Valuing Private Equity *

Morten Sorensen       Neng Wang       Jinqiang Yang

September 28, 2012

Abstract

To evaluate private equity (PE) performance, we solve a portfolio-choice model for a risk-averse investor (LP), investing in a PE fund, managed by a general partner (GP). Our model captures: illiquidity, non-diversifiable risk and incomplete markets, GP compensation, GP value added, and leverage. We derive tractable formulas for the certainty-equivalent valuation of the PE investment. Importantly, we find that costs of illiquidity and non-diversifiable risk are high and comparable to the costs of GP compensation. Interestingly, leverage reduces these costs. Our analysis suggests that conventional interpretations of PE performance measures are optimistic. On average, LPs may just break even.

Keywords: Private equity, alternative investments, illiquidity, portfolio choice, asset allocation, management fees, carried interest, incomplete markets.

JEL Classification: G11, G23, G24.

*We thank Andrew Ang, Ulf Axelson, Peter DeMarzo, Wayne Ferson, Larry Glosten, Stefan Hirth, Alexander Ljungqvist, Andrew Metrick, Berk Sensoy, René Stulz, Suresh Sundaresan, Mike Weisbach, Mark Westerfield, and seminar participants at AEA/ASSA 2012, Columbia Business School, EFA 2012, London Business School, Norwegian School of Economics, Ohio State University, and Stanford Graduate School of Business for helpful discussions and comments. Send correspondence to: Morten Sorensen, Columbia Business School and NBER, email: ms3814@columbia.edu; Neng Wang, Columbia Business School and NBER, email: neng.wang@columbia.edu; and Jinqiang Yang, Shanghai University of Finance and Economics (SUFE), email: yang.jinqiang@mail.shufe.edu.cn.
Institutional investors allocate substantial fractions of their portfolios to alternative investments. Yale University’s endowment targets a 63% allocation, with 34% to private equity, 20% to real estate, and 9% to natural resources. The California Public Employees’ Retirement System (CalPERS) allocates 14% of its $240B pension fund to private equity and 10% to real assets. More generally, allocations by public pension funds range from 0% for Georgia’s Municipal Retirement System, which is prohibited by law from making alternative investments, to 46% for the Pennsylvania State Employees’ Retirement System. At the sovereign level, China’s $482B sovereign wealth fund (CIC) recently reduced its allocation to public equity to 25%, which falls below its 31% allocation to alternative (“long-term”) investments. Given the magnitude and diversity of these allocations, it is clearly important to understand the economic value and performance of alternative investments compared to traditional, traded stocks and bonds. This study focuses on private equity (PE) investments, specifically investments by a limited partner (LP) in a PE fund, including buyout (BO), venture capital (VC), and real estate funds. Similar issues arise for investments in infrastructure, natural resources, and other alternative assets.

To value PE investments and evaluate their performance, we develop a model of the LP’s portfolio-choice problem that captures four key institutional features of PE investments. First, PE investments are illiquid and long term. PE funds have ten-year maturities and the secondary market for PE positions is opaque, making it difficult for LPs to rebalance their PE investments. Second, PE investments are risky. Part of this risk is spanned by publicly-traded liquid assets and hence commands the standard risk premium for systematic risk exposure. The combination of the remaining unspanned risk and illiquidity means that markets are incomplete and induces the LP to demand an additional premium. Third, the management of the PE fund is delegated to a general partner (GP), who receives both an annual management fee, typically 1.5%–2% of the committed capital, and a performance-based incentive fee (carried interest), typically 20% of profits. Intuitively, management fees resemble a fixed-income stream and the carried interest

---


2See Gompers and Lerner (2002) and Metrick and Yasuda (2010, 2011) for detailed discussions of the institutional features of these investments.
resembles a call option. Fourth, to compensate the LP for bearing the unspanned illiquidity risk as well as management and performance fees, the GP must generate sufficient excess return (alpha) by effectively managing the fund’s assets.

Our model delivers a tractable solution and intuitive expression for the LP’s certainty-equivalent valuation of the PE investment. When markets are incomplete, the standard law-of-one-price valuation framework does not apply. Instead, we derive a non-linear differential equation for the certainty-equivalent valuation, and obtain analytical solutions for the optimal hedging portfolio and consumption rules. Unlike the standard Black-Scholes (1973) formula, our framework incorporates alpha, management fees, carried interest, and the non-linear pricing of unspanned illiquidity risk. However, as an important special case when markets are complete, with no management fees, no carried interest, and no alpha, our model recovers the Black-Scholes formula.

We calibrate the model and use the certainty-equivalent valuation to infer the alpha that the GP must generate for the LP to break even. Break-even alphas range from 2.61% to 3.08% in our baseline calibration with the typical 2/20 compensation contract and no leverage. Surprisingly, we find that leverage reduces the (unlevered) break-even alpha. Axelson, Jenkinson, Stromberg, and Weisbach (2011) report a historical average debt to equity (D/E) ratio of 3.0 for BO transactions. In our baseline calibration, increasing the D/E ratio to 3.0 reduces the break-even alpha to 1.00%–2.05%. The benefits of leverage are twofold: First, for a given size of the LP’s investment, leverage increases the total size of PE assets for which the GP generates alpha, effectively reducing the fees per dollar of unlevered assets. Second, leverage allows better-diversified creditors to bear some of the risks of the unlevered PE asset. The cost of leverage is that it increases the risk and volatility of the LP’s (levered) claim, because the PE investment is junior to the creditors. In our calibrations, the positive effects dominate. This may provide an answer to the “PE leverage puzzle” from Axelson, Jenkinson, Stromberg, and Weisbach (2011). They find that the credit market is the primary predictor of leverage used in PE transactions, and that PE funds appear to use as much leverage as tolerated by the market.\(^3\) This behavior is inconsistent with standard theories of

\(^3\)In their conclusion, Axelson, Jenkinson, Stromberg, and Weisbach (2011) state that “the factors that predict capital structure in public companies have no explanatory power for buyouts. Instead, the main factors that do affect the capital structure of buyouts are the price and availability of debt; when credit is abundant and cheap, buyouts become more leveraged […] Private equity practitioners often state that they use as much leverage as they can to maximize the expected returns on each deal. The main constraint they face, of course, is the capital market, which
capital structure (see also Axelson, Stromberg, and Weisbach 2009). In our model it is optimal.

Finally, our model produces tractable expressions for the performance measures used in practice. Given the difficulties of estimating traditional risk and return measures such as CAPM alphas and betas, several alternative measures have been adopted such as the Internal Rate of Return (IRR), Total Value to Paid-In capital (TVPI) multiple, and Public Market Equivalent (PME). While these alternative measures are easier to compute, they are more difficult to interpret. Harris, Jenkinson, and Kaplan (2011) report a value-weighted average PME of 1.27 and conclude that “buyout funds have outperformed public markets in the 1980s, 1990s, and 2000s.” Whether or not this outperformance is sufficient to compensate LPs for the illiquidity and other frictions can be evaluated within our model. Given the break-even alpha, we calculate the corresponding break-even values of the IRR, TVPI, and PME measures. We find that these break-even values are close to their empirical counterparts. Our baseline calibration gives a break-even PME of 1.30, suggesting that the empirical average of 1.27 is just sufficient for LPs to break even on average. While the exact break-even values depend on the specific calibration, the general message is that the traditional interpretation of these performance measures may be misleading.

The closest work is Metrick and Yasuda (2010) who calculate present values of the different parts of the GP’s compensation, including management fees, carried interest, and the hurdle rate. Several other empirical studies also evaluate PE performance. Ljungqvist and Richardson (2003) use detailed cash flow information to document the draw down and capital return schedules for PE investments and calculate their excess return. Kaplan and Schoar (2005) analyze the persistence of PE performance and returns, assuming a beta of one. Cochrane (2005) and Korteweg and Sorensen (2011) estimate the risk and return of VC investments in a CAPM model after adjusting for sample selection. Gompers, Kovner, Lerner, and Scharfstein (2008) investigate the cyclicality of VC investments and public markets. Phalippou and Gottschalg (2009) evaluate reporting and limits at any particular time how much private equity sponsors can borrow for any particular deal.”

As explained below, the PME is calculated by dividing the present value (PV) of the cash flows distributed to the LP by the PV of the cash flows paid by the LP, where the PV is calculated using the realized market return as the discount rate. A PME exceeding one is typically interpreted as outperformance relative to the market.

Kaplan and Schoar (2005) find substantial persistence in the performance of subsequent PE funds managed by the same PE firm, indicating that PE firms differ in their quality and ability to generate returns. Lerner, Schoar, and Wongsunwai (2007) find systematic variation in PE performance across LP types, suggesting that LPs differ in their ability to identify and access high-quality PE firms. Hence, some specific LPs may consistently outperform (or underperform) the average.

Our analysis also relates to the literature about valuation and portfolio choice with illiquid assets, such as restricted stocks, executive compensation, non-traded labor income, illiquid entrepreneurial businesses, and hedge fund lock-ups. For example, Svensson and Werner (1993), Duffie, Fleming, Soner, and Zariphopoulou (1997), Koo (1998), and Viceira (2001) study consumption and portfolio choices with non-tradable labor income. Kahl, Liu, and Longstaff (2003) analyze a continuous-time portfolio choice model with restricted stocks. Chen, Miao, and Wang (2010) and Wang, Wang, and Yang (2012) study entrepreneurial firms under incomplete markets. For hedge funds, Goetzmann, Ingersoll, and Ross (2003), Panageas and Westerfield (2009), and Lan, Wang, and Yang (2012) analyze the impact of management fees and high-water mark based incentive fees on leverage and valuation. Ang, Papanikolaou, and Westerfield (2012) analyze a model with an illiquid asset that can be traded and rebalanced at Poisson arrival times. We are unaware, though, of any existing model that fully captures the illiquidity, unspanned risk, managerial skill (alpha) and compensation features of PE investments. Capturing these institutional features in a model that is sufficiently tractable to evaluate actual PE performance is a main contribution of this study.

1 Model

An institutional investor (LP) with an infinite horizon invests in three assets: the risk-free asset, public equity, and private equity. The risk-free asset pays a constant interest rate, \( r \). Public equity can be interpreted as a position in the public market portfolio. Its value, \( S \), follows the geometric Brownian motion (GBM):

\[
\frac{dS_t}{S_t} = \mu S dt + \sigma S dB_t^S,
\] (1)
where $B_t^S$ is a standard Brownian motion, and $\mu_S$ and $\sigma_S$ are the constant drift and volatility parameters. The Sharpe ratio is:

$$\eta = \frac{\mu_S - r}{\sigma_S}.$$  

(2)

### 1.1 PE investment

The PE investment is a one-shot investment for the LP. At time 0, the LP makes an initial PE investment $I_0$ (this is not the “committed capital,” $X_0$, as defined below). The GP leverages $I_0$ with external debt of $D_0$ to acquire $A_0 = I_0 + D_0$ worth of PE assets. Let $l = D_0/I_0$ denote the D/E ratio. In Section 7, we consider optimal choices of $I_0$ and $l$. For now, take them as given. The distinction between the PE investment and the PE asset is important. The PE investment is the LP’s investment in the PE firm, including subsequent management fees and performance fees paid to the GP. The PE assets represent the total amount of underlying (unlevered) corporate assets owned by the PE fund.\(^6\)

**PE asset.** The PE asset is illiquid and must be held to maturity, $T$ (typically ten years). Between times 0 and $T$, its value follows the GBM,

$$\frac{dA_t}{A_t} = \mu_A dt + \sigma_A dB_t^A,$$

(3)

where $B_t^A$ is a standard Brownian motion, $\mu_A$ is the drift, and $\sigma_A$ is the volatility. At time $T$, the PE asset is liquidated for total proceeds of $A_T$, which are divided between the LP and GP, as specified below. After time $T$, the LP only invests in the market portfolio and the risk-free asset, reducing the problem to the standard Merton (1971) portfolio problem. We can interpret $A_t$ as the “mark-to-market” value. It is the value of the PE asset if it were publicly traded at time $t$. This “mark-to-market” value differs from the LP’s economic value of the PE asset for several reasons, including the cost of illiquidity and the benefit of the value added by the GP over the remaining life of the PE investment.

\(^6\)In reality, PE funds have several LPs, which typically share the value of the fund pro rata. We can interpret the LP in the model as representing the aggregate collection of LPs. Alternatively, we can interpret $A_0$ as a given LP’s share of the total fund. In other words, if the total fund is $\overline{A}_0$, and a given LP owns the share $s$ of the total fund, then $A_0 = s\overline{A}_0$. 

5
**PE risk.** The correlation between $B^S_i$ and $B^A_i$ is denoted $\rho$. When $|\rho| < 1$, the PE risk is not fully spanned, markets are incomplete, and the LP cannot fully hedge the PE investment by dynamically trading the public market portfolio and risk-free asset.

We decompose the total volatility of the PE asset, $\sigma_A$, into the volatility spanned by the public market portfolio, $\rho \sigma_A$, and the unspanned volatility, given as:

$$ \epsilon = \sqrt{\sigma_A^2 - \rho^2 \sigma_A^2}. \quad (4) $$

We define the (unlevered) beta of the PE asset relative to the public market portfolio as:

$$ \beta = \frac{\rho \sigma_A}{\sigma_S}, \quad (5) $$

and rewrite the unspanned volatility as:

$$ \epsilon = \sqrt{\sigma_A^2 - \beta^2 \sigma_S^2}. \quad (6) $$

Unspanned volatility introduces an additional risk into the LP’s overall portfolio. The spanned component of the volatility, $\rho \sigma_A$, and the unspanned part, $\epsilon$, play distinct roles in the LP’s certainty-equivalent valuation, and the LP requires different premia, derived below, for bearing these risks.

**PE alpha.** Our analysis allows GP to add value in two ways. First, the GP may manage the PE asset efficiently, causing it to appreciate faster and generate an excess return (alpha) relative to the market. Second, the GP may acquire the PE asset at a discount relative to its fair market value, generating a “one-time” alpha. We postpone the analysis of the one-time alpha to Section 6, and focus on the normal alpha in the baseline specification of the model.\(^7\) The excess return (alpha) of the PE asset with respect to the public market portfolio is defined as:

$$ \alpha = \mu_A - r - \beta (\mu_S - r). \quad (7) $$

\(^7\)Our model also allows the GP to add value by leveraging the PE asset with “cheap” debt. In this analysis, however, we only consider debt priced in equilibrium. Ivashina and Kovner (2010) provide empirical evidence of cheap debt financing of PE transactions.
We interpret alpha as a measure of the GP’s managerial skill. With appropriate data, the alpha and beta can be estimated by regressing the excess returns of the PE asset on the excess returns of the public market portfolio. Note that alpha and beta are defined relative to the public market portfolio and not to the LP’s entire portfolio, which also contains the PE investment. Empirical studies of PE performance measure PE risks relative to the public market portfolio in this way, and adopting this definition allows us to use existing estimates in our calibration. Defining alphas and betas relative to the total portfolio, containing both public and private equity, is impractical because the aggregate value of PE assets is difficult to estimate and may require a different valuation framework, as indicated by our analysis.

1.2 GP compensation

The GP receives ongoing management fees and performance-based carried interest. The annual management fee is specified as a fraction \( m \) (typically 2\%) of committed capital. Committed capital, denoted \( X_0 \), is the sum of total (not discounted) management fees paid over the life of the PE investment and the initial investment, \( I_0 \), given as:

\[
X_0 = mT X_0 + I_0.
\]  

For example, committed capital of \( X_0 = \$125 \) and \( m = 2\% \) of management fees imply an annual fee of \$2.5. Over ten years, total management fees are \$25, leaving \( I_0 = \$100 \) for the initial investment. With leverage of \( l = 3 \), this initial investment enables the GP to acquire \( A_0 = \$400 \) worth of the underlying PE assets. Leverage allows the GP to manage more assets per dollar of management fees charged. Without leverage, the GP would charge an annual fee of 2.5\% \( (= \$25/\$100) \) of PE assets under management. With \( l = 3 \), this fee declines to 0.625\% \( (= \$25/\$400) \).

In addition to management fees, the GP receives carried interest. The carried interest is performance based, and defined by a schedule known as the “waterfall.” The LP’s payoff is illustrated in Figure 1, and the regions of the waterfall are given as follows.
Region 0: Debt Repayment ($A_T \leq Z_0$). Our model applies to general forms of debt, but for simplicity we consider balloon debt with no intermediate payments. The principal and accrued interest are due at maturity $T$. The debt is risky. Let $y$ denote the yield for the debt, which we derive below to ensure creditors break even. At maturity $T$, the payment to the lender is:

$$D(A_T, T) = \min \{A_T, Z_0\},$$

where $Z_0 = D_0 e^{yT}$ is the sum of principal and compound interest. Until the debt is fully repaid, the LP and GP collect nothing.

Region 1: Preferred Return ($Z_0 \leq A_T \leq Z_1$). After the debt is repaid, the LP receives the entire proceeds until the committed capital has been returned, possibly with a preferred (“hurdle”) return, $h$ (typically, 8%). The LP’s required amount, $F$, equals:

$$F = I_0 e^{hT} + \int_0^T mX_0 e^{hs} ds = I_0 e^{hT} + \frac{mX_0}{h} (e^{hT} - 1).$$
Without a hurdle, \( F = X_0 \), and the LP receives just the committed capital in this region. Given \( F \), the boundary for this region is \( Z_1 = F + Z_0 \), and the LP’s payoff is:

\[
LP_1(A_T, T) = \max \{ A_T - Z_0, 0 \} - \max \{ A_T - Z_1, 0 \}. \tag{11}
\]

**Region 2: Catch-Up** \((Z_1 \leq A_T \leq Z_2)\). To catch up, the next region awards the GP a substantial fraction, \( n \) (typically 100\%), of subsequent proceeds as carried interest. This region lasts until the GP’s carried interest equals a given share, \( k \) (typically, 20\%), of total profits. The boundary, \( Z_2 \), is defined as the amount of proceeds where the GP fully catches up, given by:

\[
k(Z_2 - X_0 - Z_0) = n(Z_2 - Z_1). \tag{12}
\]

The left-hand side is the GP’s share of total profits, and the right-hand side is the amount of carried interest received by the GP. Note that without a hurdle, the LP does not receive any part of the profits in the preferred return region, hence there is nothing for the GP to catch up on and the catch-up region disappears. The LP receives the residual cash flow, resembling a \((1 - n)\) share of mezzanine debt,\(^8\) given as:

\[
LP_2(A_T, T) = (1 - n) \left[ \max \{ A_T - Z_1, 0 \} - \max \{ A_T - Z_2, 0 \} \right]. \tag{13}
\]

**Region 3: Profit Sharing** \((A_T > Z_2)\). After catching up, the GP’s carried interest is simply the profit share, \( k \) (typically 20\%). Hence, the LP’s incremental payoff in this last region resembles a junior equity claim, given as:

\[
LP_3(A_T, T) = (1 - k) \max \{ A_T - Z_2, 0 \}. \tag{14}
\]

**Capital stack.** As illustrated in Figure 2, we can view the LP’s claim as consisting of three tranches, corresponding to regions 1 to 3 (the LP receives nothing in region 0) of the waterfall: (1)

\(^8\)PE funds usually have catch-up rates of \( n = 100\% \), leaving nothing for the LP in the catch-up region. For generality, we allow for \( n < 100\% \) in the analysis, even if it is rare in PE partnerships. Real estate partnerships commonly use a catch-up rate of \( n = 80\% \).
the preferred return region, corresponding to a senior claim; (2) the catch-up region, corresponding
to a mezzanine claim; and (3) the profit-sharing region, corresponding to a junior equity claim.
The LP’s total payoff, at maturity $T$, is the sum of the incremental payoffs earned in each of the
tranches:

$$LP(A_T, T) = LP_1(A_T, T) + LP_2(A_T, T) + LP_3(A_T, T), \quad (15)$$

where $LP_1(A_T, T)$, $LP_2(A_T, T)$, and $LP_3(A_T, T)$ are the LP’s incremental payoffs in the corre-
spanding regions, as defined above.

1.3 LP’s problem

Objective. The LP has standard time-additive separable utility, given by:

$$E \left[ \int_0^\infty e^{-\zeta t} U(C_t) \, dt \right], \quad (16)$$
where \( \zeta > 0 \) is the subjective discount rate and \( U(C) \) is a concave function. For tractability, we choose \( U(C) = -e^{-\gamma C}/\gamma \), where \( \gamma > 0 \) is the coefficient of absolute risk aversion (CARA).

**Liquid wealth dynamics.** We use \( W_t \) to denote the LP’s liquid wealth process, excluding the value of the PE investment. The LP allocates \( \Pi_t \) to risky public equity and the remaining \( W_t - \Pi_t \) to the risk-free asset. During the life of the PE investment, the liquid wealth evolves as:

\[
dW_t = (rW_t - mX_0 - C_t) dt + \Pi_t ((\mu_S - r)dt + \sigma_S dB_t^S), \quad t < T. \tag{17}
\]

The first term is the wealth accumulation when the LP is fully invested in the risk-free asset, net of management fees, \( mX_0 \), and consumption/expenditure, \( C_t \). The second term is the excess return from investing in public equity.

At time \( T \), when the PE asset is liquidated, the LP’s liquid wealth jumps from the pre-exit amount of \( W_{T-} \) to:

\[
W_T = W_{T-} + LP(A_T, T), \tag{18}
\]

where \( LP(A_T, T) \) is the LP’s payoff (net of fees) given in (15). After exit, the LP’s liquid wealth process is:

\[
dW_t = (rW_t - C_t) dt + \Pi_t ((\mu_S - r)dt + \sigma_S dB_t^S), \quad t \geq T. \tag{19}
\]

### 2 Solution

After the PE investment matures, the LP is left investing in public equity and the risk-free asset, reducing the problem to the Merton (1971) consumption/portfolio allocation problem. The solution to this problem is summarized in Proposition 1.

**Proposition 1** The LP’s post-exit value function is:

\[
J^* (W) = -\frac{1}{\gamma r} e^{-\gamma r (W+b)}, \tag{20}
\]
where $b$ is a constant,

$$b = \frac{\eta^2}{2\gamma r^2} + \frac{\zeta - r}{\gamma r^2}.
\tag{21}$$

**Optimal consumption, $C$, is**

$$C = r (W + b),
\tag{22}$$

and the optimal allocation to public equity, $\Pi$, is

$$\Pi = \frac{\eta}{\gamma r \sigma S}.
\tag{23}$$

To solve the LP’s problem before the PE investment matures, let $J(W, A, t)$ be the LP’s value function. Given $J^*$ from Proposition 1, this value function can be written as:

$$J(W_0, A_0, 0) = \max_{C, \Pi} \mathbb{E} \left[ \int_0^T e^{-\zeta t} U(C_t) \, dt + e^{-\zeta T} J^*(W_T) \right].
\tag{24}$$

**Certainty-equivalent valuation.** The LP’s optimal consumption and optimal allocation to risky public equity solve the Hamilton-Jacobi-Bellman (HJB) equation,

$$\zeta J(W, A, t) = \max_{\Pi, C} \left[ U(C) + J_t + (rW + \Pi (\mu_S - r) - mX_0 - C) J_W
\right.
\left. + \frac{1}{2} \Pi^2 \sigma_S^2 J_{WW} + \mu_A A J_A + \frac{1}{2} \sigma_A^2 A^2 J_{AA}
\right]
\tag{25}$$

In the Appendix, we show that the solution takes the exponential form,

$$J(W, A, t) = -\frac{1}{\gamma r} \exp \left[ -\gamma r (W + b + V(A, t)) \right],
\tag{26}$$

where $b$ is given in (21), and $V(A, t)$ is the LP’s certainty-equivalent valuation of the PE investment. We further show that $V(A, t)$ solves the partial differential equation (PDE),

$$rV(A, t) = -mX_0 + V_t + (r + \alpha) AV_A + \frac{1}{2} \sigma^2 A^2 V_{AA} - \frac{\gamma r}{2} \epsilon^2 A^2 V_A^2,
\tag{27}$$
where $\alpha$ is given by (7), and $\epsilon$ is the unspanned risk given in (6). This PDE is non-linear. The illiquidity premium is captured by the last term, which involves $V_A^2$, invalidating the standard law-of-one-price valuation. An LP with certainty-equivalent valuations of two individual PE investments of $V_1$ and $V_2$, as valued in isolation, may not value the portfolio with both investments as $V_1 + V_2$. This represents an important departure from the seminal Black-Scholes option pricing formula, which remains a linear PDE, despite the nonlinear payoff structure of call options.

The PDE (27) is solved subject to the following two boundary conditions. First, at maturity $T$, the LP’s total payoff is:

$$V(A_T, T) = LP(A_T, T), \quad (28)$$

where $LP(A_T, T)$ is given in (15). Second, when the value of the PE asset tends to zero, the valuation tends to the (negative) present value of the remaining management fees,

$$V(0, t) = \int_t^T e^{-r(T-s)}(-mX_0) ds = -\frac{mX_0}{r} \left(1 - e^{-r(T-t)}\right). \quad (29)$$

The LP must honor the remaining management fees regardless of the fund’s performance, and the resulting liability is effectively a risk-free annuity.

Note the distinction between the “mark-to-market” valuation, $A_t$, and the LP’s certainty-equivalent valuation, $V(A_t, t)$. For accounting purposes, investors generally agree on $A_t$, whereas different LPs may assign different $V(A_t, t)$ valuations to the same investment, due to LP-specific risk aversion and illiquidity discounts.

**Consumption and portfolio rules.** The LP’s optimal consumption rule is:

$$C(W, A, t) = r(W + V(A, t) + b), \quad (30)$$

wealth $W$ and the certainty equivalent of the PE investment $V(A,t)$.

The LP’s optimal allocation to the public market portfolio is:

$$\Pi(A,t) = \frac{\eta}{\gamma \sigma_S} - \beta AV_A(A,t).$$

(31)

The first term is the standard mean-variance term from (23). The second term is the intertemporal hedging demand with the unlevered $\beta$ of the PE asset given by (5). In option pricing terminology, we can view $V_A(A,t)$ as the “delta” of the LP’s valuation with respect to the value of the underlying PE asset. Greater values of $\beta$ and $V_A(A,t)$ create a larger hedging demand.

**Break-even alpha.** Following the initial investment, $I_0$, the LP assumes the liability of the ongoing management fees and receives a claim on the proceeds at maturity. Since the certainty equivalent, $V(A_0,0)$, values the LP’s final proceeds net of carried interest and management fees, the LP will voluntarily invest when $V(A_0,0) > I_0$. The LP breaks even, in certainty-equivalence terms, net of fees and accounting for both systematic and unspanned illiquidity risks, when

$$V(A_0,0) = I_0.$$  

(32)

The certainty-equivalent valuation is increasing in alpha, and we define the break-even alpha implicitly as the alpha that solves (32). The break-even alpha reflects the opportunity cost of capital of the PE investment. When the actual alpha exceeds the break-even alpha, the PE investment has positive economic value for the LP investor.

**Debt pricing.** Debt is priced from the perspective of dispersed risk-averse lenders. Dispersed lenders, each holding a vanishing fraction of the total debt, require no compensation for illiquidity, even when the debt is illiquid and must be held to maturity.10 Pricing the debt, however, still requires extending standard debt pricing models because the underlying PE asset earns excess

10Ivashina and Kovner (2010) report that the average (median) transaction involves total debt of $321M ($136M), which is syndicated to 7.0 (4) lenders, typically banks, leaving each individual loan as a small share of the lender’s total balance sheet.
risk-adjusted return (alpha). We show in the Appendix that the debt is priced by:

$$ rD(A, t) = D_t(A, t) + (r + \alpha)AD_A(A, t) + \frac{1}{2}\sigma_A^2 A^2 D_{AA}(A, t), \quad (33) $$

subject to the boundary condition (9). Despite the resemblance, this formula differs from the standard Black-Scholes-Merton pricing formula. Our model allows for a positive alpha, and the risk-adjusted drift is $r + \alpha$. For notational simplicity, let $FS(A_t, t; K)$ denote the time-$t$ value of a European call option on the underlying PE asset with strike price $K$. (“FS” refers to the full spanning case, analyzed below.) In the Appendix, we show that:

$$ FS(A_t, t; K) = e^{\alpha(T-t)}A_tN(q_1(A_t, t; K)) - Ke^{-r(T-t)}N(q_2(A_t, t; K)), \quad (34) $$

where:

$$ q_1(A_t, t; K) = q_2(A_t, t; K) + \sigma_A\sqrt{T - t}, \quad (35) $$
$$ q_2(A_t, t; K) = \frac{\ln(A_t/K) + (r + \alpha - \sigma_A^2/2)(T - t)}{\sigma_A\sqrt{T - t}}. \quad (36) $$

Due to the alpha, the “forward-looking present value” of the PE asset, at time $t$, is $A_te^{\alpha(T-t)}$, reflecting the time-$t$ value of compounding alpha for $(T - t)$ periods.11 As in the Black-Scholes formula, the risk-adjusted expected return on the debt equals the risk-free rate.

With $D_0$ as the initial principal of the debt, the debt’s initial market value is:

$$ D(A, 0) = e^{\alpha T}A - FS(A, 0; D_0e^{yT}), \quad (37) $$

The equilibrium yield, $y$, is defined as the solution to $D(A, 0) = D_0$, which ensures that the lenders break even.

---

11While the expression looks like a future value, it is the time-$t$ risk-adjusted present value, with the value of the asset compounded at rate $\alpha + r$ and subsequently discounted back at the rate $r$ under the risk-adjusted measure. That is, we have $E_t[A_{Tt}e^{-r(T-t)}] = e^{\alpha(T-t)}A_t$. 

15
3 Full Spanning

Consider first the special case where the PE risk is spanned by the public markets. Hence, $|\rho| = 1$ and $\epsilon = 0$. While markets are complete with respect to the PE risk, the PE asset still earns an excess return, $\alpha$. This is possible in equilibrium when the GPs’ alpha-generating skills are scarce, and the LP’s investment must be intermediated by the GP in order to obtain the alpha. With many LPs relative to GPs, we would expect that GPs extract this surplus as a rent to their scarce talent, leaving LPs breaking even (see Berk and Green 2004). \(^{12}\)

Closed-form solution. With complete spanning, the non-linear term in (27) disappears, and the PDE for $V(A,t)$ simplifies to:

$$ rV(A,t) = -mX_0 + V_t + (r + \alpha) AV_A + \frac{1}{2} \sigma_A^2 A^2 V_{AA} . $$ (38)

Compared to the standard Black-Scholes-Merton formula, the risk-adjusted drift changes from $r$ to $r + \alpha$, reflecting the manager’s skill and its impact on the value and risk-adjusted growth rate of the PE asset. As before, the term $-mX_0$ captures management fees, and the boundary conditions remain unchanged.

Unlike the incomplete-markets case where $|\rho| < 1$, as analyzed below, the PDE in equation (38) is a linear differential equation, and it admits a closed-form solution. We can separately value the incremental payoff in each of the tranches (see Figure 2), and the LP’s time-$t$ present values of these payoffs are given as:

$$ PV_1(A_t,t) = FS(A_t,t;Z_0) - FS(A_t,t;Z_1), $$ (39)

$$ PV_2(A_t,t) = (1-n)(FS(A_t,t;Z_1) - FS(A_t,t;Z_2)), $$ (40)

$$ PV_3(A_t,t) = (1-k)FS(A_t,t;Z_2). $$ (41)

\(^{12}\)Kaplan and Schoar (2005) find evidence of performance persistence for subsequent PE funds, suggesting that GPs do not extract the full surplus and leave some rents for the LPs. Hochberg, Ljungqvist, and Vissing-Jorgensen (2010) and Glode and Green (2011) present models where delegated investment managers, such as GPs or hedge fund managers, are unable to extract the full rent to their skills due to informational frictions.
The LP’s total valuation is:

\[ V(A_t, t) = PV_1(A_t, t) + PV_2(A_t, t) + PV_3(A_t, t) - \frac{mX_0}{r} \left( 1 - e^{-r(T-t)} \right), \]  

where the last term values the remaining management fees.

4 Incomplete Markets

With incomplete markets, the PDE in (27) for the LP’s valuation of the PE investment is straightforward to solve numerically. Where possible, we use parameters from Metrick and Yasuda (2010) for our baseline case. All parameters are annualized when applicable. Metric and Yasuda use a volatility of 60% per individual BO investment, with a pairwise correlation of 20% between any two BO investments, and report that the average BO fund invests in around 15 BOs (with a median of 12). From these figures we calculate a volatility of 25% for the total PE asset. Like Metrick and Yasuda, we use a risk-free rate of 5%.

For leverage, Axelson, Jenkinson, Stromberg, and Weisbach (2011) consider 153 BOs during 1985–2006, and report that, on average, equity accounted for 25% of the purchase price, corresponding to \( l = 3 \) in our model. For the compensation contract, we focus on the 2/20 contract (2% annual management fee and 20% carried interest) with an 8% hurdle rate, which is widely adopted by PE funds (see Metrick and Yasuda 2010), although we also consider typical deviations from these contract terms.

We set the unlevered beta of the PE asset to 0.5. This is consistent with evidence from Ljungqvist and Richardson (2003), who match companies involved in PE transactions to publicly-traded companies. They report that the average (levered) beta of the publicly-traded comparables is 1.04, implying that PE funds invest in companies with average systematic risk exposures. Since publicly-traded companies are typically financed with approximately one-third debt, the unlevered beta is around 0.66. We round this figure down, for an unlevered beta of 0.5, although we consider other levels of systematic risk below.

For the market parameters, we set the volatility of the market portfolio to \( \sigma_S = 20\% \), with
an expected return of $\mu_S = 11\%$, implying a risk premium of $\mu_S - r = 6\%$ and a Sharpe ratio of $\eta = 30\%$. These parameters imply a correlation between the PE asset and the market portfolio of $\rho = \beta \sigma_S / \sigma_A = 0.4$.

To determine reasonable values of the LP’s absolute risk aversion, $\gamma$, and the initial investment, $I_0$, we derive the following invariance result:

**Proposition 2** Define $a = A/I_0$, $x_0 = X_0/I_0$, $z_0 = Z_0/I_0$, $z_1 = Z_1/I_0$, and $z_2 = Z_2/I_0$. It is straightforward to verify that $V(A,t) = v(a,t) \times I_0$, where $v(a,t)$ solves the ODE,

$$rv(a,t) = -mx_0 + v_t + (r + \alpha) av_a(a,t) + \frac{1}{2} \sigma^2 a^2 v_{aa}(a,t) - \frac{\gamma I_0}{2} r e^2 a^2 v_a(a,t)^2, \quad (43)$$

subject to the boundary conditions,

$$v(a,T) = \max\{a - z_0, 0\} - n \max\{a - z_1, 0\} + (n - k) \max\{a - z_2, 0\}, \quad (44)$$
$$v(0,t) = -\frac{mx_0}{r} \left(1 - e^{-r(T-t)}\right). \quad (45)$$

The $v(a,t)$ is the LP’s certainty-equivalent valuation per dollar initially invested. Proposition 2 shows that $v(a,t)$ depends only on the product $\gamma I_0$, not on $\gamma$ and $I_0$ individually. In other words, the LP’s certainty-equivalent valuation $V(A,t)$ is proportional to the invested capital $I_0$, holding $\gamma I_0$ constant.

Given the invariance result, $\gamma I_0$ can be approximated as follows. Let $\gamma_R$ denote relative risk aversion. By definition, $\gamma_R = \gamma C_t$. Substituting the expressions in equations (30) and (21) for $C_t$ and $b$, and assuming that the LP’s time preference equals the risk-free rate ($\zeta = r$) gives:

$$\gamma_R = \gamma r (W_t + V(A,t) + b) = \gamma I_0 r \frac{W_t + V(A,t)}{I_0} + \frac{\eta^2}{2r}. \quad (47)$$

Approximating $V(A,t)$ with $V(A_0,0)$ and $W_t$ with $W_0$, we get:

$$\gamma I_0 = \frac{\gamma_R - \frac{\eta^2}{2r}}{r} \left(\frac{I_0}{W_0 + V(A,0)}\right). \quad (48)$$
With this approximation, $\gamma I_0$ is determined by the LP’s initial PE allocation (in parentheses) and relative risk aversion, $\gamma_R$. Informally, we interpret the resulting CARA preferences as a local approximation to the CRRA preferences implied by $\gamma_R$. We interpret $\gamma I_0$ as the LP’s effective risk aversion. An LP with larger relative risk aversion or greater PE exposure has greater effective risk aversion. When the allocation tends to zero or the preferences tend to risk neutral, the effective risk aversion tends to zero. With $\eta = 30\%$, $r = 5\%$, $\gamma_R = 2$, and assuming a PE allocation of $I_0/(W_0 + V(A,0)) = 25\%$, we get $\gamma I_0 = 5.5$. Correspondingly, we consider three levels of effective risk aversion: $\gamma I_0 \to 0_+$ for an effectively risk-neutral LP,¹³ a “moderate” effective risk aversion of $\gamma I_0 = 2$, and a “high” effective risk aversion of $\gamma I_0 = 5$.

Cost of fees and illiquidity. Tables 1 presents break-even alphas calculated for various levels of effective risk aversion and leverage. The baseline cases assume an unlevered beta of 0.5, management fees of $m = 2\%$, carried interest of $k = 20\%$, and a hurdle of $h = 8\%$, as discussed above. The first row of Table 1 shows break-even alphas for an LP with $\gamma I_0 = 0_+$, which is effectively risk neutral (corresponding to the full spanning case analyzed above). An effectively risk-neutral LP requires no premium for illiquidity and unspanned risk and the reported break-even alphas of 2.61\% (with $l = 0$) and 1.00\% (with $l = 3$) reflect the opportunity cost of just management fees and carried interest.

Moving down in Table 1, as $\gamma I_0$ increases either because of an increase in the LP’s risk aversion or a greater PE allocation, the required break-even alpha increases as well, reflecting the increasing cost of illiquidity and unspanned risk. Without leverage, though, this increase is modest. In the first column of Table 1, the break-even alpha increases from 2.61\% to 3.08\% and 3.74\% when $\gamma I_0$ increases from 0_+ to 2 and 5. With leverage, the increase is more substantial, because leverage increases the risk and volatility of the PE investment, and the break-even alpha increases from 1.00\% to 2.05\% and 3.33\%. Hence, with a moderate effective risk aversion ($\gamma I_0 = 2$), the costs of illiquidity and unspanned risk are similar to the combined costs of management fees and carried

---

¹³Our model does not allow the LP to be risk neutral ($\gamma = 0$). Since public equity yields a higher expected rate of return than the risk-free rate, a risk-neutral agent would hold an infinite position in the public market portfolio. The limiting solution for $\gamma \to 0_+$ remains valid, though, and we denote the corresponding limit of the effective risk aversion as $\gamma I_0 = 0_+$. In this case, the LP is effectively risk neutral and the required illiquidity premium disappears. The model solution when $\gamma = 0_+$ is the same as in the full-spanning case. Technically, the PDE (27) becomes linear and identical to the one for the full-spanning case.
Table 1: Break-even alphas for different levels of effective risk aversion and leverage. Other parameter values are $\beta = 0.5$, $k = 0.2$, $m = 2\%$, and $h = 8\%$. The baseline case is in bold.

<table>
<thead>
<tr>
<th>$\gamma I_0$</th>
<th>$l = 0$</th>
<th>$l = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0_+$</td>
<td>2.61%</td>
<td>1.00%</td>
</tr>
<tr>
<td>2</td>
<td>3.08%</td>
<td>2.05%</td>
</tr>
<tr>
<td>5</td>
<td>3.74%</td>
<td>3.33%</td>
</tr>
</tbody>
</table>

For a high level of effective risk aversion ($\gamma I_0 = 5$), the cost of illiquidity is more than three times the combined costs of management fees and carried interest.

Leverage substantially reduces the break-even alpha. Given the size of the LPs investment, $I_0$, the main advantage of increasing leverage is that it increases the total amount of PE assets, $A_0$, managed by the GP, enabling the GP to earn alpha on this larger asset base. As a secondary effect, holding the total amount of the PE assets constant, leverage is still beneficial, because it transfers risk to creditors, who are better diversified. Hence, the creditors do not demand the same illiquidity risk premium as the LP demands, and they have a lower cost of capital. The cost of leverage is that it increases both the idiosyncratic and systematic risks faced by the risk-averse LP investor. In our baseline calibrations, the positive effects dominate. Increasing leverage reduces the LP’s opportunity cost of capital and lowers the required break-even (unlevered) alpha from the GP.

Management fees and carried interest. Table 2 reports break-even alphas for different compensation contracts. In Panel A, first note that absent management fees and carried interest, a risk-neutral LP requires no alpha. Since this LP requires no compensation for unspanned illiquidity risks and absent fees, there is nothing left that requires compensation. The cost of increasing management fees, $m$, from 0 to 2\% is 0.48\%–0.56\% (depending on $k$). The cost of increasing carried interest, $k$, from 0 to 20\% is 0.47\%–0.52\% (depending on $m$). Hence, for a risk-neutral LP, the costs of management fees and carried interest are similar in magnitude.

In Panel B of Table 2, we see that increasing the effective risk aversion from $\gamma I_0 = 0$ to $\gamma I_0 = 2$...
Table 2: Break-even alphas for different levels of effective risk aversion, $\gamma I_0$, carried interest $k$, and management fees, $m$. Other parameters are $\beta = 0.5$, $h = 8\%$, and $l = 3$. The baseline case is in bold.

<table>
<thead>
<tr>
<th></th>
<th>$\gamma I_0 = 0$</th>
<th>$\gamma I_0 = 2$</th>
<th>$\gamma I_0 = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$k = 0$</td>
<td>$k = 20%$</td>
<td>$k = 30%$</td>
</tr>
<tr>
<td>$m = 0$</td>
<td>0%</td>
<td>0.47%</td>
<td>0.78%</td>
</tr>
<tr>
<td>$m = 1.5%$</td>
<td>0.34%</td>
<td>0.85%</td>
<td>1.18%</td>
</tr>
<tr>
<td>$m = 2%$</td>
<td>0.48%</td>
<td>1.00%</td>
<td>1.34%</td>
</tr>
<tr>
<td>$m = 0$</td>
<td>1.04%</td>
<td>1.44%</td>
<td>1.68%</td>
</tr>
<tr>
<td>$m = 1.5%$</td>
<td>1.47%</td>
<td>1.87%</td>
<td>2.32%</td>
</tr>
<tr>
<td>$m = 2%$</td>
<td>1.63%</td>
<td><strong>2.05%</strong></td>
<td>2.53%</td>
</tr>
<tr>
<td>$m = 0$</td>
<td>2.24%</td>
<td>2.56%</td>
<td>2.77%</td>
</tr>
<tr>
<td>$m = 1.5%$</td>
<td>2.78%</td>
<td>3.11%</td>
<td>3.33%</td>
</tr>
<tr>
<td>$m = 2%$</td>
<td>3.00%</td>
<td>3.33%</td>
<td>3.54%</td>
</tr>
</tbody>
</table>
increases the cost of the 2% management fee to 0.59%–0.85% and slightly decreases the cost of the 20% carried interest to 0.40%–0.42%. Intuitively, for a risk-averse LP the risk-free management fees are more expensive relative to the risky carried interest. These figures are consistent with Metrick and Yasuda (2010), who calculate the present values of management fees and carried interest, and also find that the cost of the management fee is twice the cost of carried interest. In Panel C, for a high level of effective risk aversion, the cost of the 2% management fee increases further to 0.76%–0.77%, while the cost of carried interest declines to 0.32%–0.33%.

The break-even alphas in Table 2 allow us to evaluate the trade-off between management fees and carried interest. A common choice is between a 2/20 (2% management fee and 20% carried interest) and a 1.5/30 compensation contract. Panel A of Table 2 shows that an LP that is risk neutral (γI0 = 0) or moderately risk averse (γI0 = 2) prefers the 2/20 contract. An LP with high risk aversion (γI0 = 5) is indifferent between these two contracts. This comparison, however, holds alpha fixed. If the higher carried interest can screen for better GPs, or if it incentivizes a GP to produce greater alpha, the trade-off may change. For an effectively risk-neutral LP, we see that the higher carried interest must increase alpha from 1.00% to 1.18%. For a moderately risk-averse LP, the alpha must increase from 2.05% to 2.32%. Hence, for these LPs, the 1.5/30 compensation contract is preferable when the more aggressive incentives increase alpha by 13%-18%.

**Buyout versus venture capital.** Although the analysis focuses on leveraged buyouts, it is useful to contrast the results to those for VC investments. VC funds typically make unlevered investments in early-stage start-up companies. Empirically, start-up companies have been found to have substantially higher systematic risk than the mature companies acquired by buyout funds (Robinson and Sensoy 2012; and Korteweg and Sorensen 2011). As a starting point, we calibrate our model to VC investments by assuming l = 0 and β = 1. The compensation contract is unchanged from previously, with carried interest of k = 20%, management fees of m = 2%, and a hurdle rate of 8%. While positive hurdle rates are less common for venture capital funds (e.g., Metrick and Yasuda 2010), including the hurdle makes the BO and VC results more directly comparable.

---

15 While the results are similar, the details of the calibrations differ slightly. Metrick and Yasuda (2010) assume that the committed capital is invested gradually and include transaction fees. Further, their model assumes a risk-neutral LP and a levered beta of 1.
Table 3 reports break-even alphas for different levels of leverage and (unlevered) betas. For buyout funds, the baseline calibration assumes leverage of $l = 3$ and systematic risk of $\beta = 0.5$, and the resulting break-even alpha of 2.05% is indicated in bold. For VC funds, the calibration assumes no leverage and $\beta = 1$, resulting in a break-even alpha of 2.82%.

We note two opposing effects. Most importantly, the greater leverage used by BO funds substantially reduces the break-even alpha, because leverage reduces management fees per dollar of PE asset. Second, the higher beta reduces the unspanned risk and lowers the break-even alpha. This second effect, however, is somewhat of an artifact of the calibration, which holds total volatility constant, and restricts the (unlevered) beta to be less than 1.25. While this is not a concern for the BO calibration, a more accurate calibration of VC performance would require better estimates of the unspanned volatility given the high systematic risk of these investments.

5 Empirical Performance Measures

The alpha generated by a GP is difficult to estimate and more readily available performance measures are used in practice, such as the Internal Rate of Return (IRR), Total Value to Paid-In capital (TVPI) multiple, and Public Market Equivalent (PME). To define these measures, divide the cash flows between the LP and GP into capital calls and distributions: $Call_t$ denotes cash flows paid by the LP to the GP, and $Dist_t$ denotes cash flows returned from the GP to the LP. Then, the IRR solves $1 = \sum \frac{Dist_t}{(1+IRR)^t} / \sum \frac{Call_t}{(1+IRR)^t}$. The multiple is defined as $TVPI = \sum Dist_t / \sum Call_t$, without any adjustment for the time value of money. Finally, $PME = \sum \frac{Dist_t}{1+R_t} / \sum \frac{Call_t}{1+R_t}$, where $R_t$ is the cumulative realized return on the market portfolio up to time $t$. Informally, the PME is the
present value of returned (distributed) capital relative to the present value of the invested (called) capital, where the present values are calculated using the realized market returns as the discount rate. Empirical studies typically interpret PME \( > 1 \) as PE investments outperforming the market, implicitly assuming a (levered) beta of one, as noted by Kaplan and Schoar (2004).

There are three concerns with the PME measure. First, the denominator blends two cash flows: the investment, \( I_0 \), and the management fees, \( mX_0 \). Management fees are effectively a risk-free claim and should be discounted at the risk-free rate. Second, the numerator contains the LP’s proceeds net of carried interest, which is effectively a call option, leaving the LP’s payoff less risky than the underlying asset. Hence, it should be discounted at a lower rate. Finally, the (levered) beta of PE investments may not be one.

### 5.1 Analytical performance measures

In the model, it is straightforward to solve for the analytical counterparts to the empirical performance measures. Let the IRR be denoted \( \phi \). It solves:

\[
I_0 + \int_0^T mX_0 e^{-\phi t} dt = e^{-\phi T} E[LP(A_T, T)],
\]

which simplifies to:

\[
I_0 + \frac{mX_0}{\phi} \left( 1 - e^{-\phi T} \right) = e^{-(\phi - \mu_A)T} \left[ EC(A_0; Z_0) - nEC(A_0; Z_1) + (n - k)EC(A_0; Z_2) \right].
\]

Here, \( EC(A; K) \) is the expected payoff, not the price, of a call option with strike price \( K \) under the physical measure, given in (A.9) in the Appendix. The expression for \( EC(A; K) \) looks similar to the Black-Scholes formula, but it calculates the expected payoff under the physical, not the risk-neutral, measure.

Ex-ante expected TVPI is given by

\[
E[TVPI] = \frac{E[LP(A_T, T)]}{X_0},
\]

24
where the numerator is the LP’s expected payoff net of carried interest, and the denominator is the committed capital $X_0$. The solution is:

$$E[TVPI] = \frac{e^{\mu_A T} \left[ EC(A_0, Z_0) - nEC(A_0; Z_1) + (n-k)EC(A_0; Z_2) \right]}{X_0}. \quad (52)$$

Finally, the *ex ante* PME is:

$$\text{PME} = \frac{E \left[ e^{-\mu_S T} LP(A_T, T) \right]}{I_0 + E \left[ \int_0^T e^{-\mu_S t} mX_0 dt \right]} = \frac{e^{(\mu_A-\mu_S) T} \left[ EC(A_0; Z_0) - nEC(A_0; Z_1) + (n-k)EC(A_0; Z_2) \right]}{I_0 + \frac{mX_0}{\mu_S} \left( 1 - e^{-\mu_S T} \right)}. \quad (53)$$

### 5.2 Break-even performance

Axelson, Jenkinson, Stromberg, and Weisbach (2011) consider 153 BO transactions during 1985–2006, and find that equity accounted for 25% of the purchase price, corresponding to $l = 3$ in our model. Table 4 reports break-even values for various levels of risk aversion and leverage. The effect of leverage on the break-even alpha is substantial. The break-even alpha decreases from 2.61% to 0.46% when $l$ increases from 0 to 9. Intuitively, the benefit of leverage is a lower relative management fee per dollar of PE assets, as illustrated above. The cost of leverage is the increase in risk and volatility. In Panel A, the LP is effectively risk neutral, eliminating the cost, and the remaining benefit of leverage is reflected in the declining break-even alpha. For an LP with greater risk aversion, the cost becomes more substantial, and the decline in the break-even alpha is smaller.

### Insert Table 4 here.

Given these break-even alphas, we calculate the break-even values of the IRR, TVPI, and PME, using (50), (52), and (53). By definition, these break-even values reflect the performance required for the LP to break even economically. Table 4 confirms that the break-even alphas decrease with leverage, yet the implied empirical performance measures increase. In Panel B, the break-even alpha declines from 3.08% to 1.77%. In contrast, the break-even TVPI multiple increases from 2.16 to 6.00, the IRR increases from 8.4% to 19.1%, and the break-even PME increases from 0.78 to
2.17. In the baseline case, with \( l = 3 \), the LP's break-even IRR is 12.7\%, the break-even TVPI is 3.61, and the break-even PME is 1.30.

It is interesting to compare the theoretical break-even values to their empirical counterparts. Harris, Jenkinson, and Kaplan (2011) summarize estimates of the empirical performance measures across datasets and studies. They report average value-weighted TVPI multiples from 1.76 to 2.30. These values fall below the model's theoretical break-even values, but these theoretical values are likely overstated because our calibration assumes that the PE asset earns alpha over the entire ten-year period. In practice, individual PE deals only last 4-5 years (Stromberg 2007). Since the TVPI multiple does not adjust for the time-value of money, it is unsuited for performance comparisons within our calibration.

The IRR adjusts for the time value of money, leading to a more reasonable comparison. Harris, Jenkinson, and Kaplan (2011) report value-weighted average IRRs of 12.3\%-16.9\%, which is close to the break-even IRR of 12.7\% in our baseline case in Table 4. The IRR and TVPI, however, are absolute performance measures, which do not adjust for the market performance.

For this reason, Harris, Jenkinson and Kaplan (2011) prefer the PME measure, which is a relative performance measure. They report average value-weighted PMEs of 1.20–1.27, which are close to our theoretical break-even PME of 1.30. While a PME of 1.27 can be interpreted as PE outperforming the market, this outperformance may be just sufficient to compensate LPs for risk and illiquidity. This is consistent with standard theories of competition and entry in financial markets (e.g., Berk and Green 2004). For a less risk-averse LP, a PME of 1.27 implies positive economic performance.

For comparison, we also calculate the break-even values of the empirical performance measures under the VC calibration, with \( l = 0 \) and \( \beta = 1 \). For a moderately risk-averse LP (\( \gamma I_0 = 2 \)), the break-even IRR is 11.1\%, and the break-even PME is about one. In this calibration, a PME greater than one is indeed equivalent to the investment being economically valuable for the LP, which is consistent with the standard interpretation. This is because the levered beta is one, and

\[16\] The studies include Ljungqvist and Richardson (2003), Kaplan and Schoar (2005), Jegadeesh et al. (2009), Phalippou and Gottschalg (2009), Korteweg and Sørensen (2010), Metrick and Yasuda (2010), Robinson and Sensoy (2011), and Stucke (2011).

\[17\] Note that the break-even value of the PME is close to but not exactly equal to one. For a risk-neutral LP, the break-even PME is 0.99; for an LP with \( \gamma I_0 = 5 \), it is 1.03.
the adjustments to the discount rates in the numerator and denominator of the PME largely cancel. This suggests that a more modest performance is sufficient for LPs to break even for VC than for BO funds. Unfortunately, actual VC performance has been very modest as well, although it has varied substantially over the past decades. Kaplan, Harris, and Jenkinson (2011) report average value-weighted IRRs of 8.6%–18.7% in the 1980s, 32.5%–38.6% during the 1990s, and -0.7%–1.6% for the 2000s. For PMEs, average value-weighted estimates are 0.90–1.08 in the 1980s, 1.26–2.12 in the 1990s, and 0.84–0.95 in the 2000s. Overall, this evidence suggests that VC funds largely broke even during the 1980s, made substantial profits during the 1990s, and lost money during the 2000s.

Returning to the BO case, Table 4 further shows that the credit spread increases with leverage and declines as effective risk aversion increases. A more risk-averse LP requires a greater break-even alpha, which benefits the lenders as well, reducing their required spread. The magnitude of the equilibrium spread is consistent with actual spreads. Table 4 shows equilibrium spreads of 2.64% to 3.48% for effective risk aversions of 0 to 2. Ivashina and Kovner (2010) report average and median spreads (to LIBOR) of 3.14% and 3.00% for syndicated loans used to finance PE transactions.

6 Sources of Alpha

Our model introduces a distinction between two types of alpha: The standard alpha, which we denote “ongoing” alpha, and a new “one-time” alpha. Ongoing alpha, like the alpha in the baseline model, is earned over the life of the investment, for example by gradually improving managerial practices. The one-time alpha is earned once, regardless of the duration of the investment, for example by initially acquiring the PE assets at a discount. This distinction only arises for illiquid long-term assets, and it raises interesting issues. With ongoing alpha, investors want to hold the PE asset longer, to earn more alpha. With one-time alpha, investors want to exit sooner, to maximize IRR. The distinction also has implications for econometric models of PE performance. With ongoing alpha, longer investments have higher returns. With one-time alpha, the return is independent of the duration of the investment.

Insert Table 5 here.
To formally extend the model to capture the one-time alpha, let the GP initially acquire the PE assets at a discount, $\alpha_0$. The initial amount of the acquired PE assets, $A_0$, is then:

$$A_0(1 - \alpha_0) = (1 + l)I_0.$$  

Table 5 reports the break-even ongoing alpha, defined as before, for different levels of one-time alpha, $\alpha_0$. The first column, without one-time alpha, reproduces the break-even values from above. The second column assumes that the GP acquires the assets at an initial discount of 5%, corresponding to $\alpha_0 = 5\%$. Naturally, a positive one-time alpha means that a lower ongoing alpha is required for the LP to break even. Specifically, the break-even ongoing alpha declines by 0.50%–0.52% (depending on effective risk aversion). Earning an ongoing alpha of 0.50%–0.52% over ten years compounds to a 5.1%–5.3% total return, which is close to the initial underpricing of 5%. In other words, earning a 5% one-time alpha simply reduces the required annual ongoing alpha by roughly one-tenth of this initial 5% return. The next columns in Table 5 show that the non-linearity remains small when the one-time alpha increases further. When it increases from 5% to 10%, the break-even ongoing alpha declines by an additional 0.52%–0.54%. Overall, while the distinction between one-time and ongoing alphas may be important for other aspects of PE investments, the non-linearity in the model is small, and for valuation purposes the amount of compounded ongoing alpha is largely equivalent to the same amount of initial one-time alpha. Therefore, the reported break-even ongoing alphas from the previous analysis can be translated into their equivalent one-time alphas by simply compounding them over ten years.

### 7 Optimal Investment and Leverage

While the main focus of our analysis is valuing a given PE investment, it is useful to derive the optimal amount of PE investment and leverage within our model. For this analysis we introduce two additional assumptions. First, it is commonly thought that the (unlevered) alpha declines with the amount of assets under management (see Kaplan, Schoar 2005; and Berk and Green 2004). In
our notation, let:

\[ \alpha(A_0) = \theta_0 - \theta_1 A_0, \]  

where \( \theta_0 > 0 \) and \( \theta_1 > 0 \) captures the decreasing return. Note that we assume that alpha is determined by the initial choice of \( A_0 \) and remains constant throughout the investment period, regardless of the subsequent performance of the PE asset.

For the second assumption, we follow the standard tradeoff theory and assume that the liquidation value of the PE asset is \( \delta \in (0, 1) \) per unit of capital. The PE asset is liquidated when the fund matures with insufficient assets to cover its liabilities (i.e., when \( A_T < Z_0 \)), and the boundary condition from this equation becomes:

\[
D(A_T, T) = \begin{cases} 
\delta A_T, & A_T < Z_0, \\
Z_0, & A_T \geq Z_0.
\end{cases}
\]  

Solving the debt pricing ODE (33), we obtain the closed-form solution for the debt value,

\[
D(A_0, 0) = \delta \left[ e^{\alpha T} A_0 - FS(A_0, 0; Z_0) \right] + (1 - \delta)Z_0 e^{-rT} N(q_2(A_0, 0; Z_0)).
\]  

Liquidation by the debtholders is inefficient. It introduces a cost of leverage and restricts the optimal leverage choice.

Let the LP start with total initial wealth of \( W_0 \). The amount invested in the PE investment is \( I_0 \), and the remainder is allocated to liquid investments (public equity and the risk-free asset). The optimal amount of leverage determines the total amount of PE assets acquired. Importantly, the debt used to leverage the PE investment is borrowed against the PE asset and is non-recourse to the LP. Hence, this leverage implicitly provides downside protection for the LP. As above, after these time-0 decisions, the PE asset is held until maturity and the LP continuously rebalances the liquid investments, as well as chooses consumption/expenditure. In short, the LP solves:

\[
\max_{I_0, I, \Pi, C} J(W_0 - I_0, A_0, 0),
\]
subject to

\[(1 + l)I_0 = A_0 = I_0 + D_0 ,\]  

(59)

where \( J(W_0, A_0, 0) \) is defined in (24) and \( D_0 = D(A_0, 0) \) is given by (57). For simplicity, we only consider the case of no initial underpricing, \( \alpha_0 = 0 \). Using the value function (26), we can equivalently write the optimization problem (58) as:

\[
\max_{A_0, l} \ V(A_0, 0) - \frac{A_0}{1 + l},
\]

(60)

where \( V(A, 0) \) is the certainty-equivalent value of the PE investment. The first-order conditions with respect to the initial PE asset size, \( A_0 \), and leverage, \( l \), are:

\[
V_A(A_0, 0) - \frac{1}{1 + l} = 0, \quad (61)
\]

\[
V_l(A_0, 0) + \frac{A_0}{(1 + l)^2} = 0. \quad (62)
\]
Figure 3 illustrates the optimal choices of $A_0$ and leverage $l$. The net gain from the PE investment is $V(A_0, 0) - I_0$. In Panel A, we plot this net gain as a function of $A_0$, while holding leverage fixed at $l = 3.5$. Intuitively, a low value of $A_0$ does not allow the LP to capitalize the alpha-generating skill on a sufficiently large asset base, however, a high value of $A_0$ decreases the alpha and also increases the unspanned illiquidity risk. The LP optimally chooses $A_0 = 2.25$ in our example. Panel B plots the net gain from the PE investment as a function of leverage $l$, holding $A_0 = 2.25$. Intuitively, increasing leverage allows the LP to better diversify the unspanned illiquidity risk. On the other hand, leverage increases the expected cost of distress. In this example, the optimal balance is achieved when $l = 3.5$. Note that the net gain from the PE investment is negative for low levels of leverage. Hence, leverage may be necessary to induce LPs to make PE investments.

### 8 Conclusion

We develop an asset allocation model for an institutional investor to value PE investments. The model captures the main institutional features: (1) Inability to trade or rebalance the PE investment, and the resulting illiquidity and unspanned risks; (2) GPs creating value and generating alpha by effectively managing the underlying PE assets; (3) GP compensation, including ongoing management fees and performance-based “carried interest;” (4) Leverage and pricing of the resulting risky debt. The model delivers tractable expressions for the LP’s asset allocation and dynamic hedging of the PE investment and provides an analytical characterization of the certainty-equivalent valuation of the PE investment.

We calibrate the model and calculate the required alpha for the LP to break even in certainty-equivalent terms. Evaluating the costs in terms of break-even alphas is a contribution to the existing literature, which traditionally calculates the present values of management fees and carried interest (e.g., Metrick and Yasuda 2010). Some advantages of break-even alphas are that they are well defined under incomplete markets, they can evaluate the cost of illiquidity, and they can compare GPs with different skills and compensation contracts.

Quantitatively, we find that LPs require the GP to produce substantial alpha to compensate
for the GP’s typical 2/20 compensation contract. For a moderately risk-averse LP, the cost of illiquidity is substantial, with a magnitude similar to the combined costs of management fees and carried interest. Leverage substantially reduces the break-even alpha. Intuitively, leverage allows the GP to manage more assets and generate alpha on a greater asset base. Holding management fees fixed, leverage reduces the effective management fee per dollar of managed PE assets. The decline in the break-even alpha with increasing leverage provides a justification for the observed use of debt in PE transactions.

Empirically, PE performance is typically evaluated in terms of the IRR, TVPI, and PME. Our model delivers break-even values of these performance measures that are reasonably close to their empirical averages. A PME greater than one is usually interpreted as PE investments outperforming the market. In our model, however, this “outperformance” may be just sufficient for LPs to break even, in certainty-equivalence terms, consistent with Berk and Green (2004). LPs with lower effective risk aversion and more skilled LPs, which can exploit the performance persistence of PE firms, can earn economic rents from PE investments.

For tractability, we assume exponential preferences, which ignore the wealth effect. Despite this simplification, our model generates rich predictions for the effects of risk aversion on valuation and asset allocation. Our work highlights the importance of understanding institutional preferences when evaluating large illiquid investments. Typical institutional investors, such as pension funds and university endowments, are long lived and involve multiple constituencies, raising issues about aggregation, intergenerational discounting, and conflicts of interests between cohorts.
Appendices

A Technical details

We first derive the full spanning benchmark solution and then sketch out the derivation for the incomplete-market solution.

A.1 Full spanning

Denote $FS(A,t;K)$ as the expected discounted payoff of a call option with strike price $K$ under the risk-adjusted measure defined in the text,

$$FS(A,t;K) = \tilde{E}_t \left[ e^{-r(T-t)} \max \{ A_T - K, 0 \} \right] = e^{\alpha(T-t)} \left[ AN(q_1(A,t;K)) - Ke^{-(r+\alpha)(T-t)} N(q_2(A,t;K)) \right],$$ (A.1)

where $N(\cdot)$ is the cumulative standard normal distribution and $q_1, q_2$ are given by (35) and (36) respectively.

A.2 Incomplete-markets solution

After exiting from holding the illiquid asset, investors solve a classic Merton-type consumption and portfolio allocation problem by investing in the risk-free asset and public equity. The wealth dynamics is given by

$$dW_t = (rW_t - C_t) dt + \Pi_t \left( (\mu_S - r)dt + \sigma_S dB_t^S \right), \quad t \geq T.$$ (A.2)

Let $J^*(W)$ denote investors’ value function after time $T$, i.e.

$$J^*(W) = \max_{\Pi,C} \mathbb{E} \left[ \int_T^\infty e^{-\zeta(s-T)} U(C_s) ds \right].$$ (A.3)

The following HJB equation holds

$$\zeta J^*(W) = \max_{\Pi,C} \left( U(C) + (rW + \Pi(\mu_S - r) - C)J^*_W(W) + \frac{1}{2} \Pi^2 \sigma^2 S J^*_{WW}(W) \right).$$ (A.4)
The FOCs for Π and C are

\[
U_C(C) = J^*_W(W), \\
Π = -\frac{(μS - r)J^*_W(W)}{σ^2S J^*_WW(W)}. \tag{A.6}
\]

We conjecture that \(J^*(W)\) is given by (20). Using the FOCs (A.5) and (A.6) for C and Π, we obtain the optimal consumption and portfolio allocation given in Proposition 1.

Solution before maturity \(T\). Substituting (26) into the HJB equation (25), we obtain

\[
-\frac{ζr}{γr} = \max_{Π, C} \left\{-e^{-γ(C-r(W+b+V))} \right\} + \frac{V_t + rW + Π(μS - r) - mX_0 - C}{γ} + μA AV_A + \frac{1}{2} σ^2_A A^2 V_{AA} - \frac{γr}{2} \left(Π^2 σ^2_S + 2ρσ_S σ_A Π AV_A + σ^2_A A^2 V^2_A\right). \tag{A.7}
\]

Using the FOCs for C and Π, we have the optimal consumption and portfolio rules given in (30) and (31), respectively. After some simplifications, we obtain the ODE (27) for \(V(A_t, t)\).

Derivation for Proposition 2. Substituting \(V(A, t) = v(a, t) × I_0\) into (27) and using \(a = A/I_0\), \(x_0 = X_0/I_0\), we obtain (43). Using the expressions for \(Z_0, Z_1, Z_2,\) and (28), we have (44). Finally, substituting \(V(A, t) = v(a, t) × I_0\) and \(x_0 = X_0/I_0\) into (29), we obtain (45).

A.3 Technical details for various performance measures

Let \(EC(A; K)\) denote the expected discounted payoff, not value, of a call option with strike price \(K\) under the physical measure,

\[
EC(A; K) = \mathbb{E}_0 \left[e^{-μA T} \max \{A_T - K, 0\}\right], \tag{A.8}
\]

\[
= AN(p_1(A; K)) - Ke^{-μA T}N(p_2(A; K)), \tag{A.9}
\]

where \(p_1(A; K)\) and \(p_2(A; K)\) are given by

\[
p_1(A; K) = p_2(A; K) + σ_A √T, \tag{A.10}
\]

\[
p_2(A; K) = \frac{ln(A/K) + (μ_A - σ^2_A/2)T}{σ_A √T}. \tag{A.11}
\]
Table 4: Break-even values of empirical performance measures and the equilibrium yield (credit spread), implied by the break-even alphas for different levels of effective risk aversion, $\gamma I_0$, and leverage, $l$. Other parameters are $\beta = 0.5$, $m = 2\%$, $k = 20\%$, and $h = 8\%$. The baseline case is in bold.

<table>
<thead>
<tr>
<th>Leverage ($l$)</th>
<th>$\gamma I_0 = 0_+$</th>
<th>$\gamma I_0 = 2$</th>
<th>$\gamma I_0 = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alpha ($\alpha$)</td>
<td>2.61% 1.68% 1.00% &amp; 0.63% &amp; 0.46% 3.08% 2.46% 2.05% 1.86% 1.77% 3.74% 3.49% 3.33% 3.30% 3.28%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>IRR ($\phi$)</td>
<td>7.9% 9.6% 11.2% &amp; 12.3% &amp; 13.0% 8.4% 10.8% 12.7% 16.8% 19.1% 9.0% 12.3% 16.5% 20.6% 23.5%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Credit spread ($y - r$)</td>
<td>N/A 1.05% 3.48% &amp; 5.69% &amp; 7.14% N/A 0.86% 2.64% 3.96% 4.66% 2.16 2.72 3.61 4.83 6.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mathbb{E}[TVPI]$</td>
<td>2.07 2.43 2.81 &amp; 3.14 &amp; 3.35 2.16 2.72 3.61 4.83 6.00 0.75 0.88 1.02 1.13 1.21</td>
<td></td>
<td></td>
</tr>
<tr>
<td>PME</td>
<td>0.75 0.88 1.02 &amp; 1.13 &amp; 1.21 0.78 0.98 1.30 1.74 2.17</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5: Break-even “ongoing” alphas for different levels of effective risk aversion, $\gamma I_0$, and “one-time” alpha, $\alpha_0$. Other parameters are $\beta = 0.5$, $m = 2\%$, $k = 20\%$, $h = 8\%$, and $l = 3$. The baseline case is in bold.

<table>
<thead>
<tr>
<th>$\gamma I_0 = 0_+$</th>
<th>$\alpha_0 = 0$</th>
<th>$\alpha_0 = 5%$</th>
<th>$\alpha_0 = 10%$</th>
<th>$\alpha_0 = 20%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma I_0 = 2$</td>
<td>2.05% 1.53% 1.00% &amp; -0.18%</td>
<td>$\gamma I_0 = 5$</td>
<td>3.33% 2.83% 2.31% 1.14%</td>
<td></td>
</tr>
</tbody>
</table>
Table 6: Summary of Key Variables and Parameters

This table summarizes the symbols for the key variables in the model and baseline parameter values.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Symbol</th>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>LP’s Consumption or expenditure</td>
<td>$C$</td>
<td>Risk-free rate</td>
<td>$r$</td>
<td>5%</td>
</tr>
<tr>
<td>LP’s Value Function</td>
<td>$J$</td>
<td>Expected return of market portfolio</td>
<td>$\mu_S$</td>
<td>11%</td>
</tr>
<tr>
<td>LP’s Value Function after exiting illiquid asset</td>
<td>$J^*$</td>
<td>Expected return of PE asset</td>
<td>$\mu_A$</td>
<td></td>
</tr>
<tr>
<td>LP’s Certainty Equivalent</td>
<td>$V$</td>
<td>Volatility of market portfolio</td>
<td>$\sigma_S$</td>
<td>20%</td>
</tr>
<tr>
<td>Future value of investment and fees</td>
<td>$F$</td>
<td>Volatility of PE asset</td>
<td>$\sigma_A$</td>
<td>25%</td>
</tr>
<tr>
<td>Debt</td>
<td>$D$</td>
<td>Aggregate equity risk premium</td>
<td>$\mu_S - r$</td>
<td>6%</td>
</tr>
<tr>
<td>Wealth</td>
<td>$W$</td>
<td>Market Sharpe ratio</td>
<td>$\eta$</td>
<td>30%</td>
</tr>
<tr>
<td>Assets</td>
<td>$A$</td>
<td>Hurdle rate</td>
<td>$h$</td>
<td>8%</td>
</tr>
<tr>
<td>Brownian Motion for Market Return</td>
<td>$B^R$</td>
<td>Carried interest</td>
<td>$k$</td>
<td>20%</td>
</tr>
<tr>
<td>Brownian Motion for PE Return</td>
<td>$B^A$</td>
<td>Management fee</td>
<td>$m$</td>
<td>2%</td>
</tr>
<tr>
<td>Committed Capital</td>
<td>$X_0$</td>
<td>Catch-up rate</td>
<td>$n$</td>
<td>100%</td>
</tr>
<tr>
<td>Invested Capital</td>
<td>$I_0$</td>
<td>Life of PE investment</td>
<td>$T$</td>
<td>10</td>
</tr>
<tr>
<td>Market portfolio allocation</td>
<td>$\Pi$</td>
<td>Excess return, alpha</td>
<td>$\alpha$</td>
<td></td>
</tr>
<tr>
<td>Correlation between market and PE asset</td>
<td>$\rho$</td>
<td>0.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>PE un-levered beta</td>
<td>$\beta$</td>
<td>0.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Subjective discount rate</td>
<td>$\zeta$</td>
<td>5%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Leverage</td>
<td>$l$</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Unspanned volatility</td>
<td>$\epsilon$</td>
<td>23%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Coefficient of absolute risk aversion</td>
<td>$\gamma$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
References


Harris, Robert S., Tim Jenkinson, and Steven N. Kaplan, 2011, “Private Equity Performance: What Do We Know?,” *working paper*.


Lan, Yingcong, Neng Wang, and Jinqiang Yang, 2012, “The Economics of Hedge Funds,” *working paper*.


