Practice Midterm Problems

Problems (A)

Problem 1. The Internal Revenue Service has studied the frequency of errors in tax form B-6014, a two-page form. The IRS has determined that the probability of an error on the first page is 0.08. Given that the first page has an error, the probability that the second page also has an error is 0.25; given that the first page does not have an error the probability that the second page does is 0.05. Let X be the number of pages with errors on a randomly selected form. Find $E[X]$.

Problem 2. You invest \$3 thousand in one stock and your spouse invests \$2 thousand in another. Over the next year, each dollar invested in your pick will increase by X dollars and each dollar invested in your spouse's will increase by Y dollars; X and Y are random variables with the following properties:

- X has a mean of 0.09 and a standard deviation of 0.20.
- Y has a mean of 0.12 and a standard deviation of 0.27.
- The correlation between X and Y is 0.6.
- X and Y are normally distributed.

Your individual earnings are $3X$ thousand, your spouse's individual earnings are $2Y$ thousand and your family earnings are the sum of two.

a. What is the expected value of your family earnings?

b. What is the standard deviation of your family portfolio earnings?

c. Find the probability that your family earnings exceed \$1 thousand.

Problems (B)

Problem 1. An oil exploration company takes X-wave readings to determine where to drill for oil. The company's geologists have determined that the X-wave reading above an oil reservoir is normally distributed with a mean of 8.3 and a standard deviation of 2.1. Where there is no oil, X-wave readings are normally distributed with a mean of 5.2 and a standard deviation of 1.3.
a. The company will drill for oil anywhere it finds an X-wave reading greater than x, with x to be determined. How should x be chosen to ensure that there is a 95% probability that the company will drill when it takes a reading above an oil reservoir?
b. If your answer to (a) is adopted, what is the probability that a reading taken at a site where there is no oil will result in a decision to drill?

Problem 2. Suppose you borrow \$1000 for one year at a variable interest rate tied to the yield on government bonds. As a result, the total interest you will pay (in dollars) is a random variable X_1, having mean 60 and standard deviation 2. You invest the borrowed money. Your earnings on the investment, X_2, have mean 85 and standard deviation 8. Suppose the correlation between your earnings and the interest you pay on the loan is 0.3.

a. Your *net* earnings at the end of the year are $Y = X_2 - X_1$. Find the expected value of your net earnings.

b. Find the standard deviation of your net earnings.

c. Find the probability that you earn enough on the investment to pay the interest, assuming X_1 and X_2 are normally distributed.

Problems (C)

Problem 1. The Hermes Cafe serves fresh shark fin at lunch and dinner. The chef orders three fins each day; any fins not served are discarded at the end of the day. The joint distribution of the lunch and dinner demand for the fin is as follows:

			Dinner Y		
			0	1	2
			0.25	0.50	0.25
	0	0.20	0.20	0	0
Lunch X	1	0.50	0.05	0.40	0.05
	2	0.30	0	0.10	0.20

The table also shows the marginal distributions of the lunch and dinner demands; i.e., the row and column sums of the joint distribution.

a. Find the expected lunch demand, the expected dinner demand, and the expected total demand in a day.

b. Find the expected value of the number of stockouts per day; i.e., the expected value of the number of demands that cannot be met because the fin is sold out.

c. According to the joint distribution above, are the lunch and dinner demands independent? Explain your answer.

Problem 2. Gotham Courier Service provides same-day delivery between any two points in New York City. Gotham is concerned about having packages delivered on time. From its records, it determines that the time to pick up and deliver a package is normally distributed with a mean of 3.8 hours and a standard deviation of 0.69 hours.

a. The company wants to advertise 95% on-time performance. To achieve this, it must set its promised interval for pick-up and delivery appropriately. Find the smallest x such that 95% of packages are delivered within x hours.

b. Promising customers delivery within a fractional number of hours (e.g., 6.3 hours), is not convenient. Find the smallest whole number (integer) n such that Gotham can promise delivery within n hours and be on time in at least 95% of cases.
c. If Gotham promises delivery within n hours, with n your answer to (b), what percentage of deliveries will be on time?

Problem 3. Suppose that the expected return on ABC stock over the next year is 9% and that the standard deviation is 4%. Suppose that for QRS stock the mean is 15% and the standard deviation is 7%. Suppose the returns on the two stocks have a correlation of .06. Finally, suppose the returns of the two stocks are normally distributed. You invest $10 thousand in each stock. (For simplicity, measure everything in thousands of dollars.)

a. What is the expected value of your investment at the end of the year?

b. What is the standard deviation of the value of your investment?

c. Suppose, now, that you have a student loan for \$24 thousand. Your primary investment objective is to earn enough to pay off the loan. So, you are primarily interested in the chances that the value of your investment will increase to \$24 thousand. Let Y be the value of your investment at the end of the year. Assume that Y is normally distributed.

Write "True" or "False" in the blanks provided to indicate which of the following changes in the two stocks will *definitely* increase the probability that Y exceeds \$24 thousand. (It is not necessary to calculate any probabilities.)

________ An increase in the expected returns with everything else held fixed.

________ An increase in the standard deviations with everything else held fixed.

________ An increase in the correlation with everything else held fixed.

Problems (D)

Problem 1. To help determine which tax returns to audit, the IRS uses an automated screening process that assigns a score to each return. The higher scoring returns are the ones deemed most likely to contain significant underpayment and are therefore most likely to be audited. From random audits, the IRS has determined that scores among returns with significant underpayment (SU returns) are normally distributed with mean 140 and standard deviation 20; scores among other returns (non-SU returns) are normally distributed with mean 100 and standard deviation 15. Overall, 8% of returns are SU returns.

The IRS will form a pool of potential returns to be audited, and then audit 5% of the pool. The pool will consist of all returns scoring x or higher, with x to be determined.

a. Find the value of x that will ensure that 85% of SU returns will be put in the pool.

b. If your answer to part (a) is adopted, what proportion of non-SU returns will be audited?

c. What proportion of all returns have a score of 135 or higher?

Problem 2. The Uris Index has a return of X_1 and the Hermes Index has a return of X_2, both over the next year. Suppose these random returns have the following features:

- X_1 has a mean of 0.15 and a standard deviation of 0.04.
- X_2 has a mean of 0.19 and a standard deviation of 0.08.
- The correlation between the two returns is 0.40.
- The returns are normally distributed.

a. A pension fund manager will invest a fraction p of the fund's wealth in X_1 and the remainder in X_2. Let Y be the pension fund's return. How should p be chosen to achieve an expected return $E[Y]$ of 0.16?

b. What is the resulting standard deviation of Y, i.e., of the pension fund's return?

Problems (E)

Problem 1. The foreign exchange trading arm of a bank holds 10 million units of currency A and owes 5 million units of currency B. Let

$$
\begin{aligned}
X_A &= \text{one-day change in number of dollars per million units of currency A;} \\
X_B &= \text{one-day change in number of dollars per million units of currency B.}
\end{aligned}
$$

Then the one-day change in the dollar value of the portfolio is $Y = 10X_A - 5X_B$. Suppose X_A has mean zero and standard deviation 0.25; suppose X_B has mean zero and standard deviation 0.10; and suppose X_A and X_B have correlation 0.65.

a. Find the expected value of Y.
b. Find the standard deviation of Y.
c. Now let $Y_1, Y_2, \ldots, Y_{20}$ be the changes in the value of the portfolio over the next 20 days. Suppose all the Y_is are independent and have the same mean and variance as the Y in parts (a) and (b). Let

$$Y_{\text{total}} = Y_1 + Y_2 + \cdots + Y_{20}$$

be the total change in the value of the portfolio over the next 20 days. Find the standard deviation of Y_{total}.

Problem 2. At a different bank, the one-day change in the dollar value of a foreign exchange portfolio is normally distributed with a mean of \$1.5 million and a standard deviation of \$9.7 million.

a. Find the value x such that the probability that the portfolio will lose more than x dollars in one day is 5%.

b. For the x you found in part (a), what is the probability that the portfolio will *increase* in value by more than x dollars in one day?

Problems (F)

Problem 1. A sales representative currently has contract proposals submitted to two companies, pending approval. The rep's assessment of the chances of approval for each of the proposals, and the corresponding values of the sales are as follows:

Company	Probability of Approval	Sales Revenue
A	0.25	\$30,000
B	0.4	\$15,000

The two companies will make their decisions independently; either, both, or neither may approve. The decisions will be made by the end of the month. Let X be the revenue brought in by the sales rep this month (from the pending contracts). Find the expected value of X. Find the standard deviation.

Problem 2. Stock 1 has an annual return X_1 that is normally distributed with mean 15% and standard deviation 20%. Stock 2 has an annual return X_2 that is normally distributed with mean 25% and standard deviation 35%. The correlation between the two returns is 0.20. Investor X puts \$10 thousand in Stock 2. Investor Y puts \$5 thousand in Stock 1 and \$5 thousand in Stock 2.

(a). Find the standard deviation of Investor Y's earnings.

(b). Find the standard deviation of the combined earnings of Investors X and Y.

Solutions (A)

1. There a four possible outcomes: error on first page only, error on both pages, error on second page only, no errors. Calculate the probabilities and the corresponding values of X as follows:

	Probability	X
1st only	(.08)(1-.25)=.06	1
both	(.08)(.75) = .02	2
2nd only	(1-.08)(.05) = .046	1
neither	(1-.08)(1-.05)=.874	0

Thus,

$$E[X] = (.06)(1) + (.02)(2) + (.046)(1) = 0.146.$$

2.a. $E[3X + 2Y] = 3E[X] + 2E[Y] = 3(.09) + 2(.12) = .51$.

2.b. $Var[3X + 2Y] = 9\sigma_X^2 + 4\sigma_Y^2 + 2(3)(2)\sigma_X\sigma_Y\rho = 1.04$. StdDev$[3X + 2Y] = \sqrt{1.04} = 1.02$.

2.c. Need to find $P(Earnings > 1)$ which is $P(3X + 2Y > 1)$. From (a) and (b) we know that Earnings are normal with mean 0.51 and standard deviation 1.02. Standardizing, we find that the probability we need is $P(Z > (1 - 0.51)/1.02)$ which is $P(Z > 0.48)$ which is $1 - 0.6844 = 0.3156$.

Solutions (B)

1.a. When a reading is taken above oil, $X \sim N(8.3, (2.1)^2)$. To ensure a 95% chance of drilling, need to set x so that $P(X > x) = 0.95$ (because the company drills whenever $X > x$). This requires that x be below the mean of X and the 95% requirement implies (from the table) that it should be 1.645 standard deviations below the mean: $x = 8.3 - 1.645 \cdot 2.1 = 4.845$.

1.b. Need to find $P(X > 4.845)$ where now $X \sim N(5.2, (1.3)^2)$. Standardizing, we find that this is the same as $P(Z > [4.845 - 5.2]/1.3) = P(Z > -0.27)$, $Z \sim N(0, 1)$. From the table we find this is 0.6064.

2.a. $E[Y] = E[X_2 - X_1] = E[X_2] - E[X_1] = 85 - 60 = 25$.

2.b. $Var[Y] = Var[X_2 + (-1)X_1] = \sigma_{X_1}^2 + \sigma_{X_2}^2 - 2\sigma_{X_1}\sigma_{X_2}\rho = 58.4$. So, $\sigma_Y = \sqrt{58.4} = 7.64$.

2.c. Need to find the probability that X_2 exceeds X_1 which is equivalent to having $X_2 - X_1 > 0$, i.e., $Y > 0$. Standardizing, we find that $P(Y > 0) = P(Z > (0 - 25)/7.64) = P(Y > -3.27) = 0.9995$.

Solutions (C)

1.a. $E[X] = 0 \cdot 0.2 + 1 \cdot 0.5 + 2 \cdot 0.3 = 1.1$. $E[Y] = 0 \cdot 0.25 + 1 \cdot 0.5 + 2 \cdot 0.25 = 1.0$. $E[X + Y] = E[X] + E[Y] = 2.1$.

1.b. The number of stockouts is 1 if the total demand is 4 and 0 otherwise. The probability of a total demand of 4 is given in the table as 0.20. Thus, the expected value is $0.2 \cdot 1 + (1 - 0.2) \cdot 0 = 0.2$.

1.c. The demands are not independent because the values in the table are not the products of the marginal probabilities. For example, $P(X = 0, Y = 0) = 0.20$ does not equal $P(X = 0)P(Y = 0) = 0.2 \cdot 0.25 = .05$, as would be required for independence.

2.a. Since $P(Z \leq 1.645) = .95$, we need to set the guaranteed level at 1.645 standard deviations above the mean: $x = 3.8 + (1.645)(.69) = 4.93$.

2.b. From 2a we know that nothing less than 4.93 will do; the best we can promise is 5 hours.

2.c. If they promise 5 hours, the on-time probability is $P(X < 5)$ which is $P(Z < [5 - 3.8]/0.69) = P(Z < 1.74) = 95.91\%$.

3.a. The value of your investment (in thousands of dollars) is

$$Y = 10(1 + X_A) + 10(1 + X_Q),$$

where the returns X_A, X_Q have the means, standard deviations and correlation given in the problem. Thus, $E[Y] = 10(1 + E[X_A]) + 10(1 + E[X_Q]) = 22.4$

3.b. To find the standard deviation, we first find the variance:

$$\begin{aligned}\sigma_Y^2 &= Var[10X_A + 10X_Q] \\ &= 100\sigma_A^2 + 100\sigma_Q^2 + 2(10)(10)\rho\sigma_A\sigma_Q \\ &= .6836,\end{aligned}$$

so $\sigma_Y = .8268$.

3c. Notice that

$$P(Y \geq 24) = P\left(Z \geq \frac{24 - 10(1 + \mu_A) - 10(1 + \mu_Q)}{\sqrt{100\sigma_A^2 + 100\sigma_Q^2 + 200\rho\sigma_A\sigma_Q}}\right).$$

So, anything that definitely decreases

$$\frac{24 - 10(1 + \mu_A) - 10(1 + \mu_Q)}{\sqrt{100\sigma_A^2 + 100\sigma_Q^2 + 200\rho\sigma_A\sigma_Q}}$$

definitely increases $P(Y \geq 24)$. It follows that the right answers are T, T, T. When you want to maximize the chances of hitting a high return, volatility helps; that's why reducing risk is incorrect for this problem.

Solutions (D)

1.a. Need to find x so that $P(X > x) = 0.85$, where $X \sim N(140, 20^2)$. Standardizing, we find that this is the same as $P(Z > (x - 140)/20) = 0.85$. From the table, we find that $(x - 140)/20$ must be -1.04. Thus, $x = 140 - (1.04)(20) = 119.2$.

1.b. Now we need to find $P(Y > x)$, with $x = 119.2$ and $Y \sim N(100, 15^2)$; i.e., $P(Z > (119.2 - 100)/15)$, which is $P(Z > 1.28) = 1 - P(Z < 1.28) = 1 - .8997 = .1003$. This is the proportion in the pool. Of these, 5% will be audited, so the answer is $(.05)(.1003) = .00502$.

1.c. Among SU the proportion is $P(X > 135) = P(Z > (135 - 140/20)) = P(Z > -0.25) = .5987$. Among non-SU the proportion is $P(Y > 135) = P(Y > (135 - 100)/15)) = P(Y > 2.33) = 1 - .9901 = .0099$. Since, overall, 8% are SU and 92% are non-SU, the overall proportion is $(.08)(.5987) + (.92)(.0099) = .057$.

2.a. Need to find p so that $pE[X_1] + (1-p)E[X_2] = 0.16$; i.e., $p(.15) + (1-p)(.19) = .16$. So, $p = 0.75$.

2.b. $Var[pX_1 + (1 - p)X_2] = p^2\sigma_1^2 + (1 - p)^2\sigma_2^2 + 2p(1 - p)\sigma_1\sigma_2\rho$. Plugging in values and taking the square root we get .042.

Solutions (E)

1.a. $E[10X_A - 5X_B] = 10E[X_A] - 5E[X_B] = 10 \cdot 0 - 5 \cdot 0 = 0$.

1.b. First find the variance:

$$\begin{aligned}Var[10X_A - 5X_B] &= 100\sigma_A^2 + 25\sigma_b^2 + 2(10)(-5)\sigma_A\sigma_B\rho \\ &= 100(0.25)^2 + 25(0.10)^2 + 2(10)(-5)(0.25)(0.10)(0.65) = 5.4438\end{aligned}$$

Now take the square root to get a standard deviation of 2.2079.

1.c. By independence,

$$\begin{aligned} Var[Y_{\text{total}}] &= Var[Y_1 + Y_2 + \cdots + Y_{20}] \\ &= Var[Y_1] + Var[Y_2] + \cdots Var[Y_{20}] \\ &= 20Var[Y] \end{aligned}$$

so the standard deviation of Y_{total} is $\sqrt{20}\sigma_Y = \sqrt{20} \cdot 2.2079 = 9.87$.

2.a. Let X be change in portfolio value, $X \sim N(1.5, (9.7)^2)$. Positive changes are gains, negative changes are losses. Need to find x so that $P(X < -x) = 0.05$. Standardizing, we find that $-x = 1.5 - 1.645(9.7) = -14.456$, so $x = 14.456$. There is only a 5% chance that the portfolio will lose more than 14.456 million dollars.

2.b. Now we need to find $P(X > 14.456)$. Standardizing, this becomes $P(Z > 1.337) = 1 - 0.909 = .091$.

Solutions (F)

1. There are four possible outcomes: neither approves, A only, B only, or both approve. The corresponding revenues are 0, 30, 15, and 45 (in thousands). Because the decisions of A and B are independent, the probabilities are $(1 - 0.25)(1 - 0.40) = 0.45$, $0.25(1 - 0.4) = 0.15$, $(1 - 0.25)0.4 = .30$, and $0.25(.4) = 0.10$. Thus

$$E[X] = (.45)0 + (.15)30 + (.30)15 + (.10)45 = 13.5.$$

Similarly,

$$E[X^2] = (.45)0 + (.15)900 + (.30)225 + (.10)2025 = 405.$$

So the standard deviation is $\sqrt{405 - (13.5)^2} = 14.92$.

2.(a) We have

$$\sigma_Y = \sqrt{25\sigma_1^2 + 25\sigma_2^2 + 2(5)(5)\sigma_1\sigma_2\rho} = \sqrt{25(.20)^2 + 25(.35)^2 + 2(5)(5)(.20)(.35)(.20)} = \sqrt{4.7625} = 2.18.$$

2.(b). The combined portfolio is $5X_1 + 15X_2$. Its standard deviation is therefore

$$\sqrt{5^2\sigma_1^2 + 15^2\sigma_2^2 + 2(5)(15)\sigma_1\sigma_2\rho} = \sqrt{25(.20)^2 + 225(.35)^2 + 2(5)(15)(.20)(.35)(.20)} = \sqrt{30.66} = 5.54.$$