B6014-Managerial Statistics

Solution to Gotham Bank Investment Services (B)

The following is a solution to the questions at the end of the Gotham Bank Investment Services (B) case. Questions (a)–(c) are fairly straightforward and are essentially what we have done in class. Questions (d)–(e) require a little bit of calculus and are therefore optional.

Let \( G \) be the return on the GM stock, let \( I \) be the return on the IBM stock, and let \( A \) be the return on the ALCOA stock.

(a) If you invest \$1 in GM, you expect a return of 13.6\% in one year. Thus your dollar is expected to grow to \((1 + 0.136) = \$1.136\).

(b) Let \( R \) be the random variable representing the return on a portfolio consisting of equal amounts in each stock. Then \( R \) can be written as:

\[
R = \frac{1}{3}G + \frac{1}{3}I + \frac{1}{3}A.
\]

Therefore:

\[
E(R) = E\left(\frac{1}{3}G + \frac{1}{3}I + \frac{1}{3}A\right)
\]

\[
= \frac{1}{3}E(G) + \frac{1}{3}E(I) + \frac{1}{3}E(A)
\]

\[
= \frac{1}{3}(0.136) + \frac{1}{3}(0.221) + \frac{1}{3}(0.129)
\]

\[
= 0.1622 = 16.22\%.
\]

The variance \( \sigma^2(R) \) can be found using:

\[
\sigma^2(R) = \left(\frac{1}{3}\right)^2(0.275)^2 + \left(\frac{1}{3}\right)^2(0.294)^2 + \left(\frac{1}{3}\right)(0.243)^2 + 2\left(\frac{1}{3}\right)(\frac{1}{3})(0.275)(0.294)(+0.354)
\]

\[
+ 2\left(\frac{1}{3}\right)(\frac{1}{3})(0.275)(0.294)(+0.362) + 2\left(\frac{1}{3}\right)(\frac{1}{3})(0.294)(0.243)(+0.357)
\]

\[
= 0.04197.
\]

Then \( \sigma(R) = 0.2049 \) or 20.49\%.

The interesting thing about this portfolio is that it has a higher expected return than the GM stock alone (16.2\% versus 13.6\%) and has a lower standard deviation (20.49\% versus 27.5\%). This portfolio is more efficient than the portfolio made up of GM stock alone.

(c) Let \( S \) be the return on the portfolio: 25\% in GM, 65\% in IBM and 10\% in AA. Then

\[
S = 0.25G + 0.65I + 0.10A.
\]
The expected return is:

\[
E(S) = E(0.25G + 0.65I + 0.10A) \\
= 0.25E(G) + 0.65E(I) + 0.10E(A) \\
= 0.25(0.136) + 0.65(0.221) + 0.10(0.129) \\
= 0.1908 = 19.08\%. 
\]

The variance can be found using:

\[
\sigma^2(S) = (0.25)^2(0.275)^2 + (0.65)^2(0.294)^2 + (0.10)^2(0.243)^2 + \\
+ 2(0.25)(0.65)(0.275)(0.294)(+0.354) + 2(0.25)(0.10)(0.275)(0.243)(+0.362) + \\
+ 2(0.65)(0.10)(0.294)(0.243)(+0.357) \\
= 0.05571. 
\]

Then \( \sigma(S) = 0.2360 \) or 23.60\%.

This portfolio has a higher expected return but also a higher standard deviation than the portfolio of part (b).

(d) Let \( x \) be the fraction invested in ALCOA, let \( y \) be the fraction invested in GM. Then 
\( 1 - (x + y) = 1 - x - y \) is the fraction invested in IBM. Let \( R \) be the return on this portfolio. Then

\[
R = xA + yG + (1 - x - y)I. 
\]

The expected return is:

\[
E(R) = E \left( xA + yG + (1 - x - y)I \right) \\
= xE(A) + yE(G) + (1 - x - y)E(I) \\
= x(0.129) + y(0.136) + (1 - x - y)(0.221) \\
= 0.221 - 0.092x - 0.085y. 
\]

We can set this to the desired value of 18\% and try to figure out what \( y \) is (as a function of \( x \)). Set:

\[
0.18 = 0.221 - 0.092x - 0.085y. 
\]

This leads to:

\[
y = \frac{41 - 92x}{85}. 
\]  \hspace{1cm} (1)

In order for the weights to represent a feasible portfolio, they must all be non-negative and at most 1. Therefore, we need the following:

- Condition #1: \( 0 \leq x \leq 1 \),
- Condition #2: \( 0 \leq y \leq 1 \),
- Condition #3: \( 0 \leq 1 - x - y \leq 1 \).
We will always choose \( x \) to satisfy \#1, and therefore we need only consider conditions \#2 and \#3.
For condition \#2, consider:

\[
0 \leq \frac{y}{1 - 92x} \leq 1 \\
0 \leq \frac{41 - 92x}{85} \leq 1 \\
0 \leq 41 - 92x \leq 85 \\
-41 \leq -92x \leq 44, \\
\frac{-44}{92} \leq x \leq \frac{41}{92} \\
0 \leq x \leq \frac{41}{92}.
\]

For condition \#3, consider:

\[
0 \leq 1 - x - y \leq 1 \\
0 \leq 1 - x - \frac{41 - 92x}{85} \leq 1 \\
0 \leq \frac{85 - 85x - 41 + 92x}{85} \leq 1 \\
0 \leq 7x + 44 \leq 85 \\
-\frac{44}{7} \leq x \leq \frac{41}{7}.
\]

One can see from the above derivation that the value \((7x + 44)/85\) represents the fraction we are investing in IBM (as a function of \(x\)).

Summarizing, the largest value possible for \(x\) is \(\min\{1, 41/92, 41/7\} = 41/92 = 44.57\%\), where \(41/92\) comes from equation (2) and \(41/7\) comes from equation (3). The smallest value possible for \(x\) is 0. Any \(x\) in the range between 0.0\% and 44.57\% defines a feasible portfolio with average return 18\%.

(e) Let’s now express the standard deviation of the family of portfolios we developed in (d). Since \(y\) now depends on \(x\) according to equation (1), each portfolio depends only on the value of \(x\), we therefore denote the portfolio returns by the random variable \(R_x\). As a function of \(x\) and the individual stock random variables, \(R_x\) is the following:

\[
R_x = \left(\frac{41 - 92x}{85}\right)G + \left(\frac{7x + 44}{85}\right)I + (x)A.
\]

By design, \(E(R_x) = 18\%\). What is \(\sigma(R_x)\) as a function of \(x\)? Let’s first calculate the variance:

\[
\sigma^2(R_x) = \left(\frac{41 - 92x}{85}\right)^2(0.275)^2 + \left(\frac{7x + 44}{85}\right)^2(0.294)^2 + x^2(0.243)^2 \\
+ 2\left(\frac{41 - 92x}{85}\right)\left(\frac{7x + 44}{85}\right)(0.275)(0.294)(+0.354) \\
+ 2\left(\frac{41 - 92x}{85}\right)(x)(0.275)(0.243)(+0.362) \\
+ 2\left(\frac{7x + 44}{85}\right)(x)(0.294)(0.243)(+0.357).
\]
Doing all these calculations, we get:

\[
\sigma^2(R_x) = 0.094962x^2 - 0.05165x + 0.05505,
\]

and

\[
\sigma(R_x) = \sqrt{0.094962x^2 - 0.05165x + 0.05505}.
\]

(4)

To minimize the standard deviation (as a function of \(x\)) we need only minimize what is inside the square root in equation (4) and also keep track of our range for \(x\): \(0 \leq x \leq 0.4457\). What is the value of \(x\) that minimizes the expression inside the square root? Call this \(x^*\). By taking the derivative and setting to zero, we get:

\[
2(0.094962)x^* - 0.05505 = 0 \quad \implies \quad x^* = \frac{0.05505}{2(0.094962)} \approx 0.290.
\]

Fortunately this falls in our allowable range for \(x\) and therefore it is not only the value that minimizes the standard deviation but it also represents a feasible portfolio. This corresponds to the following portfolio:

<table>
<thead>
<tr>
<th>Stock</th>
<th>Fraction Invested</th>
</tr>
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<tbody>
<tr>
<td>GM</td>
<td>(\frac{11 - 9x^<em>}{8x^</em>} = 16.8%)</td>
</tr>
<tr>
<td>IBM</td>
<td>(\frac{7x^* + 44}{8x^*} = 54.2%)</td>
</tr>
<tr>
<td>ALCOA</td>
<td>(x^* = 29.0%)</td>
</tr>
</tbody>
</table>

The standard deviation of this portfolio can be found by plugging \(x^*\) into equation (4). This gives:

\[
\sigma(R_{0.290}) = \sqrt{0.094962(0.290)^2 - 0.05165(0.290) + 0.05505} = \sqrt{0.048058} = 21.9\%.
\]

This portfolio also has an expected return of 18% as desired. Therefore this portfolio is on the efficient frontier because there is no way to get a return of 18% with a standard deviation equal to or lower than 21.9\% (with only these three stocks).