{17}\)

### 7 The value of dynamically adjusting leverage

Leverage in our model creates value because the manager has skills and there is an optimal scale of investments. We can decompose the PV of managerial rents \(f(w)\) and the total fund value \(v(w)\) into an unlevered component and the net increase in value due to the use of leverage. First, we summarize the results in the benchmark case where the manager takes no leverage, i.e. a pure valuation model as in GIR.

#### 7.1 A pure valuation model \((\pi(w) = 1)\): GIR

GIR incorporates the effects of the HWM on the valuation of management fees and incentive fees. But their model does not allow for managerial leverage decisions. Their model can thus be viewed as a special case of ours with \(\pi(w)\) fixed at unity for all \(w\). The following theorem summarizes the main results in GIR.

**Theorem 3** Fixing \(\pi(w) = 1\) at all times and for all \(w\), we have the following closed-form solutions for various value functions:

\[
\begin{align*}
    n_{\text{gir}}(w) &= \frac{k(w^n - b^{n-\xi}w^\xi)}{\eta(k + 1) - 1 - b^{n-\xi}(\zeta(1 + k) - 1)},  \\
    f_{\text{gir}}(w) &= \frac{c}{c + \delta + \lambda - \alpha(1)}w + \frac{(\delta + \lambda - \alpha(1))k + (\zeta(1 + k) - 1)cb^{1-\xi}}{(c + \delta + \lambda - \alpha(1))(\eta(k + 1) - 1 - b^{n-\xi}(\zeta(1 + k) - 1))}w^n \\
    e_{\text{gir}}(w) &= \frac{\delta + \lambda}{c + \delta + \lambda - \alpha(1)}w - \frac{(\delta + \lambda)k + (\zeta(1 + k) - 1)(c - \alpha(1))b^{1-\xi}}{(c + \delta + \lambda - \alpha(1))(\eta(k + 1) - 1 - b^{n-\xi}(\zeta(1 + k) - 1))}w^n \\
    &\quad + \frac{b^{n-\xi}(\delta + \lambda)k + (\eta(1 + k) - 1)(c - \alpha(1))b^{1-\xi}}{(c + \delta + \lambda - \alpha(1))(\eta(k + 1) - 1 - b^{n-\xi}(\zeta(1 + k) - 1))}w^\xi.
\end{align*}
\]

\(^{17}\)Due to space constraints, we leave out the details of the analysis when we adjust the equilibrium compensation contract. The details are available upon request.
where $\alpha(1) = \theta_0 - \theta_1$, $\eta$ and $\zeta$ are given by

$$
\eta = \frac{1}{2} - \frac{\alpha(1) + r - g - c}{\sigma^2} + \sqrt\left(\frac{1}{2} - \frac{\alpha(1) + r - g - c}{\sigma^2}\right)^2 + \frac{2(r + \lambda - g + \delta)}{\sigma^2} > 1, \tag{50}
$$

and

$$
\zeta = \frac{1}{2} - \frac{\alpha(1) + r - g - c}{\sigma^2} - \sqrt\left(\frac{1}{2} - \frac{\alpha(1) + r - g - c}{\sigma^2}\right)^2 + \frac{2(r + \lambda - g + \delta)}{\sigma^2} < 0. \tag{51}
$$

In addition, we have $m_{gir}(w) = f_{gir}(w) - n_{gir}(w)$ and $v_{gir}(w) = e_{gir}(w) + f_{gir}(w)$.

In our model, the manager creates value in two ways: the alpha-generating technology and the use of leverage. Note that M&M does not hold in our model. Using the GIR as the benchmark, we can quantify the value purely due to leverage.

### 7.2 The value of leverage

Let $\Delta f(w) = f(w) - f_{gir}(w)$ denote the difference between the PV of total fees $f(w)$ in our model and $f_{gir}(w)$, the PV of total fees in GIR. Similarly, let $\Delta v(w) = v(w) - v_{gir}(w)$ denote the difference between the PV of total fund’s value $v(w)$ in our model and $v_{gir}(w)$, the PV of total fund’s value in GIR. Table 2 is based on the same set of calculations as Table 1.

Panel A of Table 2 calculates the net present value (NPV) of leverage for various levels of ownership $\phi$ with the liquidation boundary fixed at $b = 0.5$. First, by construction, $f_{gir}(1)$ and $v_{gir}(1)$ in the GIR benchmark are independent of managerial ownership $\phi$: $f_{gir}(1) = 0.15$ and $v_{gir}(1) = 0.86$ for all $\phi$. Second, incentive alignments between the manager and investors improve with managerial ownership $\phi$. As a result, the net gain due to leverage for the total fund value, $\Delta v(1)$, increases with ownership $\phi$. However, the net gain due to leverage for the value of total fees, $\Delta f(1)$, decreases.

Panel B of Table 2 calculates the NPV of leverage for various levels of liquidation boundary $b$ with managerial ownership fixed at $\phi = 0.1$. For the GIR benchmark, $f_{gir}(1)$ decreases with liquidation boundary $b$. The manager becomes much more prudently in choosing leverage due to concerns of losing future fees, and as a result, the NPV of leverage for the manager $\Delta f(1)$ drops significantly from 0.43 to 0.098 when the liquidation boundary $b$ increases from 0 to 0.75. Similarly, the value of leverage for the total fund, $\Delta v(1)$, also decreases significantly, from 0.80 to 0.17 as we increase $b$ from 0 to 0.75.
Table 2: The value of leverage for various levels of managerial ownership $\phi$ and liquidation boundaries $b$.

This table reports the PV of total fees $f_{\text{gir}}(1)$ and the total fund’s value $v_{\text{gir}}(1)$ in the GIR benchmark and then quantifies the NPV of leverage by calculating $\Delta f(1) = f(1) - f_{\text{gir}}(1)$ and $\Delta v(1) = v(1) - v_{\text{gir}}(1)$ for various levels of managerial ownership $\phi$ and liquidation boundaries $b$. The managerial compensation contract is fixed at $c = 2.61\%$ and $k = 20\%$ for all cases. Other parameter values are: $r = 5\%$, $\sigma = 9\%$, $\lambda = 5\%$, $g = 5\%$, $\delta = 5\%$, $\theta_0 = 2.5\%$, and $\theta_1 = 0.25\%$.

<table>
<thead>
<tr>
<th>Panel A. Liquidation boundary $b = 0.5$</th>
<th>Gir Benchmark</th>
<th>NPV of Leverage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Managerial ownership $\phi$</td>
<td>$f_{\text{gir}}(1)$</td>
<td>$v_{\text{gir}}(1)$</td>
</tr>
<tr>
<td>0</td>
<td>0.1459</td>
<td>0.8580</td>
</tr>
<tr>
<td>0.05</td>
<td>0.1459</td>
<td>0.8580</td>
</tr>
<tr>
<td>0.10</td>
<td><strong>0.1459</strong></td>
<td><strong>0.8580</strong></td>
</tr>
<tr>
<td>0.25</td>
<td>0.1459</td>
<td>0.8580</td>
</tr>
<tr>
<td>0.50</td>
<td>0.1459</td>
<td>0.8580</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B. Managerial ownership $\phi = 0.1$</th>
<th>Gir Benchmark</th>
<th>NPV of Leverage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liquidation boundary $b$</td>
<td>$f_{\text{gir}}(1)$</td>
<td>$v_{\text{gir}}(1)$</td>
</tr>
<tr>
<td>0</td>
<td>0.1792</td>
<td>0.8230</td>
</tr>
<tr>
<td>0.25</td>
<td>0.1735</td>
<td>0.8289</td>
</tr>
<tr>
<td><strong>0.5</strong></td>
<td><strong>0.1459</strong></td>
<td><strong>0.8580</strong></td>
</tr>
<tr>
<td>0.75</td>
<td>0.0850</td>
<td>0.9220</td>
</tr>
</tbody>
</table>

8 Conclusions

Hedge fund managers are paid via both management fees and high-powered incentive fees. Management fees are often specified as a fraction of the assets under management (AUM), and incentive fees are indexed to the stochastic and endogenously evolving high-water mark (HWM), which is effectively the running maximum of the AUM. While this compensation structure potentially rewards managerial talent, it also invites agency conflicts between the manager and investors. Incentive fees provide the manager a sequence of call options on the AUM and hence encourage the manager to take on excessive risk. However, management fees
provide an unlevered equity-like financial claim in the fund and thus mitigate managerial risk seeking induced by incentive fees. The manager dynamically manages leverage to maximize the present value of total fees, and investors rationally anticipate dynamically changing managerial incentives and price accordingly their investments in the fund.

The two key state variables in our model are the AUM and the HWM. In our homogenous framework, the ratio between the AUM and the HWM, $w$, determines the fund’s dynamic leverage decisions and various value functions including those for management and incentive fees. As the fund’s performance improves, i.e. $w$ increases, the manager increases leverage as incentive fees become deeper in the money. When the performance deteriorates, i.e. $w$ decreases, the manager reduces leverage as management fees become more important, *ceteris paribus*. This performance-dependent dynamic leverage strategy has important implications on valuing management fees and incentive fees. While both management fees and incentive fees contribute significantly to managerial compensation in general, they carry different weights in contributing to total managerial rents under different circumstances.

In our model, leverage contributes significantly to the present value of managerial rents and the total fund value because leverage allows the manager to capitalize on skills using a larger pool of capital in a non-M&M world. Granting investors options to liquidate the fund substantially curtails the manager’s risk-taking incentives, effectively making the manager behave in a risk-averse manner. Having the manager co-invest in the fund as a limited partner also helps to align incentives and reduce agency conflicts.

In our model, the manager’s skill is known. However, in reality, investors may learn about managerial skills. Moreover, managerial skills may depend on macro conditions and can be time-varying. Extending the model to allow investors to learn about time-varying managerial skills is left for future research. Additionally, the recent crisis reveals that market liquidity and funding liquidity are first-order issues (Brunnermeier and Pedersen (2009)), and that funding costs can increase significantly in crisis periods. Anticipating these liquidity problems, the manager rationally chooses the fund’s leverage and prudently manages risk in a state contingent way (Dai and Sundaresan (2010)). In future work, we plan to integrate market and funding illiquidity into our framework.

In order to make dynamic leverage and valuation results practical and operational, we have taken real-world hedge fund management compensation contracts as given and focus on
the space of compensation contracts which feature management and incentive fees. Within
the set of contracts featuring management and incentive fees, we are able to provide a direct
link from managerial skills to their compensation contracts. For example, in our model, a
more talented manager charges more. James Simons’ Renaissance charges a management fee
at 5% of the AUM and an incentive fee at 36%, while the industry standard charges man-
agement and incentive fees at 2% and 20%, respectively. Developing an industry equilibrium
model of hedge funds with heterogenous managerial skills is an interesting topic which we
leave for future research. We expect that in equilibrium, managers with better skills run
larger funds and potentially charge higher fees.
References


Appendices

A Technical details

**Proposition 1.** Without incentive fees, the manager chooses optimal investment \( \pi^* \) to maximize his value \( F^*(W) \) by solving the following HJB equation:

\[
(r + \lambda)F^*(W) = \max_{\pi} \ cW + (\pi \alpha(\pi) + r - \delta - c)WF^*_W(W) + \frac{1}{2}\pi^2\sigma^2 W^2 F^*_W(W).
\]  

(A.1)

With no incentive fees, we can conjecture that \( F^*(W) = zW \), where \( z \) is a constant to be determined. The FOC for leverage is given by

\[
\alpha(\pi^*) + \pi^*\alpha'(\pi^*) = 0,
\]

(A.2)

Substituting the optimal investment leverage into the HJB equation (A.1), we have

\[
F^*(W) = M^*(W) = \frac{c}{c + \delta + \lambda - \alpha^*} W,
\]

(A.3)

where \( \alpha^* \) is given in (13). We then obtain explicit linear value functions for \( E^*(W) \) and \( V^*(W) \) given in Proposition 1.

**Theorem 1.** Using the FOC for \( \pi \) in the HJB equation (18), we have

\[
\pi = \frac{\alpha(\pi)}{-\sigma^2 W F_{WW}(W, H) F_W^{-1}(W, H) - \alpha'(\pi)}.
\]

(A.4)

We have also verified the second-order condition. We conjecture that the value function \( F(W, H) \) takes the following homogeneous form in \( W \) and \( H \):

\[
F(W, H) = f(w)H.
\]

(A.5)

The above conjecture implies the following results: \( F_W(W, H) = f'(w) \), \( F_{WW}(W, H) = f''(w)/H \), and \( F_H(W, H) = f(w) - wf'(w) \). Substituting (A.4) into the HJB equation (18) and boundary conditions (20), we obtain (21)-(25) in Theorem 1.
Proposition 2. Applying the Ito’s formula to (3) and (4), we obtain the following dynamics for $w$:

$$dw_t = dW_t - dH_t = \mu_w(w)dt + \pi(w_t)\sigma dB_t - dJ_t,$$  \hspace{1cm} (A.6)

where $\mu_w(w)$ is given in (27). Applying the standard differential equation pricing methodology to the present values $M(W,H)$ given in (5), $N(W,H)$ given in (6) and $E(W,H)$ given in (8), we obtain (28), (29), and (30) for $m(w)$, $n(w)$, and $e(w)$, respectively.

Now consider the boundary behavior at $W_t = H_t$. We use the same argument as the one for $F(W,H)$. Consider the scenario where the asset value increases by $\Delta H$ over a small time interval $\Delta t$, the high-water mark is then re-set to $H + \Delta H$. Because value functions are continuous, we thus have

$$M(H + \Delta H, H) = M(H + \Delta H - k\Delta H, H + \Delta H),$$  \hspace{1cm} (A.7)

$$N(H + \Delta H, H) = k\Delta H + N(H + \Delta H - k\Delta H, H + \Delta H),$$  \hspace{1cm} (A.8)

$$E(H + \Delta H, H) = E(H + \Delta H - k\Delta H, H + \Delta H).$$  \hspace{1cm} (A.9)

Therefore, by taking limits, we have:

$$kM_W = M_H, \quad kN_W = k + N_H, \quad kE_W = E_H.$$  \hspace{1cm} (A.10)

Similarly, using the property that value functions are homogeneous with degree one in $W$ and $H$, we may write $M(W,H) = m(w)H$, $N(W,H) = n(w)H$, and $E(W,H) = e(w)H$. Finally, $W = 0$ is an absorbing state and therefore $M(0,H) = N(0,H) = E(0,H) = 0$. Substituting these results into valuation equations, we obtain results in Proposition 2.

Theorem 3. GIR is a special case of our model where the investment strategy fixed at $\pi(w) = 1$ for all $w$. Using our earlier results, we calculate $f(w)$ as follows:

$$(r + \lambda - g + \delta)f(w) = cw + (\alpha(1) + r - g - c)wf'(w) + \frac{\sigma^2w^2f''(w)}{2},$$  \hspace{1cm} (A.11)

where $\alpha(1) = \theta_0 - \theta_1$. The linear ODE (A.11) has the following closed-form solution:

$$f(w) = xw + y_1w^n + y_2w^\xi,$$  \hspace{1cm} (A.12)
where \(x, y_1\) and \(y_2\) are constant coefficients which we will determine next, and \(\eta\) and \(\zeta\) are given by (50) and (51), respectively. Note that \(\eta\) and \(\zeta\) are the positive and negative roots of the following quadratic equation, respectively:

\[
(r + \lambda - g + \delta) = (\alpha(1) + r - g - c)z + \frac{\sigma^2}{2}z(z - 1).
\]  

(A.13)

To ensure convergence, we require \(c + \delta + \lambda - \alpha(1) > 0\), i.e. equivalently \(\eta > 1\). Using the boundary conditions \(f(b) = 0\) and \(f(1) = (k + 1)f'(1) - k\), we obtain

\[
x = \frac{c}{c + \delta + \lambda - \alpha(1)},
\]

(A.14)

\[
y_1 = \frac{(\delta + \lambda - \alpha(1))k + (\zeta(1 + k) - 1)cb^{1-\zeta}}{(c + \delta + \lambda - \alpha(1))(\eta(k + 1) - 1 - b^{\eta-\zeta}(\zeta(1 + k) - 1))},
\]

(A.15)

\[
y_2 = -\frac{b^{\eta-\zeta}(\delta + \lambda - \alpha(1))k + (\eta(1 + k) - 1)cb^{1-\zeta}}{(c + \delta + \lambda - \alpha(1))(\eta(k + 1) - 1 - b^{\eta-\zeta}(\zeta(1 + k) - 1))}.
\]

(A.16)

Closed-form expressions for other value functions can be similarly derived.