COLUMBIA BUSINESS SCHOOL

Value at Risk (B)

Portfolio Standard Deviation

The RiskMetrics methodology for calculating a portfolio's value at risk is based on the assumption that a portfolio's profit and loss over the VaR horizon (e.g., one day, two weeks, or one month) conforms to a normal distribution. A consequence of this key assumption is that the VaR is simply a multiple of the portfolio's standard deviation. The appropriate multiplier depends on the desired "confidence level" (e.g., 95% or 99%), with a higher confidence level requiring a larger multiplier.

The RiskMetrics VaR Calculator is primarily a tool for computing a portfolio's standard deviation. The calculator's interface allows the user to specify the portfolio's positions; this information is combined with the RiskMetrics database to calculate the standard deviation. The VaR is then determined as a multiple of the standard deviation, but this step is almost incidental—the heart of the calculation lies in determining the portfolio standard deviation.

How, then, does the VaR Calculator combine historical data with specific information about a portfolio's composition to determine the portfolio standard deviation? From the historical data, RiskMetrics estimates standard deviations and correlations for the returns on hundreds of asset prices, indices, interest rates, and exchange rates. These standard deviations and correlations are combined with a portfolio's specific composition to arrive at the portfolio standard deviation through a fairly simple formula. The method is illustrated in Figure 1, which shows an example from the RiskMetrics Technical Document. We will develop the formula in greater detail.

Scaling a Portfolio

Before proceeding to the case of a portfolio with two or more assets, we examine a portfolio with just one asset. Consider a long position in gold currently worth $10 million. Suppose we want a one-day VaR at 99% confidence and the VaR calculator produces $171,000. What would the VaR be for a $20 million position in gold?

It is intuitively reasonable to expect the VaR to be twice as large—after all, any change in the market price of gold that results in the first portfolio losing x dollars also results in the second portfolio losing $2x$ dollars. Indeed, the VaR calculator would report a one-day 99% VaR of $342,000 in this case.

What would the VaR be for a portfolio that is *short* $20 million worth of gold? Shorting an asset involves borrowing it and selling it, usually with the expectation that it can be repurchased in the future at a lower price to repay the loan. In practice, there are both legal restrictions and costs associated with shorting; but if we set these aside, the short position can be described as holding $-$20 million worth of gold. What is the VaR for this negative position? Under the assumption of normally distributed changes in the price of gold, an increase by any amount is just as likely as a decrease of equal magnitude. Accordingly, the VaR for $-$20 million in gold is the same as the VaR for a long position of $20 million.

Recall that in the RiskMetrics methodology the VaR is simply a multiple of the portfolio standard deviation: VaR$= k$StdDev, where the multiplier k depends only on the confidence level. The examples above of how VaR changes when we scale a portfolio (by a factor of 2 in the first example, by a factor of -2 in the second) are direct consequences of how the standard deviation scales with the portfolio.

Let

$$X = \text{change in value of \$10 million gold portfolio over the next day}$$

and observe that X is a random variable. The change in value over the next day of a $20 million gold portfolio is $2X$. As in the discussion of VaR above,

$$\text{StdDev}[2X] = 2\text{StdDev}[X].$$

Similarly, the change in value of a $20 million short position is $-2X$ and

$$\text{StdDev}[-2X] = 2\text{StdDev}[X].$$

The rule for any factor b and any random variable X is

$$\text{StdDev}[bX] = |b|\text{StdDev}[X].$$

An immediate consequence of this is

$$\text{Variance}[bX] = b^2\text{Variance}[X].$$

• **Example 2:** Let's complicate matters somewhat. You are a USD-based corporation and hold a DEM 140 million position in the 10-year German government bond. What is your VaR over a 1-day horizon period, again, given that there is a 5% chance of understating the realized loss?

What is your exposure?	The only difference versus the previous example is that you now have both interest rate risk on the bond and FX risk resulting from the DEM exposure. The exposure is still USD 100 million but it is now at risk to two market risk factors.
What is your risk?	If you use an estimate of 10-year German bond standard deviation of 0.605%, you can calculate:

Interest rate risk: $100 million $\times 1.65 \times 0.605\%$ = $999,000
FX Risk: $100 million $\times 1.65 \times 0.565\%$ = $932,000

Now, the total risk of the bond is not simply the sum of the interest rate and FX risk because the correlation[2] between the return on the DEM/USD exchange rate the return on the 10-year German bond is relevant. In this case, we estimated the correlation between the returns on the DEM/USD exchange rate and the 10-year German government bond to be –0.27. Using a formula common in standard portfolio theory, the total risk of the position is given by:

$$[1.1] \quad \text{VaR} = \sqrt{\sigma^2_{\text{Interest rate}} + \sigma^2_{\text{FX}} + (2 \times \rho_{\text{Interest rate, FX}} \times \sigma_{\text{Interest rate}} \times \sigma_{\text{FX}})}$$

$$\text{VaR} = \sqrt{(0.999)^2 + (0.932)^2 + (2 \times -0.27 \times 0.999 \times 0.932)}$$
$$= \$ 1.168 \text{ million}$$

To compute VaR in this example, RiskMetrics provides users with the VaR of interest rate component (i.e., 1.65×0.605), the VaR of the foreign exchange position (i.e., 1.65×0.565) and the correlation between the two return series, –0.27.
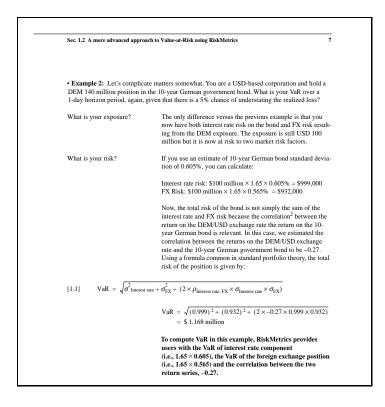
Figure 1: Example from the *RiskMetrics Technical Document*

A Portfolio with Two Assets

Consider next a portfolio with $10 million in the S&P500 and $6 million in the Nikkei 225, a Japanese stock index.[1] To calculate the VaR for this portfolio, RiskMetrics must first find the standard deviation of the change in portfolio value.

On September 23, 1999, the RiskMetrics web page reported the following information for a one-day VaR horizon:

S&P500 price volatility	1.198560%
Nikkei price volatility	1.443259%
Correlation	$-.110735$

The first number means that the one-day change in value of a $1 position in the S&P500 has a standard deviation of $0.01198560. Under the assumption of normality, there is approximately a 95% chance that a position in the S&P500 worth $1 today will be worth between $0.9760288 (i.e.,

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[1]Strictly speaking, these are indices and not assets so one cannot literally own a dollar amount of either. However, it is possible to take a position in a stock index either through futures contracts on the indices or by holding the stocks that make up the index. For simplicity, we ignore this distinction and treat the indices as though they were asset prices.

$1−2 · \$0.01198560$) and \$1.0239712 (i.e., \$1+2 · \$0.01198560). The Nikkei volatility has the same interpretation for a \$1 position in the Nikkei. The reported correlation is the correlation between the changes in the two indices over a single day.

The RiskMetrics VaR calculator combines these three numbers (estimated from historical data) with the portfolio positions of \$10 million and \$6 million specified as inputs. To explain how, we first let

$$X = \text{one-day change in value of \$1 position in S\&P500}$$

and

$$Y = \text{one-day change in value of \$1 position in Nikkei.}$$

Then

$$
\begin{aligned}
\Delta\text{Portfolio} &= \text{one-day change in portfolio value} \\
&= (10 \text{ million})X + (6 \text{ million})Y.
\end{aligned}
$$

We need to find StdDev[ΔPortfolio].

For this, we use the following general formulas:

$$\text{Variance}[aX + bY] = a^2\text{Variance}[X] + b^2\text{Variance}[Y] + 2ab\text{Covariance}[X, Y]$$

$$\text{StdDev}[aX+bY] = \sqrt{a^2\text{Variance}[X] + b^2\text{Variance}[Y] + 2ab\text{Covariance}[X, Y]}.$$

Using σ_X^2 and σ_Y^2 to denote the variances of X and Y and using ρ to denote their correlation, we can rewrite these formulas as

$$\text{Variance}[aX + bY] = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_X\sigma_Y\rho$$

and

$$\text{StdDev}[aX + bY] = \sqrt{a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_X\sigma_Y\rho}. \tag{1}$$

(Recall that Covariance[X, Y] $= \sigma_X\sigma_Y\rho$.)

We now apply this formula to ΔPortfolio with

$$
\begin{aligned}
a &= 10 \text{ million} \\
b &= 6 \text{ million} \\
\sigma_X &= 0.01198560 \\
\sigma_Y &= 0.01443259 \\
\rho &= -.110735.
\end{aligned}
$$

The result is

$$
\begin{aligned}
&\text{StdDev}[\Delta\text{Portfolio}] \\
&= \sqrt{\begin{array}{l}(10 \text{ million})^2(0.01198560)^2 + (6 \text{ million})^2(0.01443259)^2 \\ +2(10 \text{ million})(6 \text{ million})(0.01198560)(0.01443259)(-.110735)\end{array}} \\
&= 0.139877 \text{ million,}
\end{aligned}
$$

or \$139,877.

The calculation works the same way for a one-month horizon. The portfolio coefficients a, b are unchanged in this case, but the statistical estimates provided by RiskMetrics become

$$
\begin{aligned}
\sigma_X &= 0.0569597 \\
\sigma_Y &= 0.06627205 \\
\rho &= -.052458.
\end{aligned}
$$

We now get

$$
\begin{aligned}
&\text{StdDev}[\Delta\text{Portfolio}] \\
&= \sqrt{\begin{aligned}&(10 \text{ million})^2(0.0569597)^2 + (6 \text{ million})^2(0.06627205)^2 \\ &+ 2(10 \text{ million})(6 \text{ million})(0.0569597)(0.06627205)(-.52458)\end{aligned}} \\
&= 0.677340 \text{ million},
\end{aligned}
$$

or \$677,340.

The Diversification Effect

To appreciate formula (1) for the portfolio standard deviation, consider the standard deviation of each piece of the portfolio separately. Using a one-day horizon, we have

$$
\begin{aligned}
\text{StdDev of S\&P500 position} &= (\$10 \text{ million})(0.01198560) = \$119,856 \\
\text{StdDev of Nikkei position} &= (\$6 \text{ million})(0.01443259) = \$86,596.
\end{aligned}
$$

Summing these, we get \$206,451. But we calculated the portfolio standard deviation to be just \$139,877. The fact that the portfolio standard deviation is lower than the sum of the individual standard deviations reflects the risk-reducing effect of diversification. Observe from (1) that the more negatively correlated the two assets, the greater the risk reduction[2]

The diversification effect is present even if the two assets are positively correlated, so long as $\rho < 1$; i.e., so long as the assets are not perfectly correlated, there is some risk reduction from distributing a portfolio between the assets rather than investing fully in a single asset. This is a consequence of the mathematical fact that

$$
\sqrt{a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_X\sigma_Y\rho} \leq |a|\sigma_X + |b|\sigma_Y.
$$

The expression on the left is the sum of the standard deviations of aX and bY; the expression on the right is the standard deviation of $aX + bY$.

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[2]When ρ is negative, and a and b are positive, $2ab\sigma_X\sigma_Y\rho$ is negative. In this case, with a larger $|\rho|$ we are subtracting more from the term inside the square root in (1).

Multiple Assets

The RiskMetrics VaR calculator handles with a large number of assets using a natural generalization of formula (1). Suppose the portfolio has n assets and let

$$
\begin{aligned}
a_i &= \text{portfolio's dollar position in } i\text{th asset}, \quad i = 1, \ldots, n; \\
X_i &= \text{one-day change of \$1 position in } i\text{th asset}, \quad i = 1, \ldots, n.
\end{aligned}
$$

The one-day change in portfolio value is

$$
a_1 X_1 + a_2 X_2 + \cdots + a_n X_n.
$$

Let $\sigma_i = \text{StdDev}[X_i]$ and let ρ_{ij} denote the correlation between X_i and X_j. Then the variance of the change in portfolio value is

$$
\begin{aligned}
& a_1^2 \sigma_1^2 + a_1 a_2 \sigma_1 \sigma_2 \rho_{12} + \cdots + a_1 a_n \sigma_1 \sigma_n \rho_{1n} \\
+ \ & a_2 a_1 \sigma_2 \sigma_1 \rho_{21} + a_2^2 \sigma_2^2 + \cdots + a_2 a_n \sigma_2 \sigma_n \rho_{2n} \\
+ \ & \cdots \\
+ \ & a_n a_1 \sigma_n \sigma_1 \rho_{n1} + a_n a_2 \sigma_n \sigma_2 \rho_{n2} + \cdots + a_n^2 \sigma_n^2.
\end{aligned}
$$

Taking the square root yields the standard deviation. This formula is the key to the RiskMetrics VaR Calculator.