Valuing private equity*

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December 13, 2011

Abstract

To understand the pricing and performance of private equity (PE), we analyze the incomplete-markets portfolio choice problem facing a risk-averse limited partner (LP) investing in liquid stocks and bonds along with an illiquid PE investment. A general partner (GP) manages the PE asset and generates alpha on it and charges management and performance fees via carried interest in return. Our complete-markets benchmark extends the Black-Scholes option pricing framework to allow for positive alpha, which is necessary to justify managerial fees. For our incomplete-markets framework, we derive tractable formulas for the LP's portfolio weights and certainty equivalent valuation for the PE asset. The key term that accounts for the discount due to illiquidity and idiosyncratic risk reflects the fundamental difference between our valuation framework and the standard Black-Scholes framework. We find that the typically observed 2-20 compensation contract implies a large alpha for the LP to break even, but leveraging the PE asset substantially reduces this break-even alpha. Our model also allows us to quantify the increase in performance measures, break-even alpha being a natural one, as the GP increases fees. For our baseline calibration, our model implied performance measures are in line with empirical measures. We also evaluate the limitations of standard PE performance measures from a certainty equivalent perspective, and propose a new Adjusted Public-Market-Equivalent (PME) measure.

Keywords: idiosyncratic risk premium, illiquidity, hedging, real options, PE, hurdle rate, management fees, performance fees, carried interest, catch up, alpha

JEL Classification: G11, G23, G24,

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*Preliminary and Incomplete. Comments are welcome.

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1 Introduction

There is considerable controversy about the risks and returns of investments in private equity (PE) funds. PE is different from traditional assets, because the funds are illiquid and investments in PE funds are usually held to maturity, rendering traditional models of valuations and portfolio choice inadequate to describe the risk and return of PE investments.

We present a model of PE investing where a limited partner (LP) allocates capital between a liquid risk-free asset and a risky stock (such as the market portfolio) as well as an illiquid PE investment. The liquid investments are continuously rebalanced, but the PE investment remains fixed once the LP acquires it. The LP's return from the PE investment is calculated net of the general partner (GP)'s compensation, which includes ongoing management fees and the carried interest. Carried interest is a profit share that may serve as an incentive or performance fee by giving the GP exposure to the upside. Intuitively, the carried interest resembles a call option on the underlying investment (the LP is short this call option), while the management fee resembles a fixed income claim. For an LP to break even, the GP must generate sufficient excess return (alpha) to compensate the LP for the costs of management fees, carried interest, idiosyncratic risk, and illiquidity.

In our baseline model with complete markets, valuing private equity is essentially equivalent to valuing public equity, and we can appeal to the standard dynamic replicating portfolio arguments underlying the Black and Scholes (1973) and Merton (1973) framework. Since our model requires that the GP can add value, we extend the standard Black-Scholes pricing formula to allow for a positive alpha in the process for the underlying asset, and we derive an explicit solution that captures the optionality of the compensation arising from the waterfall.

In this complete-market benchmark with alpha, we derive the required alpha that the GP must generate for the LP to break even. In our baseline calibration, for an un-levered PE investment, this break-even alpha equals 2.6% annually. Importantly, leveraging the PE investment substantially reduces this break-even alpha. To illustrate, with an debt-equity ratio of three, the GP only needs to generate an alpha of 1% for LP to break even. Intuitively, leverage allows the GP to manage more assets and to generate alpha on a greater asset base. Since management fees are based on the LP's committed capital, leverage reduces the *effective* management fee per dollar of managed PE assets. The decline in the break-even alpha with leverage provides a new justification for the observed use of debt in PE transactions (given the existing structure of management contracts). Note that the break-even alpha is independent of the (unlevered) beta of the underlying PE asset, because the GP's compensation is effectively a *derivative* on the underlying PE asset, and beta thus has no role on the pricing of GP's compensation when market are complete (but not when markets are incomplete).

With incomplete markets, the return to the PE asset is not spanned by traded assets, and the LP demands an additional risk premium for bearing non-diversifiable idiosyncratic risk. We derive a (non-linear) differential equation for the certainty equivalent valuation of the PE investment. In addition to the terms in the complete-market benchmark, this differential equation contains an additional term that prices the idiosyncratic risk of the investment. Despite its richness, the model delivers a tractable solution and intuitive expressions for the certainty equivalent value of the PE investment. We further obtain closed-form expressions for the optimal hedging portfolio and consumption rules under incomplete markets and illiquidity.

Quantitatively, we show that both risk aversion and leverage have significant effects on the break-even alpha under incomplete markets. To illustrate, with an unlevered beta of the underlying PE asset of one-half and an initial debt-equity ratio of three, the LP's break-even alpha increases from 1.0% to 2.1% (110 basis points) as our proxy for relative risk aversion increases from (effectively) zero to two. The effect of leverage is also significant. Continuing with the LP with a proxy of relative risk aversion of two and an unlevered beta of one-half, the break-even alpha increases from 2.1% (from before) to 3.1% (100 basis points) when the debt-equity ratio decreases from three to zero.

We also quantify the effects of changing management compensation structure on the break-even alpha. For common amounts of leverage (a debt-equity ratio around three), starting from a typical 2-20 compensation contract, increasing the management fees from 2.0% to 2.5% is roughly comparable increasing the incentive fees (carry) from 20% to 25%. Both require demand approximately a 20-25 basis point increase in the alpha generated by the GP for the LP to break even.

Estimates of PE performance reported in the literature, such as internal rate of return (IRR), total-value-to-paid-in-capital (TVPI), and public-market equivalent (PME), are typically interpreted under a standard CAPM model, where possible. We provide a series of adjustments to these measures to interpret them in terms of the actual economic value earned by the LP investor after accounting for the effects of management fees, carried interest, and illiquidity/non-diversifiable risks, and we relate observed values for these empirical performance measures to our break-even alpha.

Our paper extends the literature about the risk and return of PE investments. Kaplan and Schoar (2005) analyze returns to PE investments using a standard CAPM model, assuming a beta of one. Cochrane (2005) and Korteweg and Sorensen (2011) estimate the risk and return of venture capital investments in a CAPM model. Most closely related to our work is Metrick and Yasuda (2010), who value various features of the general partners' compensation, such as management fees, carried interest, and hurdle rate, assuming a risk-neutral investor and simulating cash flows using empirical distributions. Choi, Metrick, and Yasuda (2011) extend Metrick and Yasuda (2010) and compare the relative values of the fair-value-test (FVT) carry scheme to other benchmark carry schemes.

There is also a related literature about the valuation of illiquid assets, although no paper specifically models the LP's portfolio problem incuding PE investments. Ang, Papanikolaou and Westerfield (2010) solve for the optimal asset allocation when investing in an illiquid asset along with liquid stocks and bonds, but in their setup the illiquid asset is traded and rebalanced at random intervals, following a Poisson arrival process. Chen, Miao, and Wang (2010) derive an optimal capital structure model for entrepreneurial firms under incomplete markets where the under-diversified entrepreneur trades off the diversification benefits of external financing (such as risky debt) against the costs of external financing (such as the distress cost of debt).

2 Model

Liquid investments. As in Merton (1971), the LP has an infinite horizon and continuously rebalances a portfolio with a risk-free asset and the risky market portfolio. The risk-free asset pays interest at a constant rate r. The return of the risky market portfolio is independently and identically distributed, given by

$$dR_t = \mu_R dt + \sigma_R dB_t^R \,, \tag{1}$$

where B^R is a standard Brownian motion, and μ_R and σ_R are constant mean and volatility parameters. Let η denote the Sharpe ratio of the market portfolio,

$$\eta = \frac{\mu_R - r}{\sigma_R} \tag{2}$$

Illiquid PE investment. In addition to the liquid investment, the LP invests I_0, at time 0, in a PE limited partnership (LP cannot short the PE investment, so $I_0 > 0$). The PE investment is controlled by a general partner (GP), typically the management company of a private equity firm. The GP uses this amount to acquire A_0 worth of privately held assets. Without leverage, $A_0 = I_0$. With leverage, considered in Section 6, $A_0 > I_0$. The PE investment is illiquid and must be held until a fixed maturity date T,[1] at which point the investment is exited and the assets are sold for A_T. The proceeds are shared between the LP and GP as specified below. Subsequently, the LP will only invest in liquid assets, i.e. the risky market portfolio and the risk-free assets.

During the life of this PE investment ($0 \leq t \leq T$), the value of the PE assets follows a geometric Brownian motion (GBM)

$$dA_t/A_t = \mu_A dt + \sigma_A dB_t^A \,, \tag{3}$$

where B^A is a standard Brownian motion, μ_A is the drift, and σ_A is the volatility. The returns of the PE asset and market portfolio are correlated, and ρ is the correlation coefficient between the two Brownian motions B^A and B^R. Whenever $|\rho| \neq 1$, markets are incomplete, and the investor cannot completely hedge the PE investment risk.

The volatility for the PE asset's systematic risk is $\rho\sigma_A$, and the volatility of the PE asset's idiosyncratic risk, ϵ, is

$$\epsilon = \sigma_A\sqrt{1-\rho^2} = \sqrt{\sigma_A^2 - \beta^2\sigma_R^2} \,. \tag{4}$$

In the second equality, β is the unlevered beta of the PE asset, defined as in the CAPM setting as

$$\beta = \frac{\rho\sigma_A}{\sigma_R} \,. \tag{5}$$

Analogously, alpha is defined as,

$$\alpha = \mu_A - r - \beta(\mu_R - r) \,. \tag{6}$$

[1] We assume that the value of liquidation is sufficiently low that it is never optimal to liquidate the asset before maturity T, and we ignore the secondary market in which the LP can potentially sell the investment at a significant discount.

Under complete markets no arbitrage implies no risk-adjusted excess returns, hence $\alpha = 0$. In our incomplete-markets framework, the LP must be compensated for holding non-diversifiable and idiosyncratic risks, along with fees, so $\alpha > 0$. Below we calculate the minimum alpha required for the LP to break even.

Management fees. The GP receives both management fees and incentive fees ("carried interest"). Management fees are charged on an ongoing basis, and the committed capital, X_0, is the sum of the investment, I_0, and the accumulated management fees paid over the life of the PE investment. The annual management fee is given as the fraction m, typically 2%, of the committed capital, hence

$$I_0 + mTX_0 = X_0 \,, \tag{7}$$

implying $X_0 = I_0/(1 - mT)$. To illustrate, a ten-year PE investment I_0 of \$10 with a 2% annual management fee implies a total committed capital X_0 of \$12.5.

Waterfall. At maturity T, the GP and LP share the proceeds based on a schedule (or waterfall), with three regions. The boundaries between the regions, defined shortly, are denoted F and Z. When the PE investment is liquidated at time T, A_T is the total proceeds and $A_T - X_0$ is the profits.

The first region, $A_T \leq F$, is the preferred return region. The LP receives the entire proceeds, and the GP receives no carried interest. The boundary F is the amount where the LP achieves an IRR equal to a specified hurdle return h, typically 8%. Therefore,

$$F = I_0 e^{hT} + \int_0^T mX_0 e^{hs} ds = I_0 e^{hT} + \frac{mX_0}{h}(e^{hT} - 1) \,. \tag{8}$$

Note that $h = 0$ implies $F = X_0$, and $h > 0$ implies $F > X_0$.

The second region, $F \leq A_T \leq Z$, is the GP's catch-up region. The GP receives carried interest equal to a fraction n, typically 100%, of the net proceeds $A_T - F$. The LP receives the remainder, if any. This region lasts until the GP fully catches up, meaning that the GP's carried interest equals a given fraction k, typically 20%, of total profits $A_T - X_0$. Hence, the second boundary Z solves

$$k(Z - X_0) = n(Z - F). \tag{9}$$

With a positive h, the GP only fully catches up when $n > k$. When $n \leq k$, we set $Z = \infty$.[2]

The third region, $A_T > Z$, is the profit-sharing region. The GP has fully caught up, and the total carried interest is simply a share of total profits, $k \times (A_T - X_0)$. The LP receives the remaining proceeds, $X_0 + (1-k)(A_T - X_0)$.

Liquid wealth dynamics. Let W denote the LP's liquid wealth process, including both the market portfolio and risk-free asset but excluding the illiquid PE investment. The investor allocates the amount Π_t to the market portfolio and the remaining $W_t - \Pi_t$ to the risk-free asset. The LP's liquid wealth dynamics are then given by

$$dW_t = (rW_t - mX_0 - C_t)\, dt + \Pi_t \left((\mu_R - r)dt + \sigma_R dB_t^R \right) . \tag{10}$$

The first term gives the rate of wealth accumulation when the LP fully invests in the risk-free asset (net of the management fee, mX_0, and consumption/expenditure C). The second term is the excess return from investing in the risky market portfolio.

LP's voluntary participation. With incomplete markets standard no-arbitrage pricing does not apply, and we value the LP's illiquid PE investment in terms of its certainty equivalent. Let $V(A_0, 0)$ be the certainty equivalent at time 0, taking into account management and performance fees as well as illiquidity. The voluntary participation constraint for the LP is

$$V(A_0, 0) \geq I_0 \,. \tag{11}$$

Note $V(A_0, 0)$ must exceed the invested capital I_0, not total committed capital X_0, because the costs of management fees (and carried interest) are included in V. Without leverage, $A_0 = I_0$. With leverage, as in Section 6, A_0 exceeds I_0 by the amount of debt. In both cases, the participation constraint (11) holds.

LP's objective. The LP has a standard time-additive separable utility function given by

$$\mathbb{E}\left[\int_0^\infty e^{-\zeta t} U\left(C_t\right) dt \right] , \tag{12}$$

where $\zeta > 0$ is the subjective discount rate and $U(C)$ is a concave function. For tractability, we choose $U(C) = -e^{-\gamma C}/\gamma$, where $\gamma > 0$ is the coefficient of absolute risk aversion (CARA).

[2]When $k = n$ and $h = 0$ (implying $F = X_0$) any Z solves (9). In this case, the sharing rule in the catch-up and profit regions are identical, and the choice of Z is irrelevant.

Summary of frictions. The model contains three frictions: First, the investor cannot continuously trade the PE position. After deciding the initial exposure to PE, this position is held to maturity. This creates illiquidity risk due to irreversibility and inability to rebalance the PE position. Second, the PE position is large. The LP is not fully diversified and must be compensated for non-diversifiable risk. Third, the investor pays management and performance fees to the GP. Hence, to justify the PE investment, the GP must generate value and the PE asset must outperform the market, after adjusting for risk and fees.

3 Complete-markets solution

First consider the case with complete markets (CM). The LP's optimization problem can be decomposed into two separate ones: total wealth maximization and utility maximization. By dynamically trading a few long-lived assets, the PE investment can be perfectly replicated and thus investors demand neither idiosyncratic nor illiquidity risk premia. With a constant interest rate r and a constant Sharpe ratio $\eta = (\mu_R - r)/\sigma_R$, the capital asset pricing model (CAPM) holds. With CM, there cannot be an excess returns, adjusting for systematic risks. Hence, $\alpha = 0$, and the equilibrium expected rate of return μ_A for the PE asset is $\mu_A = r + \beta(\mu_R - r)$.

Using the standard dynamic replicating portfolio argument, as in the Black-Scholes setting, we obtain an ODE for the LP's value of the PE investment, $V(A,t)$, as

$$rV(A,t) = -mX_0 + V_t(A,t) + rAV_A(A,t) + \frac{1}{2}\sigma_A^2 A^2 V_{AA}(A,t) \,. \tag{13}$$

The term $-mX_0$ captures management fees. The other terms are standard.

Boundary Conditions. The LP's payoff can be viewed as the sum of three distinct claims or tranches, corresponding to the three regions in the waterfall: (1) the LP's senior preferred return claim, (2) the GP's catch-up region corresponding to a mezzanine claim for the LP, and (3) the profit-sharing region corresponding to a junior equity claim. When valuing the LP's total investment, these three claims are combined to form the LP's boundary condition. Hence, at maturity time T, the LP's total payoff is

$$V(A_T,T) = LP_1(A_T,T) + LP_2(A_T,T) + LP_3(A_T,T) \,, \tag{14}$$

where $LP_1(A_T, T)$, $LP_2(A_T, T)$, and $LP_3(A_T, T)$ are the incremental payoffs in the three regions of the waterfall, as described next.

In the preferred return region, the LP's payoff is given by

$$LP_1(A_T, T) = \min\{A_T, F\} , \tag{15}$$

where F is given in (8) and can be interpreted as the principal of the senior debt claim in the Black-Scholes capital structure terminology. In the catch-up region, the incremental payoff resembles a $(1-n)$ fraction of mezzanine debt, given by

$$LP_2(A_T, T) = (1-n)\left(\max\{A_T - F, 0\} - \max\{A_T - Z, 0\}\right) . \tag{16}$$

Here, Z is given in (9) and can be interpreted as the sum of the principals of the senior and mezzanine debt claims. Finally, in the profit-sharing region, the incremental payoff is a junior claim, resembling a $(1-k)$ equity share, given by

$$LP_3(A_T, T) = (1-k)\max\{A_T - Z, 0\} . \tag{17}$$

Analogously, we could express the GP's total payoff as the sum of the incremental payoffs in the three regions. It is simpler, however, to give the GP's payoffs as the remaining proceeds, $A_T - V(A_T, T)$.

LP is assumed to honor the management fees regardless of the fund's performance. Therefore, the LP's liability equals the present value (PV) of the remaining management fees,

$$V(0, t) = \int_t^T e^{-r(T-s)}(-mX_0)ds = -\frac{mX_0}{r}\left(1 - e^{-r(T-t)}\right) < 0 . \tag{18}$$

Solution. We use $BS(A_t, t; K)$ to denote the Black-Scholes call option pricing formula, given in (A.4) of the appendix. Under CM, we can apply the Black-Scholes pricing methodology to value each tranche separately, as

$$V^*(A_t, t) = LP_1^*(A_t, t; F) + LP_2^*(A_t, t; F) + LP_3^*(A_t, t; F) - \frac{mX_0}{r}\left(1 - e^{-r(T-t)}\right) , \tag{19}$$

where

$$LP_1^*(A_t, t) = A_t - BS(A_t, t; F) , \tag{20}$$

$$LP_2^*(A_t, t) = (1-n)(BS(A_t, t; F) - BS(A_t, t; Z)) , \tag{21}$$

$$LP_3^*(A_t, t) = (1-k)BS(A_t, t; Z) . \tag{22}$$

Here, LP_1^*, LP_2^*, and LP_3^* are the LP's values of the incremental payoffs in the three tranches, representing the preferred-return, catch-up, and profit-sharing regions. The last term in (20) is the cost of the management fees, which is effectively a fixed-income liability. Under complete markets there can be no risk-adjusted excess return ($\alpha = 0$). Hence, $V^*(A_t, t) < A_t$, and a rational LP will not allocate any capital to the PE investment.

4 Incomplete-markets solution

Turning to the incomplete-markets setting with non-diversifiable risk and illiquidity, we first consider the LP's decision problem after exiting the PE investment and then analyze the problem before exit. Here, we consider an unlevered PE asset, but we return to leverage in Section 6.

4.1 The decision problem after exiting the PE investment

After exiting the illiquid PE investment, the LP investor faces a classic Merton consumption/portfolio allocation problem. The standard solution to this problem is summarized in Proposition 1. We use the value function $J^*(W)$ from this proposition when analyzing the pre-exit decisions.

Proposition 1 *The LP's post-exit value function takes the following form:*

$$J^*(W) = -\frac{1}{\gamma r} e^{-\gamma r (W+b)} . \tag{23}$$

where b is the constant given by

$$b = \frac{\eta^2}{2\gamma r^2} + \frac{\zeta - r}{\gamma r^2} . \tag{24}$$

The optimal consumption, C, is given by

$$C = r(W + b) . \tag{25}$$

The optimal allocation to the risky market portfolio, Π, is given by

$$\Pi = \eta / (\gamma r \sigma_R) . \tag{26}$$

4.2 Optimality and private valuation ($t \leq T$)

Let $J(W, A, t)$ denote the LP's pre-exit value function. The LP's consumption C and allocation to the risky market portfolio Π solve the Hamilton-Jacobi-Bellman (HJB) equation,

$$\begin{aligned}\zeta J(W, A, t) &= \max_{\Pi, C} U(C) + J_t + (rW + \Pi(\mu_R - r) - mX_0 - C)J_W \\ &\quad + \frac{1}{2}\Pi^2\sigma_R^2 J_{WW} + \mu_A A J_A + \frac{1}{2}\sigma_A^2 A^2 J_{AA} + \rho\sigma_R\sigma_A \Pi A J_{WA} \,. \end{aligned} \tag{27}$$

In the appendix, we show that the LP's value function has the exponential form,

$$J(W, A, t) = -\frac{1}{\gamma r}\exp\left[-\gamma r\left(W + b + V(A, t)\right)\right] , \tag{28}$$

where the constant b is given in (24) and $V(A, t)$ is the LP's private certainty equivalent valuation for the PE investment net of fees. We characterize $V(A, t)$ shortly.

The LP's optimal consumption C is given by

$$C(W, A, t) = r\left(W + b + V(A, t)\right) , \tag{29}$$

which is a version of the permanent-income/precautionary-saving models.[3] Comparing to (25), we see that the LP's total certainty equivalent wealth is the sum of the liquid wealth W and certainty equivalent of the illiquid asset $V(A, t)$.

The LP's optimal portfolio rule is

$$\Pi(A, t) = \frac{\eta}{\gamma r \sigma_R} - \beta A V_A(A, t) \,, \tag{30}$$

where the first term is the standard mean-variance demand, the second term is the intertemporal hedging demand, and the unlevered β of the underlying PE asset is given by (5). Following the option pricing terminology, we denote $V_A(A, t)$ the delta of the LP's PE investment with respect to A, the value of the underlying asset. The higher the β and the delta $V_A(A, t)$, the higher the hedging demand.

We next turn to the LP's private valuation of the PE investment $V(A, t)$. In the appendix, we show that $V(A, t)$ solves the nonlinear ordinary differential equation (ODE):

$$rV(A, t) = -mX_0 + V_t + (r + \alpha)\, AV_A + \frac{1}{2}\sigma_A^2 A^2 V_{AA} - \frac{\gamma r}{2}\epsilon^2 A^2 V_A^2 \,, \tag{31}$$

[3]See Friedman (1957) and Hall (1978) for seminal contributions. Caballero (1991) and Wang (2006) derive explicitly solved optimal consumption rules under incomplete markets with CARA utility. Miao and Wang (2007) derive the optimal American-style growth option exercising problems under incomplete markets. Chen, Miao, and Wang (2010) integrate the incomplete-markets real options framework of Miao and Wang (2007) into Leland (1994) to analyze entrepreneurial default, cash-out, and credit risk implications.

where ϵ is the idiosyncratic volatility for the PE investment given in (4) and α is given by (6). The nonlinear ODE (31) is solved subject to (14) and (18), the same boundary conditions as in the complete-markets case.

Unlike the complete-markets case, ODE (31) is non-linear because its last term involves V_A^2, a non-linear function of V_A. Incomplete markets frictions, non-diversification and illiquidity, invalidate the law-of-one-price valuation paradigm. That is, an LP whose certainty equivalent of two individual PE investments, in isolation, are V_1 and V_2, does not value the portfolio consisting of these two investments at $V_1 + V_2$.

5 Results

Parameter choices. As described in the previous section, we can solve for the LP's valuation of the PE investment given a set of parameter values. When possible, we use parameter values from Metrick and Yasuda (2010) (henceforth MY) to help compare the results. All parameters are annualized when applicable. MY use a volatility of 60% per individual buyout investment, with a pairwise correlation of 20% between any two buyout investments. MY report that the average buyout fund invests in a portfolio of around 15 buyouts (with a median of 12). Based on their estimates, we use a volatility of 25% for the underlying PE asset.

As MY, we use a risk-free rate of 5%. For parameters not in MY, we use a volatility of $\sigma_R = 20\%$ and an expected return of $\mu_R = 11\%$ for the market portfolio, implying a risk premium of $\mu_R - r = 6\%$ and a Sharpe ratio of $\eta = 30\%$. Initially, we set the unlevered β of the PE asset to 0.5, implying a correlation between the PE asset and the market portfolio of $\rho = \beta\sigma_R/\sigma_A = 0.4$. We also consider other values of β for comparative statics.

We next turn to risk aversion γ and the initial investment I_0. To assess reasonable combinations of γ and I_0, we rely on the following invariance result.

Proposition 2 *Define $\overline{\gamma} = \gamma I_0$, $a = A/I_0$, $x_0 = X_0/I_0$, $z = Z/I_0$ and $f = F/I_0$. It is straightforward to verify that $V(A,t) = v(a,t) \times I_0$, where $v(a,t)$ solves the following ODE,*

$$rv(a,t) = -mx_0 + v_t + (r+\alpha)\, av_a(a,t) + \frac{1}{2}\sigma_A^2 a^2 v_{aa}(a,t) - \frac{\overline{\gamma} r}{2}\epsilon^2 a^2 v_a(a,t)^2\,, \tag{32}$$

subject to the boundary conditions,

$$v(a,T) = a - n(\max\{a-f,0\} - \max\{a-z,0\}) - k\max\{a-z,0\}, \tag{33}$$

$$v(0,t) = -\frac{mx_0}{r}\left(1-e^{-r(T-t)}\right). \tag{34}$$

Proposition 2 shows that $v(a,t)$ depends only on $\overline{\gamma} = \gamma I_0$, not γ and I_0 separately. This means that, holding $\overline{\gamma}$ fixed, the LP's certainty equivalent valuation $V(A,t)$ is proportional to the invested capital I_0.[4] To illustrate, holding $\overline{\gamma}$ constant, if an LP values a (scalable) PE investment with invested capital $I_0 = \$100$M at $V_0 = \$120$M, then our model implies that the value of the same PE investment with $I_0 = \$100$ is $V_0 = \$120$. This invariance is desirable from a valuation perspective but does not hold under incomplete markets in general.

Informally, we may interpret $\overline{\gamma} = \gamma I_0$ as the LP's relative risk aversion, focusing on the PE investment. The standard approximation of the relative risk aversion is γW_0 where W_0 is the LP's initial total wealth, and when calibrating the model, it is desirable to target a given relative risk aversion. Our invariance result implies that targeting $\overline{\gamma}$ is equivalent to targeting the standard relative risk aversion measure, γW_0, via $\overline{\gamma} = \gamma I_0 = (\gamma W_0) \times (I_0/W_0)$.[5] To illustrate, if we target a relative risk aversion coefficient of $\gamma W_0 = 5$ and assume that the PE investment is 40% of the LP's total wealth, then $\overline{\gamma} = 5 \times 40\% = 2$, regardless of the amount of invested capital I_0.

The invariance result also allows us rewrite the portfolio rule (30) as

$$\frac{\Pi(A,t)}{I_0} = \frac{\eta}{\overline{\gamma} r \sigma_R} - \beta a v_a(a,t). \tag{35}$$

The left-hand side of (35) is the ratio between the allocation to the risky market portfolio and the PE investment I_0. On the right-hand side, the first term is the standard mean-variance term from Merton (1971), and the second term is the dynamic hedging of the PE investment.

5.1 Cost of investing with unskilled manager

Applying the invariance result, we focus on the properties of $v(a,t)$ and its delta, $v_a(a,t)$ (note $V_A(A,t) = v_a(a,t)$). Figure 1 illustrates the case where the GP is unskilled, $\alpha = 0$.

[4]This does not imply the law of one price, which does not hold for this model, as discussed earlier.

[5]A similar invariance result holds if we use W_0 instead of I_0 in Proposition 2. We choose I_0, because it simplifies the exposition of our results.

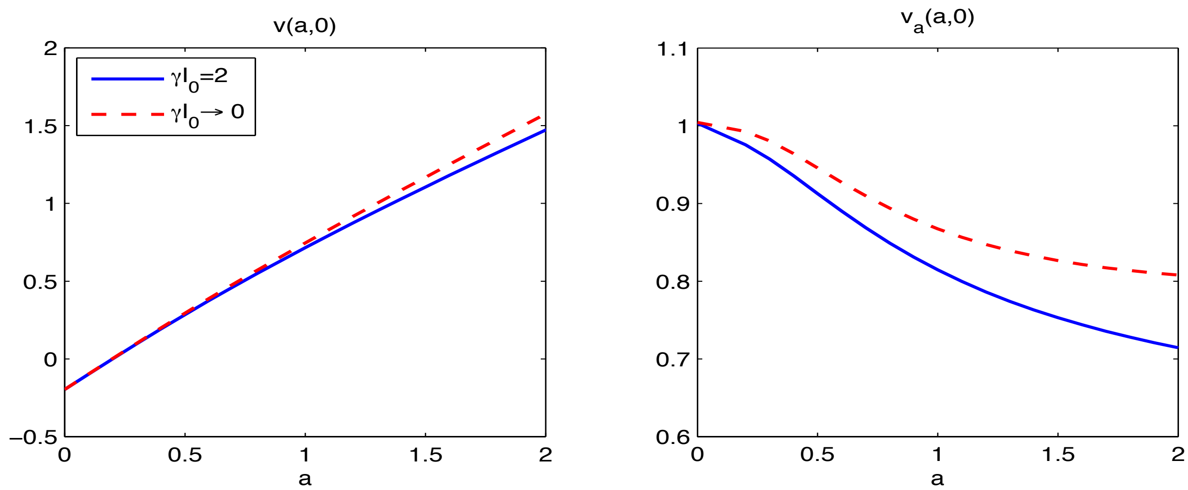

Figure 1: Certainty equivalent valuation $v(a, 0)$ and delta $v_a(a, 0)$ when $\alpha = 0\%$

With CM, the idiosyncratic risk premium for the LP vanishes, and the valuation coincides with the Black-Scholes pricing formula (20).[6] With no excess returns ($\alpha = 0$) but the costs of management fees and carried interest, $v(1, 0) < 1$, meaning that a rational LP would not voluntarily invest in PE. In our baseline calibration, $v(1, 0) = 0.746$, stating that the LP loses 25.4% of the invested capital I_0 when the GP has no skill. Of these 25.4%, the 19.7% are due to management fees, and the remaining 5.7% are due to incentive fees.

In other words, without skill, an LP's initial investment of $I_0 = \$100$ (equivalent to committed capital of $X_0 = \$125$ with \$25 of management fees paid over a ten-year period) has an economic value of \$74.6 (net of fees and carried interest) to the LP. Management fees and carried interest imply that an LP investing with unskilled GP loses 25 cents for each dollar invested in the underlying PE investment, I_0. As discussed below, to break even ($v(1, 0) = 1$), the LP requires $\alpha = 2.61\%$.

A risk-averse LP demands an additional idiosyncratic risk premium. For $\overline{\gamma} = 2$, $v(1, 0) = 0.715$. This extra 3.1% discount reflects the idiosyncratic risk premium. Hence, management fees, carried interest and idiosyncratic risk premium jointly imply that a risk-averse LP with $\overline{\gamma} = 2$ investing with unskilled GP loses 29 cents for each dollar invested in the underlying PE investment, I_0. In this case, the break-even alpha is 3.08%.

[6]The limit case when $\gamma \to 0$ has the same certainty equivalent valuation as the complete-markets case does. See Chen, Miao and Wang (2010) for a similar observation.

Panel B in Figure 1 shows that a more risk averse LP has a lower delta $v_a(a,t)$, which implies that the LP's certainty equivalent is less sensitive to changes in the value of the underlying asset. Intuitively, the more risk averse LP values the PE asset less and has a flatter valuation $v(a,t)$. Hence, we have the seemingly counter-intuitive result that a more risk averse LP has a lower hedging demand.

5.2 Performance measures

Because of management fees, carried interest, and idiosyncratic risks, the LP will only invest with skilled managers. We define the break-even alpha as the minimal level of alpha that the GP must generate in order for the LP to participate. This break-even alpha solves $I_0 = V(A_0, 0)$, which is equivalent to $v(1,0) = 1$. Given $\overline{\gamma}$, the break-even alpha is independent of the amount of invested capital I_0. In the baseline calibration, the break-even alphas are 2.61% for $\overline{\gamma} = 0$ and 3.08% for $\overline{\gamma} = 2$.

The alpha generated by a GP is difficult to observed directly, and more readily observable performance measures are used in practice. Harris, Jenkinson, and Kaplan (2011) summarize studies with empirical estimates of the three most common performance measures:[7] the internal rate of return (IRR), the total-value-to-paid-in-capital multiple (TVPI), and the public market equivalent (PME).

To define these performance measures divide the cash flows between the LP and GP into capital calls and distributions. Capital calls, $Call_t$, are cash flows from the LP to the GP, and distributions, $Dist_t$, are those from the GP to the LP. The performance measures are then defined as follows: IRR solves $1 = \sum \frac{Dist_t}{\mathrm{IRR}^t} / \sum \frac{Call_t}{\mathrm{IRR}^t}$, TVPI $= \sum Dist_t / \sum Call_t$, and PME $= \sum \frac{Dist_t}{r_t} / \sum \frac{Call_t}{r_t}$, where r_t is the cumulative realized return on the market portfolio up to time t. PME is the value of distributed capital relative to called capital, discounted by the realized market return. Assuming $\beta = 1$, the empirical studies typically interpret PME > 1 as the PE investment outperforming the market.

Analytical performance measures. In our model, we can solve for the analytical counterparts to these performance measures. The LP's required IRR for the PE asset, denoted

[7] These studies include Driessen, Lin, Phalippou (2011), Jegadeesh et al. (2009), Kaplan and Schoar (2005), Korteweg and Sorensen (2010), Ljungqvist and Richardson (2003), Metrick and Yasuda (2010), Phalippou and Gottschalg (2009), Robinson and Sensoy (2011), and Stucke (2011).

ϕ, solves,

$$I_0 + \int_0^T mX_0 e^{-\phi t} dt = e^{-\phi T} \mathbb{E}\left[LP_1(A_T, T) + LP_2(A_T, T) + LP_3(A_T, T)\right] , \tag{36}$$

which simplifies to

$$\begin{aligned} & I_0 + \frac{mX_0}{\phi}\left(1 - e^{-\phi T}\right) = \\ & e^{-(\phi-\mu_A)T}\left[A_0 - n\left(EC(A_0; F) - EC(A_0; Z)\right) - kEC(A_0; Z)\right] . \end{aligned} \tag{37}$$

Here, $EC(A; K)$ is the expected payoff of a call option with strike price K under the physical measure, as given in (A.14) in the Appendix. The expression for $EC(A; K)$ looks similar to the Black-Scholes formula, but it calculates the expected payoff of a call option under the physical measure, not the risk-neutral one.

The ex-ante expected TVPI is defined as

$$\mathbb{E}[\text{TVPI}] \quad = \quad \frac{\mathbb{E}\left[LP_1(A_T, T) + LP_2(A_T, T) + LP_3(A_T, T)\right]}{X_0} , \tag{38}$$

where the numerator is the LP's expected payoff net of carried interest, and the denominator is the total committed capital. The solution is

$$\mathbb{E}[\text{TVPI}] \quad = \quad \frac{e^{\mu_A T}\left[A_0 - n\left(EC(A_0; F) - EC(A_0; Z)\right) - kEC(A_0; Z)\right]}{X_0} . \tag{39}$$

In our model, the PME is defined as

$$\begin{aligned} \text{PME} \quad &= \quad \frac{\mathbb{E}\left[e^{-\mu_R T}\left(LP_1(A_T, T) + LP_2(A_T, T) + LP_3(A_T, T)\right)\right]}{I_0 + \mathbb{E}\left[\int_0^T e^{-\mu_R s} mX_0 ds\right]} \\ &= \quad \frac{e^{(\mu_A - \mu_R)T}\left[A_0 - n\left(EC(A_0; F) - EC(A_0; Z)\right) - kEC(A_0; Z)\right]}{I_0 + \frac{mX_0}{\mu_R}\left(1 - e^{-\mu_R T}\right)} . \end{aligned} \tag{40}$$

There are three concerns with the standard PME measure. First, the denominator combines two types of cash flows, the investment I_0 and the management fees. Management fees are effectively a risk-free claim and should be discounted at a rate close to the risk-free rate. Second, the numerator contains the total proceeds net of carried interest. The carried interest is effectively a call option, making the LP's total payoff at maturity less risky than the underlying asset. Hence, it should be discounted at a lower rate than the underlying PE investment. Finally, the beta of the PE investment may not equal one.

To address these concerns, we define the adjusted PME as follows,

$$\text{Adj. PME} = \frac{\widetilde{\mathbb{E}}\left[e^{-rT}\left(LP_1(A_T,T) + LP_2(A_T,T) + LP_3(A_T,T)\right)\right]}{I_0 + \widetilde{\mathbb{E}}\left[\int_0^T e^{-rs} m X_0 ds\right]}. \tag{41}$$

Here, $\widetilde{\mathbb{E}}[\,\cdot\,]$ denotes the expectation under the risk-adjusted measure, analogous to the Black-Scholes option-pricing methodology.[8] We treat management fees as a risk-free claim, discounted at the risk-free rate. We discount the carried interest and the underlying PE investment, taking into account their different risks. Finally, we allow the underlying PE investment to have arbitrary beta. Our adjusted PME captures the systematic risks of these different cash-flow components. It does not capture the effects of idiosyncratic risk. More precisely, when an LP's risk aversion approaches zero, an adjusted PME exceeding one is equivalent to positive performance. With risk aversion, an adjusted PME exceeding one is a necessary, but not sufficient, condition for positive performance.

In our model, the adjusted PME is

$$\text{Adj. PME} = \frac{e^{\alpha T}\left[A_0 - n(RC(A_0;F) - RC(A_0;Z)) - kRC(A_0;Z)\right]}{I_0 + \frac{mX_0}{r}\left(1 - e^{-rT}\right)}, \tag{42}$$

where $RC(A;K)$ is the expected payoff of a call option under the risk-adjusted measure, given in (A.18) of the Appendix.

5.3 Break-even performance

Table 1 reports break-even values of the various performance measures. The break-even alpha solves $v(1,0) = 1$ for various levels of risk aversion and beta. The break-even values of IRR, TVPI, PME, and adjusted PME are implied by the corresponding break-even alpha.

Insert Table 1 here.

Effectively risk-neutral LP. For an effectively risk-neutral LP ($\overline{\gamma} \to 0$), the break-even alpha is 2.61%. The GP must generate this excess return to compensate the LP for management fees and carried interest. This break-even alpha is independent of β, because

[8] Under the risk-adjusted measure $\widetilde{E}[\,\cdot\,]$, the calculation is analogous to the expectation in Black-Scholes, but it allows for positive alpha. With CM and no alpha, this pricing formula is the Black-Scholes formula.

alpha is the risk-adjusted excess return and there is no premium for idiosyncratic risk. The break-even IRR, however, increases with beta, because IRR does not account for risk. Quantitatively, the break-even IRR increases in lockstep with β. For example, as β increases from 0.5 to 1, the systematic risk premium increases by $0.5 \times 6\% = 3\%$, and the IRR also increases by this amount. Similarly, the break-even TVPI and PME also increase with β, reflecting the increasing expected return.

Empirically, a PME exceeding one is typically interpreted as earning positive economic value for the LPs. Table 1 shows that the conventional interpretation of this measure is misleading, when β differs from one. For example, with $\beta = 0$, the PME only needs to exceed 0.57 for the LPs to earn positive economic value.

In contrast, the break-even value of the adjusted PME, our newly constructed measure, equals one at all levels of β, because this measure appropriately accounts for the systematic risks of different tranches (e.g. management fees, carried interest) of the cash flows.

Hence, an effectively risk-neutral LP participates in the PE investment as long as the investment's alpha exceeds the break-even alpha, 2.61%, or the adjusted PME exceeds one, regardless of the systematic risk exposure. In contrast, the interpretations of IRR, TVPI, and conventional PME measures depend on the investment's beta.

Risk-averse LP. A risk-averse LP demands additional premium for bearing idiosyncratic risks. As the LP's risk aversion increases, the required compensation increases, and the break-even alpha increases. For example, with $\beta = 0.5$, the break-even alpha, increases by 47 basis points, from 2.61% to 3.08%, as $\overline{\gamma}$ increases from 0 to 2. Correspondingly, the break-even IRR increases by almost the same amount. The break-even values of the remaining measures also increase with risk aversion. For the adjusted PME, the break-even value now exceeds one due to the idiosyncratic-risk premium. To illustrate, as $\overline{\gamma}$ increases from 0 to 2, the break-even adjusted PME increases from 1 to 1.04.

In Table 1, the total volatility is constant at $\sigma_A = 25\%$. Recall that the systematic component of the volatility is $\rho\sigma_A = \beta\sigma_R$ and thus the idiosyncratic volatility is $\epsilon = \sqrt{\sigma_A^2 - \beta^2\sigma_R^2}$. As beta increases, the idiosyncratic volatility ϵ decreases, which causes the break-even alpha to decline. To illustrate, for $\overline{\gamma} = 2$, the break-even alpha decreases by 56 basis points from 3.17% to 2.61% as beta increases from 0 to 1.25.

In our calibration, $\beta = 1.25$ is the case with no idiosyncratic risk, $\epsilon = 0$. With no idiosyncratic risk premium, the break-even alpha equals 2.61%, regardless of risk aversion $\overline{\gamma}$. The other performance measures, imputed from this break-even alpha, also remain unchanged.

Absent leverage, the primary determinants of the break-even alpha appear to be the management fees and carried interest. The idiosyncratic risk has a smaller effect.

6 Leverage

Motivating Leverage The use of leverage in PE has presented a puzzle. Axelson, et al. (2011) show that the patterns of leverage in PE deals is largely independent of the patterns of publicly traded companies. PE firms appear to be primarily driven by credit markets, and employ as much leverage as creditors permit. Under the standard Modigliani-Miller, argument, greater leverage allows shareholders to earn a higher expected return but at a higher risk, these effects exactly offset, leaving the shareholders indifferent. To overcome this, Axelson, Stromberg, and Weisbach (2009) present a model of ex-ante and ex-post discipline to explain the use of leverage. Our results provide a simpler explanation. With incomplete markets and GP ability, the benefit of leverage is that it allows the GP to apply their value-adding ability across a wider asset base, and the cost of leverage is that it exposes to LP to greater idiosyncratic and systematic risks. But there is no reason to expect these two effects to cancel. Our findings suggest that the first effect dominates, and a simple explanation for the use of leverage is that it permits GPs to apply their value adding ability across a wider asset base.[9] The economic cost of the corresponding increases in risk appears comparatively modest.

6.1 Pricing with leverage

At time 0, let l denote the debt-equity ratio. For a given amount of invested capital I_0 (equity), the value of debt equals $D_0 = lI_0$, and the total PE asset equals $A_0 = (l+1)I_0$. We next price debt from the perspective of well diversified risk-averse investors.

Debt pricing. Our model applies to general forms of debt, but for simplicity we consider balloon debt with no intermediate payments and where the principal and all accrued interest

[9] Our model assumes constant returns to scale of the GP's value-adding ability. This assumption is not essential for the argument, and the same logic applies with decreasing returns.

are due at maturity T. Let y denote the yield for the debt, which we derive below. The payment at maturity T is the sum of principal and compounded interest,

$$D(A_T, T) = \min\left\{A_T, D_0 e^{yT}\right\} . \tag{43}$$

Intuitively, the expected rate of return on debt is higher than the risk-free rate because it is a risky claim on the underlying PE asset. Additionally, the PE asset has alpha, which implies that this debt has a higher value than the one implied by the standard Black-Scholes pricing formula. In the appendix, we show that debt is priced by

$$rD(A,t) = D_t(A,t) + (r+\alpha)AD_A(A,t) + \frac{1}{2}\sigma_A^2 A^2 D_{AA}(A,t)\,, \tag{44}$$

subject to the boundary condition (43). Despite the resemblance to the Black-Scholes debt pricing formula, our debt pricing formula is fundamentally different. Unlike Black-Scholes, our model allows for positive α, and hence the risk-adjusted drift is $r+\alpha$. A positive alpha is critical, because it allows the LP to break even given fees, illiquidity, and idiosyncratic risk. Like Black-Scholes, the risk-adjusted expected return on the risky debt equals the risk-free rate. The market value of debt at time 0 is then given by

$$D_0(A) \quad = \quad e^{\alpha(T-t)}\left[A - RC(A; D_0 e^{yT})\right] , \tag{45}$$

where $RC(A;\cdot)$ is the expected payoff of a call option under the risk-adjusted measure, given in (A.18) in the Appendix.

We solve for the equilibrium yield $y(A_0, T)$ as follows. We calculate the candidate final payoff in (43) using the original debt D_0 and a candidate y. The resulting candidate value of debt at time zero $\widehat{D}_0$ then follows from the pricing formula (45), and the equilibrium yield is determined when $\widehat{D}_0 = D_0$.

Boundary conditions with leverage. With leverage, the LP's claim is junior to the debt, but it can still be valued as three tranches over three regions. For the preferred-return region, due to the seniority of debt, the LP's payoff equals the difference between the payoffs of two call options,

$$LP_0(A_T, T) = \max\left\{A_T - D_0 e^{yT}, 0\right\} - \max\left\{A_T - \left(D_0 e^{yT} + F\right), 0\right\} , \tag{46}$$

where F is given in (8).

In the catch-up region, the LP's incremental payoff equals the fraction $(1 - n)$ of the difference between the payoffs of two other call options,

$$LP_1(A_T, T) = (1 - n) \left[\max\left\{A_T - D_0 e^{yT} - F, 0\right\} - \max\left\{A_T - Z_{Lev}, 0\right\}\right] , \tag{47}$$

where Z_{Lev} solves

$$k\left(Z_{Lev} - D_0 e^{yT} - X_0\right) = n\left(Z_{Lev} - D_0 e^{yT} - F\right) . \tag{48}$$

Here, Z_{Lev} is the level of A_T such that the GP catches up.

Finally, in the profit region, the LP's incremental payoff equals the fraction $(1 - k)$ of the payoff of the option with strike price Z_{Lev},

$$LP_2(A_T, T) = (1 - k) \max\left\{0, A_T - Z_{Lev}\right\} . \tag{49}$$

As in the case without leverage, the LP's total payoff equals $V(A_T, T) = LP_0(A_T, T) + LP_1(A_T, T) + LP_2(A_T, T)$.

In the appendix, we derive expressions for the performance measures, IRR, TVPI, PME, and adjusted PME with leverage. These are analogous to the expressions without leverage, albeit more complex. We use these formulas in the following analysis.

6.2 Effects of Risk Aversion and Risk

Tables 2 and 3 report break-even values of the performance measures for various degrees of leverage. Because of leverage, the break-even alpha now solves $v(1 + l, 0) = 1$, and the break-even IRR, TVPI, PME, and adjusted PME are implied by this break-even alpha.

Insert Tables 2 and 3 here.

Effectively risk-neutral LP. We first consider the effects of systematic risk on the performance measures. For an effectively risk-neutral LP, the idiosyncratic risk is not priced. Tables 2 and 3 report break-even performance measures when initial debt-equity ratio of 1 and 3 ($l = 1, 3$), respectively. The patterns are largely similar. Quantitatively, the effects are stronger with higher leverage. Unlike the case without leverage, we now have the credit spread for debt.

The first column in Table 2, with $l = 1$, shows that the break-even alpha is 1.68% compared to 2.61% without leverage, a reduction of 93 basis points. This break-even alpha is independent of beta, because alpha measures the excess return adjusted for systematic risk. The credit spread $y - r$ for debt is 1.05%. Importantly, this spread is also independent of beta because debt is priced as a derivative on the underlying PE asset with alpha.

As the unlevered β increases from 0.5 to 1, the PE asset's systematic risk premium increases by $0.5 \times 6\% = 3\%$, and consequently the break-even IRR for the LP increases by 4.2%, which is higher than 3% due to leverage. The break-even TVPI and PME are more sensitive to changes in β with leverage than without. For example, as we increase beta from 0.5 to 1, without leverage, the TVPI increases from 2.07 to 2.73. With leverage, however, the TVPI increases from 2.43 to 3.60, which is much more significant. Moreover, the PME increases from 0.88 to 1.30, and the usual interpretation of a PME exceeding one as equivalent to outperformance may be more misleading with leverage. For the adjusted PME, the break-even value equals one at all levels of β, because this measure appropriately accounts for systematic risk and leverage.

Risk-averse LP. As risk aversion increases, the LP demands additional idiosyncratic risk premium, and the break-even alpha increases. In Table 2, with $\beta = 0.5$, the break-even alpha increases by 78 basis points, from 1.68% to 2.46%, as $\overline{\gamma}$ increases from 0 to 2. Correspondingly, the break-even IRR increases by 1.2%, and the break-even values of the remaining measures also increase. In contrast, the credit spread declines, because a higher alpha increases the debt value, implying a lower spread. In the table, total volatility σ_A is constant, hence an increase in beta reduces idiosyncratic volatility, which also reduces the break-even alpha. The effect of beta on the break-even alpha is also significant. For example, as we increase the unlevered beta from 0.5 to 1, the break-even alpha decreases from 2.46% to 2.03% for $\overline{\gamma} = 2$. Finally, as the case without leverage, $\beta = 1.25$ implies no idiosyncratic risk, and in this case all performance measures remain independent of risk aversion. Table 3 shows that all of these effects become more pronounced as leverage increases.

In sum, with leverage, the management fees and carried interest are no longer the primary determinants of the break-even alpha. The break-even values now also heavily depend on the magnitudes of risk aversion and beta.

6.3 Effect of Leverage

Insert Tables 4 and 5 here.

Tables 4 demonstrates the effects of leverage for an effectively risk-neutral LP ($\overline{\gamma} \to 0$). We consider various leverage levels of l from 0 to 9. The effect of leverage on the break-even alpha is substantial. The break-even alpha decreases from 2.61% when $l = 0$ to 1.00% when $l = 3$, and this annual alpha compounds over the life of the fund. Note that the decline on the break-even alpha is independent of the unlevered β. The economic intuition for this decline is as follows. With greater leverage, the GP can apply the alpha across more assets, and hence a lower alpha is required to generate sufficient returns to compensate the LPs. The classical Modigliani-Miller argument would say that the LPs should be indifferent to leverage, but this argument does not hold with positive alpha and fees. Increasing leverage allows the GP to generate alpha for a larger amount of assets and charge lower fees per dollar of assets under management. To illustrate, by simply leveraging three times, the unlevered return for the asset value A_T that the GP needs to generate in order for the LP to break even is reduced by 15%.

The adjusted PME equals one for all cases, because this measure appropriately accounts for the systematic risk and the effectively risk neutral LP demands no idiosyncratic risk premium. The credit spread increases with the amount of leverage. The leverage effect on credit spread is significant. For example, the credit spread is 3.48% when $l = 3$.

For beta greater than zero, the IRR, TVPI, and PME increase with leverage. With $l = 3$ and the unlevered $\beta = 0.5$, the LP's IRR equals 11.2% increasing from 7.9% by 3.3%. The TVPI increases to 2.81 from 2.07 when $l = 0$. The PME increases to 1.02 from 0.75 when $l = 0$. For the special case with $\beta = 0$, levered beta equals zero and hence IRR, TVPI and PME do not change with leverage (see Panel A).

Importantly, as the break-even alpha, credit spread is independent of the unlevered beta because debt is effectively priced as a derivative on the underlying PE asset. However, unlike the standard derivatives analysis, our underlying PE asset is not tradable and hence admits a positive alpha. Other performance measures including IRR, TVPI and PME all increase with the unlevered beta.

6.4 Comparison to empirical findings

We now compare the break-even values of the performance measures to the actual performance of PE investors. Axelson, Jenkinson, Stromberg, and Weisbach (2007) consider 153 buyouts during 1985–2006, and report that, on average, equity accounted for only 25% of the purchase price, which corresponds to $l = 3$ in our model. Turning to the unlevered beta, a number of studies report levered betas ranging from a low of 0.7–1.0 (Jegadeesh, Kraussl, and Pollet (2010)) to a high of 1.3 (Driessen, Lin, and Phalippou (2011)). For a levered beta of one and assuming a debt beta of zero for simplicity, we obtain an implied unlevered beta of 0.25. This estimate seems unreasonably low because typical PE funds target firms whose risks are comparable to publicly traded ones. Using an average debt-equity ratio of 0.5 for a representative firm with levered beta of one, we obtain an approximate implied unlevered firm beta of 2/3. Hence, we use an unlevered beta of 0.5, which seems a reasonable starting point.

We view a sensible range of LP's risk aversion is $0 \leq \overline{\gamma} \leq 5$, with a preferable value around 1-2. These choices of leverage, unlevered beta, and risk aversion point to the second and third columns of Panel B in Table 3. The implied break-even values for the IRR are 12.7–13.8%, and for the TVPI are 3.24-3.61. Harris, Jenkinson, and Kaplan (2011) report separate performance figures for various datasets, with average value-weighted IRRs of 12.3-16.9%, which compare favorably to our breakeven IRR of 12.7% and 13.8% for $\overline{\gamma} = 1, 2$. Their value-weighted TVPI multiple, however, ranges from 1.76 to 2.30, which is somewhat lower than our break-even figure. The empirical and theoretical IRR and TVPI measures are difficult to compare, however, since they are absolute performance measures, and hence they are sensitive to the realized market performance over the life of the fund. In our calibration, we assume an excess market return of 6%, but actual market returns have varied substantially over the past decades.

The PME is closer to a relative performance measure. Harris, Jenkinson, and Kaplan (2011) report average PMEs of 1.20–1.27. These figures are very close to our break-even values of 1.17–1.30. While an average PME greater than one is often interpreted as evidence that LPs have outperformed the market, this out-performance appears to be almost exactly equal to the amount required to compensate a risk-averse LPs for illiquidity and idiosyncratic risk in our model.

7 Management fees and carried interest

We first quantify the effects of changing management fees and carried interest on the break-even alpha for the complete-markets case. Then, we turn to the incomplete-markets setting.

7.1 Complete markets

Table 6 presents changes in the break-even alpha as management and incentive fees change, focusing on the complete-markets case (or equivalently, the case where the LP is close to being risk neutral). Panel A shows the impact of management fees for different levels of the debt-equity ratio l. Without leverage ($l = 0$), the break-even alpha moves almost in lockstep with m. As m increases from 1.5% to 2.0% and again from 2.0% to 2.5%, in 50 basis-point steps, the break-even alpha increases from 2.1% to 2.6% and again from 2.6% to 3.1%, in 50 basis-point steps. With a debt-equity ratio of three ($l = 3$), the break-even alpha becomes less sensitive to m. The break-even alpha only increases by about 15-16 basis points from 0.85% to 1.00% and again from 1.00% to 1.16%, as m increases from 1.5% to 2.0% and again from 2.0% to 2.5%.

Panel B illustrates the impact of the carried interest, holding management fees fixed at $m = 2\%$. Without leverage ($l = 0$), the break-even alpha is less sensitive to k. As k increases from 20% to 25% and again from 25% to 30%, the break-even alpha increases from 2.61% to 2.83% and again to 3.07%, i.e.g, by 22-24 basis-point increments. The effect of a 5% increase of the incentive fee (carry) k has roughly half the effect on the break-even alpha as a 50 basis-point increase in the management fee m.

With a debt-equity ratio of three ($l = 3$), the break-even alpha becomes less sensitive to changes in k. In our example, the break-even alpha increases by about 16-18 basis points from 1% to 1.16% and further to 1.34%, as k increases from 20% to 25% and again to 30%. Thus, with a debt-equity ratio of three, the effect of a 5% increase in the incentive fee (carry) k has roughly the same effect on the break-even alpha as a 50 basis-point increase in the management fee m. Note that the break-even alpha is independent of beta under complete markets, although this independence no longer holds under incomplete markets.

These results provide an alternative comparison for different combinations of management versus incentive fees (carry). The results summarize a compensation structure into the required threshold of managerial skills, as measured by the break-even alpha, needed for the LP to break even. With a leverage of three, starting from a typical 2-20 compensation contract, an increase in the management fees m from 2.0% to 2.5% is roughly comparable to an increase in the incentive fees (carry) k from 20% to 25%.

7.2 Incomplete markets

Table 7 shows how the break-even alpha relates to management and incentive fees for a risk-averse LP with a relative risk aversion around two. With incomplete markets, the break-even alpha depends on the beta for the underlying PE asset, and we focus on the case of $\beta = 0.5$. Panel B shows the impact of management fees for different levels of the debt-equity ratio l. Without leverage ($l = 0$), the break-even alpha moves almost in lockstep with m. As m increases from 1.5% to 2.0% and again from 2.0% to 2.5%, in 50 basis-point steps, the break-even alpha increases from 2.56% to 3.08% and again from 3.08% to 3.65%, in 52-57 basis points increments. With a debt-equity ratio of three ($l = 3$), the break-even alpha is less sensitive to m. The break-even alpha now only increases by about 18-20 basis points, from 1.87% to 2.05% and again from 2.05% to 2.25%, as m increases from 1.5% to 2.0% and again from 2.0% to 2.5%.

Panel B in Table 8 illustrates the impact of increasing carried interest, fixing management fees at $m = 2\%$ and the unlevered beta at $\beta = 0.5$. Without leverage ($l = 0$), the break-even alpha is less sensitive to k than to m. As k increases from 20% to 25% and again from 25% to 30%, the break-even alpha increases from 3.08% to 3.29% and again from 3.29% to 3.50%, i.e., in 21 basis-point increments. The effect of a 5% increase of the incentive fee (carry) k has roughly 40% of the effect on the break-even alpha as a 50 basis-point increase in the management fee m.

Leverage makes the break-even alpha less sensitive to changes in k. In our example, with a debt-equity ratio of three ($l = 3$), the break-even alpha increases by about 13-14 basis points from 2.05% to 2.18% and again from 2.18% to 2.32%, as k increases from 20% to 25% and again from 25% to 30%. With a debt-equity ratio of three, the effect of a 5% increase in the incentive fee (carry) k has roughly comparable, but slightly smaller effects on the break-even alpha as a 50 basis-point increase in the management fee m (about 14 basis points versus 18-20 basis points).

To a first-order approximation, our incomplete-markets results corroborate the complete-markets results. Specifically, starting from a typical 2-20 compensation contract, an increase in the management fees m from 2.0% to 2.5% is roughly comparable to an increase in the incentive fees (carry) k from 20% to 25%. Both changes require an additional 20-25 basis points of alpha generated by the GP on the PE asset for the LP to remain indifferent.

8 Conclusion

We develop a first model of the asset allocation problem facing an institutional investor (LP), who invests in liquid assets (the composition between the risky market portfolio and the risk-free asset) as well as an illiquid long-term private equity (PE) investment. The model captures rich specifications of PE investments and managerial compensation structure. We incorporate two salient features of compensation, management fees and carried interest, as well as other related features including preferred returns, hurdle rate, and catch-up provisions. In addition to paying fees, the LP also incurs under-diversification costs from the significant idiosyncratic risk exposure to the PE investment as the PE asset is illiquid and markets are incomplete. In order for the GP to justify fees and LPs to break even, the GP needs to generate alpha and create value from the PE assets composed of the underlying portfolio companies.

Despite these rich institutional features, we derive tractable formulas for the optimal asset allocation rule and provide an analytical characterization for the LP's certainty equivalent valuation of the PE investment. As a first pass, we may view the GP's compensation, management fees and carried interest as a fixed-income claim and a call-option-type contract on the underlying PE investment, respectively. Using this analogy, we may apply the standard Black-Scholes option pricing formula to price various "tranches" of the cash flows that are distributed to the GP and the LP based on the contractual agreement between the two.

However, this Black-Scholes-based exercise misses at least two key institutional features for the PE. First, the GP has alpha which the standard Black-Scholes formula rules out by assumption. Second, the LP demands an idiosyncratic risk premium which is also ruled out by the complete-markets assumption in the Black-Scholes framework. Our formula for the LP's certainty equivalent valuation of the PE asset captures both the GP's alpha and the LP's idiosyncratic risk premium. Implied by the LP's optimal asset allocation rule, our valuation model generalizes the Black-Scholes option pricing formula with two new terms:

(1) the alpha term, which augments the risk-adjusted drift of the PE asset from the risk-free rate r to $r+\alpha$, and (2) the idiosyncratic risk premium term, which depends on the LP's risk aversion and idiosyncratic volatility. Because of incomplete markets, the law of one prices no longer holds in our model. Given the prevalent usage of leverage in the PE industry, we further extend our model to allow for external debt and price the debt accordingly.

Quantitatively, we find that common variations of the management fees and carried interest lead to significant changes in the break-even alpha, representing the required value added by the GP for LPs to break even. In our calibration, for an un-levered PE investment, this break-even alpha equals 2.6% annually. Importantly, leveraging the PE investment substantially reduces this break-even point. With an debt-equity ratio of three, the GP only needs to generate an alpha of 1.0% for LP to break even. Intuitively, leverage allows the to GP manage more assets and generate alpha on a greater asset base. Since management fees are calculated from the LP's committed capital, leverage reduces the *effective* management fee per dollar of managed PE assets. The decline in the break-even alpha with increasing leverage provides a new justification for the observed use of debt in PE transactions (given the existing structure of management contracts). Additionally, we show that both risk aversion and leverage have significant effects on the break-even alpha under incomplete markets.

Moreover, we quantify the effects of changing management compensation structure on the break-even alpha for the manager. We find that for commonly adopted leverage (with a debt-equity ratio around three for example), starting from a typical 2-20 compensation contract, an increase in the management fees m from 2.0% to 2.5% is roughly comparable to an increase in the incentive fees (carry) k from 20% to 25%.

Three new insights emerge from our analysis. First, we present a new risk-adjusted PME measure and shows that this measure can diverge substantially from the standard PME measures, raising concerns about the usual interpretation of a PME exceeding one as indicating outperforming the market. Even when the beta of the overall cash flows equals one (as typically assumed), the risks of the calls and distributions are substantially different, leading to a biased measure. Our results indicate that this bias may be substantial, and our new adjusted PME measure may better reflect the economic value of PE investments.

Second, the existing literature has largely evaluated the compensation structure by calculating the PV of the different components and comparing changes in these PVs as the term changes, for example by increasing the carried interest from 20% to 30%. These PVs and their changes are difficult to interpret, however, and we recast this analysis in terms of a more meaningful trade off. Specifically, our model naturally frames the question in terms of how much greater value a GP must generate, in terms of alpha, for an LP to be indifferent to a change in the terms.

Third, our results provide a new explanation for the leverage puzzle in PE. The existing literature has struggled with explaining the leverage patterns in PE. Axelson et al. (2011) report that leverage in PE appears largely unrelated to leverage patterns for publicly traded companies. To explain this apparent puzzle, Axelson, Stromberg, and Weisbach (2009) propose a model of ex-ante and ex-post monitoring. Our results, however, provide a different explanation. Our analysis indicates that a substantial benefit of leverage is that it allows GPs to apply their value-adding ability across a greater amount of assets, and hence effectively lowers the management fees per unit of assets under management.

The model assumes that all capital is invested initially and all exits are realized at maturity. The analysis does not capture the dynamics of investment decisions over the life of a fund as well as issues surrounding the valuation of unfounded liabilities (in our model, management fees are simply valued as a risk-free annuity). While it is straightforward to allow for deterministic investment dynamics as in Metrick and Yasuda (2010), modeling stochastic investment and exit processes, possibly depending on the market return, introduces substantial additional complexity.

Appendices

A Technical details

We first derive the complete-markets benchmark solution and then sketch out the derivation for the incomplete-market solution.

A.1 For complete-markets benchmark

$$
\begin{aligned}
V^*(A_t,t) &= \widetilde{\mathbb{E}}_t\left[\int_t^T e^{-r(s-t)}(-mX_0)ds + e^{-r(T-t)}V^*(A_T,T)\right] && \text{(A.1)}\\
&= \frac{mX_0}{r}\left(e^{-r(T-t)}-1\right) + e^{-r(T-t)}\widetilde{\mathbb{E}}_t\left[\min\{A_T,F\}+(1-k)\max\{0,A_T-Z\}\right]\\
&\quad + e^{-r(T-t)}\widetilde{\mathbb{E}}_t\left[(1-n)\left(\max\{A_T-F,0\}-\max\{A_T-Z,0\}\right)\right] && \text{(A.2)}
\end{aligned}
$$

Simplifying, we have

$$
V^*(A_t,t) = -\frac{mX_0}{r}\left(1-e^{-r(T-t)}\right) + A_t - n\left[BS(A_t,t;F)-BS(A_t,t;Z)\right] - kBS(A_t,t;Z)\,, \tag{A.3}
$$

where $BS(A_t,t;K)$ is the Black-Scholes call option pricing formula,

$$
BS(A_t,t;K) = A_tN(d_1(t;K)) - Ke^{-r(T-t)}N(d_2(t;K))\,, \tag{A.4}
$$

with

$$
\begin{aligned}
d_1(t;K) &= d_2(t;K) + \sigma_A\sqrt{T-t}\,, && \text{(A.5)}\\
d_2(t;K) &= \frac{\ln(\frac{A_t}{K}) + \left(r-\frac{\sigma_A^2}{2}\right)(T-t)}{\sigma_A\sqrt{T-t}}\,. && \text{(A.6)}
\end{aligned}
$$

A.2 Incomplete-markets solution

Solution after maturity T. After exiting from holding the illiquid asset, investors solve a classic Merton-type consumption and portfolio allocation problem by investing in the risk-free asset and the risky market portfolio. The wealth dynamics is given by

$$
dW_t = (rW_t - C_t)\,dt + \Pi_t\left((\mu_R - r)dt + \sigma_R dB_t^R\right)\,, \quad t \geq T\,. \tag{A.7}
$$

Let $J^*(W)$ denote investors' value function after time T, i.e.

$$J^*(W) = \max_{\Pi, C} \mathbb{E}\left[\int_T^\infty e^{-\zeta(s-T)} U\left(C_s\right) ds\right]. \tag{A.8}$$

The following HJB equation holds

$$\zeta J^*(W) = \max_{\Pi, C} \; U(C) + (rW + \Pi(\mu_R - r) - C) J^*(W)' + \frac{1}{2}\Pi^2 \sigma_R^2 J^*(W)''. \tag{A.9}$$

The FOCs for Π and C are

$$U_C(C) = J^*(W)', \tag{A.10}$$

$$\Pi = -\frac{(\mu_R - r) J^*(W)'}{\sigma_R^2 J^*(W)''}. \tag{A.11}$$

We conjecture that $J^*(W)$ is given by (23). Using the FOCs (A.10) and (A.11) for C and Π, we obtain the optimal consumption and portfolio allocation given in Proposition 1.

Solution before maturity T. Substituting (28) into the HJB equation (27), we obtain

$$\begin{aligned} -\frac{\zeta}{\gamma r} &= \max_{\Pi, C} -\frac{e^{-\gamma(C - r(W+b+V))}}{\gamma} + V_t + rW + \Pi(\mu_R - r) - mX_0 - C + \mu_A A V_A \\ &\quad + \frac{1}{2}\sigma_A^2 A^2 V_{AA} - \frac{\gamma r}{2}\left(\Pi^2 \sigma_R^2 + 2\rho\sigma_R\sigma_A \Pi A V_A + \sigma_A^2 A^2 V_A^2\right) \end{aligned} \tag{A.12}$$

Using the FOCs for C and Π, we have the optimal consumption and portfolio rules given in (29) and (30), respectively. After some algebras, we have ODE (31).

Derivation for Proposition 2. Substituting $V(A,t) = v(a,t) \times I_0$ into (31) and using $\overline{\gamma} = \gamma I_0$, $a = A/I_0$, $x_0 = X_0/I_0$, we obtain (32). Using (8), (9), and (14), we have (33). Finally, substituting $x_0 = X_0/I_0$ into (18), we obtain (34).

A.3 Technical details for various performance measures

For results in Section 5.2. Let $EC(A;K)$ denote the expected payoff of a call option with strike price K under the physical measure,

$$EC(A;K) = \mathbb{E}_0\left[e^{-\mu_A T}\max\left\{A_T - K, 0\right\}\right], \tag{A.13}$$

$$= AN(p_1(K)) - Ke^{-\mu_A T} N(p_2(K)), \tag{A.14}$$

where $p_1(K)$ and $p_2(K)$ are given by

$$p_1(K) = p_2(K) + \sigma_A\sqrt{T}\,, \tag{A.15}$$

$$p_2(K) = \frac{\ln(\frac{A}{K}) + \left(\mu_A - \frac{\sigma_A^2}{2}\right)T}{\sigma_A\sqrt{T}}\,. \tag{A.16}$$

Denote $RC(A;K)$ as the expected payoff of a call option with strike price K under the risk-adjusted measure defined in the text,

$$RC(A;K) = \widetilde{\mathbb{E}}_0\left[e^{-rT}\max\{A_T - K, 0\}\right]\,, \tag{A.17}$$

$$= e^{\alpha T}\left(AN(q_1(K)) - Ke^{-(r+\alpha)T}N(q_2(K))\right), \tag{A.18}$$

where

$$q_1(K) = q_2(K) + \sigma_A\sqrt{T}\,, \tag{A.19}$$

$$q_2(K) = \frac{\ln(\frac{A}{K}) + \left(r + \alpha - \frac{\sigma_A^2}{2}\right)T}{\sigma_A\sqrt{T}}\,. \tag{A.20}$$

For Section 6 with leverage. The market value of debt at time t is

$$D(A,t) = e^{\alpha(T-t)}A - \left[Ae^{\alpha(T-t)}N(\widehat{d}_1) - D_0e^{yT}e^{-r(T-t)}N(\widehat{d}_2)\right]\,, \tag{A.21}$$

where

$$\widehat{d}_1 = \widehat{d}_2 + \sigma_A\sqrt{T-t}\,, \tag{A.22}$$

$$\widehat{d}_2 = \frac{\ln(\frac{A}{D_0e^{yT}}) + \left(r + \alpha - \frac{\sigma_A^2}{2}\right)(T-t)}{\sigma_A\sqrt{T-t}}\,. \tag{A.23}$$

Incorporating the risky debt, we define the *ex-ante* TVPI as

$$\mathbb{E}[\text{TVPI}] = \frac{\mathbb{E}\left[LP_1(A_T,T) + LP_2(A_T,T) + LP_3(A_T,T)\right]}{X_0} \tag{A.24}$$

$$= \frac{e^{\mu_A T}\left[EC(A_0;D_0e^{yT}) - n\left(EC(A_0;F + D_0e^{yT}) - EC(A_0;Z_{Lev})\right) - kEC(A_0;Z_{Lev})\right]}{X_0}.$$

Similarly, the LP's ex-ante IRR solves the equation

$$I_0 + \int_0^T mX_0e^{-\phi t}dt = e^{-\phi T}\mathbb{E}\left[LP_1(A_T,T) + LP_2(A_T,T) + LP_3(A_T,T)\right]\,, \tag{A.25}$$

which simplifies to

$$
\begin{aligned}
& I_0 + \frac{mX_0}{\phi}\left(1 - e^{-\phi T}\right) \\
& = e^{(\mu_A - \phi)T}\left[EC(A_0; D_0 e^{yT}) - n\left(EC(A_0; F + D_0 e^{yT}) - EC(A_0; Z_{Lev})\right) - kEC(A_0; Z_{Lev})\right] .
\end{aligned} \tag{A.26}
$$

The PME is then given by

$$
\begin{aligned}
\text{PME} &= \frac{\mathbb{E}\left[e^{-\mu_R T}\left[LP_1(A_T, T) + LP_2(A_T, T) + LP_3(A_T, T)\right]\right]}{\mathbb{E}\left[\int_0^T e^{-\mu_R s} mX_0 ds + I_0\right]} \\
&= \frac{e^{(\mu_A - \mu_R)T}\left[EC(A_0; D_0 e^{yT}) - n\left(EC(A_0; F + D_0 e^{yT}) - EC(A_0; Z_{Lev})\right) - kEC(A_0; Z_{Lev})\right]}{I_0 + \frac{mX_0}{\mu_R}\left(1 - e^{-\mu_R T}\right)} .
\end{aligned} \tag{A.27}
$$

The adjusted ex-ante PME is given by

$$
\begin{aligned}
\text{Adjusted PME} &= \frac{\widetilde{\mathbb{E}}\left[e^{-rT}\left[LP_1(A_T, T) + LP_2(A_T, T) + LP_3(A_T, T)\right]\right]}{\widetilde{\mathbb{E}}\left[\int_0^T e^{-rs} mX_0 ds + I_0\right]} \\
&= \frac{e^{\alpha T}\left[RC(A_0; D_0 e^{yT}) - n\left(RC(A_0; F + D_0 e^{yT}) - RC(A_0; Z_{Lev})\right) - kRC(A_0; Z_{Lev})\right]}{I_0 + \frac{mX_0}{r}\left(1 - e^{-rT}\right)} .
\end{aligned} \tag{A.28}
$$

Table 1: Effects of risk aversion on performance measures: The case with no leverage

		Panel A.	$\beta = 0$	
Risk Aversion ($\overline{\gamma}$)	0_+	1	2	5
Alpha (α)	2.61%	2.90%	3.17%	3.93%
IRR (ϕ)	5.0%	5.29%	5.6%	6.3%
$\mathbb{E}$[TVPI]	1.58	1.62	1.66	1.78
PME	0.57	0.59	0.60	0.64
Adj. PME	1.00	1.03	1.05	1.13

		Panel B.	$\beta = 0.5$	
Risk Aversion ($\overline{\gamma}$)	0_+	1	2	5
Alpha (α)	2.61%	2.85%	3.08%	3.74%
IRR (ϕ)	7.9%	8.2%	8.4%	9.0%
$\mathbb{E}$[TVPI]	2.07	2.12	2.16	2.30
PME	0.75	0.77	0.78	0.83
Adj. PME	1.00	1.02	1.04	1.11

		Panel C.	$\beta = 1$	
Risk Aversion ($\overline{\gamma}$)	0_+	1	2	5
Alpha (α)	2.61%	2.71%	2.82%	3.12%
IRR (ϕ)	10.8%	10.9%	11.1%	11.3%
$\mathbb{E}$[TVPI]	2.73	2.75	2.78	2.86
PME	0.99	0.99	1.00	1.03
Adj. PME	1.00	1.01	1.02	1.05

		Panel D.	$\beta = 1.25$	
Risk Aversion ($\overline{\gamma}$)	0_+	1	2	5
Alpha (α)	2.61%	2.61%	2.61%	2.61%
IRR (ϕ)	12.3%	12.3%	12.3%	12.3%
$\mathbb{E}$[TVPI]	3.13	3.13	3.13	3.13
PME	1.13	1.13	1.13	1.13
Adj. PME	1.00	1.00	1.00	1.00

Table 2: Effects of risk aversion on performance measures: The case with $l = 1$

		Panel A.	$\beta = 0$	
Risk Aversion ($\overline{\gamma}$)	0_+	1	2	5
Alpha (α)	1.68%	2.16%	2.60%	3.79%
IRR (ϕ)	5.0%	5.9%	6.6%	8.5%
Credit spread ($y - r$)	1.05%	0.93%	0.83%	0.61%
$\mathbb{E}$[TVPI]	1.58	1.71	1.83	2.19
PME	0.57	0.62	0.66	0.79
Adj. PME	1.00	1.08	1.16	1.39

		Panel B.	$\beta = 0.5$	
Risk Aversion ($\overline{\gamma}$)	0_+	1	2	5
Alpha (α)	1.68%	2.08%	2.46%	3.49%
IRR (ϕ)	9.6%	10.2%	10.8%	12.3%
Credit spread ($y - r$)	1.05%	0.95%	0.86%	0.66%
$\mathbb{E}$[TVPI]	2.43	2.57	2.72	3.13
PME	0.88	0.93	0.98	1.13
Adj. PME	1.00	1.07	1.13	1.33

		Panel C.	$\beta = 1$	
Risk Aversion ($\overline{\gamma}$)	0_+	1	2	5
Alpha (α)	1.68%	1.86%	2.03%	2.51%
IRR (ϕ)	13.8%	14.0%	14.3%	14.9%
Credit spread ($y - r$)	1.05%	1.00%	0.96%	0.85%
$\mathbb{E}$[TVPI]	3.60	3.68	3.77	4.01
PME	1.30	1.33	1.36	1.45
Adj. PME	1.00	1.03	1.06	1.14

		Panel D.	$\beta = 1.25$	
Risk Aversion ($\overline{\gamma}$)	0_+	1	2	5
Alpha (α)	1.68%	1.68%	1.68%	1.68%
IRR (ϕ)	15.7%	15.7%	15.7%	15.7%
Credit spread ($y - r$)	1.05%	1.05%	1.05%	1.05%
$\mathbb{E}$[TVPI]	4.33	4.33	4.33	4.33
PME	1.56	1.56	1.56	1.56
Adj. PME	1.00	1.00	1.00	1.00

Table 3: Effects of risk aversion on performance measures: The case with $l = 3$

		Panel A.	$\beta = 0$	
Risk Aversion ($\overline{\gamma}$)	0_+	1	2	5
Alpha (α)	1.00%	1.67%	2.22%	3.71%
IRR (ϕ)	5.0%	7.1%	8.6%	12.1%
Credit spread ($y - r$)	3.48%	2.91%	2.53%	1.74%
$\mathbb{E}$[TVPI]	1.57	1.91	2.20	3.07
PME	0.57	0.69	0.79	1.11
Adj. PME	1.00	1.21	1.39	1.94
		Panel B.	$\beta = 0.5$	
Risk Aversion ($\overline{\gamma}$)	0_+	1	2	5
Alpha (α)	1.00%	1.58%	2.05%	3.33%
IRR (ϕ)	11.2%	12.7%	13.8%	16.5%
Credit spread ($y - r$)	3.48%	2.98%	2.64%	1.91%
$\mathbb{E}$[TVPI]	2.81	3.24	3.61	4.66
PME	1.02	1.17	1.30	1.68
Adj. PME	1.00	1.18	1.34	1.80
		Panel C.	$\beta = 1$	
Risk Aversion ($\overline{\gamma}$)	0_+	1	2	5
Alpha (α)	1.00%	1.28%	1.50%	2.11%
IRR (ϕ)	16.6%	17.2%	17.7%	18.9%
Credit spread ($y - r$)	3.48%	3.23%	3.04%	2.60%
$\mathbb{E}$[TVPI]	4.70	5.00	5.24	5.89
PME	1.70	1.81	1.89	2.13
Adj. PME	1.00	1.08	1.16	1.36
		Panel D.	$\beta = 1.25$	
Risk Aversion ($\overline{\gamma}$)	0_+	1	2	5
Alpha (α)	1.00%	1.00%	1.00%	1.00%
IRR (ϕ)	19.0%	19.0%	19.0%	19.0%
Credit spread ($y - r$)	3.48%	3.48%	3.48%	3.48%
$\mathbb{E}$[TVPI]	5.96	5.96	5.96	5.96
PME	2.15	2.15	2.15	2.15
Adj. PME	1.00	1.00	1.00	1.00

Table 4: Effects of leverage: The case with risk aversion $\overline{\gamma} = 0_{+}$

		Panel A.	$\beta = 0$		
Leverage (l)	0	1	3	6	9
Alpha (α)	2.61%	1.68%	1.00%	0.63%	0.46%
IRR (ϕ)	5.0%	5.0%	5.0%	5.0%	5.0%
Credit spread ($y - r$)	0	1.05%	3.48%	5.69%	7.14%
$\mathbb{E}$[TVPI]	1.58	1.58	1.58	1.58	1.58
PME	0.57	0.57	0.57	0.57	0.57
Adj. PME	1.00	1.00	1.00	1.00	1.00

		Panel B.	$\beta = 0.5$		
Leverage (l)	0	1	3	6	9
Alpha (α)	2.61%	1.68%	1.00%	0.63%	0.46%
IRR (ϕ)	7.9%	9.6%	11.2%	12.3%	13.0%
Credit spread ($y - r$)	0	1.05%	3.48%	5.69%	7.14%
$\mathbb{E}$[TVPI]	2.07	2.43	2.81	3.14	3.35
PME	0.75	0.88	1.02	1.13	1.21
Adj. PME	1.00	1.00	1.00	1.00	1.00

		Panel C.	$\beta = 1$		
Leverage (l)	0	1	3	6	9
Alpha (α)	2.61%	1.68%	1.00%	0.63%	0.46%
IRR (ϕ)	10.8%	13.8%	16.6%	18.7%	20.0%
Credit spread ($y - r$)	0	1.05%	3.48%	5.69%	7.14%
$\mathbb{E}$[TVPI]	2.73	3.60	4.70	5.77	6.53
PME	0.99	1.30	1.70	2.08	2.36
Adj. PME	1.00	1.00	1.00	1.00	1.00

		Panel D.	$\beta = 1.25$		
Leverage (l)	0	1	3	6	9
Alpha (α)	2.61%	1.68%	1.00%	0.63%	0.46%
IRR (ϕ)	12.3%	15.7%	19.0%	21.6%	23.1%
Credit spread ($y - r$)	0	1.05%	3.48%	5.69%	7.14%
$\mathbb{E}$[TVPI]	3.13	4.33	5.96	7.62	8.84
PME	1.13	1.56	2.15	2.75	3.19
Adj. PME	1.00	1.00	1.00	1.00	1.00

Table 5: Effects of leverage: The case with risk aversion $\overline{\gamma} = 2$

		Panel A.	$\beta = 0$		
Leverage (l)	0	1	3	6	9
Alpha (α)	3.17%	2.60%	2.22%	2.00%	1.97%
IRR (ϕ)	5.6%	6.6%	8.6%	11.0%	13.22%
Credit spread ($y - r$)	0	0.83%	2.53%	3.81%	4.40%
$\mathbb{E}$[TVPI]	1.66	1.83	2.20	2.75	3.42
PME	0.60	0.66	0.79	1.00	1.23
Adj. PME	1.05	1.16	1.39	1.74	2.16
		Panel B.	$\beta = 0.5$		
Leverage (l)	0	1	3	6	9
Alpha (α)	3.08%	2.46%	2.05%	1.86%	1.77%
IRR (ϕ)	8.4%	10.8%	13.8%	16.8%	19.1%
Credit spread ($y - r$)	0	0.86%	2.64%	3.96%	4.66%
$\mathbb{E}$[TVPI]	2.16	2.72	3.61	4.83	6.00
PME	0.78	0.98	1.30	1.74	1.17
Adj. PME	1.04	1.13	1.34	1.66	2.00
		Panel C.	$\beta = 1$		
Leverage (l)	0	1	3	6	9
Alpha (α)	2.82%	2.03%	1.50%	1.20%	1.08%
IRR (ϕ)	11.1%	14.3%	17.7%	20.6%	22.7%
Credit spread ($y - r$)	0	0.96%	3.04%	4.78%	5.76%
$\mathbb{E}$[TVPI]	2.78	3.77	5.24	6.94	8.49
PME	1.00	1.36	1.89	2.51	3.07
Adj. PME	1.02	1.06	1.16	1.30	1.46
		Panel D.	$\beta = 1.25$		
Leverage (l)	0	1	3	6	9
Alpha (α)	2.61%	1.68%	1.00%	0.63%	0.46%
IRR (ϕ)	12.3%	15.7%	19.0%	21.6%	23.1%
Credit spread ($y - r$)	0	1.05%	3.48%	5.69%	7.14%
$\mathbb{E}$[TVPI]	3.13	4.33	5.96	7.62	8.84
PME	1.13	1.56	2.15	2.75	3.19
Adj. PME	1.00	1.00	1.00	1.00	1.00

Table 6: Effects of management and incentive fees on break-even alpha: The case with risk aversion $\overline{\gamma} = 0_+$

Panel A. k=0.2			
l	$m = 1.5\%$	$m = 2\%$	$m = 2.5\%$
0	2.11%	2.61%	3.14%
1	1.41%	1.68%	1.99%
3	0.85%	1.00%	1.16%

Panel B. m=2%			
l	$k = 0.2$	$k = 0.25$	$k = 0.3$
0	2.61%	2.83%	3.07%
1	1.68%	1.90%	2.13%
3	1.00%	1.16%	1.34%

Table 7: Effects of management fees on break-even alpha: The case with risk aversion $\overline{\gamma} = 2$

Panel A. $\beta = 0$			
l	$m = 1.5\%$	$m = 2\%$	$m = 2.5\%$
0	2.65%	3.17%	3.74%
1	2.29%	2.60%	2.93%
3	2.04%	2.22%	2.43%

Panel B. $\beta = 0.5$			
l	$m = 1.5\%$	$m = 2\%$	$m = 2.5\%$
0	2.56%	3.08%	3.65%
1	2.16%	2.46%	2.79%
3	1.87%	2.05%	2.25%

Panel C. $\beta = 1$			
l	$m = 1.5\%$	$m = 2\%$	$m = 2.5\%$
0	2.31%	2.82%	3.37%
1	1.74%	2.03%	2.35%
3	1.34%	1.50%	1.68%

Table 8: Effects of incentive fees on break-even alpha: The case with risk aversion $\overline{\gamma} = 2$

Panel A. $\beta = 0$			
l	$k = 0.2$	$k = 0.25$	$k = 0.3$
0	3.17%	3.38%	3.59%
1	2.60%	2.78%	2.98%
3	2.22%	2.35%	2.49%

Panel B. $\beta = 0.5$			
l	$k = 0.2$	$k = 0.25$	$k = 0.3$
0	3.08%	3.29%	3.50%
1	2.46%	2.65%	2.85%
3	2.05%	2.18%	2.32%

Panel C. $\beta = 1$			
l	$k = 0.2$	$k = 0.25$	$k = 0.3$
0	2.82%	3.03%	3.26%
1	2.03%	2.23%	2.45%
3	1.50%	1.65%	1.81%

Table 9: Summary of Key Variables and Parameters

This table summarizes the symbols for the key variables in the model and baseline parameter values.

Variable	Symbol	Parameter	Symbol	Value
LP's Consumption or expenditure	C	Risk-free rate	r	5%
LP's Value Function	J	Expected return of market portfolio	μ_R	11%
LP's Value Function after exiting illiquid asset	J^*	Expected return of PE asset	μ_A	
LP's Certainty Equivalent	V	Volatility of market portfolio	σ_R	20%
Future value of investment and fees	F	Volatility of PE asset	σ_A	25%
Debt	D	Aggregate equity risk premium	$\mu_R - r$	6%
Wealth	W	Market Sharpe ratio	η	30%
Assets	A	Hurdle rate	h	8%
Brownian Motion for Market Return	B^R	Carried interest	k	20%
Brownian Motion for PE Return	B^A	Management fee	m	2%
Committed Capital	X_0	Catch-up rate	n	100%
Invested Capital	I_0	Life of PE investment	T	10
Market portfolio allocation	Π	Idiosyncratic risk premium	α	
		Correlation between market and PE asset	ρ	0.4
		Systematic Risk	β	0.5

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