Dynamics of Entrepreneurship under Incomplete Markets

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Abstract

An entrepreneur faces substantial non-diversifiable business risk and liquidity constraints. We provide an analytically tractable dynamic incomplete-markets framework in which to study the entrepreneur’s interdependent business start-up, capital accumulation/asset sales, portfolio allocation, consumption, and business exit decisions. We show that business risks and liquidity constraints have economically significant effects. The entrepreneur invests substantially less in his business than an otherwise identical public firm does. Non-diversifiable idiosyncratic risk and liquidity constraints give rise to precautionary demand and cause the marginal value of wealth to be greater than unity. The investment decisions for an entrepreneurial firm differ fundamentally from those for public firms as in standard q theories of investment. The exit option provides flexibility and insurance to the entrepreneur. Moreover, it generates a convexity effect of volatility on business investment and portfolio choice. The entrepreneur’s entry decision depends on his outside option, the fixed start-up cost and the endogenously determined marginal value of wealth under incomplete markets. We also provide an operational framework to calculate both systematic and idiosyncratic risk components of the entrepreneurial project’s cost of capital. Despite various margins that the entrepreneur has to manage his risk exposure, both systematic and idiosyncratic risks have substantial quantitative effects on the entrepreneur’s welfare and have potentially important implications on economic growth.

Keywords: idiosyncratic risk premium, hedging, Tobin’s q, liquidity, credit constraint, precautionary saving, portfolio choice, investment, entry and exit

JEL Classification: G11, G31, E2

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1 Introduction

An entrepreneur owns a business and assumes significant accountability for the risks and outcome of the business. Casual observations and empirical studies have shown that active businesses account for a large fraction of entrepreneurs’ total wealth and that entrepreneurial firms tend to have highly concentrated ownership.\footnote{For example, see Moskowitz and Vissing-Jorgensen (2002), and Gentry and Hubbard (2004).} Moreover, entrepreneurs often face liquidity constraints (Evans and Jovanovic (1989) and Cagetti and De Nardi (2006)).

In entrepreneurial firms, frictions such as borrowing constraints and lack of diversification invalidate the classic complete-markets and implied standard law-of-one-price-based valuation analysis. An entrepreneur’s non-diversifiable position in his investment project causes his business decisions (capital accumulation and business entry/exit) and household decisions (consumption, saving, and asset allocation) to be highly interdependent. In a recent survey of research on entrepreneurship in macroeconomics, Quadrini (2009) discusses the importance of borrowing constraints and lack of diversification on entrepreneurship as a career choice, entrepreneurial saving/investment, and economic development/growth.

We model the impact and interactions of these two conceptually important frictions, liquidity constraints and lack of diversification, on entrepreneurial entry, capital accumulation/asset sale, consumption, portfolio allocation, and exit (via liquidation). We then use the entrepreneurial optimal decision rules to provide an intertemporal valuation framework for the entrepreneurial firm and calculate the implied risk/return trade-off and the cost of capital. We provide an operational framework to quantify the impact of non-diversifiable risk and liquidation constraints on the cost of capital and certainty equivalent for the entrepreneur.

In order to capture the entrepreneur’s dynamic investment decision, we build our model of entrepreneurship using a workhorse dynamic $q$ theory of investment (Hayashi (1982)). Unlike the neoclassic framework of Hayashi (1982), our model predicts that non-diversifiable risk influences investment and drives a wedge between the entrepreneur’s marginal $q$ and average $q$. In addition to being a producer, an entrepreneur is also a household who makes intertemporal consumption-saving and portfolio allocation decisions. We build on the complete-markets classic portfolio choice framework of Merton (1971), but extend it to allow for non-diversifiable risk and a limited ability to borrow. Doing as such allows us to capture non-diversifiable
risk and precautionary saving motives as in modern consumption literature.\textsuperscript{2}

Finally, we augment the integrated intertemporal consumption-saving, portfolio choice, and investment framework with endogenous entry and exit decisions. Allowing for endogenous entry into entrepreneurship, we capture the agent’s career choice between being an entrepreneur and taking his outside option. Moreover, by endowing the agent with a liquidation option to exit entrepreneurship, we provide one channel for the entrepreneurship to manage downside business risk exposure. This exit option in turn has implications on the entrepreneur’s decision making (e.g., investment, consumption, and portfolio allocation) and subjective valuation of the firm.

Non-diversifiable risk under incomplete markets makes the risk-return tradeoff analysis for the entrepreneurial firm different from that of an otherwise identical (public) firm.\textsuperscript{3} Consider a project valuation exercise. For simplicity, assume that the standard capital asset pricing model (CAPM) holds if this project belongs to a public firm owned by diversified investors. The standard diversification-based law of one price valuation tells us that there is no idiosyncratic risk premium. However, if the same project were carried out by an entrepreneurial firm where the owner is fully exposed to the project’s idiosyncratic and systematic risks, we naturally expect that there is an additional “private” equity idiosyncratic risk premium. How do we calculate the cost of capital for entrepreneurial firms? What are the determinants of this idiosyncratic risk premium? We provide an operational framework to answer these questions. Our calibration exercise suggests that the idiosyncratic risk is likely to be economically important for non-diversified entrepreneurs.

We consider an infinitely-lived agent who maximizes utility over his intertemporal consumption. He has a non-expected recursive utility featuring constant relative risk aversion and constant elasticity of intertemporal substitution, i.e., the agent has an Epstein-Zin utility including the widely used isoelastic utility as a special case. Separating elasticity of intertemporal substitution and risk aversion allows us to examine the effects of these two important preference parameters on different economic variables.

At time zero, he has a discrete career choice to make: whether to become a self-employed


\textsuperscript{3}Of course, ownership structure is endogenous. We are doing a “comparative” analysis where two firms are identical in all aspects other than the ownership structure.
entrepreneur or to take his outside option (e.g. by being a worker earning wages and facing complete markets). His entrepreneurial project/idea is defined by capital accumulation technology and a production function. Capital is productive but exposes the entrepreneur to non-diversifiable business risk. Moreover, to be an entrepreneur, he needs to pay a fixed entry cost. The entrepreneur optimally chooses the initial firm size to balance the positive value creation effect of productive capital with the negative impact of lower liquid wealth on the entrepreneur’s ability to diversify risk and smooth intertemporal consumption. Lenders are perfectly competitive and earn zero profits.

After setting up his firm, he starts to accumulate capital by paying additional convex adjustment costs as in the standard $q$ theory of investment (Lucas and Prescott (1971) and Hayashi (1982)). Physical capital generates stochastic cash flows. The entrepreneur also has an option at his chosen time to liquidate the project and to convert illiquid capital into liquid wealth (i.e. his liquidation option is of “American” style). The liquidation option provides some downside risk protection for the entrepreneur’s wealth management and consumption smoothing.

In addition to the project, the entrepreneur also has liquid financial wealth which he can invest in both a risk-free asset and a risky asset (e.g. market portfolio) as in a standard consumption-portfolio choice problem (e.g. Merton (1971)). Assume that return of the risky market portfolio is imperfectly correlated with the business risk. By dynamically trading the market portfolio, the entrepreneur can hedge the systematic component of his business risk, but not the idiosyncratic risk component of his business. Moreover, the entrepreneur can borrow up to a fraction of the illiquid capital stock. This debt is secured by physical capital.

We find that the entrepreneur significantly underinvests in his business due to non-diversifiable risk compared with the complete-markets benchmark. The entrepreneur also rationally scales back his consumption. His private valuation of the business is significantly lower than the complete-markets valuation. Moreover, we show that the marginal value of savings can be significantly higher than unity. The entrepreneur optimally chooses his liquidation policy to manage his downside risk. As a result, the entrepreneur may not exhaust his debt capacity.

The entrepreneur’s significant non-diversifiable business risk exposure also has important
implications on his portfolio allocation decisions. The entrepreneur reduces his total allocation to the risky market portfolio. However, given that his certainty equivalent wealth is also lower under incomplete markets due to non-diversifiable risk, the allocation to the risky market portfolio as a fraction of the entrepreneur’s certainty equivalent wealth may be increasing. This is likely to happen when the entrepreneur is close to liquidating his business. The liquidation option generates a convexity effect and makes the entrepreneur effectively less risk averse and hence holds a more risky market portfolio, *ceteris paribus*.

We find that there are significant utility costs for the entrepreneur to bear non-diversifiable idiosyncratic risk. For an expected-utility entrepreneur whose coefficient of relative risk aversion is equal to two and has no liquid wealth, the subjective valuation of the entrepreneurial business is about 11% lower than the complete-markets benchmark.

**Related Literature.** Our paper links to several strands of literature in finance, macroeconomics, and entrepreneurship. First, we contribute to the economics literature on entrepreneurship.

Hall and Woodward (2008) provide a quantitative analysis for the lack of diversification of venture-capital-backed entrepreneurial firms. Heaton and Lucas (2004) show that risky non-recourse debt (of the limited-liability entrepreneurial firm) provides diversification benefits for the entrepreneur in a static setting, and analyze the interaction between capital budgeting, capital structure, and portfolio choice for the entrepreneur. Chen, Miao, and Wang (2010) introduce non-diversifiable risk into a workhorse dynamic corporate finance model (Leland (1994)) to develop an incomplete-markets framework of capital structure tradeoff theory for entrepreneurial firms. They show that more risk averse entrepreneurs borrow more in order to lower their business risk exposure. For analytical tractability, Chen, Miao, and Wang (2010) adopt negative exponential utility, while this paper uses non-expected Epstein-Zin utility. Herranz, Krasa, and Villamil (2009) assess quantitatively the impact of legal institutions on entrepreneurial firm dynamics. They find that more risk averse entrepreneurs default more, but carry less debt. Heaton and Lucas (2000) find that households with high and variable business income hold less wealth in the stock market.

Evans and Jovanovic (1989) show the importance of personal wealth and liquidity constraints for entrepreneurship.\(^4\) Cagetti and De Nardi (2006) construct a model of en-

entrepreneurship with entry, exit and investment decisions and quantify the importance of borrowing constraints on aggregate capital accumulation and wealth distribution. Hurst and Lusardi (2004) challenge the importance of liquidity constraints and also provide some evidence that the start-up sizes of entrepreneurial firms tend to be small. Our model analyzes the effects of borrowing constraints and non-diversifiable risk in an incomplete-markets environment and accounts for both the entry of the entrepreneur and the initial size of entrepreneurial firm.

Second, this paper complements the voluminous literature on capital accumulation and the \( q \) theory of investment. Almost all models in the \( q \) theory literature are applicable only to publicly traded firms where non-diversifiable idiosyncratic business risk carries no risk premium for firms owned by risk-neutral investors. We extend the \( q \) theory of investment to account for non-diversifiable idiosyncratic risk and credit constraints under incomplete markets. In addition, the firm in our model has American-style entry and exit options. We show that the impact of non-diversifiable idiosyncratic risk on capital accumulation is both conceptually important and also quantitatively significant in an incomplete-markets framework. We provide the counterparts of marginal \( q \) and average \( q \) for private firms owned and operated by non-diversified entrepreneurs.

Third, our paper contributes to the literature on portfolio choice with non-tradable income risk by allowing for an endogenous non-marketable income stream from business operations via endogenous capital accumulation, business entry and exit decisions. The flexibility to influence business income and the endogeneity of business entry/exit decision provide more channels for the entrepreneur to manage risk. Most papers in the literature take non-tradable income as exogenously given.\(^5\)

Fourth, our paper also contributes to the macroeconomic literature on permanent-income and buffer-stock savings.\(^6\) We show that entry/exit options generate a convexity effect. As a result, the entrepreneur’s certainty equivalent wealth in liquid wealth is convex before becoming an entrepreneur and/or when the entrepreneur is close to liquidating the firm. These options significantly alter the entrepreneur’s decision making. Fifth, a subset of our


model’s economic results is also related to the real options analysis under incomplete markets. Miao and Wang (2007) and Hugonnier and Morellec (2007) are examples studying the impact of non-diversifiable risk on real options exercise. These papers show that the non-diversifiable risk significantly alters real option exercising.

Finally, our model also relates to recent work on dynamic corporate finance. Bolton, Chen, and Wang (2010) analyze the firm’s optimal investment, financing, and risk management decisions and Tobin’s $q$ for financially constrained firms. Unlike their paper, which applies to publicly traded firms facing financial constraints, our model focuses on entrepreneurial firms where the key friction is non-diversifiable risk exposure. DeMarzo, Fishman, He, and Wang (2010) integrate a dynamic moral hazard framework of DeMarzo and Fishman (2007b) and DeMarzo and Sannikov (2006) with the neoclassical $q$ theory of investment (Hayashi (1982)). They derive an optimal dynamic contract and provide financial implementation.\footnote{DeMarzo and Fishman (2007a) analyze the impact of agency on investment dynamics in discrete time.}

\section{The model}

First, we introduce the agent’s preferences. Then, we set up the agent’s optimization problem by describing his career choice, intertemporal consumption and saving, portfolio choice/hedging, business investment and exit decisions.

\textbf{Preferences.} The agent has a preference featuring both constant relative risk aversion and constant elasticity of intertemporal substitution (Epstein and Zin (1989) and Weil (1990)). We use the continuous-time formulation of this non-expected utility introduced by Duffie and Epstein (1992a). That is, the agent has a recursive preference defined as follows:

$$J_t = \mathbb{E}_t \left[ \int_t^{\infty} f(C_s, J_s) ds \right],$$

where $f(C, J)$ is known as the normalized aggregator for consumption $C$ and the agent’s utility $J$. Duffie and Epstein (1992a) show that $f(C, J)$ for Epstein-Zin non-expected homothetic recursive utility is given by

$$f(C, J) = \frac{\zeta}{1 - \psi^{-1}} \frac{C^{1-\psi^{-1}} - ((1 - \gamma)J)^{\chi}}{((1 - \gamma)J)^{\chi-1}},$$

\footnote{DeMarzo and Fishman (2007a) analyze the impact of agency on investment dynamics in discrete time.}
where
\[
\chi = \frac{1 - \psi^{-1}}{1 - \gamma}.
\] (3)

The parameter \( \psi > 0 \) measures the elasticity of intertemporal substitution, and the parameter \( \gamma > 0 \) is the coefficient of relative risk aversion. The parameter \( \zeta > 0 \) is the agent’s subjective discount rate.

The widely used time-additive separable constant-relative-risk-averse (CRRA) utility is a special case of the Duffie-Epstein-Zin-Weil recursive utility specification where the coefficient of relative risk aversion is equal to the inverse of the elasticity of intertemporal substitution \( \psi \), i.e. \( \gamma = \psi^{-1} \) implying \( \chi = 1 \).8

For the general recursive utility given in (1)-(2) with \( \gamma \neq 1/\psi \), \( f(C, J) \) is not separable in \( C \). The agent’s current utility depends on his consumption and future utility in a non-separable way, and thus his preference is not one in the expected utility class. Separating the coefficient of relative risk aversion coefficient \( \gamma \) from the elasticity of intertemporal substitution \( \psi \) allows us to analyze the impact of changing one parameter on the agent’s decision making and subjective valuation without changing the other. Therefore, it allows us to separately quantify the effects of intertemporal substitution and risk attitude. Despite being a more general preference specification, the scale-invariance property of this recursive utility proves useful in keeping our model analysis tractable (as in Merton (1971) and Duffie and Epstein (1992b), for example).

Career choice and initial firm size. The agent is endowed with an entrepreneurial idea and initial wealth \( W_0 \). The entrepreneurial idea is defined by a productive capital accumulation technology and production function. (We will shortly return to the detailed specification of the entrepreneurial idea.) To implement the entrepreneurial idea, he needs to pay a one-time fixed start-up cost \( \Phi \) at his choosing starting (stochastic) time \( T^0 \). The entrepreneur will also optimally choose the initial firm size (i.e. capital stock \( K_{T^0} \)) at the time of entry \( T^0 \). So the entry decision has two margins: timing and the initial size. One example is being a taxi/limo driver. The agent can first start with a used car. After he

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8For this special case, we have \( f(C, J) = U(C) - \zeta J \), where \( U(C) \) is the expected CRRA utility with \( \gamma = \psi^{-1} \) and hence \( U(C) = \zeta C^{1-\psi^{-1}}/(1-\psi^{-1}) \). Note that for CRRA utility, \( f(C, J) \) is additively separable. By integrating (1) forward for this CRRA special case, we obtain \( J_t = \max_C E_t [ \int_t^\infty e^{-\zeta(s-t)} U(C(s)) ds ] \).
builds up his savings, he can potentially upgrade his vehicle. With sufficient savings, he may potentially start running a small limo service. Allowing for a choice in size lowers the effect of the wealth constraint on being an entrepreneur.

Before becoming an entrepreneur, the agent can take an alternative job (e.g. to be a worker), therefore, being an entrepreneur is a discrete career decision. That is, we do not allow the agent to be a part-time entrepreneur and a part-time worker at the same time.\(^9\)

To highlight the risky nature of entrepreneurial activities, we assume that being an entrepreneur offers potentially higher rewards at greater risk than being a worker. Hamilton (2000) finds that earnings of the self-employed are smaller on average and have higher variance than earnings of workers (using data from Survey of Income and Program Participation). To contrast the earnings profile differences between being an entrepreneur and being a worker, we assume that the outside option (by being a worker) gives the agent a constant flow of income at the rate of \(r \Pi\). Therefore, by choosing his outside option, the agent obtains the present value of his outside option \(\Pi\).

At the optimally chosen (stochastic) entry time \(T^0\), he incurs total outflow \((K_{T^0} + \Phi)\). He finances this amount out of his savings \(W_{T^0}\), where \(T^0\) denotes the time instantaneously prior to the entrepreneur’s entry time \(T^0\). If his savings is not sufficient, he finances the rest via (collateralized) borrowing.\(^{10}\) Therefore, his wealth immediately upon entry is given by \(W_{T^0} = W_{T^0} - (K_{T^0} + \Phi)\), which can be negative (i.e. borrowing to be an entrepreneur).

Lenders make zero profit in competitive capital markets. Debt is collateralized by physical capital. If the entrepreneur reneges on debt, creditors can always liquidate the firm’s capital and recover a constant fraction \(l > 0\) per unit of capital. Therefore, the borrower has no incentive to default on debt and he can borrow up to \(lK\) at the risk-free rate by using the firm’s capital as the collateral.

We will show that initial wealth \(W_0\) plays a role in how long it takes the agent to become an entrepreneur and the choice of the firm’s initial size. Borrowing constraints and precautionary motives under incomplete markets are both conceptually and quantitatively important. Moreover, these two frictions interact and generate economically significant feedback effects on entrepreneurship.

\(^9\)This is a standard and reasonable assumption. For example, see Vereshchagina and Hopenhayn (2009) for a dynamic career choice model featuring the same assumption.

\(^{10}\)We use \(W_{T^0}\) to denote the agent’s wealth at time \(T^0\), just prior to the stochastic entry time \(T^0\).
Entrepreneurial idea: capital investment and production technology. The entrepreneurial idea is defined by a production function and capital accumulation technology. Let $I$ denote the firm’s gross investment. As is standard in capital accumulation models, the change of the firm’s capital stock $K$ is given by the difference between gross investment and depreciation, in that

$$dK_t = (I_t - \delta K_t) \, dt, \quad t \geq 0,$$

where $\delta \geq 0$ is the rate of depreciation.

The firm’s operating revenue over time period $(t, t + dt)$ is proportional to its time-$t$ capital stock $K_t$, and is given by $K_t dA_t$, where $dA_t$ is the firm’s productivity shock over the period $(t, t+dt)$. The productivity shock $dA_t$ is assumed to be independently and identically distributed (i.i.d.), and is given by

$$dA_t = \mu_A dt + \sigma_A dZ_t,$$

where $Z$ is a standard Brownian motion, $\mu_A > 0$ is the mean of the productivity shock, and $\sigma_A > 0$ is the volatility of the productivity shock. Intuitively, a higher value of $\mu_A$ implies more productive capital. We interpret $K dA$ as the firm’s revenue after payments for labor, other factors of production, and operating costs, and hence may be negative. The firm’s only explicitly specified factor of production in the model is capital. Its operating profit $dY_t$ over the same period $(t, t + dt)$ is given by

$$dY_t = K_t dA_t - I_t dt - G(I_t, K_t) dt,$$

where $G(I, K)$ is the adjustment cost not captured by the cost of purchasing the investment good.

Following the neoclassical investment literature (Hayashi (1982)), we assume that the firm’s adjustment cost is homogeneous of degree one in $I$ and $K$. We thus may write the adjustment cost in the following homogeneous form

$$G(I, K) = g(i) K,$$

where $i$ is the firm’s investment capital ratio ($i = I/K$), and $g(i)$ is an increasing and convex function. We choose this simple adjustment cost function because it yields a rather simple benchmark result under complete markets as in Hayashi (1982): Tobin’s average $q$ is equal
to marginal $q$ under perfect capital markets where the Modigliani-Miller theorem applies. As we will show, the non-diversifiable risk will drive a wedge between the equivalent of Tobin’s $q$ and marginal $q$ for entrepreneurial firms. That is, this homogeneity assumption allows us to trace out the impact of non-diversifiable risk and borrowing constraints on the entrepreneur’s real investment and (subjective) valuation. For simplicity, we assume that $g(i)$ is quadratic, in that

$$g(i) = \frac{\theta i^2}{2},$$

where the parameter $\theta$ measures the degree of the adjustment cost. A higher value $\theta$ implies a more costly adjustment process.\(^{11}\)

The entrepreneur has an option to liquidate his firm at his choosing time. That is, the entrepreneur has a perpetual American style liquidation option. Liquidation gives a terminal value $lK$, where $l > 0$, a constant, is the exit value for a unit of capital. Let $T^l$ denote the entrepreneur’s optimally chosen stochastic liquidation time. To focus on the economically interesting case, we consider the case where capital is productive. Then, in the absence of financial frictions (e.g. the Modigliani-Miller world), liquidating productive capital is never optimal because it destroys value. However, when markets are not complete and the entrepreneur is poorly diversified, liquidation provides an important channel for the entrepreneur to manage his downside business risk exposure.

Our production specification features both the widely used “$AK$” technology\(^ {12}\) and the adjustment cost technology found in macroeconomics literature. While simple, our production/capital accumulation technology specification is a reasonable starting point and is also analytically tractable. Eberly, Rebelo, and Vincent (2009) provide empirical evidence in support of the neoclassic benchmark of the Hayashi (1982) model with serially correlated productivity shocks using Compustat for public firms. Next, we turn to the agent’s financial investment opportunities.

**Financial investment opportunities.** As in Merton (1971), the agent has standard financial investment opportunities: a risk-free asset which pays a constant rate of interest $r$ and a risky asset, e.g. a market portfolio. Assume that the incremental return $dR_t$ of

\(^{11}\)The key is a convex and homogeneous adjustment cost function, not the specific quadratic form.

\(^{12}\)Cox, Ingersoll, and Ross (1985) feature an equilibrium production economy with the “$AK$” technology. See Jones and Manuelli (2005) for a recent survey on endogenous growth models.
the market portfolio over time period \( dt \) is independently and identically distributed (i.i.d.) returns, i.e.

\[
dR_t = \mu_R dt + \sigma_R dB_t,
\]

where \( \mu_R \) and \( \sigma_R \) are constant mean and volatility parameters of the market portfolio return process, and \( B \) is a standard Brownian motion. Let

\[
\eta = \frac{\mu_R - r}{\sigma_R}
\]

denote the Sharpe ratio of the market portfolio. Assume that the entrepreneur’s business risk and the return of the market portfolio is correlated with coefficient \( \rho \). As long as \( |\rho| \neq 1 \), markets are incomplete and the entrepreneur cannot completely hedge his business risk. Non-diversifiable risk will thus play a role in the entrepreneur’s decision making and subjective valuation.

Let \( W \) and \( X \) denote the agent’s financial wealth and the amount of investment in the risky asset, respectively. Thus, \( (W - X) \) is the remaining amount invested in the risk-free asset. Without entrepreneurial business (i.e. \( t \leq T^0 \) and \( t \geq T^l \)), his wealth accumulation is given by

\[
dW_t = r (W_t - X_t) dt + \mu_R X_t dt + \sigma_R X_t dB_t - C_t dt, \quad t < T^0 \text{ and } t > T^l. \tag{11}
\]

While being an entrepreneur, his liquid financial wealth \( W \) evolves as follows:

\[
dW_t = r (W_t - X_t) dt + \mu_R X_t dt + \sigma_R X_t dB_t - C_t dt + dY_t, \quad T^0 \leq t \leq T^l. \tag{12}
\]

The difference between (12) and (11) is the last term \( dY_t \), which gives the endogenous stochastic profit flow from business operations.

If the agent chooses to become an entrepreneur, he can borrow against the physical capital \( K \) at all times. That is, his wealth \( W \) is allowed to be negative. To be consistent with the assumption that entrepreneurial borrowing is risk-free, we require the liquidation value of his business, \( lK \), to be greater than his outstanding liability, in that

\[
W_t \geq -lK_t, \quad T^0 \leq t \leq T^l. \tag{13}
\]

While the entrepreneur is allowed to borrow up to \( lK_t \) at the risk-free rate \( r \), he may rationally choose not to exhaust his debt capacity due to his precautionary motive, meaning that the
borrowing constraint (13) may not bind. That is, rationality may serve as a tighter natural constraint. Without physical capital as collateral, the agent cannot borrow, i.e. \( W_t \geq 0 \) for \( t \leq T^0 \) and \( t \geq T^l \).

**The agent’s optimization problem.** The agent maximizes his utility defined in (1)-(2). The timeline of his decisions can be described in five steps. First, before becoming an entrepreneur (i.e. \( t \leq T^0 \)), he collects his income as a worker, chooses his consumption \( C \) and portfolio allocation amount \( X \) to the market portfolio. Second, he chooses the optimal stochastic timing \( T^0 \) to become an entrepreneur, and the initial size of his business \( K_{T^0} \). Doing so, he incurs the fixed start-up cost \( \Phi \), and finances the total costs \( (K_{T^0} + \Phi) \) with his savings and potentially some borrowing up to his borrowing limit collateralized by capital. Third, after setting up the firm, he chooses consumption \( C \), real investment \( I \), allocation amount \( X \) in the market portfolio subject to the collateralized borrowing limit (13), and his financial and business investment opportunities stated above. Fourth, he optimally chooses his exit strategy by choosing the stochastic liquidation time \( T^l \) for the capital stock. Finally, after liquidating capital, he collects the proceeds from liquidation and adds to his financial wealth. He then retires, allocates his wealth between the risk-free and the risky market portfolio, and finances his consumption out of his financial wealth.

### 3 Benchmark: Complete markets

Before turning to the incomplete-markets framework, we first construct a complete-markets setup which serves as a natural benchmark for us to understand our incomplete-markets results.\(^\text{13}\) With complete markets, the entrepreneur’s intertemporal optimization problem can be decomposed into two separate ones: wealth maximization and utility maximization. Using the homogeneity property of our model, we show that optimal consumption, allocation to the risky market portfolio, and investment are all proportional to the capital stock \( K \). Whenever applicable, we use the lower case to denote the corresponding variable in the upper case, scaled by \( K \). For example, we use \( w, c, x, \) and \( i \) to denote liquid wealth-capital.

\(^{13}\)To complete markets in the setup sketched out in Section 2, we introduce the second (non-redundant) tradable risky financial asset to dynamically span the entrepreneur’s idiosyncratic risk exposure. Without loss of generality, we choose the newly introduced asset to be the one which is only subject to the idiosyncratic risk of the entrepreneur’s business. This second asset earns no risk premium because all systematic risk is spanned by the risky market portfolio by construction.
ratio \( w = W/K \), consumption-capital ratio \( c = C/K \), total allocation to the risky market portfolio per unit of capital \( x = X/K \), and investment-capital ratio \( i = I/K \), respectively. The following proposition summarizes our main results under complete markets. Technical conditions and derivation details are given in the appendix.

**Proposition 1** The entrepreneur’s value function \( J^{FB}(K,W) \) is given by

\[
J^{FB}(K,W) = \left( bP^{FB}(K,W) \right)^{1-\gamma} \frac{1}{1-\gamma},
\]

where the total wealth \( P^{FB}(K,W) \) is given by the sum of \( W \) and firm value \( Q^{FB}(K) \):

\[
P^{FB}(K,W) = W + Q^{FB}(K) = W + q^{FB} K,
\]

and

\[
b = \zeta \left[ 1 + \frac{1-\psi}{\zeta} \left( r - \zeta + \frac{\eta^2}{2\gamma} \right) \right]^{1-\psi}.
\]

Firm value \( Q^{FB}(K) \) is equal to \( q^{FB} K \), where Tobin’s \( q, q^{FB}, \) is given by

\[
q^{FB} = 1 + \theta i^{FB},
\]

where the first-best investment-capital ratio \( i^{FB} \) is given by

\[
i^{FB} = (r + \delta) - \sqrt{(r + \delta)^2 - \frac{2}{\theta} (\mu_A - \rho \eta \sigma_A - (r + \delta))}.
\]

The optimal consumption \( C \) is proportional to \( K \), i.e. \( C(K,W) = c^{FB}(w)K \), where

\[
c^{FB}(w) = m^{FB} \left( w + q^{FB} \right),
\]

and \( m^{FB} \) is the marginal propensity to consume (MPC) and is given by

\[
m^{FB} = \zeta + (1-\psi) \left( r - \zeta + \frac{\eta^2}{2\gamma} \right).
\]

The optimal allocation amount to the risky market portfolio \( X \) is also proportional to the capital stock \( K \), i.e. \( X(K,W) = x^{FB}(w)K \), where

\[
x^{FB}(w) = \left( \frac{\mu_R - r}{\gamma \sigma_R^2} \right) \left( w + q^{FB} \right) - \frac{\rho \sigma_A}{\sigma_R} \sigma_R.
\]

The capital asset pricing model (CAPM) holds for the firm with its expected return given by

\[
\xi^{FB} = r + \beta^{FB} (\mu_R - r),
\]

13
where the firm’s beta, $\beta^{FB}$ is constant and given by

$$\beta^{FB} = \frac{\rho \sigma_A}{\sigma_R q^{FB}}.$$  

Equations (17) and (18) give the first-best Tobin’s $q$ and investment-capital ratio, respectively. The adjustment cost makes Tobin’s $q$ different from unity because installed capital earns rents due to the adjustment costs. Average $q$ is equal to marginal $q$ in our complete-markets benchmark due to the homogeneity property as in Hayashi (1982), where it is assumed that the firm is risk neutral. We explicitly account for the effect of systematic risk premium on firm investment and value, i.e. Tobin’s $q$. As in CAPM, the term $\rho \sigma_A$ in (18) accounts for the systematic risk premium.

With complete markets, both liquid financial wealth and business wealth are tradable and marked to market. The entrepreneur’s own valuation of his business is thus equal to the market value (i.e. law of one price holds). Hence, the entrepreneur’s total wealth is given by the sum of firm value and his liquid wealth, i.e. $P(K, W) = p^{FB}(w)K$, where $p^{FB}(w) = w + q^{FB}$. Equation (19) gives the optimal consumption rule, effectively the permanent-income hypothesis under complete markets. For each unit of total wealth, the entrepreneur consumes $m^{FB}$, which is the MPC out of total wealth. If the elasticity of intertemporal substitution is unity ($\psi = 1$), the MPC out of wealth under complete markets $m^{FB}$ is equal to the agent’s discount rate $\xi$. In general, the MPC also depends on the risk-free rate $r$, the elasticity of intertemporal substitution $\psi$, the coefficient of risk aversion $\gamma$, and the Sharpe ratio $\eta$, which is the ratio between the expected excess return $\mu_R - r$ of the market portfolio, and the volatility of the market portfolio return $\sigma_R$.

The entrepreneur’s investment amount $X$ in the market portfolio is proportional to the capital stock $K$. Equation (21) gives $x(w)$, the portfolio allocation to the market portfolio per unit of capital. The first term in (21) is the well-known mean-variance portfolio allocation rule adapted to allow for the presence of a productive business project. The second term in (21) is the hedging term induced by the correlation between the entrepreneur’s production technology and the return of the market portfolio. The risky production technology induces two effects on the entrepreneur’s portfolio choice even under complete markets. First, the entrepreneur’s diversification motive induces him to consider the correlation between his production technology and the risky market portfolio (the hedging demand component). Having productive capital and risky production technology also increases his “total” wealth.
and hence increases the entrepreneur’s total investment in the risky asset (the $q^{FB}$ component in the first term).

Anticipating the important and distinct roles that systematic and idiosyncratic volatility play, we decompose the total volatility of the productivity shock into systematic and idiosyncratic components. The systematic volatility is equal to $\rho \sigma_A$ and the idiosyncratic component of the volatility is given by

$$\epsilon = \sqrt{1 - \rho^2 \sigma_A}.$$  \hspace{1cm} (24)

As the standard finance theory implies, the idiosyncratic volatility $\epsilon$ carries no risk premium and plays no role under complete markets.

## 4 Incomplete-markets model solution after entry

Having characterized the complete-markets solution, we now analyze the effects of non-diversifiable risk under incomplete markets. We proceed by backward induction. First, consider the agent’s decision problem after liquidating his business. We then derive the entrepreneur’s interdependent consumption, portfolio choice, and business investment/exit decisions when he has non-diversifiable risk exposure to his investment project.

### The agent’s decision problem after exiting entrepreneurship.

After exiting from entrepreneurship by liquidating all capital stock, the entrepreneur is no longer exposed to the business risk. He faces a classic Merton consumption/portfolio allocation problem with non-expected recursive utility. The solution is effectively the same as the complete-markets results in Proposition 1 (but without physical capital). We summarize the results as a corollary to Proposition 1.

**Corollary 1** The entrepreneur’s value function takes the following homothetic form:

$$V(W) = \frac{(bW)^{1-\gamma}}{1-\gamma},$$  \hspace{1cm} (25)

where $b$ is a constant given in (16). The optimal consumption $C$ and allocation amount $X$ in the risky market portfolio are respectively given by

$$C = m^{FB}W,$$  \hspace{1cm} (26)

$$X = \left(\frac{\mu_R - r}{\gamma \sigma_R^2}\right)W,$$  \hspace{1cm} (27)

where $m^{FB}$ is the MPC out of wealth and is given in (20).

We will use the value function given in (25) when analyzing the agent’s pre-exit decisions.

**The entrepreneur’s decision problem while running his business.** Let $J(K,W)$ denote the entrepreneur’s value function. The entrepreneur chooses consumption $C$, real investment $I$, and the allocation to the risky market portfolio $X$ by solving the following Hamilton-Jacobi-Bellman (HJB) equation:

$$
0 = \max_{C,I,X} f(C,J) + (I - \delta K)J_K + (rW + (\mu_R - r)X + \mu_AK - I - G(I,K) - C)J_W \\
+ \left( \frac{\sigma_A^2K^2 + 2\rho\sigma_A\sigma_RKX + \sigma_R^2X^2}{2} \right) J_{WW}.
$$

(28)

The entrepreneur’s first-order condition (FOC) for consumption $C$ is:

$$
f_C(C,J) = J_W(K,W).
$$

(29)

The above condition states that the marginal utility of consumption (the left side of (29)) is equal to the marginal utility of savings $J_W$.

The FOC with respect to real investment $I$ gives

$$
(1 + G_I(I,K))J_W = J_K.
$$

(30)

To increase capital stock by one unit, the entrepreneur needs to forgo $(1 + G_I(I,K))$ units of consumption. The marginal cost of a unit of investment good is marginal utility of wealth $J_W$. Therefore, the entrepreneur’s marginal cost of investing is given by the product of $(1 + G_I(I,K))$ and the marginal utility of savings $J_W$. The marginal benefit of increasing capital stock by a unit is $J_K$. At optimality, the entrepreneur equates the two sides of (30).

The FOC with respect to portfolio choice $X$ is given by

$$
X = - \frac{\mu_R - r}{\sigma_R^2 J_{WW}} \frac{J_W}{J_W} - \frac{\rho\sigma_A}{\sigma_R} K.
$$

(31)

The firm term in (31) is the mean-variance demand and the second term captures the hedging demand.

Exploiting the homogeneity of the entrepreneur’s value function, we conjecture that the value function $J(K,W)$ takes the following form:

$$
J(K,W) = \frac{(bP(K,W))^{1-\gamma}}{1 - \gamma},
$$

(32)
where $b$ is given in (16). Comparing the value functions with and without the business project, (32) with (25), we may intuitively refer to $P(K, W)$ as the entrepreneur’s certainty equivalent wealth. That is, by certainty equivalent wealth, we mean the minimal amount of wealth so that the agent is willing to permanently give up his business project and all his liquid wealth $W$.

Let $W$ denote the *endogenous* lower boundary at which the entrepreneur liquidates the firm. Let $w$ denote the ratio between $W$ and capital stock $K$, i.e. $w = W/K$. The following theorem summarizes the entrepreneur’s optimal consumption, portfolio allocation, business investment/liquidation decision rules and his certainty equivalent wealth. Note that in equilibrium, the entrepreneur will not be indebted for more than $|W|$, i.e. the maximal equilibrium debt capacity is $|w|$ per unit of physical capital $K$.

**Theorem 1** The entrepreneur liquidates his business if his wealth-capital ratio is sufficiently low, i.e. $w \leq \overline{w}$, where $\overline{w}$ is optimally determined below. He continues operating business provided that $w \geq \overline{w}$. The certainty equivalent wealth per unit of physical capital $p(w) \equiv P(K, W)/K$ solves the following ordinary differential equation (ODE):

$$
0 = \frac{m^{FB}p(w)w^{1-\psi} - \psi p(w)}{\psi - 1} - \delta p(w) + (r + \delta)wp'(w) + (\mu_A - \rho\eta\sigma_A)p'(w)
+ \frac{(p(w) - (w + 1)p'(w))^2}{2\theta p'(w)} + \frac{\eta^2 p(w)p'(w)}{2h(w)} - \frac{\epsilon^2 h(w)p'(w)}{2p(w)}, \quad \text{if } w \geq \overline{w},
$$

(33)

where $\epsilon$ is the idiosyncratic volatility given in (24) and $h(w)$ is given by

$$
h(w) = \gamma p'(w) - \frac{p(w)p''(w)}{p'(w)}. \quad \text{(34)}
$$

When $w$ approaches $\infty$, $p(w)$ approaches complete-markets certainty equivalent wealth-capital ratio, i.e.

$$
\lim_{w \to \infty} p(w) = w + q^{FB}.
$$

(35)

Finally, the ODE (33) for $p(w)$ satisfies the following conditions at the endogenously chosen liquidation boundary $\overline{w}$:

$$
p(w) = w + l,
$$

(36)

$$
p'(w) = 1,
$$

(37)

17
However, if the conditions (36)-(37) do not permit an interior solution satisfying \( w > -l \), the optimal liquidation boundary is then given by the maximal borrowing capacity, i.e. \( w = -l \).

The dynamics of the wealth-capital ratio \( w \) are given by

\[
dw_t = \mu_w(w_t)dt + \sigma_R x(w_t)dB_t + \sigma_A dZ_t,
\]

where the drift \( \mu_w(w) \) gives the expected change of \( w \) and is given by

\[
\mu_w(w) = (r + \delta - i(w))w + (\mu_R - r)x(w) + \mu_A - i(w) - g(i(w)) - c(w).
\]

The next proposition summarizes the entrepreneur’s optimal decisions.

**Proposition 2** The entrepreneur’s optimal consumption-capital ratio \( c = C/K \), investment-capital ratio \( i = I/K \), and market portfolio allocation-capital ratio \( x = X/K \) depend on financial wealth-capital ratio \( w = W/K \), and are respectively given by

\[
c(w) = m^{FB} p(w)(p'(w))^{-\psi},
\]

\[
i(w) = \frac{1}{\theta} \left( \frac{p(w)}{p'(w)} - w - 1 \right),
\]

\[
x(w) = -\frac{\rho \sigma_A}{\sigma_R} + \frac{\mu_R - r}{\sigma_R^2} \frac{p(w)}{h(w)},
\]

where \( h(w) \) is given in (34).

We will provide economic interpretations of the decision rules in the next section.

5. Incomplete-markets model results after entry

We now explore the implications of our incomplete-markets model of Section 4. We choose parameter values as follows and whenever applicable, all parameters are annualized. The risk-free interest rate is \( r = 4.6\% \). The aggregate equity risk premium is \( (\mu_R - r) = 6\% \) (Mehra and Prescott (1985)), and the annual volatility of the market portfolio return is \( \sigma_R = 0.2 \) implying the Sharpe ratio for the aggregate stock market \( \eta = (\mu_R - r)/\sigma_R = 30\% \). The annual subjective discount rate is set to equal to the risk-free rate, i.e. \( \zeta = r = 4.6\% \), so that the entrepreneur’s intertemporal decisions are not driven by the wedge \( (\zeta - r) \), which measures the entrepreneur’s impatience.
On the real investment side, our model is a version of the neoclassic $q$ theory of investment (i.e. Hayashi (1982)) augmented with a systematic risk premium. Using the sample of large firms in Compustat from 1981 to 2003, Eberly, Rebelo, and Vincent (2009) provide empirical evidence in support of Hayashi (1982). Using their work as a guideline, we choose: the expected productivity $\mu_A = 20\%$, and the volatility of productivity shocks $\sigma_A = 10\%$. Fitting the first-best $q_{FB}$ and the first-best investment-capital ratio $i_{FB}$ to the sample averages (1.3 for average $q$ and 15% for the investment-capital ratio, respectively), we obtain the adjustment cost parameter value $\theta = 2$ and the rate of depreciation for capital stock: $\delta = 12.5\%$. The imputed adjustment cost parameter ($\theta = 2$) is in the range of estimates used in the literature.\textsuperscript{14} We choose the liquidation parameter value $l = 0.9$, i.e. the firm recovers 90 cents on a dollar of the book value of capital upon liquidation (Hennessy and Whited (2005)).

We consider two widely used values for the coefficient of relative risk aversion: $\gamma = 2$ and $\gamma = 4$. We set the elasticity of intertemporal substitution to be $\psi = 0.5$, so that the first case corresponds to the expected utility with $\gamma = 1/\psi = 2$, and the second case (i.e. $\gamma = 4$, $\psi = 0.5$) maps to non-expected utility with $\gamma > 1/\psi$. Finally, we set the correlation between the risky market portfolio return and the entrepreneur’s business to zero, i.e. $\rho = 0$.

### 5.1 Decomposing the entrepreneur’s welfare

We measure the entrepreneur’s welfare via his value function, which is given in (14). The entrepreneur’s value function $J(K,W)$ is homogeneous of degree $(1 - \gamma)$ in his certainty equivalent wealth $P(K,W)$. The certainty equivalent wealth is the minimal amount of transfer payment to the agent for him to voluntarily and permanently give up his wealth $W$ and the illiquid business project. Therefore, we may equivalently quantify the agent’s welfare via his certainty equivalent wealth $P(K,W)$.

**Certainty equivalent wealth and private enterprise value.** The entrepreneur’s certainty equivalent wealth $P(K,W)$ is homogenous of degree one in $K$ and $W$. Therefore, we may write $P(K,W) = p(w)K$ and focus on the certainty equivalent wealth-capital ratio: $p(w)$.

\textsuperscript{14}See Whited (1992), Hall (2004), Riddick and Whited (2009), and Eberly, Rebelo, and Vincent (2009).
In corporate finance, enterprise value is defined as firm value excluding liquid financial assets (e.g. cash and liquid assets in the market portfolio). For the entrepreneurial firm, we may similarly define the entrepreneur’s private enterprise value \( Q(K,W) \) as the entrepreneur’s certainty equivalent wealth in excess of liquid financial wealth \( W \):

\[
Q(K,W) = P(K,W) - W. \tag{43}
\]

Dividing the private enterprise value \( Q(K,W) \) by illiquid physical capital \( K \), we have

\[
q(w) = \frac{Q(K,W)}{K} = p(w) - w. \tag{44}
\]

If the firm is owned by diversified investors and is managed in their interest, \( q(w) \) defined in (44) then corresponds to Tobin’s average \( q \) (as in corporate finance for publicly traded firms). However, for a firm owned and managed by an entrepreneur, (44) captures the impact of non-diversifiable risk on the entrepreneur’s subjective valuation. A natural and intuitive interpretation of \( q \) defined in (44) is the “private” average \( q \). Unlike the standard definition of Tobin’s \( q \) under complete markets, our \( q \) defined in (44) depends on the entrepreneur’s preferences.

For Figures 1-5, we graph for two levels of risk aversion \( \gamma = 2 \) and \( \gamma = 4 \). The top left and top right panels of Figure 1 plot \( p(w) \) and \( q(w) \), respectively. Note that \( p(w) \) and \( q(w) = p(w) - w \) differ only by \( w \), a 45 degree line out of the origin, and thus effectively convey the same information. Graphically, it is easier to read Panel B of Figure 1 for \( q(w) \) than Panel A for \( p(w) \). Without loss of generality, we may focus the remaining discussions on \( q(w) \). For a given \( \gamma \), the higher the wealth-capital ratio \( w \), the less the entrepreneur is concerned about non-diversifiable idiosyncratic risk, implying a higher value of \( q(w) \), i.e. \( q'(w) > 0 \). In the limit (i.e. \( w \to \infty \)), the entrepreneur’s “total” wealth effectively all comes from liquid wealth, and therefore non-diversifiable idiosyncratic risk has effectively no effect on entrepreneurial decisions and the model solution approaches the complete-markets one. In that case, \( \lim_{w \to \infty} q(w) \to q^{FB} \), where \( q^{FB} = 1.31 \) for our baseline calculation. Note that the first-best complete-markets Tobin’s \( q \), \( q^{FB} \), is independent of entrepreneurial preferences (complete-markets Arrow-Debreu separation results). Under incomplete markets, the convergence to the complete-markets benchmark quantitatively requires a relatively high value of \( w \). With \( w = 3 \), \( q(3) \) is significantly below the complete-markets benchmark values \( q^{FB} \). The less risk averse the entrepreneur, the higher Tobin’s \( q \), \( q(w) \).
Figure 1: Certainty equivalent wealth $p(w)$, private enterprise value ratio $q(w)$, marginal value of wealth $P_W(K,W)$ and marginal value of capital $P_K(K,W)$.

More interestingly, $q(w)$ is not globally concave. (The second derivatives and hence concavity properties for $p(w)$ and $q(w)$ are the same because $q(w) = p(w) - w$.) While the entrepreneur is risk averse, it does not necessarily imply that $q(w)$ is concave. Indeed, the risk-averse entrepreneur’s value function $J(K,W)$ is concave in certainty equivalent wealth $P(K,W)$ and is also concave in $W$. Figure 1 shows that $q(w)$ is concave in $w$ for sufficiently high $w$, i.e. $w \geq \tilde{w}$ where $\tilde{w}$ is the inflection point at which $p''(\tilde{w}) = q''(\tilde{w}) = 0$. For sufficiently low $w$, $(w \leq \tilde{w})$, $q(w)$ is convex.

The endogenous liquidation boundary $w$ captures the entrepreneur’s optimal liquidation decision. The entrepreneur’s ability to eliminate his exposure to the non-diversifiable idiosyncratic business risk by liquidating his business generates optionality and hence convexity in
The exit option is particularly valuable for low values of \( w \). As a result, \( q(w) \) is convex in \( w \) for sufficiently low \( w \), i.e. \( w \leq \bar{w} \). Costly liquidation of capital stock when \( w \) is low in our model provide a downside risk protection for the entrepreneur in terms of his utility and consumption smoothing. Recall that debt is fully collateralized and hence is risk-free in our model. Our argument for convexity is different from the argument that risky debt may create state-contingent insurance as in Zame (1993), Heaton and Lucas (2004), and Chen, Miao, and Wang (2009). However, both arguments rely on “put” options (a liquidation option in our model and a default option in risky debt models) providing downside protection for the entrepreneur under incomplete markets. We now decompose the certainty equivalent wealth \( P(K, W) \).

**Decomposing the certainty equivalent wealth \( P(K, W) \).** Using the homogeneity property and the Euler’s theorem, we write \( P(K, W) \) as follows:

\[
P(K, W) = P_K(K, W)K + P_W(K, W)W,
\]

where the marginal value of wealth \( P_W(K, W) \) is given by

\[
P_W(K, W) = p'(w),
\]

and the marginal value of physical capital is given by

\[
P_K(K, W) = p(w) - wp'(w).
\]

It is immediate to see \( Q_K(K, W) = P_K(K, W) \) and \( Q_W(K, W) = P_W(K, W) - 1 \), or equivalently \( q'(w) = p'(w) - 1 \). We may refer to \( q'(w) \) as the net private marginal value of cash.

For public firms owned by diversified investors, the marginal increase of firm value associated with a unit increase of capital is often referred to as marginal \( q \). For a firm owned and managed by a non-diversified entrepreneur, the marginal increase of the entrepreneur’s certainty equivalent wealth associated with a unit increase of capital, \( P_K(K, W) \), is the natural counterpart to the marginal \( q \) for public firms. We refer to \( P_K(K, W) \) as the private (i.e. subjective) marginal \( q \) for the entrepreneurial firm.

Similarly, for public firms, the marginal increase of firm value associated with a unit increase of cash is referred to as marginal value of cash (as in Bolton, Chen, and Wang (2009)). From the non-diversified entrepreneur’s perspective, the marginal increase of his
certainty equivalent wealth associated with a unit increase of wealth, \( P_W(K,W) \), is the natural counterpart to the marginal value of cash for public firms. We refer to \( P_W(K,W) \) as the private (i.e. the entrepreneur’s) marginal value of wealth.

We now turn to the marginal value of wealth \( P_W(K,W) \) and the marginal value of capital \( P_K(K,W) \).

**The entrepreneur’s marginal value of wealth** \( P_W(K,W) \). The lower left panel of Figure 1 plots the marginal value of wealth: \( P_W(K,W) = p'(w) \). In perfect capital markets, wealth has no value beyond its market value: \( P_W(K,W) = 1 \). With incomplete markets, the marginal value of liquidity \( P_W(K,W) \) is generally greater than unity because liquid wealth mitigates the negative impact of financial frictions on business investment and consumption. Figure 1 shows that \( p'(w) \) is equal to unity at the optimal liquidation boundary \( (p'(w) = 1) \), increases with \( w \) up to the (endogenously determined) inflection point \( \tilde{w} \) (at which \( p''(\tilde{w}) = 0 \)), decreases with \( w \) for \( w \geq \tilde{w} \), and finally approaches unity when \( w \to \infty \). Frictions (incomplete markets and borrowing constraints) increase the shadow costs for both consumption and productive investment and make \( p'(w) \geq 1 \).

Informal analysis may have led us to conclude that less constrained entrepreneurs (i.e. those with higher wealth) value their wealth less and hence the marginal value of cash \( P_W(K,W) \) shall globally decrease with wealth (i.e. \( p''(w) < 0 \)). This is incorrect as we see from Figure 1. The convexity of \( p(w) \) (i.e. \( p''(w) > 0 \) for \( w \leq \tilde{w} \)) arises because the liquidation option provides a valuable exit option for non-diversified entrepreneurs under incomplete markets. This, in turn, lowers the marginal value of wealth since exiting the business implies that the entrepreneur is no longer worried about non-diversifiable risk.

**The entrepreneur’s marginal value of capital** \( P_K(K,W) \). The lower right panel of Figure 1 plots \( P_K(K,W) \), also referred to as the private marginal \( q \). At the optimal liquidation boundaries, the marginal value of capital is equal to the liquidation recovery value per unit of capital, i.e. \( P_K(K,W) = l = 0.9 \), where \( l = 0.9 \) in our baseline calculation. When \( w \) is sufficiently high (i.e. \( w \to \infty \)), the impact of non-diversifiable risk vanishes, i.e. \( p(w) \to w + q^{FB} \) and \( P_K(K,W) \to q^{FB} \).

Perhaps surprisingly, the private marginal \( q \) is not monotonic in the natural financial constraint measure: wealth-capital ratio \( w \). One seemingly natural but loose intuition is
that the (private) marginal $q$ increases with financial slack (i.e. wealth-capital ratio $w$). Presumably, less financially constrained entrepreneurs face lower costs of investment and hence have higher marginal $q$. However, this intuition in general does not hold. Using the analytical formula (47) for private marginal $q$, we obtain:

$$\frac{dP_K(K,W)}{dw} = -wp''(w).$$

(48)

Therefore, the sign of $dP_K(K,W)/dw$ depends on both the sign of $w$ and of the concavity of $p(w)$. When the entrepreneur’s wealth is positive ($w > 0$) and $p(w)$ is concave, the private marginal $q$, $P_K(K,W)$, increases with $w$. See the right end of Figure 1 (i.e. when $w$ is sufficiently large). When the entrepreneur is in debt ($w < 0$) and $p(w)$ is convex, $P_K(K,W)$ also increases with $w$. See the left end of Figure 1, provided that the endogenous liquidation boundary $w < 0$. In the intermediate region of $w$, the private marginal $q$ may decrease with financial slack $w$ (for example, $w < 0$ with $p''(w) < 0$).

Perhaps also surprisingly, the private marginal $q$, $P_K(K,W)$, can sometimes be higher than $q^{FB}$, Tobin’s marginal, which is also the average $q$ under the first-best benchmark. To illustrate this seemingly counter-intuitive result, it is useful to rewrite the private marginal $q$ as follows:

$$P_K(K,W) = q(w) - w(p'(w) - 1).$$

(49)

The above formula states that the private marginal $q$, $P_K(K,W)$, is equal to the difference between (i) the private average $q$, $q(w) = p(w) - w$ and (ii) a term $w(p'(w) - 1)$, which captures the impact of financial constraints. Recall two general results of our model: (i) the private average $q$ under incomplete markets is lower than the first-best benchmark: $q(w) \leq q^{FB}$ and (ii) the marginal value of cash $P_W(K,W)$ is greater than unity: $p'(w) \geq 1$. Using the above two general results, we show that the wedge between private marginal $q$ and private average $q$, $P_K(K,W) - q(w)$, can be either negative or positive depending on the sign of $W$.

For entrepreneurs with positive wealth ($W > 0$), increasing capital $K$ mechanically lowers $w = W/K$, which measures the degree of financial constraints. Strengthening of the financial constraint in turn lowers the marginal product of capital which gives rise to the prediction that the private marginal $q$, $P_K(K,W)$, is below $q(w)$, the private average $q$. This gives rise to a negative wedge $P_K(K,W) - q(w)$. However, for an entrepreneur in debt ($W < 0$),
investment increases \( w = W/K \) (by moving towards zero from the left of the origin) and hence relaxes the financial constraint. This implies a positive wedge \( P_K(K,W) - q(w) \). Third, the effect of relaxing financial frictions (due to non-diversifiable risk) can be so significant that the private marginal \( q \), which measures marginal value of capital, can exceed the first-best benchmark level: \( q_{FB} \) (see the second term in (49)). Figure 1 shows that for both \( \gamma = 2 \) and \( \gamma = 4 \), the private marginal \( q \) is higher than \( q_{FB} = 1.31 \) for some values of \( w < 0 \).

### 5.2 Business investment and exit decisions

We now turn to the entrepreneur’s business investment decisions. We rewrite the FOC for the entrepreneur’s investment-capital ratio \( i(w) = I/K \) in his firm as follows:

\[
(1 + \theta i(w)) P_W(K,W) = P_K(K,W) .
\]  

(50)

To install a unit of capital, the entrepreneur needs to incur cost \( 1 + \theta i(w) \) at the margin. The incremental cost \( \theta i(w) \) is the marginal adjustment cost (e.g. over-time marginal labor costs and marginal installation costs) beyond the capital purchase cost. Moreover, the marginal cost of using a unit of liquid wealth from the entrepreneur’s perspective is \( P_W(K,W) = p'(w) \). Therefore, using the “chain” rule argument, we need to incorporate both the marginal production cost \( (1 + \theta i(w)) \) and the marginal financing cost \( P_W(K,W) \) to obtain the marginal cost of installing a unit of capital, which is given by \( (1 + \theta i(w)) P_W(K,W) \), i.e. the left side.
of (50). The right side is the marginal value of capital $P_K$. The entrepreneur equates the two sides of (50) by optimally choosing his investment. The FOC (50) states that the entrepreneur’s optimal investment decision depends on the ratio between the private marginal $q$, $P_K(K,W)$, and the private marginal value of wealth (his savings), $P_W(K,W)$. Both the private marginal $q$ and private marginal value of savings $P_W$ are endogenously determined. Moreover, they are highly correlated.

The top left and right panels in Figure 2 plot the investment-capital ratio $i(w)$ and $i'(w)$, the sensitivity of $i(w)$ with respect to $w$. Non-diversifiable idiosyncratic business risk induces underinvestment: $i(w) < i^{FB} = 0.15$. The underinvestment result (relative to the first-best MM benchmark) is common in incomplete-markets models.

More interestingly and less intuitively, investment-capital ratio is not monotonic in $w$. That is, investment may decrease with wealth! This seemingly counter-intuitive result directly follows from the convexity of the entrepreneur’s certainty equivalent wealth $p(w)$ in $w$. We may characterize the sensitivity of investment with respect to his savings as follows:

$$i'(w) = -\frac{p(w)p''(w)}{\theta(p'(w))^2}. \quad (51)$$

Using the above result, we see that whenever $p(w)$ is concave, investment increases with his savings, the intuitive part of the results. However, whenever $p(w)$ is convex, investment decreases with savings. Put differently, underinvestment is less of a concern when the entrepreneur is closer to liquidating his business. This is because distorting investment (underinvestment) is suboptimal in the absence of frictions. The entrepreneur has less incentives to distort investment if the likelihood of exiting incomplete markets and hence avoiding idiosyncratic business risk is high enough (close to liquidation).

The entrepreneur always has the option to exit his business by liquidating capital. The downside risk is thus capped by his exit option. After exiting, the entrepreneur is then only exposed to systematic shocks. This exit option induces a convexity effect of volatility on

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15Bolton, Chen, and Wang (2010) derive a similar FOC for investment in a dynamic corporate finance framework when cash/credit is the marginal source of financing for a financially constrained firm. In an optimal dynamic contracting framework, DeMarzo, Fishman, He, and Wang (2010) also derive a similar FOC with endogenous financial constraint. However, the economic settings, economic interpretations and financial frictions are different across these papers. In Bolton, Chen, and Wang (2010), the friction is costly external financing. In DeMarzo, Fishman, He, and Wang (2010), the friction is a dynamic moral hazard issue (such as hidden cash flows or effort choices by the agent). In our paper, it is the non-diversifiable (i.e. non-spanned) idiosyncratic risk under incomplete markets. Of course, at a micro-founded level, these frictions are potentially related.
p(w) for sufficiently low w. Note that the entrepreneur’s value function J(K, W) is concave in certainty equivalent wealth P(K, W) and is also concave in W. That is, the entrepreneur is risk averse (JWW < 0), but volatility increases his certainty equivalent wealth in the region w ≤ ˜w, where ˜w is defined by p"(˜w) = 0.

Now we turn to the entrepreneur’s liquidation decision. Because diversification benefits are more important for more risk-averse entrepreneurs, a more risk-averse entrepreneur liquidates capital earlier in order to avoid idiosyncratic risk exposure and achieve full diversification. For example, indeed, the optimal liquidation boundaries are w = −0.8 and w = −0.65 for γ = 2 and γ = 4, respectively. The (American) option to convert an illiquid risky business into liquid financial assets is more valuable for more risk-averse entrepreneurs. Note that the borrowing constraint does not bind even for a less risk-averse entrepreneur (e.g. γ = 2). The entrepreneur rationally liquidates his business before he exhausts his debt capacity w ≥ −l = −0.9 to ensure that his wealth does not fall too low. The entrepreneur does not exhaust his borrowing capacity because of his risk aversion. While borrowing more to invest is desirable in terms of generating positive value for (outside diversified) investors, doing so may be too risky for non-diversified entrepreneurs. Moreover, anticipating that the liquidation option will soon be exercised, the entrepreneur has less incentive to distort investment further when w is close to the liquidation boundary. This anticipation effect explains the non-monotonicity result for i(w) in w.

Next, we turn to the entrepreneur’s portfolio choice decisions for his liquid assets.

5.3 Optimal portfolio allocation

The entrepreneur’s portfolio allocation in the market portfolio per unit of physical capital, x(w), has two components: (i) the hedging component −ρσA/σR as in Merton (1971) and (ii) the mean-variance demand term given by ησ−1p(w)/h(w). The hedging term captures the standard diversification arguments. In this analysis, we will focus on the economically more interesting component: the mean-variance demand term.

Under complete markets, the “total” wealth-capital ratio is pFB(w) = w + qFB. The demand for the risky market portfolio, xFB(w), increases linearly in pFB(w). The two straight lines in the lower left panel of Figure 2 depict this linear relation for γ = 2 and γ = 4. As mean-variance investors in Merton (1971), the less risk-averse entrepreneur demands more
of the risky market portfolio, ceteris paribus.

The mean-variance demand, $(\mu_R - r)\sigma_R^2 p(w)/h(w)$, generalizes the seminal result in Merton (1971) to an incomplete-markets setting with non-tradable assets and idiosyncratic risk. The entrepreneur’s portfolio demand differs in two aspects: (i) replacing total wealth $w + q^{FB}$ by certainty equivalent wealth $p(w)$ and (ii) replacing risk aversion $\gamma$ by $h(w) = \gamma p'(w) - p(w)p''(w)/p'(w)$ (which is also given in (34)). At least for the purpose of understanding the entrepreneur’s portfolio allocation, a natural interpretation of $h(w)$ is his effective risk aversion due to precautionary motive and borrowing constraints under incomplete markets.

Figure 3 plots $h(w)$ for $\gamma = 2, 4$. In the limit $w \to \infty$, idiosyncratic business risk has no role and hence markets are effectively complete for the entrepreneur. As a result, $h(w)$ approaches the complete-markets value, i.e. $h(w) \to \gamma$. Perhaps surprisingly, when the entrepreneur is close to liquidating his firm (i.e. $w \to w$), effective risk aversion $h(w)$ is lower than $\gamma$. This can be seen from $h(w) = \gamma - (w+l)p''(w) < \gamma$. That is, the convexity effect near the liquidation boundary $w$ implies this result. When the liquidation option is sufficiently in the money, the entrepreneur behaves in a less risk averse manner than without the liquidation option under complete markets (after controlling for the direct effect of incomplete markets.
Figure 4: Market portfolio allocation-capital ratio $x(w)$ and its sensitivity $x'(w)$.

on certainty equivalent wealth). Figure 3 shows that $h(w)$ is not monotonic in $w$. For most values of $w$ (other than near the liquidation boundary), the effective risk aversion $h(w)$ is indeed higher than the risk aversion coefficient $\gamma$, consistent with our intuition.

Incomplete markets influence the entrepreneur’s demand for the risky market portfolio via two channels: certainty equivalent wealth $w$ and effective risk aversion $h(w)$. Even near the liquidation boundary (i.e. for low values of $w$), the reduction in $p(w)$ is big enough to offset the decrease in $h(w)$ and hence demand for the risky market portfolio is lower than that in the complete-markets benchmark. The left panel of Figure 4 plots the demand for the risky market portfolio $x(w)$ under incomplete markets and shows that it is lower than that under complete markets for both $\gamma = 2$ and $\gamma = 4$.

It is worth noting that $x'(w)$, the sensitivity of $x(w)$ with respect to wealth $w$, can be substantial (for example, in excess of unity). Moreover, $x'(w)$ can even be negative. That is, increasing wealth may lead to the entrepreneur’s reduced investment in the risky market portfolio. This seemingly counterintuitive result stems from two competing forces which determine the entrepreneur’s portfolio demand: the certainty equivalent wealth $p(w)$ and the effective risk aversion $h(w)$. While increasing $w$ unambiguously makes the entrepreneur’s certainty equivalent wealth $p(w)$ higher, increasing $w$ may also make the entrepreneur effectively more risk averse, i.e. $h(w)$ may also increase with $w$. If increasing $w$ leads to a lower percentage increase in $p(w)$ than the percentage increase in $h(w)$, increasing $w$ lowers $x(w)$. 

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Figure 5: Consumption-capital ratio \( c(w) \) and the MPC out of wealth \( C_w = c'(w) \).

When \( x \) increases from the liquidation boundary, the entrepreneur can become sufficiently more risk averse (the liquidation option is sufficiently out of the money) and thus lowers his portfolio demand (despite higher \( p(w) \)). This argument shows the importance of exit options (flexibility) on the entrepreneur’s portfolio investment strategy. The right panel of Figure 4 shows that the sensitivity \( x'(w) \) is negative near the liquidation boundary, increases with \( w \), turns positive, and eventually falls with \( w \) but remains positive for sufficiently high \( w \).

Next, we turn to the impact of incomplete markets on the entrepreneur’s consumption decisions.
5.4 Optimal consumption and the MPCs

Recall that the consumption-capital ratio is given by \( c(w) = m^{FB}p(w)(p'(w))^{-\psi} \). Consumption depends not only on the certainty equivalent wealth \( p(w) \) but also on the marginal value of wealth \( p'(w) \). Consumption \( c(w) \) under incomplete markets is lower than the complete-markets level \( c^{FB}(w) \), i.e. \( c(w) < c^{FB}(w) \), for two reasons: the certainty equivalent wealth \( p(w) \) is lower and the marginal value of wealth \( p'(w) \) is higher. Consumption is more costly under incomplete markets than under complete markets even after controlling for the difference in certainty equivalent wealth (due to incomplete markets).

The upper left panel of Figure 5 plots the optimal consumption-capital ratio \( c(w) \) under both complete and incomplete markets for \( \gamma = 2, 4 \), and confirm the result that \( c(w) < c^{FB}(w) \) for all \( w \) and for \( \gamma = 2, 4 \). The two straight lines in the upper left panel are for \( c^{FB}(w) = m^{FB}(w + q^{FB}) \) for \( \gamma = 2, 4 \) under complete markets. The MPC \( m^{FB} = 0.057 \) for \( \gamma = 2 \), which is higher than \( m^{FB} = 0.052 \) for \( \gamma = 4 \). For both \( \gamma = 2 \) and \( \gamma = 4 \), the elasticity of intertemporal substitution \( \psi \) is set at 0.5, which is less than unity; the wealth effect thus dominates the substitution effect for both cases. Given \( w \), the less risk-averse entrepreneur consumes more today.

The upper right panel of Figure 5 plots the MPC out of wealth: \( C_W(K,W) = c'(w) \). Intuitively, the MPC out of wealth is positive. However, the MPC \( C_W = c'(w) \) is not monotonic in \( w \). The MPC \( C_W(K,W) \) first increases with wealth and then decreases with wealth.

Another way to state the above result is to report consumption \( C(K,W) \) per unit of certainty equivalent wealth \( P(K,W) \). Let \( m(w) \) denote this ratio, i.e. \( m(w) = C(K,W)/P(K,W) \). It is immediate to show that

\[
m(w) = \frac{C(K,W)}{P(K,W)} = \frac{c(w)}{p(w)} = m^{FB}(p'(w))^{-\psi}.
\]  

(52)

Note that \( m(w) \) is always lower than the MPC under complete markets, i.e. \( m(w) \leq m^{FB} \). See the lower left panel of Figure 5. Liquid wealth provides additional benefits for the entrepreneur to buffer against his business income shocks under incomplete markets (i.e. \( P_W = p'(w) \geq 1 \)). The entrepreneur has an additional motive to save and consumption is more costly under incomplete markets. Note that the sensitivity \( m'(w) \) is also not monotonic (see the lower right panel of Figure 5). Indeed, \( m'(w) \) has the same sign as that of \(-p''(w)\).
We know that $p(w)$ is not globally concave and hence $m(w)$ is not monotonic.

6 Entrepreneurial entry: Career choice and firm size

In the previous section, we have studied the agent’s decision making and subjective valuation after he becomes an entrepreneur. However, whether the agent decides to be an entrepreneur is an endogenous decision. Entry into entrepreneurship is a first-order question. Limited wealth often constrains the agent from becoming an entrepreneur and also lowering the initial firm size. Anticipating the importance of the wealth constraint, the agent may rationally save more, building up his wealth so that he can become an entrepreneur in the future and also potentially choose a bigger initial firm size. This leads to the optimal entry timing and initial firm size decisions to which we now turn.

We consider two cases with increasing realism: first a time-0 binary career decision and then a richer model allowing for the choice of entry timing. For both cases, we will also study the determinants of the initial firm size.

6.1 When career choice is “now or never”: A binary decision

First consider the case where the agent has a static time-0 binary choice to be an entrepreneur or not. If he does not become an entrepreneur and instead chooses his outside option which has a present value $\Pi$. He then solves an intertemporal consumption/portfolio problem similar to Merton (1971) but with recursive utility. His value function is $V(W + \Pi)$ where $V(\cdot)$ is given in (25) with certainty equivalent wealth $(W_0 + \Pi)$.

If he decides to be an entrepreneur, he incurs a fixed start-up cost $\Phi$ and then chooses the initial size $K_0$. Thus, after starting his business, his liquid wealth immediately drops to $W_0 - (\Phi + K_0)$, which can be negative (i.e. in debt). The entrepreneur is allowed to borrow provided that his debt value is less than the liquidation value of capital, which in turn ensures that debt is fully collateralized and hence is risk-free. This implies the following borrowing constraint at time 0: $W_0 - (\Phi + K_0) \geq -lK_0$. Equivalently, we write this constraint as follows:

$$W_0 \geq \Phi + (1 - l)K_0.$$  \hspace{1cm} (53)

To rule out the case where the entrepreneur makes instant profits by starting up his business
and then immediately liquidating capital for profit, we require the liquidation parameter $l$ to be less than unity: $l < 1$.

The agent’s value function is given by $J(K_0, W_0 - (\Phi + K_0))$ and his certainty equivalent wealth is $P(K_0, W_0 - (\Phi + K_0)) = p(w_0 - 1 - \Phi/K_0)K_0$. He chooses $K_0$ to maximize his value function and equivalently to maximize his certainty equivalent wealth by solving

$$\max_{K_0} P(K_0, W_0 - (\Phi + K_0)),$$

subject to (53) and the agent’s optimization after becoming an entrepreneur. Let $K_0^*$ denote the optimal initial capital stock from the entrepreneur’s optimal tradeoff.

Finally, the agent compares the certainty equivalent wealth $P(K_0^*, W_0 - (\Phi + K_0^*))$ by being an entrepreneur with that from not being an entrepreneur: $W + \Pi$, and makes the career decision. The following theorem summarizes the main results.

**Theorem 2** At time 0, an agent with initial wealth $W_0$ chooses to be an entrepreneur if and only if $W_0$ is greater than the threshold level $\bar{W}_0$ given by

$$\bar{W}_0 = \frac{\Phi p'(w^*) + \Pi}{p'(w^*) - 1},$$

where $w^*$ is the solution of the following equation:

$$p'(w^*) = \frac{p(w^*)}{1 + w^*}.$$  

The entrepreneurial firm’s initial firm size $K_0^*$ is given by

$$K_0^* = \frac{W_0 - \Phi}{1 + w^*}. $$

The entrepreneur’s certainty equivalent wealth is then given by

$$P(K_0^*, W_0 - \Phi - K_0^*) = p(w^*)K_0^* = p'(w^*) (W_0 - \Phi),$$

where the firm’s start-up $w^*$ is given by (56). After starting up the firm, consumption, portfolio allocation, and firm investment/liquidation decisions are given by Theorem 1.

We now analyze the firm’s optimal initial size $K_0^*$, and the initial certainty equivalent wealth $P(K_0^*, W_0 - \Phi - K_0^*)$ as functions of the agent’s initial wealth $W_0$ for two levels of risk aversion: $\gamma = 2, 4$ (see Figure 6). First, risk attitude plays a significant role in
determining entrepreneurship: The threshold initial wealth $W_0$ to become an entrepreneur increases significantly from 2.86 to 4.60 when risk aversion increases from 2 to 4. Second, for less risk-averse entrepreneurs, initial wealth also has a much more significant impact on firm size $K_0^*$. For example, the initial firm size $K_0^*$ by 1.57 with each dollar increase in $W_0$ with $\gamma = 2$, which is much higher than 1.07, the marginal increase of the initial firm size $K_0^*$ with respect to $W_0$ for $\gamma = 4$. Finally, the marginal effect of initial wealth $W_0$ is higher for less risk-averse entrepreneurs (e.g. 1.2 for $\gamma = 2$ and 1.12 for $\gamma = 4$ as seen from the right panel). Note that it is not just the liquidity constraint per se, but rather the interaction between the liquidity constraint and the precautionary saving motive under incomplete markets that has significant effects on the entry decision of entrepreneurship and the initial project size.

### 6.2 When career choice is flexible: Optimal entry timing

We now allow the agent to choose the optimal entry time. The agent receives cash flows at a rate $r\Pi$ if and only if he takes the outside option. When becoming an entrepreneur, he quits his outside option and only collects income from his business as described earlier. The agent can hold his investment opportunity/option. We will show that this additional flexibility allows the agent to build up financial strength, which is highly valuable. For analytical
tractability, we assume becoming an entrepreneur is irreversible.

Let \( F(W) \) denote the agent’s value function before becoming an entrepreneur. Using the similar argument as in our earlier analysis, we conjecture that \( F(W) \) is given by

\[
F(W) = \frac{(bE(W))^{1-\gamma}}{1-\gamma},
\]

where \( b \) is the constant given by (16) and \( E(W) \) can be interpreted as the agent’s certainty equivalent wealth.

We will show that the entrepreneurship decision is characterized by a time-invariant endogenously determined cutoff threshold \( \hat{W} \), an American-style option exercising rule. At any time \( t \), provided that \( W_t \geq \hat{W} \), the agent immediately enters entrepreneurship and makes intertemporal decisions as summarized in Theorems 1 and 2. Otherwise, the agent takes his outside option. We next summarize the main results in the “waiting” region for career choice.

**Theorem 3** Provided that \( W \leq \hat{W} \), the agent’s certainty equivalent wealth \( E(W) \) solves:

\[
0 = \frac{m^{FB}E(W)(E'(W))^{1-\psi}}{\psi - 1} - \psi E(W) + r(W + \Pi)E'(W) + \frac{\eta^2}{2} \frac{E(W)E'(W)^2}{\gamma E'(W)^2 - E(W)E''(W)},
\]

with the following boundary conditions

\[
E(\hat{W}) = p'(w^*)(\hat{W} - \Phi),
\]

\[
E'(\hat{W}) = p'(w^*),
\]

\[
E(-\Pi) = 0.
\]

where \( w^* \) is given in Theorem 2. The agent’s consumption and portfolio rules are given by

\[
C = b^{1-\psi} \xi^\psi E(W)E'(W)^{-\psi},
\]

\[
X = \frac{\mu_R - r}{\sigma_R^2} \frac{E(W)E'(W)}{\gamma E'(W)^2 - E(W)E''(W)},
\]

The value-matching condition (61) states that the agent’s certainty equivalent wealth \( E(W) \) is continuous at the endogenously determined cutoff level \( \hat{W} \). The smooth-pasting condition (62) gives the agent’s optimal indifference condition between being an entrepreneur or not with wealth \( \hat{W} \). Finally, if the agent is in debt with amount \( \Pi \), he will never get out
of the debt region and cannot pay back the fixed start-up cost. Therefore, the option value to be an entrepreneur is zero and the certainty equivalent wealth is zero as given by (63).

We next analyze the agent’s certainty equivalent wealth $E(W)$ before becoming an entrepreneur for two levels of risk aversion: $\gamma = 2, 4$ (see Figure 7). First, the less risk-averse agent is more entrepreneurial. For example, the threshold wealth $\hat{W}$ is 5.66 for $\gamma = 4$, which is higher than 4.3 for $\gamma = 2$. Unlike a typical real options problem, ours features incomplete markets. This means that less risk-averse agents value the investment option more and hence exercise it earlier. Second, less risk-averse agents also value the future investment opportunity more because they demand a lower idiosyncratic risk premium as we will show shortly. Finally, for all levels of $W$, $E'(W)$ is greater for less risk-averse agents. The marginal effect of wealth is greater for less risk-averse agents for the two reasons we have stated earlier.

Figure 8 quantifies the value of “entry timing flexibility” by using our two formulations of entrepreneurship entry. For both $\gamma = 2$ and $\gamma = 4$, the convex curves in Figure 8 correspond to the the one with full timing flexibility (i.e. “American” version), while the straight lines give the “now-or-never” version at time 0. First, it is immediate to see that timing flexibility is valuable. Second, the value of timing flexibility is highest when the agent’s wealth is in the intermediate range where building more financial strength substantially lowers the risk
premium for the business project and hence enhances welfare. When the option value is sufficiently close to being in the money or deep out of the money, the wedge between the two versions of entrepreneurship entry is small.

We next study the risk premium implications for entrepreneurial firms.

7 Idiosyncratic risk premium for an entrepreneurial firm

A fundamental issue in entrepreneurial finance is to determine the cost of capital for private firms owned by non-diversified entrepreneurs. Intuitively, the entrepreneur demands both the systematic risk premium and an additional idiosyncratic risk premium which accounts for his non-diversifiable risk exposure. Compared to an otherwise identical public firm held by diversified investors, the cost of capital should be higher for the entrepreneur. Next, we provide a procedure to calculate the cost of capital for the entrepreneurial firm.

Let $\xi$ denote the constant yield (internal rate of return) for the entrepreneurial firm until liquidation. That is, $\xi$ as a function of the initial wealth-capital ratio $w_0 = W_0/K_0$, $\xi(w_0)$,
The idiosyncratic risk premium: $\alpha(w)$. We plot the results for two levels of risk aversion $\gamma = 2$ and $\gamma = 4$.

solves the following valuation equation:

$$Q(K_0, W_0) = \mathbb{E} \left[ \int_0^\tau e^{-\xi(w_0)t} dY_t + e^{-\xi(w_0)\tau} lK_\tau \right] ,$$

where $\tau$ is the stochastic liquidation time. The right side of (66) is the present discounted value (PDV) of the firm’s operating cash flow plus the PDV of the liquidation value using the same discount rate $\xi(w_0)$. The left side is the "private" enterprise value $Q(K_0, W_0)$ that we have obtained earlier using the entrepreneur’s optimality. Recall that the firm’s discount rate under complete markets, $\xi^{FB}$, is given in (22). We thus measure the idiosyncratic risk premium as the wedge between $\xi(w_0)$ and $\xi^{FB}$.

$$\alpha(w_0) = \xi(w_0) - \xi^{FB} = \xi(w_0) - r - \beta^{FB}(\mu_R - r) .$$

There is much debate in the empirical literature about the significance of this private equity risk premium. For example, Moskowitz and Vissing-Jorgensen (2002) document the risk-adjusted returns to investing in U.S. nonpublicly traded equity are not higher than the returns to private equity.

Our model provides an analytical formula to calculate this private equity idiosyncratic risk premium. Figure 9 plots the idiosyncratic risk premium for two levels of risk aversion.
\(\gamma = 2, 4\). For sufficiently high levels of initial wealth-capital ratio \(w_0\), the idiosyncratic risk premium \(\alpha(w_0)\) eventually disappears (approaching zero). Moreover, this premium \(\alpha(w_0)\) is higher for more risk-averse agents. Intuitively, the more risk-averse entrepreneur demands a higher rate of return and hence a higher \(\alpha(w_0)\) for bearing non-diversifiable risk. For entrepreneurs with positive wealth, we do not find a significant idiosyncratic risk premium. For both \(\gamma = 2\) and \(\gamma = 4\), the annual idiosyncratic risk premia are less than 1%.

### 8 Comparative analysis

We now analyze the effects of various structural parameters such as the elasticity of intertemporal substitution \(\psi\), idiosyncratic volatility \(\epsilon\), the adjustment cost parameter \(\theta\), and the liquidation parameter \(l\) on the entrepreneur’s decision making and business valuation. For all the figures, we fix \(\gamma = 2\) and use the other parameter values given in the baseline model (Section 4).

In the preceding analysis, we have shown that risk aversion has substantial effects. Next, we analyze the impact of elasticity of intertemporal substitution \(\psi\). Epstein-Zin-Weil recursive utility allows us to separate the effects of elasticity from risk aversion.

**Elasticity of intertemporal substitution \(\psi\).** In asset pricing (long-run risk) literature, a high elasticity of intertemporal substitution is often used following the important contribution by Bansal and Yaron (2004), in order to generate stylized asset pricing facts. However, there is much disagreement about the value of the elasticity of intertemporal substitution among economists. To illustrate the effects of this important parameter, we consider two commonly used but significantly different values for elasticity of intertemporal substitution: \(\psi = 0.25\) and \(\psi = 2\). First, recall that under complete markets, elasticity has a significant effect on consumption. For example, with \(\psi = 2\), the MPC out of certainty equivalent wealth \(m^*\) is only 0.014 per year, which is substantially lower than the MPC \(m^* = 0.072\) when elasticity is \(\psi = 0.25\). Intuitively, the entrepreneur with high elasticity (\(\psi = 2\)) is willing to reduce current consumption in order to build up wealth for future consumption. Importantly, wealth accumulation is significantly faster with high elasticity \(\psi\) under complete markets.

Insert Figure 10 here.
From Figure 10, we find that for a given value of \( w \), the elasticity of intertemporal substitution \( \psi \) does not have quantitatively significant effects on the private enterprise value \( q(w) \), the marginal value of liquidity \( P_W \), the investment-capital ratio \( i(w) \), and the market portfolio allocation-capital ratio \( x(w) \). However, similar to the complete-markets setting, the elasticity of intertemporal substitution \( \psi \) has important implications on the entrepreneur’s capital accumulation and hence on his long-run consumption, investment, and welfare.

**Idiosyncratic volatility \( \epsilon \).** We next show that the effects of the idiosyncratic volatility \( \epsilon \) are quantitatively significant. In Figure 11, we plot for two levels of idiosyncratic volatility: \( \epsilon = 0.1 \) and \( \epsilon = 0.2 \). First, the entrepreneur values the firm significantly less when facing greater idiosyncratic volatility. For a given level of \( w \), the entrepreneur invests significantly less in the firm (i.e. lowering \( i(w) \)), liquidates the project earlier, and allocates less liquid wealth to the aggregate market portfolio (i.e. lowering \( x(w) \)), and consumes less due to the “background” risk of business (i.e. lowering \( c(w) \)). Finally, the figure shows that the idiosyncratic volatility naturally has a significant effect on the idiosyncratic risk premium. For example, when doubling the idiosyncratic volatility from 10% to 20%, the annual idiosyncratic risk premium for an entrepreneur with no liquid wealth \( (w = 0) \) increases from 0.5% to 2.3%.

Insert Figure 11 here.

**Adjustment cost parameter \( \theta \).** We next demonstrate the effects of the adjustment cost parameter \( \theta \). Whited (1992) estimates the parameter value to be around two.\(^{16}\) Eberly, Rebelo and Vincent (2009) use an extended Hayashi (1982) model and provide a larger empirical estimate of this parameter value (close to seven) for large Compustat firms. Based on these studies, we choose \( \theta = 2 \) and \( \theta = 8 \) in Figure 12 for illustrative purposes. Our main results are as follows. First, the entrepreneur’s private valuation \( q(w) \) and investment \( i(w) \) depend on \( \theta \) especially when \( w \) is high (approaching the complete-markets benchmark). This is essentially the result inherited from the first-best benchmark. The higher the adjustment cost, the lower the enterprise value. Second, the adjustment cost has almost no effects on the entrepreneur’s consumption and portfolio choice. Finally, it also has little effects on the

\(^{16}\)Hall (2001, 2004) argue that the parameter \( \theta \) is small using U.S. aggregate data.
idiosyncratic risk premium. There is much smoothing going on due to the convex adjustment costs.

Insert Figure 12 here.

**Liquidation parameter** $l$. Finally, we study the effects of the liquidation parameter $l$. We plot for two values: $l = 0.6$ and $l = 0.9$ in Figure 13. We show that the liquidation parameter has a quantitatively significant impact on the entrepreneur’s decision making and private valuation of the firm particularly when the entrepreneur is short of liquidity/wealth (i.e. the left ends of the curves in each panel). Increasing the liquidation value $l$ allows the entrepreneur to borrow more (higher debt capacity). Hence, the entrepreneur exits from his business later. In addition, while running the business, the entrepreneur invests more, consumes more, and allocates more to the market portfolio because of the lower downside liquidation risk. As a result, a higher liquidation parameter value also makes the entrepreneur bear a lower idiosyncratic risk premium. However, when the liquidation option is sufficiently out of the money, i.e. when the entrepreneur is not short of liquidity (a sufficiently high $w$), liquidation has almost no effects on entrepreneurial decision making and valuation.

Insert Figure 13 here.

9 Conclusion

This paper provides an incomplete-markets framework to analyze the entrepreneur’s interdependent business entry, capital accumulation/growth, portfolio choice, consumption, and business exit decisions. We show that the entrepreneur rationally chooses to reduce his business investment and consumption compared with the complete-markets setting where he can fully capitalize his business income. We find that non-diversifiable idiosyncratic risk influences the private valuation of a business and the implied cost of capital for private businesses. We provide a procedure to compute the private equity idiosyncratic risk premium (Moskowitz and Vissing-Jorgensen (2002)). Moreover, we find that the marginal value of liquidity can be significant for private firms. For example, the entrepreneur reduces his investment in the risky market portfolio when he faces non-diversifiable business risk. Our
paper also shows the importance of non-diversifiable risk on entrepreneurial activities. The entrepreneur’s liquidation option substantially enhances the entrepreneur’s ability to manage downside risk, and generates interesting convexity effects. We also find significant effects of the agent’s wealth on entrepreneurship entry and analytically characterize the option value of building financial strength before entering entrepreneurship.

For tractability, we have made some simplifying assumptions. For example, we have ignored the entrepreneur’s legal entity choice. In reality, the entrepreneur can choose to set up a limited liability corporation to manage his business risk exposure. We have also simplified external financing options and ignored agency conflicts between the entrepreneur and financiers. We leave these important issues for future research (See Zwiebel (1996) and Morellec (2004) on managerial agency).

Finally, our model is a single agent’s intertemporal decision problem. To shed further lights on the impact of entrepreneurship on wealth distribution, economic growth and policy implications, it is desirable to construct a general equilibrium incomplete-markets model. Our analytically tractable and reasonably realistic decision theoretic model may provide one natural starting point for the general equilibrium analysis.

\[^{17}\text{See Aiyagari (1994) and Huggett (1993) for foundational incomplete-markets equilibrium (Bewley) models and Cagetti and De Nardi (2006) for an application with entrepreneurship.}\]
References


Technical Appendix

A  Theorem 1, Proposition 1, and Proposition 2

We conjecture that the value function is given by (14). We thus have

\[ J_K(K,W) = b^{1-\gamma}(p(w)K)^{-\gamma}(p(w) - wp'(w)), \]  
(A.1)

\[ J_W(K,W) = b^{1-\gamma}(p(w)K)^{-\gamma}p'(w), \]  
(A.2)

\[ J_{WW}(K,W) = b^{1-\gamma} \left( \frac{(p(w)K)^{-\gamma}p''(w)}{K} - \gamma(p(w)K)^{-\gamma-1}(p'(w))^2 \right), \]  
(A.3)

The first-order conditions (FOCs) for \( C \) and \( X \) are

\[ f_C(C,J) = J_W(K,W), \]  
(A.4)

\[ X = -\frac{\rho A}{\sigma R} K + \frac{(r - \mu_R)J_W(K,W)}{\sigma^2_{JWW}(K,W)}. \]  
(A.5)

Using the homogeneity property of \( J(K,W) \), we obtain the following for \( c(w) \) and \( x(w) \):

\[ c(w) = b^{1-\psi}\zeta^\psi p(w)(p'(w))^{-\psi}, \]  
(A.6)

\[ x(w) = -\frac{\rho A}{\sigma R} + \frac{\mu_R - r}{\sigma^2_{h(w)}} p(w). \]  
(A.7)

Substituting \( c(w) \) into (2), we have

\[ f(C,J) = \frac{\zeta}{1-\psi} \left( \frac{(bp(w)K)^{1-\gamma}bp'(w))^{1-\psi}}{\zeta^{1-\psi}} - (bp(w)K)^{1-\gamma} \right). \]  
(A.8)

Substituting (A.1), (A.2), (A.3), (A.7) and (A.8) into (28) and simplifying, we obtain

\[ 0 = \max_i \left( \frac{\zeta^\psi(bp'(w))^{1-\psi}}{\psi - 1} - \frac{\psi \zeta}{\psi - 1} \right) p(w) + (i - \delta)(p(w) - wp'(w)) \]

\[ + (rw + \mu_A - \rho \eta \sigma_A - i - g(i)) p'(w) + \frac{\eta^2 p(w)p'(w)}{2h(w)} + \frac{\epsilon^2 h(w)p'(w)}{2p(w)}, \]  
(A.9)

where \( h(w) \) is given in (34). Using the FOCs for investment-capital ratio \( i \), we obtain (41). Substituting it into (A.9), we obtain the ODE (33).

Using Itô’s formula, we obtain the following dynamics for the entrepreneur’s wealth-capital ratio \( w \):

\[ dw_t = d \left( \frac{W_t}{K_t} \right) = \frac{dW_t}{K_t} - \frac{W_t}{K_t^2} dK_t = \mu_w(w_t)dt + \sigma_R x(w_t)dB_t + \sigma_A dZ_t, \]  
(A.10)
where $\mu_w(w)$ is given by (39).

Now consider the lower liquidation boundary $W$. When $W \leq W$, the entrepreneur liquidates the firm. Using the value-match condition at $W$, we have

$$J(K, W) = V(W + lK),$$

(A.11)

where $V(W)$ given by (25) is the agent’s value function after retirement and with no business. The entrepreneur’s optimal liquidation strategy implies the following smooth-pasting condition at the endogenously determined liquidation boundary $W$:

$$J_W(K, W) = V_W(W + lK).$$

(A.12)

We conjecture that $W = \underline{w}K$. Substituting this conjecture, using (A.11) and (A.12), and simplifying, we obtain the scaled value-matching and smooth pasting conditions given in (36) and (37), respectively.

**Complete-markets benchmark solution.** When $w$ approaches infinity, markets are effectively complete. Non-diversifiable risk no longer matters for investment and consumption. Therefore, firm value approaches the complete-markets value and $\lim_{w \to \infty} J(K, W) = V(W + q^{FB}K)$, which implies (35). The certainty equivalent wealth $P(K, W)$ is equal to the sum of the financial wealth $W$ and complete-markets firm value $q^{FB}K$, in that

$$P(K, W) = W + q^{FB}K.$$  

(A.13)

Equivalently, we have $p(w) = w + q^{FB}$. Substituting this linear relation into (33), taking the limit $w \to \infty$, we obtain the following equation:

$$0 = \left(\frac{\psi b^{1-\psi} - \psi \zeta}{\psi - 1} + \frac{\eta^2}{2\gamma}\right)(w + q^{FB}) + (i^{FB} - \delta)q^{FB} + rw + \mu_A - \rho \eta \sigma_A - i^{FB} - q(i^{FB}).$$

(A.14)

In order for the above to hold, we require that both the linear term coefficient and the constant term are equal to zero. This gives rise to

$$b = \zeta \left[1 + \frac{1 - \psi}{\zeta} \left(r - \zeta + \frac{\eta^2}{2\gamma}\right)\right]^{1-\psi},$$

(A.15)

$$m^{FB} = b^{1-\psi} \zeta = \zeta + (1 - \psi) \left(r - \zeta + \frac{\eta^2}{2\gamma}\right).$$

(A.16)
We may now write (A.14) as follows:

\[
rq^{FB} = \mu_A - \rho \eta \sigma_A - i^{FB} - g(i^{FB}) + (i^{FB} - \delta)q^{FB}. \tag{A.17}
\]

Using the FOC for investment, i.e. \(i^{FB} = (q^{FB} - 1) / \theta\), we obtain (18).

Next, we calculate the expected rate of return for the firm. We have

\[
dR_t^{FB} = \frac{dY_t + dQ_t^{FB}}{Q_t^{FB}} = \frac{\mu_A dt + \sigma_A dZ_t - i^{FB} K_t dt - g(i^{FB}) K_t dt}{Q_t^{FB}} + \frac{q^{FB} dK_t}{Q_t^{FB}}, \tag{A.18}
\]

\[
= \left( r + \frac{\rho \eta \sigma_A}{q^{FB}} \right) dt + \frac{\sigma_A}{q^{FB}} dZ_t. \tag{A.19}
\]

Therefore, the expected return \(\mu_r^{FB}\) is

\[
\mu_r^{FB} = r + \frac{\rho \eta \sigma_A}{q^{FB}}. \tag{A.20}
\]

The beta is thus given by

\[
\beta^{FB} = \frac{\rho \sigma_A}{\sigma_R} \frac{1}{q^{FB}}. \tag{A.21}
\]

The systematic risk premium is therefore given by

\[
\beta^{FB}(\mu_R - r) = \frac{\rho \eta \sigma_A}{q^{FB}}. \tag{A.22}
\]

\section{B \hspace{1em} Theorem 2 and Theorem 3}

\textbf{Theorem 2.} The entrepreneur chooses initial firm size \(K_0^*\) to maximize utility yielding the following FOC:

\[
P_K(K_0^*, W_0 - \Phi - K_0^*) = P_W(K_0^*, W_0 - \Phi - K_0^*). \tag{B.1}
\]

The above condition (B.1) states that the marginal value of capital \(P_K(K_0^*, W_0 - \Phi - K_0^*)\) is equal to the marginal value of wealth \(P_W(K_0^*, W_0 - \Phi - K_0^*)\) at the optimally chosen \(K_0^*\). Simplifying (B.1) gives (56), which characterizes the optimal initial wealth-capital ratio \(w_0^* \equiv (W_0 - \Phi)/K_0^* - 1 = w^*\) immediately after the firm is set up. Note that \(w^*\) is independent of the fixed start-up cost \(\Phi\) and outside option value \(\Pi\).

Second, using the Euler’s theorem, we write \(P(K_0^*, W_0 - \Phi - K_0^*)\) as follows:

\[
P(K_0^*, W_0 - \Phi - K_0^*) = P_K^* \times K_0^* + P_W^* \times (W_0 - \Phi - K_0^*) = p'(w^*) (W_0 - \Phi), \tag{B.2}
\]
where the second equality follows from (B.1). For an entrepreneur, his certainty equivalent is
given by the marginal value of wealth \( p'(w^*) \) multiplied by \( (W_0 - \Phi) \), the agent’s initial wealth after paying the fixed start-up cost \( \Phi \). Intuitively, if \( J(K^*_0, W_0 - (\Phi + K^*_0)) > V(W_0 + \Pi) \) the agent chooses to become an entrepreneur immediately, otherwise he will choose his outside option. So the threshold level \( \bar{W} \) should satisfy \( J(K^*_0, \bar{W} - (\Phi + K^*_0)) = V(\bar{W} + \Pi) \) which gives (55).

**Theorem 3.** Before the agent becomes an entrepreneur, the wealth dynamics evolve as follows:

\[
dW_t = r(W_t - X_t)dt + \mu_R X_t dt + \sigma_R X_t dB_t + r \Pi dt - C_t dt.
\] (B.3)

Let \( F(W) \) and \( E(W) \) denote the agent’s value function and certainty equivalent wealth before becoming entering entrepreneurship. Using the standard principle of optimality for recursive utility (Duffie and Epstein (1992)), the following HJB equation holds:

\[
0 = \max_{C, X} f(C, F) + (rW + (\mu_R - r)X + r \Pi - C)F'(W) + \frac{\sigma_R^2 X^2}{2} F''(W). \quad (B.4)
\]

The FOCs for \( C \) and \( X \) are given by

\[
F'(W) = f_C(C, F), \quad (B.5)
\]

\[
X(W) = \frac{(r - \mu_R)F'(W)}{\sigma_R^2 F''(W)} = \frac{(\mu_R - r)E(W)E'(W)}{\sigma_R^2(\gamma E(W)^2 - E(W)E''(W))}. \quad (B.6)
\]

Using the conjectured value function (59) and simplifying, we obtain

\[
C(W) = b^{1-\psi} \zeta^\psi E(W)(E'(W))^{-\psi}, \quad (B.7)
\]

\[
f(C, F) = \frac{\zeta}{1 - \psi - 1} \left[ \frac{(bE(W))^{1-\gamma}(bE'(W))^{1-\psi}}{\zeta^{1-\psi}} - (bE(W))^{1-\gamma} \right]. \quad (B.8)
\]

Substituting these results into (B.4), we obtain

\[
0 = \frac{m^F E(W)(E'(W))^{1-\psi} - \psi E(W)}{\psi - 1} + r(W + \Pi)E'(W) + \frac{\eta^2}{2} \gamma E'(W)^2 - E(W)E''(W) \cdot (B.9)
\]

Finally, we solve for the optimal level of wealth \( W \) for the agent to become an entrepreneur: \( \hat{W} \). Using the standard optimal stopping argument, we use the following value
match and smooth-pasting conditions at $\hat{W}$, and obtain the following results:

\begin{align*}
F(\hat{W}) &= J(K^*, \hat{W} - \Phi - K^*), \quad \text{(B.10)} \\
F'(\hat{W}) &= J_w(K^*, \hat{W} - \Phi - K^*), \quad \text{(B.11)}
\end{align*}

Using the conjectures (58) and (59), we obtain the following conditions in terms of certainty equivalent wealth at $\hat{W}$:

\begin{align*}
E(\hat{W}) &= p'(w^*) \left( \hat{W} - \Phi \right), \quad \text{(B.12)} \\
E'(\hat{W}) &= p'(w^*), \quad \text{(B.13)}
\end{align*}

where $w^*$ is defined by (56). Finally, we have the absorbing condition: $E(-\Pi) = 0.$
Figure 10: The effects of elasticity of intertemporal substitution $\psi$. 
Figure 11: The effects of idiosyncratic volatility $\epsilon$. The correlation coefficient: $\rho = 0$. Other parameter values are the same as in the baseline model (Section 4).
Figure 12: The effects of the adjustment cost parameter $\theta$. 
Figure 13: The effects of the liquidation parameter $l$. 