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Contingent Capital with a Capital-Ratio Trigger

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Contingent capital in the form of debt that converts to equity when a bank faces financial distress has been proposed as a mechanism to enhance financial stability and avoid costly government rescues. Specific proposals vary in their choice of conversion trigger and conversion mechanism. We analyze the case of contingent capital with a capital-ratio trigger and partial and ongoing conversion. The capital ratio we use is based on accounting or book values to approximate the regulatory ratios that determine capital requirements for banks. The conversion process is partial and ongoing in the sense that each time a bank's capital ratio reaches the minimum threshold, just enough debt is converted to equity to meet the capital requirement, so long as the contingent capital has not been depleted. We derive closed-form expressions for the market value of such securities when the firm's asset value is modeled as geometric Brownian motion, and from these we get formulas for the fair yield spread on the convertible debt. A key step in the analysis is an explicit expression for the fraction of equity held by the original shareholders and the fraction held by converted investors in the contingent capital.

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1. Introduction

Several proposals for enhancing the stability of the financial system include requirements that banks hold some form of contingent capital, meaning equity that becomes available to a bank in the event of a crisis or financial distress. Variants of this idea differ in the choice of trigger for the activation of contingent capital and in how the capital is held before a triggering event. The Dodd–Frank Act calls for regulators to study the potential effectiveness of contingent capital, and specific definitions for triggering events were put forward in a recent consultative document issued by the Basel Committee on Banking Supervision (2010).

Flannery (2005) proposed reverse convertible debentures—a form of debt that converts to equity if a bank's capital ratio falls below a threshold. His proposal uses a capital ratio based on the market value of the bank's equity and the book value of its debt. Flannery (2009) updated the proposal and renamed the securities contingent capital certificates. Kashyap et al. (2008) proposed a "lock box" to hold bank funds that would be released in the event of a crisis; in this proposal, the trigger is a systemic event, and not a risk of bankruptcy at an individual institution. McDonald (2011) and the Squam Lake Working Group (2009) proposed contingent capital with a trigger that depends on the health of both an individual bank and the banking system as a whole. The convertible securities designed by the U.S. Treasury for its Capital Assistance Program may be viewed as a type of contingent capital in which banks hold the option to convert preferred shares to common equity and find it advantageous to do so if their share price drops sufficiently low; this contract was studied by Glasserman and Wang (2011).

Alternative proposals for the design of contingent capital have led to work on valuation. McDonald (2011) priced contingent capital with a dual trigger through joint simulation of a bank's stock price and a market index. Pennacchi (2010) compared several cases by simulation in a jump-diffusion model of a bank's assets. Albul et al. (2010) obtained closed-form pricing expressions under the assumption that all debt has infinite maturity and that the conversion trigger is defined by a threshold level of assets. Raviv (2004) also used an asset-level trigger and obtains closed-form expressions with finite-maturity debt. Von Furstenberg (2011) built a binomial tree for the evolution of a bank's capital ratio. Sundaresan and Wang (2010) showed that setting the conversion trigger at a level of the stock price may result in multiple solutions or no solution for the market price of the stock and convertible debt, raising questions about the viability of contracts designed with market-based triggers. Koziol and Lawrenz (2010) noted that contingent capital can increase incentives for risk taking by making bankruptcy more remote. Among recent alternatives to the mechanisms considered in these papers, Duffie (2010) proposed mandatory rights offerings by banks facing financial distress, McAndrews (2010) proposed a combination of a rights offering and convertible debt, and Pennacchi et al. (2010) suggested bundling contingent capital with buyback options for equity holders. Brennan and de Longevialle (2010) estimated the overall potential size of the contingent capital market at one trillion dollars and discussed investor perspectives on some alternative features.

We develop a model to study contingent capital in the form of debt that converts to equity based on a capital-ratio trigger. The bank is required to hold a minimum ratio of equity to total assets (equivalently, it faces an upper bound on leverage); if its asset value drops too low, part of its debt converts to equity to maintain the required capital ratio. Our setting is thus similar to Flannery's (2005, 2009), though he compared the market value of equity to the book value of debt.

Existing regulatory capital requirements for banks are based primarily on book values. Under Basel rules, banks must maintain regulatory capital equal to at least 8% of their risk-weighted assets. U.S. banks also face an overall capital-to-assets constraint with a minimum of 3% and a threshold of 5% to qualify as "well capitalized." All of these ratios are based on regulatory accounting measures of debt and capital rather than the market price of a bank's stock. Existing issuances to date—the contingent core capital ("CoCo") bonds issued by Lloyd's Banking Group in November 2009, mortgage lender Yorkshire Building Society in December 2009, and Credit Suisse in February 2011, and the principal write-down bonds issued by Rabobank in March 2010—all use triggers based on regulatory capital ratios and not market prices. Flannery (2005, 2009) and Pennacchi et al. (2010) advocate the use of market data because it is continuously updated, forward looking, and less vulnerable to accounting manipulation, while noting concerns that market values could potentially be manipulated to trigger conversion. The results of Sundaresan and Wang (2010) show that defining an internally consistent market-based trigger can be problematic. Because there are good arguments for both market-value and book-value triggers, both types of securities merit investigation; because the two require somewhat different analysis, here we limit ourselves to book-value capital ratios.

A distinguishing feature of our analysis is that we model partial and ongoing conversion of contingent capital as a bank's capital ratio declines, consistent with Flannery's (2005) original proposal. (Acharya et al. (2010, p. 166) call this *progressive* conversion.) Previous models have relied on the assumption that convertible debt is converted in its entirety as soon as a threshold is hit. Instead, we assume just enough conversion takes place to maintain the minimum capital ratio required, leading to a process of continuous conversion. This partial conversion process lends itself to a somewhat larger tranche of convertible debt than all-at-once conversion would, and it makes the full tranche truly contingent, with each layer converted only as needed. With all-at-once conversion, most of the debt is converted too early (or too late).

Partial conversion has important implications for investors: as contingent capital converts to equity, bond holders become shareholders and thus share in any costs or benefits to shareholders of subsequent conversion. We will show that increasing the minimum capital requirement has the effect of slowing conversion and thus shifts more of the dilution cost from conversion to investors who became shareholders through earlier conversion of debt. A higher capital ratio can therefore benefit the original shareholders if the loss in asset value is sufficiently large; the value of the convertible debt need not be monotone in the required capital ratio.

We undertake our valuation in a structural model, starting from the firm's assets. The firm's capital structure is comprised of senior (unconvertible) debt, contingent capital, and equity. Market values of debt and equity are determined, as usual, by viewing these as claims on the assets; but the book value of debt is calculated by discounting future coupon and principal payments at the yield at which the bond was issued, consistent with accounting rules. We use the resulting book values in our capital ratio. The market and book values of debt must agree at issuance and at maturity, and we incorporate this constraint in our analysis to fix the coupon rates. In our framework, investors in contingent capital hold claims on four types of payments: coupons on unconverted debt, the remaining principal on convertible debt, dividends earned through debt converted to equity, and the value of this equity at the maturity of the debt. We value the contingent capital as the sum of the values of these payments.

Once the contingent capital is exhausted, we assume that a failure to meet the minimum capital requirement results in a seizure and liquidation by regulators. Liquidation occurs prior to bankruptcy in the sense that a bank has positive equity when it first breaches its capital ratio. We incorporate potential liquidation costs for shareholders and also for bond holders in our valuation. Indeed, these costs have a significant impact on our valuations, as does asset volatility. Asset volatility affects both the likelihood of conversion of debt to equity and the upside potential of equity following conversion.

The rest of this paper is organized as follows. Section 2 presents our model of the firm and the conversion of debt to equity, and §3 examines how equity is allocated between converted shareholders and the original shareholders as the value of the firm's assets evolve. Section 4 introduces dividends. Section 5 details the cash flows paid to investors in the firm's senior debt, contingent capital, and equity, and §6 presents explicit expressions for the values of these cash flows. Section 7 closes the model by solving for the coupons on the two types of debt to equate market and book values at issuance; from these we get the yield spread on contingent capital. Section 8 extends the model to distinguish between market and book value of assets. Section 9 illustrates our results through numerical examples. Detailed calculations leading to our valuation formulas are deferred to appendices.

2. Model of the Firm

Our model of the firm (or bank) builds on a long line of research on capital structure that includes Merton (1974), Black and Cox (1976), Leland (1994), and numerous subsequent papers. This approach starts by modeling the dynamics of a firm's assets and then prices debt and equity as claims on those assets. In Merton (1974), the firm defaults at the maturity of the debt if its asset value is less than the face value of the debt. In Black and Cox (1976), bankruptcy occurs when asset value drops to an exogenous reorganization boundary, and in Leland (1994), the time of default is chosen strategically by shareholders. In our setting, we will need to provide a corresponding prescription for the conversion of contingent capital to equity, as well as specifying a trigger for liquidation of the firm. We interpret the liquidation event as resulting from seizure by regulators when the firm is unable to sustain its capital requirement, which, by design, occurs prior to a traditional bankruptcy event.

Our starting point is a stochastic process V_t that models the book value of the firm's assets; this process drives the required level of capital in our model, just as accounting-based measures of asset value drive capital requirements in practice. For tractability, we take V_t to be geometric Brownian motion,

$$\frac{dV_t}{V_t} = (r-\delta)\,dt + \sigma\,dW_t, \tag{1}$$

where W is a standard Brownian motion, and δ is a constant payout rate to the firm's security holders. In §2.1, we calculate book values for senior and convertible debt; subtracting the book value of debt from the book value of assets leaves Q_t, the book value of shareholder's equity, which is our measure of capital. (In practice, regulatory capital also includes certain debt instruments not captured in our model.) Our minimum capital requirement is expressed as a lower bound on Q_t/V_t.

We use these book values to model capital requirements and the conversion of debt to equity. But for valuation, we need to calculate market values: we take the market value of a security to be the expected discounted value of cash flows received by investors, irrespective of book values. In the basic version of our model, we assume that the market value of the firm's assets equals the book value V_t—in other words, we assume the bank uses mark-to-market accounting for its assets.[1] In the more general version of our model (introduced in §8), we represent market and book values of assets through correlated geometric Brownian motions, thus allowing an imperfect relationship between the two and creating some uncertainty about how much market value will be realized when a liquidation is triggered by a book-value-based capital ratio.

In either version of the model, we calculate market values for senior and convertible debt as contingent claims on the market value of assets. We pin down the market values of these contingent claims with the constraint that market and book values of debt must coincide at issuance and at maturity: when debt is issued, its book value is recorded at its selling price (market value), and when it matures, its book value and market value equal the final payment of principal and interest. In short, we use the book value of assets to drive the conversion of contingent capital, and we use the market value of assets to drive the valuation of contingent capital. Keeping track of these two notions of value is essential to pricing securities that depend on an accounting-based trigger.

Our model entails several idealizations and simplifications. We assume that capital ratios can be observed continuously; in practice, regulatory capital is calculated quarterly, but large banks routinely calculate internal "economic capital" on a daily basis, so the necessary information could in principle be monitored for regulatory purposes to trigger conversion. A limitation of our model is that it does not allow for jumps in asset value—a large jump could potentially wipe out all the contingent capital and leave the firm bankrupt. This type of event is beyond the scope of our model.

[1] This would be the case under Financial Accounting Standard 157. Even prior to this proposed rule, using data from 2001–2005, Calomiris and Nissim (2007) reported that for many bank assets (in contrast to those of nonfinancial firms), book value is indeed close to fair value.

2.1. Debt

The firm issues ordinary senior debt as well as junior convertible debt. Both types of debt are issued at time zero and mature at time $T > 0$. The senior debt has a face or par value of D (due at time T) and a continuous coupon rate of c_2, meaning that it pays $c_2 D$ per unit of time. The debt is issued at a price of D_0. From an accounting perspective, the effective interest rate for the debt is the discount rate d_2 that equates the cash raised (D_0) to the present value of future payments promised on the debt; i.e., the value of d_2 that solves

$$D_0 = De^{-d_2 T} + \int_0^T c_2 D e^{-d_2 s}\, ds = D\left[e^{-d_2 T}\left(1 - \frac{c_2}{d_2}\right) + \frac{c_2}{d_2}\right].$$

The book value of the debt at any intermediate date t, $0 < t < T$, is then

$$D_t = D\left[e^{-d_2(T-t)}\left(1 - \frac{c_2}{d_2}\right) + \frac{c_2}{d_2}\right] \tag{2}$$

if the firm has not yet failed. In other words, throughout the life of the debt, book value is calculated by discounting remaining payments at the effective interest rate at which the debt was originally issued.

In the absence of any other type of debt, we would model default as occurring the first time the value of the firm's assets fall below the boundary defined by D_t, $0 \le t \le T$. This is an instance of the mechanism used by Black and Cox (1976), though they used an exponential boundary, which corresponds to setting $c_2 = 0$. The boundary in Black and Cox (1976) is often interpreted as a protective debt covenant, and that interpretation could be applied here. In the case of a regulated bank, which is our focus, the boundary will serve to define a minimum capital requirement the bank must maintain, rather than a privately negotiated covenant. The capital requirement will set the liquidation boundary higher (by the amount of the required capital buffer) than the default boundary (2). The bank is seized by regulators before bankruptcy if the capital requirement is not maintained.

Next we introduce convertible debt with a face value of B, a continuous coupon rate c_1, and maturity T, issued at time zero at a price of B_0. The assumption that all of the debt has the same maturity T is a simplifying idealization. The effective interest rate d_1 equates B_0 to the present value of the promised payments of coupon and principal,

$$B_0 = Be^{-d_1 T} + \int_0^T c_1 B e^{-d_1 s}\, ds = B\left[e^{-d_1 T}\left(1 - \frac{c_1}{d_1}\right) + \frac{c_1}{d_1}\right].$$

As part of the original contingent capital issuance converts to equity, the remaining principal decreases, but we apply the same effective interest rate d_1 to calculate the book value of the debt outstanding. If the remaining principal at time t is $\tilde{B}_t$, then the book value at time t is

$$B_t = \tilde{B}_t\left[e^{-d_1(T-t)}\left(1 - \frac{c_1}{d_1}\right) + \frac{c_1}{d_1}\right]. \tag{3}$$

We take up the conversion mechanism that determines $\tilde{B}_t$ in the next subsection.

Equations (2) and (3) take the coupon rates c_1 and c_2 as given. As part of our analysis, we will solve for the values of c_1 and c_2 that make the values of the two types of debt consistent with the overall value of the firm. In particular, we will choose c_1 and c_2 to ensure that the initial values B_0 and D_0 are consistent with market values of debt given the face amounts B and D and the dynamics of the firm's asset value.

2.2. Conversion from Debt to Equity

We denote by V_t the book value of the firm's assets at time t. Subtracting the firm's debt from its assets at time t leaves

$$Q_t = V_t - B_t - D_t; \tag{4}$$

we refer to Q_t as capital, shareholder's equity, or simply equity, but it should be interpreted as a book value or regulatory measure and not as the market value of equity, because (2) and (3) are accounting based measures of debt. Indeed, the goal of our analysis is to calculate market values based on the contractual terms of the contingent capital.

The firm is required to maintain a capital ratio of at least α, $0 < \alpha < 1$, which imposes the constraint

$$Q_t \ge \alpha V_t \quad \text{or} \quad (1-\alpha)V_t \ge B_t + D_t.$$

For example, to model a bank that is required to hold capital equal to 5% of assets, we would set $\alpha = 0.05$.[2] As V fluctuates, a bank could be in danger of violating this requirement; the contingent capital converts from debt to equity (decreasing B_t and increasing Q_t) to maintain the constraint as long as possible. Flannery (2005) introduced this mechanism using the market value of equity, rather than regulatory capital, to drive conversion.

Before formalizing the conversion mechanism in our model, we consider the example in Figure 1. Part (a) of the figure shows an initial balance sheet with 100 in assets, 60 in senior debt and 30 in convertible debt, leaving 10 in shareholder's equity. For

[2] We can model a capital requirement tied to risk-weighted assets, rather than total assets, by adjusting the value of α. The average ratio of risk-weighted assets to total assets over all FDIC banks was 70%–75% during 2003–2010, so a capital requirement of 8% of risk-weighted assets could be approximated by a requirement of 5%–6% of total assets. For the largest bank holding companies, the asset ratio is 40%–60%, corresponding to a lower value of α. The adjustment in α could be tailored to a specific institution based on its mix of assets.

Figure 1 **(a) Initial Balance Sheet with a 10% Capital Ratio Satisfied; (b) After a Drop in Asset Value; (c) After Conversion of Debt to Equity Restoring the 10% Capital Ratio**

(a)

Assets	Liabilities
$V = 100$	$D = 60$
	$B = 30$
	$Q = 10$

(b)

Assets	Liabilities
$V = 95$	$D = 60$
	$B = 30$
	$Q = 5$

(c)

Assets	Liabilities
$V = 95$	$D = 60$
	$B = 25.5$
	$Q = 9.5$

simplicity, we consider a minimum capital requirement of 10%, which is just met in (a). In (b), the firm's assets drop to a value of 95; the loss of 5 is absorbed by equity. To meet the capital requirement, the firm converts 4.5 of convertible debt to equity to arrive at the balance sheet in (c), which again just meets the capital requirement.

In our model, V evolves continuously in time with continuous paths, and we will derive the process of minimal conversion under which conversion takes place precisely at those times t at which $Q_t = \alpha V_t$; i.e., times at which $(1-\alpha)V_t = B_t + D_t$. We will assume throughout that the bank is initially well capitalized in the sense that $Q_0 > \alpha V_0$.

In terms of the amount $\tilde{B}_t$ of principal remaining (not converted) at time t, the capital constraint is

$$(1-\alpha)V_t \geq \tilde{B}_t\left[e^{-d_1(T-t)}\left(1-\frac{c_1}{d_1}\right)+\frac{c_1}{d_1}\right] + D\left[e^{-d_2(T-t)}\left(1-\frac{c_2}{d_2}\right)+\frac{c_2}{d_2}\right]. \quad (5)$$

Once the contingent capital is exhausted, the constraint becomes $(1-\alpha)V_t \geq D_t$. Let τ_b denote the first time $(1-\alpha)V_t = D_t$, at which point the firm is seized by regulators. Define L_t by setting

$$(1-\alpha)L_t = \max_{0\leq s\leq t}\left\{\left(B+\frac{D[e^{-d_2(T-s)}(1-c_2/d_2)+c_2/d_2]-(1-\alpha)V_s}{[e^{-d_1(T-s)}(1-c_1/d_1)+c_1/d_1]}\right)^{+}\right\}. \quad (6)$$

Then we show below that $(1-\alpha)L_t$ is the cumulative amount of principal converted up to time t. More precisely, we claim that if we set $\tilde{B}_t = B-(1-\alpha)L_t$, then (5) is satisfied for all $t\in[0,\tau_b]$, and $(1-\alpha)L_t$ is the least amount of conversion that meets this condition.

Equation (6) simplifies when both kinds of debt have constant book value. This holds when the debt is issued at par (i.e., $B_0 = B$ and $D_0 = D$) so the coupon rates coincide with the effective interest rates, meaning that $c_1 = d_1$ and $c_2 = d_2$. In this case, Equation (6) simplifies to

$$(1-\alpha)L_t = \left(B+D-(1-\alpha)\min_{0\leq s\leq t} V_s\right)^{+}. \quad (7)$$

The conversion process in this case becomes easier to visualize if we introduce two thresholds,

$$a = \frac{B+D}{1-\alpha}, \quad b = \frac{D}{1-\alpha}. \quad (8)$$

Under our standing assumption that the capital constraint is satisfied at time zero, $V_0 > a$. Conversion starts when V first hits a. Subsequently, at each instant at which V hits a level lower than any previously reached, additional contingent capital is converted to satisfy the constraint. Once V hits b (which happens at τ_b), the contingent capital has been fully converted (see Figure 2). The process L is given by

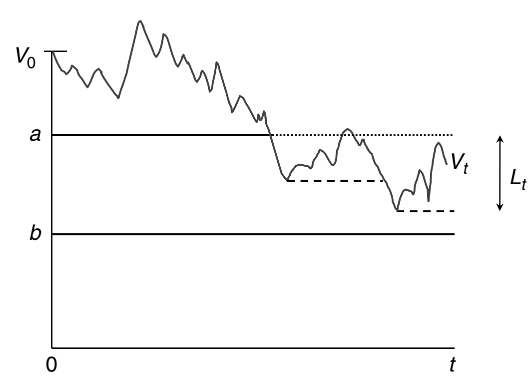
$$L_t = \min\left\{\left(a-\min_{0\leq s\leq t} V_s\right)^{+},\ a-b\right\}, \quad \text{for all } t\in[0,T]. \quad (9)$$

The width $a-b$ is $(1-\alpha)$ times the face value B of contingent capital. A similarly tractable case holds when the two types of debt pay no coupon and have the same effective interest rate—that is, when $c_1 = c_2 = 0$ and $d_1 = d_2 = d$.

We formalize the conversion mechanism in the following result, in which we view (6) as a mapping from a path of V to a path of L:

Proposition 2.1. *Let D, B, c_1, c_2, d_1, and d_2 be given. The function $\{L_t, t\in[0,\tau_b]\}$, defined by applying* (6) *to*

Figure 2 **Illustration of the Conversion Process**



Notes. Conversion begins when V reaches the upper boundary a. The total amount converted to time t is $(1-\alpha)L_t$, where L_t is the distance from the running minimum of V to a, capped at $a-b$.

$\{V_t, t \in [0, \tau_b]\}$, *is the only function with the following properties*:

(i) *L is increasing and continuous with* $L_0 = 0$;

(ii) $V_t - (B - (1-\alpha)L_t)(e^{-d_1(T-t)}(1 - c_1/d_1) + c_1/d_1) - D_t \geq \alpha V_t$ *for all* $t \in [0, \tau_b]$;

(iii) *L increases only when equality holds in* (ii).

Any function satisfying (i) *and* (ii) *is greater than or equal to L on* $[0, \tau_b]$.

Condition (i) is natural for the process of cumulative conversion. Condition (ii) states that conversion occurs to preserve the required capital ratio until τ_b, when the contingent capital is exhausted. Condition (iii) states that conversion occurs only as needed—when the firm is at its minimum capital requirement. The result follows from the standard reflection mapping (as in Harrison 1985, p. 21) applied to the function

$$V_t - \frac{1}{1-\alpha}\left(B\left[e^{-d_1(T-t)}\left(1 - \frac{c_1}{d_1}\right) + \frac{c_1}{d_1}\right] + D_t\right).$$

The proposition determines L only up to the time τ_b, when the contingent capital has been fully converted. Using (6) or the special case in (9), we can conveniently extend the definition of L to the interval $[0, T]$, even if $\tau_b < T$.

3. Equity Allocation

We will value the contingent capital bond by calculating the expected present value of the payments to the holder of the security. The payments include coupons (paid continuously in proportion to the unconverted debt), any remaining principal at maturity, a fraction of the firm's equity earned through conversion, and dividends paid on a fraction of equity. From the analysis in the previous section, we can determine how much of the contingent capital remains unconverted at each point in time. To value the equity component as the bond converts, we need to analyze what fraction of the firm's equity is held by investors who were converted from contingent capital holders to equity holders. We limit ourselves to the case $c_1 = d_1$ and $c_2 = d_2$, which, as explained in the previous section, equates book value to remaining face value for both kinds of debt.

To motivate the analysis that follows, consider again the example of Figure 1. Suppose, for simplicity, that the firm starts with 10 shares outstanding. By writing down 4.5 in convertible debt in (c), the firm automatically adds 4.5 to equity, but how the total equity is apportioned to the prior and new shareholders depends on how many new shares are issued in exchange for the converted debt. We introduce a conversion ratio $q > 0$, which is the book value of equity received by the contingent capital investors for each dollar of face value of debt converted. If $q = 1$, then in (c), the converted investors need to get 4.5 in book value of equity. This is accomplished by issuing them 9 shares, because they then own a fraction $9/(10+9)$ of the firm, and 9/19ths of the total equity of 9.5 is indeed 4.5. If $q = 2$, then they should get 180 shares: this gives them a fraction $180/(10+180)$ of the total equity of 9.5 for a book value of 9, which is indeed twice the book value of the debt they gave up. The dilution leaves the original shareholders with 0.5 in book value, or 1/18th of the total equity. The conversion ratio q has no effect on the total amount of equity, but it determines how the equity is divided between the original and converted shareholders. We need to keep track of this allocation to determine the market value of the convertible debt. Book value of equity is not, by itself, a direct measure of market value; but the proportions of book value of equity held by the two types of investors determine how cash flows are allocated, and the market value of the contingent capital is the expected discounted value of all cash flows received by the investors in these securities.

We will derive an expression for the amount of equity held at any time by the original equity investors. As a lead-in to the continuous-time setting, we consider a discrete-time formulation with a discrete transition over a small interval Δt and write $V_{t+\Delta t} = V_t + \Delta V_t$. Suppose (as in Figure 1(a)) that the firm is just at the capital ratio boundary at time t, and it suffers an asset loss $\Delta V_t < 0$. From (9) (and Proposition 2.1), we know that L increases when V reaches a new minimum and $\Delta L_t = -\Delta V_t$. The resulting amount of equity following conversion is given by

$$Q_{t+\Delta t} = Q_t + \Delta V_t + (1-\alpha)\,\Delta L_t = Q_t + \alpha\,\Delta V_t,$$

the minimal amount of additional equity required to preserve the capital ratio (as in Figure 1(c)).

Let Q^o denote the amount (book value) of equity held by the original shareholders, and let $\pi_t = Q_t^o/Q_t$ denote the fraction of equity they own. Suppose the conversion at time t is the first to occur, so that the equity is fully held by the original shareholders just before conversion and $Q_t^o = Q_t$. Then

$$Q_{t+\Delta t}^o = Q_t^o + \Delta V_t - (q-1)(1-\alpha)\,\Delta L_t.$$

In other words, the original shareholders absorb the full loss ΔV_t in asset value, and they lose an amount $(q-1)(1-\alpha)\,\Delta L_t$ to the new shareholders as a result of the conversion. More generally, if the original shareholders own a fraction π_t of the equity at time t, then they absorb a fraction π_t of the losses, and we have

$$Q_{t+\Delta t}^o = Q_t^o + \pi_t\big(\Delta V_t - (q-1)(1-\alpha)\,\Delta L_t\big). \tag{10}$$

To formulate a precise result, we work directly in continuous time. We defined Q_t in (4). Under our constant book-value condition, $c_1 = d_1$, $c_2 = d_2$, (4) becomes

$$Q_t = V_t - [B - (1-\alpha)L_t] - D, \tag{11}$$

and the expression

$$dQ_t = dV_t + (1-\alpha)\,dL_t \tag{12}$$

is well defined because V is geometric Brownian motion and L has increasing paths. We introduce the process Q^o by setting

$$dQ_t^o = \frac{Q_t^o}{Q_t}\big(dV_t - (q-1)(1-\alpha)\,dL_t\big), \quad 0 \le t \le \tau_b, \tag{13}$$

with initial condition $Q_0^o = Q_0$. We interpret Q^o as the equity held by the original shareholders: Equation (13) says that the change in their equity is their share of the change in asset value plus their share of the transfer to new shareholders upon conversion. Using (12) to write this equation as

$$\frac{dQ_t^o}{Q_t^o} = \frac{dQ_t}{Q_t} - q(1-\alpha)\frac{dL_t}{Q_t} \tag{14}$$

offers the following interpretation: the percentage change in the book value of equity held by the original shareholders dQ_t^o/Q_t^o equals the overall percentage change dQ_t/Q_t so long as no conversion occurs; at an instant of conversion, the percentage change in book value held by the original shareholders is reduced by the fraction of equity transferred to the new shareholders. (In Figure 1, (14) describes the transition from (a) to (c) with $q=1$, $Q_t^o = Q_t = 10$, $dQ_t = -0.5$, $dQ_t^o = -5$, and the converted amount $(1-\alpha)\,dL_t = 4.5$.) Because $dL_t = 0$ for $t > \tau_b$, based on (14) we extend Q_t^o beyond τ_b if $\tau_b < T$ by setting

$$Q_t^o = (Q_t/Q_{\tau_b})Q_{\tau_b}^o, \quad t \in (\tau_b, T], \tag{15}$$

so the fraction Q_t^o/Q_t does not change in $[\tau_b, T]$. The following result confirms that these definitions are meaningful and that they lead to an explicit solution.

Theorem 3.1. *Suppose $B_t \equiv B$ and $D_t \equiv D > 0$ for $t \in [0,T]$. Then* (14) *and* (15) *have exactly one solution, and it is given by*

$$Q_t^o = Q_t\left(\frac{a - L_t}{a}\right)^{(q(1-\alpha))/\alpha}, \quad 0 \le t \le T. \tag{16}$$

Consequently, the fraction of equity held by the original shareholders at time t is given by

$$\pi_t = \left(\frac{a-L_t}{a}\right)^{(q(1-\alpha))/\alpha} = \left(1 - \frac{(1-\alpha)L_t}{B+D}\right)^{(q(1-\alpha))/\alpha}$$
$$= \left(\min\left\{1, \frac{(1-\alpha)\min_{0\le s\le t} V_s}{B+D}\right\}\right)^{(q(1-\alpha))/\alpha}, \tag{17}$$

for $t \in [0,T]$.

A remarkable feature of (17) is that the fraction of equity held by the original shareholders at any time t depends only on the minimum asset value reached up to time t. Different paths of V may produce very different paths for the conversion process and may result in different terminal values for equity; and yet, if they reach the same minimum asset value, they leave the original shareholders owning the same fraction of the firm. The total amount of contingent capital converted to time t is $(1-\alpha)L_t$, and it is interesting that the dependence of π_t on this amount is nonlinear yet explicit.

We note some properties of (17). If $L_t = 0$ (i.e., if V never reaches the capital-ratio trigger $a = (B+D)/(1-\alpha)$ in $[0,t]$), then $\pi_t = 1$, reflecting the fact that no conversion has occurred. If $L_t = b - a$ (i.e., if V reaches the lower boundary $b = D/(1-\alpha)$ at which the required capital ratio can no longer be sustained), the contingent capital is fully exhausted, but the original shareholders are not wiped out; they own a fraction

$$\left(\frac{b}{a}\right)^{(q(1-\alpha))/\alpha} = \left(\frac{D}{B+D}\right)^{(q(1-\alpha))/\alpha} \tag{18}$$

of the remaining equity $V_t - D = \alpha D/(1-\alpha)$. The following result records the dependence of π_t on the minimum ratio α:

Corollary 3.2. *The proportion π_t of equity owned by the original shareholders is an increasing function of α with $\min_{0\le s\le t} V_s$ held fixed if*

$$\min_{0\le s\le t} V_s < \frac{\exp(-\alpha)}{1-\alpha}(B+D); \tag{19}$$

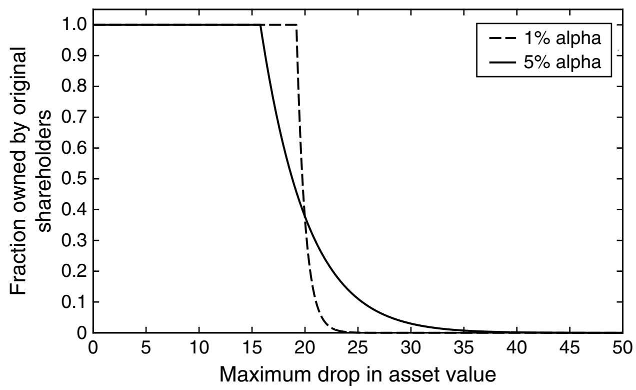
it is decreasing in α if the opposite inequality holds.

This result is easily established by differentiating the third expression for π_t given in (17). We interpret the corollary as stating, perhaps surprisingly, that a higher required capital ratio ultimately protects the original shareholders: if the loss in asset value is sufficiently large, the original shareholders keep a higher fraction of the firm under a higher (and thus more stringent) capital ratio α. Moreover, the total amount of shareholder equity Q_t is itself an increasing function of α; this follows from (9) and (11).

To interpret the condition in the corollary, recall that conversion of debt to equity begins when asset value reaches $a = (B+D)/(1-\alpha)$. For small α, $\exp(-\alpha) \approx 1$, so the threshold in (19) is nearly the same as the trigger for conversion. Thus, at higher α, conversion is triggered sooner (resulting in a lower π), but if asset value continues to decline, a higher α results in a higher fraction of equity held by the original shareholders.

This phenomenon is illustrated in Figure 3 for a firm with $D = 50$, $B = 30$, and initial asset value

Figure 3 Comparison of the Fraction π_t Held by the Original Shareholders as a Function of the Maximum Loss in Asset Value Up to Time t, for Two Values of the Capital Ratio α



$V_0 = 100$. The figure plots π_t against the maximum loss in asset value, $V_0 - \min_{0\le s\le t} V_s$ for two different values of α. Conversion begins when the loss in value reaches $V_0 - (B+D)/(1-\alpha)$, which is approximately 15.8 with $\alpha = 0.05$ and 19.2 with $\alpha = 0.01$. The higher capital ratio triggers conversion sooner; however, once conversion begins at the smaller value of α, the two curves quickly cross. Indeed, from the corollary we know that once the loss exceeds $V_0 - \exp(-0.01)(B+D)/(1-0.01) \approx 20$, any capital ratio greater than 1% keeps a higher fraction of equity with the original shareholders.

4. Dividends and Debt Service Payments

As is standard in much of the capital structure literature (e.g., Leland and Toft 1996), we will assume that the firm's assets generate cash at a rate proportional to their value (in our setting, book value), and these cash flows are used to service the firm's debt and to pay dividends to shareholders. If the firm pays out a constant fraction $\delta \in (0,1)$ of its asset value, then from time t to $t+dt$, the cash flow available will be $\delta V_t\,dt$.

With a coupon rate of c_2 and a face value of D, the senior debt requires payments at rate $c_2 D$ prior to maturity. Interest on debt is tax deductible, and we model this as in, e.g., Leland (1994) and Leland and Toft (1996): if the firm's marginal tax rate is $\kappa \in (0,1)$, it incurs an after-tax cost rate of $(1-\kappa)c_2 D$ in servicing the senior debt. We could apply different marginal tax rates κ_1, κ_2 to the two types of debt[3] to get after-tax coupon rates $(1-\kappa_i)c_i$, $i = 1, 2$; for simplicity, we use a common value κ. The outstanding convertible debt at time t is $B - (1-\alpha)L_t$, requiring an after-tax payment at rate $c_1(1-\kappa)[B-(1-\alpha)L_t]$.

The difference

$$\delta V_t - (1-\kappa)\big(c_1(B-(1-\alpha)L_t) + c_2 D\big)$$

between the rate at which cash is generated, and the rate at which it is paid to debt holders is the rate at which dividends are paid to shareholders, whenever this difference is positive. When the difference is negative, the firm is generating insufficient cash to service its debt. As is customary, we interpret a negative dividend as the issuance of a small amount of new equity, which brings cash into the firm. This cash is immediately paid out to the debt holders, so the issuance has no impact on the total amount of capital in the firm.

We will assume, in fact, that the new equity is issued to existing shareholders (as in a rights offering) and that the original and converted shareholders participate in equal proportions. Thus, the proportion π_t of the firm owned by the original shareholders is unchanged. The new shareholders then receive a net cash flow at rate

$$(1-\pi_t)\big(\delta V_t - (1-\kappa)\big(c_1(B-(1-\alpha)L_t) + c_2 D\big)\big), \quad (20)$$

regardless of whether this is positive (in which case it is a dividend) or negative (in which case it is the cost of raising equity). We will need to incorporate this stream of payments into our overall valuation of the contingent capital.

Two parameter ranges for the coupon and payout rates merit special mention. We know that as long as the firm has not exhausted its convertible debt, it can maintain the minimum capital ratio by converting debt into equity; that is, it can maintain the bound

$$(1-\alpha)V_t \ge B - (1-\alpha)L_t + D,$$

with equality holding at the instants of conversion. It follows that if

$$(1-\alpha)\delta > (1-\kappa)\max\{c_1, c_2\},$$

the firm always generates enough cash to service its debt, and shareholders always earn a dividend. In contrast, if

$$(1-\alpha)\delta < (1-\kappa)\min\{c_1, c_2\},$$

then the firm will stop paying a dividend—and will start issuing small amounts of equity—in advance of any debt converting to equity.

5. Decomposition of Payments on Convertible and Senior Debt

In this section, we decompose the payments to holders of the convertible debt into a principal payment, coupon payments, dividends on converted equity,

[3] It is unclear if coupons on contingent capital would be tax deductible under the current tax code in the United States because the conversion feature may make the debt too equity-like. This possibility could be modeled by taking $\kappa_1 = 0$. But tax rules could also be changed if regulators sought to create incentives for banks to hold more of their debt in the form of contingent capital.

and a terminal equity payment. We decompose payments on the senior debt contingent on the firm's ability to maintain the required capital ratio. These decompositions prepare the way for the valuations in the next section.

The horizon for the valuation is the smaller of the debt maturity T and the time τ_b at which V first hits $b = D/(1-\alpha)$. At τ_b, the firm has exhausted its contingent capital and can no longer sustain the required capital ratio; as before, we assume the firm is then seized by regulators and liquidated.[4] The firm still has equity at this point, but not enough to meet the capital requirement. To capture the possible loss in value from seizure, we assume that shareholders recover a random fraction $X_1 \in [0, 1]$ of the equity value at τ_b, the remaining fraction $1 - X_1$ representing a deadweight cost. (An alternative loss mechanism is the delayed recapitalization used by Peura and Keppo 2006.) Similarly, we apply a random recovery fraction of $X_2 \in [0, 1]$ to senior debt. We assume that X_1 and X_2 are independent of V but not of each other. Indeed, to enforce absolute priority of debt over equity, we need $P(X_2 = 1 \mid X_1 > 0) = 1$. Independence between (X_1, X_2) and V will imply that only the expected recovery rates $R_i = E[X_i]$, $i = 1, 2$, enter into our valuations. These can satisfy $R_1 > 0$ and $R_2 < 1$ without violating absolute priority. As just one illustration, any $0 \le R_1 \le R_2 \le 1$ can be realized as expected recovery rates while satisfying absolute priority by assigning to (X_1, X_2) the outcomes $(1, 1)$, $(0, 1)$, and $(0, 0)$ with probabilities R_1, $R_2 - R_1$, and $1 - R_2$, respectively.

5.1. Convertible Debt

We use $r > 0$ to denote a fixed (risk-free) interest rate at which to discount all payoffs for valuation. The discounted payoffs of the components of the convertible debt are as follows:

- *principal payment at maturity,*
$$e^{-rT}(B - (1-\alpha)L_T); \tag{21}$$
- *earned coupon,*
$$\int_0^T e^{-rs} c_1 (B - (1-\alpha)L_s)\, ds; \tag{22}$$
- *equity earned through conversion,*
$$e^{-rT}(1-\pi_T)\big(V_T - [(B - (1-\alpha)L_T) + D]\big)\mathbf{1}_{\{\tau_b > T\}} + e^{-r\tau_b}(1-\pi_{\tau_b}) X_1 \alpha V_{\tau_b} \mathbf{1}_{\{\tau_b \le T\}}; \tag{23}$$
- *net dividends,*
$$\int_0^{\min\{T,\tau_b\}} e^{-rt}(1-\pi_t) \cdot \big(\delta V_t - (1-\kappa)\big(c_1(B - (1-\alpha)L_t) + c_2 D\big)\big)\, dt. \tag{24}$$

[4] An alternative interpretation is that the firm undergoes a distressed sale, so the full value of the assets is not recovered, but the equity holders need not be wiped out.

In (21), $(1-\alpha)L_T$ is the total amount of debt converted to equity, so $B - (1-\alpha)L_T$ is the remaining principal at maturity. Similarly, in (22), $B - (1-\alpha)L_s$ is the remaining principal at time s, and multiplying this expression by c_1 yields the rate at which the holders of the bond earn coupons.

Equation (23) breaks down the claim on equity into two parts, depending on whether liquidation occurs before the maturity of the debt. In the first term, $\tau_b > T$, so the firm survives throughout the interval $[0, T]$. The market value of the firm's total equity at T is the difference
$$V_T - \big[(B - (1-\alpha)L_T) + D\big] \tag{25}$$
between the value of the firm's assets and the principal payments on the two kinds of debt. Here we invoke our assumption (relaxed in §8) that asset value is marked to market so that (25) is the cash paid to equity holders after retiring all debt if the assets are sold at T. A fraction $(1 - \pi_T)$ of this residual value goes to the new shareholders—those who acquired an equity stake through conversion of the contingent capital. In the second case in (23), the firm is seized and liquidated at time τ_b when the contingent capital is exhausted. At this instant, the firm just meets its capital requirement, so the residual market value is αV_{τ_b}. A fraction X_1 of this is recovered by shareholders upon liquidation, and a fraction $(1 - \pi_{\tau_b})$ of the recovered value goes to the new shareholders.

Finally, the integrand in (24) is the discounted value of the net dividend rate in (20) paid to the converted shareholders at time t. To value the contingent capital, we will need to calculate the expectations of (21)–(24).

5.2. Senior Debt

The payments on the senior debt can be decomposed similarly but more simply into principal and coupon payments. We again distinguish the cases $\tau_b \le T$ and $\tau_b > T$, the first case corresponding to seizure and liquidation of the firm. The discounted payoffs to senior debtholders are as follows:

- *earned coupon,*
$$\int_0^{\min\{\tau_b, T\}} c_2 D e^{-rs}\, ds; \tag{26}$$
- *principal,*
$$\big(e^{-rT}\mathbf{1}_{\{\tau_b > T\}} + X_2 e^{-r\tau_b}\mathbf{1}_{\{\tau_b < T\}}\big)D. \tag{27}$$

In Equation (26), coupons are paid until either the maturity of the debt at time T or the liquidation at τ_b. In (27), the principal payment is reduced from the original face value of D to $X_2 D$ in the case of liquidation, reflecting a random recovery fraction of X_2 for the senior debt and the possibility of a deadweight cost of seizure and liquidation. If $X_2 \equiv 1$, the senior debt would be entirely riskless.

6. Valuation

To calculate expectations of (21)–(27), we posit that the dynamics of the book value of the firm's assets are given by (1). Equivalently, we have, with $\mu = r-\delta-\sigma^2/2$,

$$V_t = V_0 \exp\{\mu t + \sigma W_t\}. \tag{28}$$

We are assuming that the firm's assets are marked to market, so that V also represents the market value of the firm's assets; we drop this assumption in §8. In writing the drift in (1) as $r-\delta$, we are implicitly specifying the dynamics of V under a risk-neutral pricing measure that we will use to take expectations in (21)–(26). Mathematically, this is by no means necessary—we could use any constant drift, including one that incorporates a risk premium, and modify our valuation formulas accordingly.

6.1. A Partial Transform

Inspection of the discounted payoffs in (21)–(26) and the proportion π_t in (17) indicates that the key remaining step for valuation is taking expectations involving powers of V and its running minimum, with the running minimum restricted to an interval. We therefore undertake a preliminary calculation of a general such expression, which we will then use to value the various payments.

Set

$$\tilde{W}_t = \log(V_t/V_0) \quad \text{and} \quad \tilde{m}_t = \min_{0\le s\le t} \tilde{W}_s; \tag{29}$$

then $\tilde{W}$ is a Brownian motion with drift μ and diffusion coefficient σ. Let

$$\begin{aligned} H(t,v,k,y) &= H_{\mu,\sigma}(t,v,k,y) \\ &= \mathbf{E}\big[\exp(v\tilde{W}_t + k\tilde{m}_t)\mathbf{1}\{\tilde{m}_t \le y\}\big], \\ &\qquad t,k\ge 0,\ v,y\in(-\infty,\infty). \end{aligned} \tag{30}$$

The function H depends on the parameters μ and σ through the processes $\tilde{W}$ and $\tilde{m}$; because these parameters remain fixed, we suppress this dependence and write simply $H(t,v,k,y)$ in referring to the function. The function is given explicitly in the following result.

Proposition 6.1. *The function H in* (30) *evaluates to*

$$H(t,v,k,y) = \exp(\mu v t + v^2\sigma^2 t/2)h(t,k,y), \tag{31}$$

with

$$\begin{aligned} h(t,k,y) &= \frac{2\theta}{2\theta+k\sigma^2} e^{ky+2y\theta/\sigma^2}\Phi\left(\frac{y+t\theta}{\sigma\sqrt{t}}\right) + \frac{2\theta+2k\sigma^2}{2\theta+k\sigma^2} \\ &\quad \cdot e^{k\theta t + k^2\sigma^2 t/2}\Phi\left(\frac{y-(\theta+k\sigma^2)t}{\sigma\sqrt{t}}\right), \end{aligned} \tag{32}$$

where $\theta = \mu + v\sigma^2$, and Φ is the standard normal distribution function.

With $y=0$, (30) defines the joint Laplace transform of $\tilde{W}_t$ and $-\tilde{m}_t$, and in this sense the general case in (30) defines a partial transform. In our application of the formula, y will always take the value $\log(a/V_0)$ or $\log(b/V_0)$, corresponding to the asset levels at which conversion of contingent capital starts and ends. In several cases, we need to take the difference of values of H at these two values of y with other arguments held fixed, so it will be convenient to define

$$\begin{aligned} \Delta H(t,v,k) &= H\big(t,v,k,\log(a/V_0)\big) \\ &\quad - H\big(t,v,k,\log(b/V_0)\big). \end{aligned} \tag{33}$$

6.2. Principal and Coupon Payments

The discounted expected value of the principal payment on the convertible debt is the expected value of Equation (21) and is given by

$$e^{-rT}\big(B-(1-\alpha)\mathbf{E}[L_T]\big). \tag{34}$$

Thus, to value the principal payment it suffices to find the expectation of L_T.

Proposition 6.2. *The expected present value of the contingent capital's principal payment is* (34), *where*

$$\begin{aligned} \mathbf{E}[L_t] &= aH\big(t,0,0,\log(a/V_0)\big) - bH\big(t,0,0,\log(b/V_0)\big) \\ &\quad - V_0\Delta H(t,0,1). \end{aligned}$$

This expression evaluates to

$$\begin{aligned} \mathbf{E}[L_t] &= a\Phi(\delta_{a1}^-) - b\Phi(\delta_{b1}^-) + \frac{2V_0(\mu+\sigma^2)}{(2\mu+\sigma^2)} \\ &\quad \cdot e^{t\mu+(t\sigma^2)/2}\big(\Phi(\delta_{a2}^-)-\Phi(\delta_{b2}^-)\big) + \frac{\sigma^2}{(2\mu+\sigma^2)} \\ &\quad \cdot \left(a\left(\frac{a}{V_0}\right)^{(2\mu)/\sigma^2}\Phi(\delta_{a1}^+) - b\left(\frac{b}{V_0}\right)^{(2\mu)/\sigma^2}\Phi(\delta_{b1}^+)\right), \end{aligned} \tag{35}$$

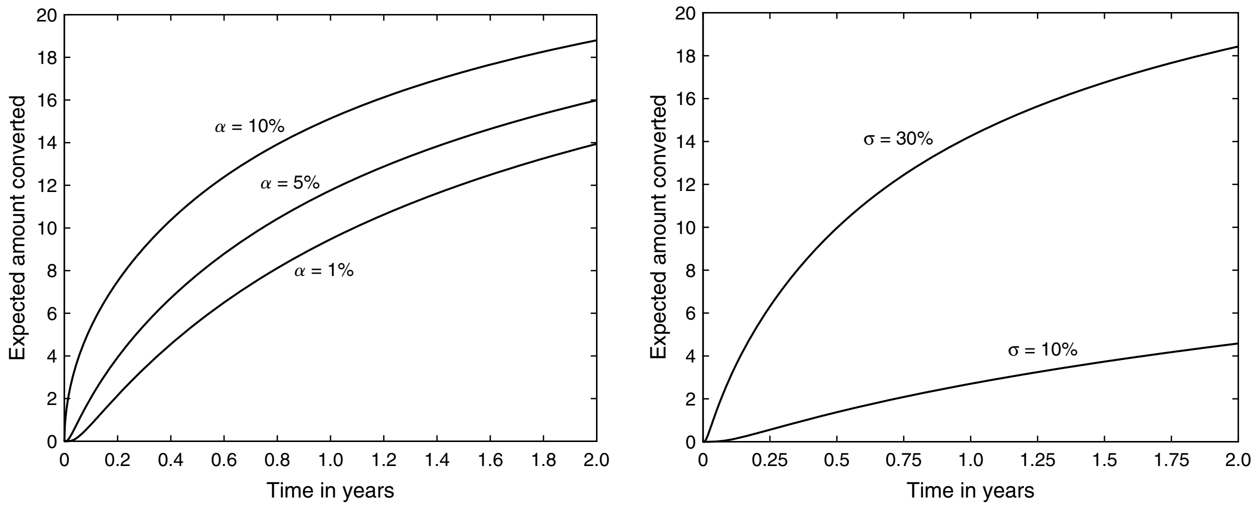
where

$$\delta_{a1}^{\pm} = \frac{\pm t\mu + \log(a/V_0)}{\sigma\sqrt{t}}, \quad \delta_{a2}^{\pm} = \frac{t(\mu+\sigma^2)\pm\log(a/V_0)}{\sigma\sqrt{t}},$$

$$\delta_{b1}^{\pm} = \frac{\pm t\mu + \log(b/V_0)}{\sigma\sqrt{t}}, \quad \delta_{b2}^{\pm} = \frac{t(\mu+\sigma^2)\pm\log(b/V_0)}{\sigma\sqrt{t}}.$$

Figure 4 plots the expected amount of contingent capital converted by time t, namely, $(1-\alpha)\mathbf{E}[L_t]$, over a two-year horizon for various levels of α and σ. Recall that $\mathbf{E}[L_t]$ depends on α through the boundaries a and b of the conversion band. The figure uses $V_0 = 100$ with $D = 60$, $B = 30$, $r = 5\%$, and $\delta = 3\%$. The left panel fixes σ at 25%, and the right panel fixes α at 5%. The curves show qualitatively different behavior near time zero: when the initial asset level is far from the conversion trigger (either because

Figure 4 **Comparison of $(1-\alpha)\mathbf{E}[L_t]$, the Expected Amount of Contingent Capital Converted by Time t, for Different Values of the Capital Ratio α (Left) and the Asset Volatility σ (Right)**



α is small or because σ is small), the expected amount converted is nearly flat for small t; the curves are steeper when the conversion trigger is closer.

The expected present value of the contingent capital coupon payments (22) is given by

$$B\frac{c_1}{r}(1-e^{-rT}) - c_1(1-\alpha)\int_0^T e^{-rt}\mathbf{E}[L_t]\,dt. \tag{36}$$

We do not have a simple expression for the integral in (36); however, because $\mathbf{E}[L_t]$ is smooth and monotone, the integral can be accurately approximated by replacing it with a sum.

6.3. Equity Earned Through Conversion

We turn now to (23), which gives the discounted terminal value of the equity acquired by the contingent capital investors through the process of conversion. We value separately the two terms in (23), the first corresponding to the firm surviving until T, the second corresponding to seizure and liquidation before T.

Proposition 6.3. *The value of the converted equity stake in the event of survival* (*the first term in* (23)) *is given by* $\exp(-rT)$ *times*

$$V_0\Delta H(T,1,0) - V_0\left(\frac{V_0}{a}\right)^{q(1-\alpha)/\alpha}\Delta H\big(T,1,q(1-\alpha)/\alpha\big)$$
$$- V_0(1-\alpha)\Delta H(T,0,1) + V_0(1-\alpha)\left(\frac{V_0}{a}\right)^{q(1-\alpha)/\alpha}$$
$$\cdot\,\Delta H\big(T,0,1+q(1-\alpha)/\alpha\big). \tag{37}$$

In the event of seizure and liquidation (*the second term in* (23)), *the value of the converted equity stake is, with* $R_1 = \mathbf{E}[X_1]$ *and* $\theta_1 = \sqrt{\mu^2+2\sigma^2 r}$,

$$R_1\alpha b\left(1-\left(\frac{b}{a}\right)^{q(1-\alpha)/\alpha}\right)\left(\frac{b}{V_0}\right)^{(\mu-\theta_1)/\sigma^2}$$
$$\cdot\, e^{-rT}H\big(T,(\theta_1-\mu)/\sigma^2,0,\log(b/V_0)\big). \tag{38}$$

6.4. Net Dividends

As discussed in §4, the difference between the total payout rate δV_t and debt service payments creates a dividend stream for equity holders, a fraction $1-\pi_t$ of which flows to investors who originally held convertible debt, as in (24). Taking the expected value of this expression, we get

$$\mathbf{E}\bigg[\int_0^{\min\{T,\tau_b\}} e^{-rt}(1-\pi_t)$$
$$\cdot(\delta V_t-(1-\kappa)(c_1(B-(1-\alpha)L_t)+c_2D))\,dt\bigg]$$
$$=\int_0^T e^{-rt}\mathbf{E}\big[(1-\pi_t)\big(\delta V_t-(1-\kappa)$$
$$\cdot(c_1(B-(1-\alpha)L_t)+c_2D)\big)\mathbf{1}\{\tau_b>t\}\big]\,dt. \tag{39}$$

The expectation inside the integral can be evaluated in closed form:

Proposition 6.4. *The expected net rate at which the contingent capital investors earn dividends* (*i.e., the expectation on the right side of* (39)) *is given by*

$$\delta V_0\Delta H(t,1,0) - (1-\alpha)(1-\kappa)(c_2-c_1)b\,\Delta H(t,0,0)$$
$$-(1-\alpha)(1-\kappa)c_1V_0\,\Delta H(t,0,1) - \left(\frac{V_0}{a}\right)^{q(1-\alpha)/\alpha}$$
$$\cdot\big[\delta V_0\,\Delta H(t,1,q(1-\alpha)/\alpha)$$
$$-(1-\alpha)(1-\kappa)(c_2-c_1)b\,\Delta H\big(t,0,q(1-\alpha)/\alpha\big)\big]$$
$$-(1-\alpha)(1-\kappa)c_1V_0\,\Delta H\big(t,0,1+q(1-\alpha)/\alpha\big). \tag{40}$$

The present value of the cumulative dividends is the time integral of this expression, which is easily and accurately approximated by a sum over a discrete set of dates.

It is also evident from this expression that the effect of the marginal tax rate κ is simply to replace each original coupon rate c_i with $(1-\kappa)c_i$. The formula remains valid if we replace $(1-\kappa)c_i$ with $(1-\kappa_i)c_i$ to allow different levels of tax deductibility of the two types of coupons.

6.5. Senior Debt

The expected value of the coupon payments (26) is given by

$$\begin{aligned}\mathbf{E}\left[\int_0^{\min\{\tau_b,T\}} c_2 D e^{-rs}\,ds\right] &= D\frac{c_2}{r}\left(1-\mathbf{E}[\exp\{-r\min\{\tau_b,T\}\}]\right)\\ &= D\frac{c_2}{r}\left(1-e^{-rT}\mathbb{P}(\tau_b>T)-\mathbf{E}\left[e^{-r\tau_b}\mathbf{1}_{\{\tau_b\le T\}}\right]\right). \qquad (41)\end{aligned}$$

Similarly, the discounted expected value of the principal payment (27) is given by

$$\begin{aligned}&D\mathbf{E}\left[e^{-rT}\mathbf{1}_{\{\tau_b>T\}}+X_2e^{-r\tau_b}\mathbf{1}_{\{\tau_b<T\}}\right]\\ &\quad= D\left(e^{-rT}\mathbb{P}(\tau_b>T)+R_2\mathbf{E}\left[e^{-r\tau_b}\mathbf{1}_{\{\tau_b\le T\}}\right]\right). \qquad (42)\end{aligned}$$

The probability $\mathbb{P}(\tau_b>T)$ coincides with $\mathbb{P}(\tilde{m}_T>\log(b/V_0))$, which can be evaluated directly using Equation (B1) in Appendix B; the expectation $\mathbf{E}[\exp(-r\tau_b)\mathbf{1}\{\tau_b\le T\}]$ is evaluated explicitly in Equation (C3) in Appendix C. With these substitutions, the total discounted expected value of the senior debt becomes

$$\begin{aligned}&D\frac{c_2}{r}+D\left(1-\frac{c_2}{r}\right)e^{-rT}\left[\Phi\left(\frac{\mu T-\log(b/V_0)}{\sigma\sqrt{T}}\right)-\left(\frac{b}{V_0}\right)^{(2\mu)/\sigma^2}\Phi\left(\frac{\mu T+\log(b/V_0)}{\sigma\sqrt{T}}\right)\right]\\ &+D\left(R_2-\frac{c_2}{r}\right)\left[\left(\frac{b}{V_0}\right)^{(\mu-\theta_1)/\sigma^2}\Phi\left(\frac{\log(b/V_0)-\theta_1T}{\sigma\sqrt{T}}\right)+\left(\frac{b}{V_0}\right)^{(\mu+\theta_1)/\sigma^2}\Phi\left(\frac{\log(b/V_0)+\theta_1T}{\sigma\sqrt{T}}\right)\right],\end{aligned}$$

where, as before, θ_1 is the square root of $2\sigma^2r+\mu^2$. The following result values the senior debt using the function H:

Proposition 6.5. *The value of the senior debt, including both coupon payments* (26) *and principal* (27), *is given, with* $\theta_1=\sqrt{2\sigma^2r+\mu}$, *by*

$$\begin{aligned}&D\frac{c_2}{r}+D\left(1-\frac{c_2}{r}\right)e^{-rT}(1-H(T,0,0,\log(b/V_0)))\\ &+D\left(R_2-\frac{c_2}{r}\right)\left(\frac{b}{V_0}\right)^{(\mu-\theta_1)/\sigma^2}\\ &\cdot e^{-rT}H(T,(\theta_1-\mu)/\sigma^2,0,\log(b/V_0)).\end{aligned}$$

7. Closing the Model: Market Yields

In our calculations, we have assumed that both the senior debt and the convertible debt are sold at par at time zero; this leads to constant book values (for the unconverted principal), (9), and the resulting tractability. In §6, we have calculated market prices for senior and convertible debt, with coupon rates assumed given. For our model to be internally consistent, we need the market prices we calculate at time zero to coincide with our assumption that the bonds sell at par. We now show that this is indeed possible and that it determines the coupon rates for both types of debt.

For the senior debt, equating the expected discounted value of the coupon and principal calculated in §6.5 to the face value D yields the coupon rate

$$c_2=r\left(1+\frac{(1-R_2)\mathbf{E}[e^{-r\tau_b}\mathbf{1}_{\{\tau_b\le T\}}]}{1-e^{-rT}\mathbb{P}(\tau_b>T)-R_2\mathbf{E}[e^{-r\tau_b}\mathbf{1}_{\{\tau_b\le T\}}]}\right).$$

R_2 should not be in the denominator

The probability and expectation in this expression are evaluated in Appendix C.4, thus allowing direct evaluation of c_2. If $R_2=1$, the coupon rate c_2 reduces to r: under our assumption that the firm is seized and liquidated when it violates its capital requirement—before insolvency—the senior debt is riskless if there is no loss of value at liquidation.

Similarly, for the convertible debt, equating our valuation (the sum of the expectations of (21)–(24)) with the face value B yields the coupon rate

$$c_1=\frac{B-A_1-A_3-A_4}{A_2+A_5},$$

the denominator should be A2 - A5

where A_1 is the expected principal in (34),

$$A_2=\frac{B}{r}(1-e^{-rT})-(1-\alpha)\int_0^T e^{-rt}\mathbf{E}[L_t]\,dt,$$

from (36), A_3 is the expected terminal equity value (the sum of (38) and $\exp(-rT)$ times (37)), and

$$A_4=\mathbf{E}\left[\int_0^{\min\{T,\tau_b\}} e^{-rt}(1-\pi_t)(\delta V_t-(1-\kappa)c_2D)\,dt\right]$$

and

$$A_5=\mathbf{E}\left[\int_0^{\min\{T,\tau_b\}} e^{-rt}(1-\pi_t)(1-\kappa)(B-(1-\alpha)L_t)\,dt\right]$$

come from the net dividends in (39). The results in §6 yield explicit expressions for A_1–A_5 and thus for the coupon rate c_1.

We view these expressions as the key practical contribution of our analysis. Given the characteristics of the firm—its asset volatility and the face value of its senior and convertible debt—these equations give the coupon rates required by the market. For debt issued at par, the coupon rate equals the yield; so, more generally, we interpret these rates as the yields required by the market for the two types of debt. These equations are therefore useful in gauging the yield required by investors in contingent capital as compensation for bearing the risk that the debt they hold converts to equity.

8. Distinguishing Market and Book Values of Assets

To this point, we have assumed that the bank's assets are marked to market so that V represents the market value of assets as well as their book value. We now extend the model to capture a stochastic relation between the two. We use A_t to denote the market value of assets. Our key assumption is that although the market and book values of assets may differ, they are sufficiently aligned to agree on whether a bank is solvent. If the bank were liquidated at time t, debt holders would be due $B_t + D$, so the bank is solvent if its assets have at least this value. Our condition, then, is that $A_t > B_t + D$ whenever $V_t > B_t + D$. To model this relationship, we introduce a second geometric Brownian motion U,

$$U_t = U_0 \exp\{\theta_u t + \sigma_u W'_t\},$$

with W' and W (the original Brownian motion driving V) having instantaneous correlation ρ. We model A_t as satisfying

$$A_t - B_t - D = U_t(V_t - B_t - D). \tag{43}$$

The process U can be roughly interpreted as a market-to-book ratio, but whereas $V_t - B_t - D$ is the book value of equity, $A_t - B_t - D$ is the difference between the market value of assets and the book value of debt. A natural choice in this setting would be to take $\theta_u = -\sigma_u^2/2$, so that $E[U_t]$ is constant, but we need not limit ourselves to this case.

In this extension of our basic model, conversion from debt to equity is still governed by book value V, just as before; but the value received by equity holders at either the maturity date T or at a seizure at τ_b now depends on the market value A. Accordingly, we modify (23) by replacing V_T with A_T and V_{τ_b} with A_{τ_b}. This case remains tractable under the parameter restriction $2\sigma^2\gamma \geq \mu^2$, where

$$\gamma = -\theta_u + \sigma_u \mu\rho/\sigma - \tfrac{1}{2}\sigma_u^2(1-\rho^2) + r.$$

In Proposition 6.3, (37) becomes

$$\begin{aligned}
&\vartheta \Delta H\Big(T, 1+\sigma_u\frac{\rho}{\sigma}, 0\Big) \\
&\quad - \vartheta\Big(\frac{V_0}{a}\Big)^{q(1-\alpha)/\alpha} \Delta H\Big(T, 1+\sigma_u\frac{\rho}{\sigma}, q(1-\alpha)/\alpha\Big) \\
&\quad - \vartheta(1-\alpha)\Delta H\Big(T, \sigma_u\frac{\rho}{\sigma}, 1\Big) \\
&\quad + \vartheta(1-\alpha)\Big(\frac{V_0}{a}\Big)^{q(1-\alpha)/\alpha} \Delta H\Big(T, \sigma_u\frac{\rho}{\sigma}, 1+q(1-\alpha)/\alpha\Big),
\end{aligned}$$

with $\vartheta = V_0 U_0 \exp((r-\gamma)T)$, and (38) becomes

$$\begin{aligned}
&R_1 U_0 \alpha b\bigg(1-\Big(\frac{b}{a}\Big)^{(q((1-\alpha)/\alpha))}\bigg) \\
&\quad \cdot \bigg[\Big(\frac{b}{V_0}\Big)^{(\rho\sigma_u/\sigma+\mu-\theta_2)/\sigma^2} \Phi\Big(\frac{\log(b/V_0)-\theta_2 T}{\sigma\sqrt{T}}\Big) \\
&\qquad + \Big(\frac{b}{V_0}\Big)^{(\rho\sigma_u/\sigma+\mu+\theta_2)/\sigma^2} \Phi\Big(\frac{\log(b/V_0)+\theta_2 T}{\sigma\sqrt{T}}\Big)\bigg].
\end{aligned}$$

These expressions are derived through a minor modification of the proof of Proposition 6.3 after making the substitution in (43).

With the condition that $\theta_u = -\sigma_u^2/2$, this extension introduces two new parameters, the "book-to-market volatility" σ_u and correlation ρ, as well as the initial value A_0. Though not directly observable, these parameters could be calibrated using market values of a firm's debt and equity and book values from financial statements. Because our model already has several parameters, in the numerical examples of the next section, we limit ourselves to the basic model in which $A_t = V_t$.

9. Example

In this section, we use numerical examples to investigate how the yields derived from our model change with parameter inputs and how the introduction of convertible debt influences the spread on senior debt. Table 1 shows the parameter values we use. The first

Table 1 Parameters for Base Case (I) and Modified Scenario (II)

		I	II
Debt over assets ratio	D/V_0	90%	
Capital adequacy ratio	α	4%	
Risk free rate	r	5%	0.5%
Volatility of asset returns	σ	8%	16%
Debt maturity	T	1.5	
Fractional payout of assets	δ	3%	1.5%
Tax rate	κ	30%	
Recovery rate for equity	R_1	30%	
Recovery rate for senior debt	R_2	95%	

set (I) is our base case and is intended to be representative of the end of 2006, before the financial crisis, based on data for the 20 largest (by assets) banks in the United States. The parameter modifications indicated under II are intended to be representative of 2009. In both cases, we consider a bank with 90% debt that is required to maintain a minimum capital ratio of 4% of assets (which corresponds to 8% of risk-weighted assets for a bank whose assets have an average risk weight of 50%). The maturity T approximates the weighted average maturity of debt for large banks, using a six-month maturity for deposits. The base case has a relatively low asset volatility of 8% (see, e.g., the estimates in Nikolova 2003), a payout rate of 3% (reflecting both interest payments and dividends), and a risk-free rate of 5%, which is very close to the average Treasury rate at the end of 2006, when the Treasury yield curve was quite flat.

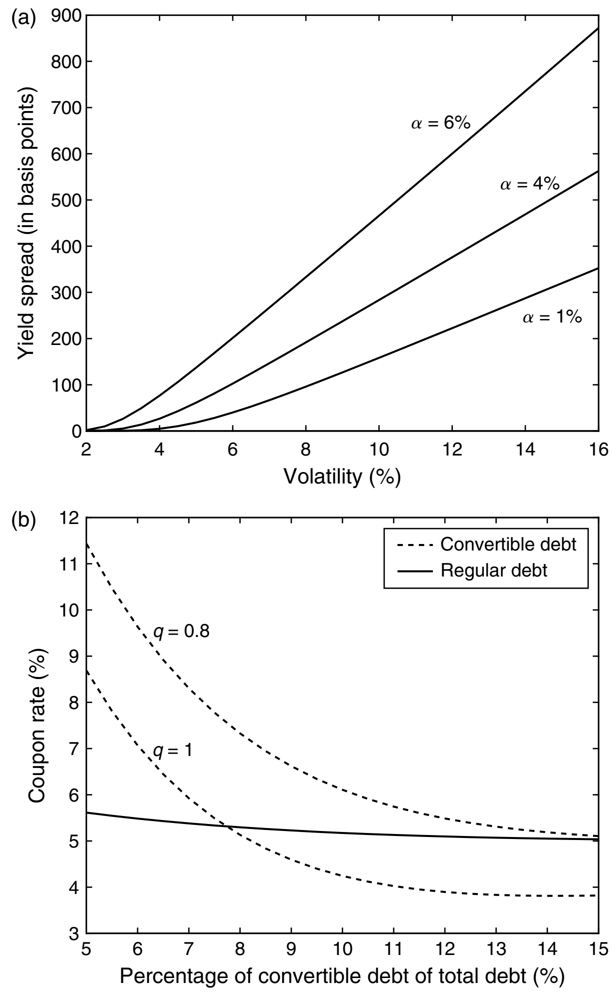
We begin with nonconvertible debt only. Recall that the coupon rate is set to price the bond at par, so the coupon and yield are equal. Figure 5(a) shows the yield spreads we obtain with our assumed recovery rate R_2 of 95%. With a 100% recovery rate, the debt would be riskless. The potential loss of 5% takes effect only in case of seizure by regulators; this occurs at a positive capital ratio, when the bank's assets still exceed the value of its debt, so the loss reflects a liquidation cost. Our base case of $\alpha = 4\%$ and $\sigma = 8\%$ produces a spread of 1.9%. As expected, the figure shows that the spread increases if we increase α or σ, because each of these changes increases the likelihood of seizure and thus of a loss from liquidation.

Figure 5(b) illustrates the effect of introducing convertible debt to the balance sheet. The total amount of debt (regular debt and convertible debt) is fixed at 90% of total asset value. We change the proportion of convertible debt from 5% to 15% of the total debt. The graph shows the required coupon rates for both the senior and the convertible debt. The coupon rate on the convertible debt depends on the loss incurred by shareholders at seizure and liquidation; we assume a 30% recovery rate, meaning that 70% of the remaining equity value at seizure is lost through liquidation. The figure shows two levels of the conversion ratio, $q = 1$ and $q = 0.8$.

The first observation is that the coupon rate on the senior debt decreases when the proportion of convertible debt increases. The contingent capital works as a cushion against liquidation; therefore, with the same recovery rate, the senior debt suffers lower liquidation costs because of the reduced likelihood of seizure, and this translates to a lower compensating coupon rate.

With only a small amount of convertible debt, the required coupon on this debt is high, and this can be understood as follows. With a thin layer of convertible debt on the balance sheet, the probability of liquidation does not change much, and if the asset level

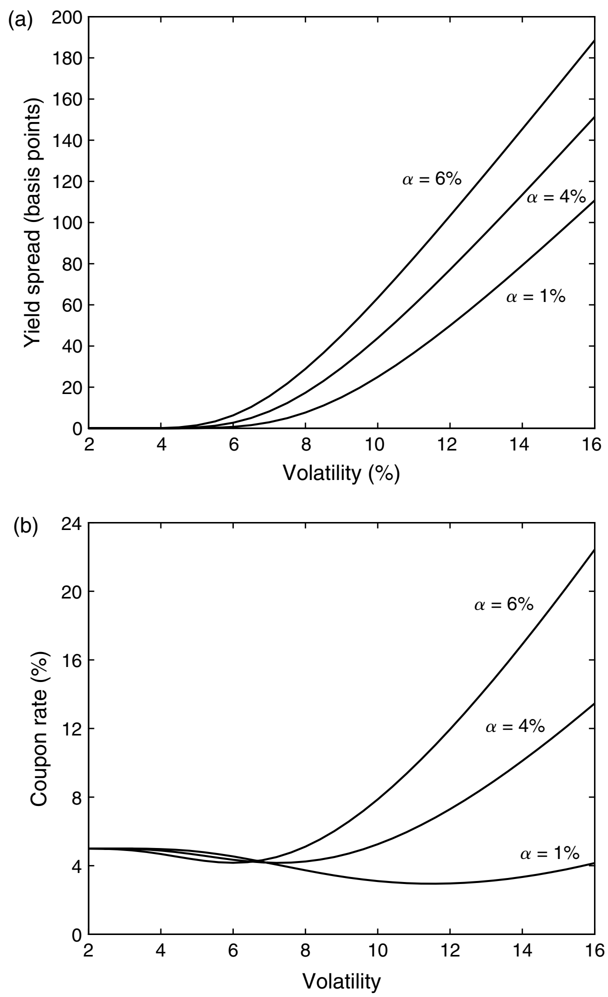
Figure 5 **(a) Sensitivity of Senior Debt to Volatility and Minimum Capital Ratio α in the Absence of Convertible Debt; (b) Coupon Rates at Different Magnitudes of Convertible Debt as a Percentage of Total Debt**



hits the conversion trigger it is very likely that the full layer of contingent capital will be converted and the liquidation boundary will be reached, leaving little chance for the converted investors to benefit from the potential upside to equity. Indeed, they are likely to incur the 30% liquidation cost to equity shortly after conversion.

However, the coupon rate decreases quickly as we thicken the layer of convertible debt. Indeed, when $q = 1$ and convertible debt makes up more than 7.8% of total debt, it earns a lower coupon than the senior debt, and at more than 8% of the total debt, its coupon drops below the risk-free rate. This pattern results from the potential upside of the equity the contingent capital investors earn through conversion. Conversion occurs precisely when the book value of equity is low, so, conditional on survival, the contingent capital investors can benefit substantially from an increase in equity value. Increasing the proportion of contingent capital widens the interval between the conver-

Figure 6 (a) Sensitivity of Senior Debt to Volatility and Minimum Capital Ratio α when 10% of Debt Is Replaced with Convertible Debt; (b) Coupon Rates for Convertible Debt

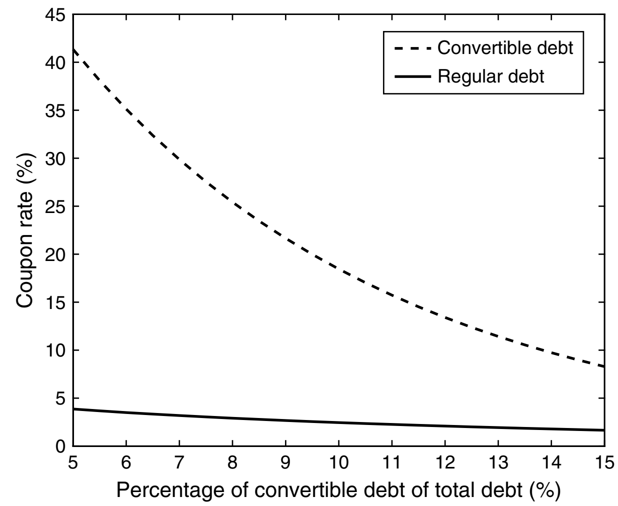


sion trigger and the liquidation trigger and increases the likelihood of an upside gain through conversion to equity. Lowering q to 0.8 reduces the upside gain from conversion and thus requires a higher coupon to keep the convertible debt priced at par.

Figure 6(a) reproduces Figure 5(a) but now with convertible debt making up 10% of total debt. The figure shows that the coupon rate for senior debt is now much less sensitive to volatility; for example, at $\alpha = 6\%$ and a volatility of 16%, the spread does not exceed 200 basis points, whereas without contingent capital it was over 800 basis points. This clearly shows the effect of the protection provided by the convertible debt.

Figure 6(b) shows the required coupon rate for convertible debt at different values of volatility and α. The graphs are more complicated and nonmonotonic in this case. This reflects the hybrid nature of the contingent capital, with both equity-like and debt-like behavior. Volatility has an adverse effect on debt and a favorable effect on equity. What we observe in Figure 6(b) is the trade-off between these two effects.

Figure 7 Coupon Rates at Different Magnitudes of Convertible Debt with Parameter Set II



Next we consider parameter set II of Table 1, based roughly on conditions in 2009. Volatility is much higher, the risk-free rate is much lower, and we have cut the payout rate δ to reflect lower dividend rates. Figure 7 shows the resulting coupon rate on senior and convertible debt, and it shows that in these new market conditions, the fair coupon rate on the convertible debt is dramatically higher. Indeed, with these parameters, the geometric Brownian motion that models assets has a negative risk-neutral drift ($r - \delta - \sigma^2/2 = -0.0228$) and a high volatility, implying a higher chance of liquidation. As the debt sells at par, higher liquidation probabilities must be compensated with higher coupon rates. Increasing the size of the convertible debt decreases the required coupon rate; but, in contrast to the previous parameter set, even at 10% convertible debt we observe very high coupon rates. We see this as reflecting the necessity of issuing contingent capital in advance of a crisis; in an environment of high volatility, investors will demand a much higher coupon unless the overall level of leverage is substantially reduced. The problem is diminished with a wider tranche of contingent capital, which provides a buffer for the senior debt and yields for the convertible debt in the range of 5%–10%.

10. Concluding Remarks

We have developed a model to value contingent capital in the form of debt that converts to equity. The key distinguishing features of our analysis are that we formulate a capital-ratio trigger and we model partial and on-going conversion. Our capital-ratio trigger approximates a regulatory capital requirement by using book values for debt and equity. Our partial conversion process allows just enough debt to convert to equity to maintain the required ratio until the contingent capital is fully exhausted. We derive closed-form expressions for yield spreads by adding a consistency requirement that market and book values of debt agree at issuance and at maturity.

Our numerical examples indicate that the fair yield for contingent capital in our model is quite sensitive to some of the model's inputs—in particular, to the size of the convertible tranche, to the volatility of the firm's assets, and to recovery rates in the event that the firm breaches its minimum capital requirement and is seized by regulators. This sensitivity—particularly to asset volatility and recovery rates, which are not directly observable and are difficult to estimate—as well as the overall complexity of the product could present obstacles to generating the investor demand that would be needed for widespread issuance of contingent convertible bonds.

Appendix A. Equity Allocation

In this appendix, we prove Theorem 3.1 under the more general assumption that V is any continuous semimartingale (as in Protter 1990, pp. 44, 114). We first show that the expression for Q^o in (16) satisfies (14). For $t \in [0, \min(\tau_b, T)]$, we have $Q_t > 0$. By (4), Q is a continuous semimartingale, and L is an increasing process, so we may take the differential of (16) to get

$$dQ_t^o = dQ_t \frac{Q_t^o}{Q_t} + Q_t d\left[\left(\frac{a - L_t}{a}\right)^{(q((1-\alpha)/\alpha))}\right],$$

and

$$d\left[\left(\frac{a - L_t}{a}\right)^{(q((1-\alpha)/\alpha))}\right] = -q\frac{1-\alpha}{\alpha}\frac{Q_t^o}{Q_t}\frac{dL_t}{a - L_t}. \tag{A1}$$

From part (iii) of Proposition 2.1, we know that if t is a point of increase (in the sense of Harrison 1985, p. xvii) of L, then $V_t - (B - (1-\alpha)L_t - D) = \alpha V_t$; in other words, $Q_t = \alpha V_t$. This expression also gives $V_t = a - L_t$. Thus, we have

$$\frac{dL_t}{a - L_t} = \frac{dL_t}{V_t} = \frac{dL_t}{Q_t/\alpha}.$$

Making this substitution in (A1) and rearranging terms, we get (14). If $\tau_b < T$, then for $t \in (\tau_b, T]$ we have $L_t = L_{\tau_b}$, and (16) is consistent with (15). Thus, Q^o in (16) solves (14) and (15).

To prove uniqueness, we use Theorem 6 on p. 194 of Protter (1990), for which we rewrite (14) and (15) as

$$Q_t^o = Q_o + \int_0^t f(s, \omega, Q_s^o)\, dZ_s,$$

with $Z_s = V_s + (q-1)(1-\alpha)L_s$ and

$$f(t, \omega, x) = \begin{cases} x/Q_t(\omega) & t \le \tau_b(\omega), \\ (b/a)^{q(1-\alpha)/\alpha} & t > \tau_b(\omega), \end{cases}$$

the second case giving $Q^o_{\tau_b}/Q_{\tau_b}$, as in (18). For each fixed x, the mapping $(t, \omega) \to f(t, \omega, x)$ is continuous in t and adapted. For each fixed (t, ω) and any real x, y,

$$\left|f(t, \omega, x) - f(t, \omega, y)\right| \le \frac{(1-\alpha)}{\alpha D}\mathbf{1}\{s < \tau_b(\omega)\}|x - y|,$$

because $Q_t \ge \alpha D/(1-\alpha)$ for $t \in [0, \tau_b)$. The conditions for Protter's (1990) theorem are thus satisfied, and uniqueness follows. The expressions in (17) for π follow directly from (14).

Appendix B. Proof of Proposition 6.1

The main objective of this section is to prove Proposition 6.1. First, we recall that, at each t, $\tilde{W}_t$ defined in (29) has a $N(\mu t, \sigma^2 t)$ distribution, and $\tilde{m}_t$ has the following distribution and density (see, e.g., Harrison 1985, p. 14), for $m \le 0$:

$$\mathbb{P}(\tilde{m}_t \le m) = \Phi\left(\frac{m - \mu t}{\sigma\sqrt{t}}\right) + \exp\left\{2\mu\frac{m}{\sigma^2}\right\}\Phi\left(\frac{m + \mu t}{\sigma\sqrt{t}}\right), \tag{B1}$$

$$f^{\mu}_{\tilde{m}_t}(m) = \frac{2}{\sigma\sqrt{2\pi t}}\exp\left\{\frac{-1}{2\sigma^2 t}(m - \mu t)^2\right\} + 2\frac{\mu}{\sigma^2}\exp\left\{2\mu\frac{m}{\sigma^2}\right\}\Phi\left(\frac{m + \mu t}{\sigma\sqrt{t}}\right). \tag{B2}$$

Now let

$$h_\mu(t, k, y) = H(t, 0, k, y) = \int_{-\infty}^{y} e^{km} f^{\mu}_{\tilde{m}_t}(m)\, dm.$$

Integration yields

$$h_\mu(t, k, y) = \frac{2\mu}{2\mu + k\sigma^2} e^{ky + 2y\mu/\sigma^2}\Phi\left(\frac{y + t\mu}{\sigma\sqrt{t}}\right) + \frac{2\mu + 2k\sigma^2}{2\mu + k\sigma^2} e^{k\mu t + k^2\sigma^2 t/2}\Phi\left(\frac{y - (\mu + k\sigma^2)t}{\sigma\sqrt{t}}\right).$$

We can now evaluate H. By the Girsanov theorem,

$$\mathbf{E}\left[e^{v\tilde{W}_t + k\tilde{m}_t}\mathbf{1}\{\tilde{m}_t \le y\}\right] = e^{v\mu t + v^2\sigma^2 t}\mathbf{E}_\theta\left[e^{k\tilde{m}_t}\mathbf{1}\{\tilde{m}_t \le y\}\right],$$

the subscript θ indicating that the expectation is taken with the drift of $\tilde{W}$ equal to $\theta = \mu + v\sigma^2$ rather than μ. The remaining expectation is given by $h_\theta(t, k, y)$.

Appendix C. Valuation Results

C.1. Proof of Proposition 6.2

We may write $L_t = (a - V_0 \exp(\tilde{m}_t))^+ \wedge (a - b)$ as

$$\begin{aligned} L_t &= (a-b)\mathbf{1}]\{\tilde{m}_T \le \log(b/V_0)\} \\ &\quad + (a - V_0 e^{\tilde{m}_T})\mathbf{1}\{\log(b/V_0) < \tilde{m}_T \le \log(a/V_0)\} \\ &= a\mathbf{1}\{\tilde{m}_T \le \log(a/V_0)\} - b\mathbf{1}\{\tilde{m}_T \le \log(b/V_0)\} \\ &\quad - V_0 e^{\tilde{m}_T}\mathbf{1}\{\log(b/V_0) < \tilde{m}_T \le \log(a/V_0)\}. \end{aligned}$$

The first expression in the proposition now follows from the definition of H in (30) and ΔH in (33). The second expression follows by making the substitutions in (31) and simplifying terms.

C.2. Proof of Proposition 6.3

We begin with the second part of the proposition, showing that (38) is the expectation of the second term in (23). By definition, we have $V_{\tau_b} = b$, and the fraction π_{τ_b} is given in (18). With these substitutions, the second term in (23) simplifies to

$$X_1 \alpha b\left(1-\left(\frac{b}{a}\right)^{(q((1-\alpha)/\alpha))}\right)e^{-r\tau_b}\mathbf{1}\{\tau_b \le T\}.$$

To calculate its expectation, we need to find $\mathbf{E}[e^{-r\tau_b}\mathbf{1}\{\tau_b \le T\}]$. By the Girsanov theorem, this expectation coincides with

$$\mathbf{E}_\theta\left[\exp\left\{-r\tau_b + \frac{\mu-\theta}{\sigma^2}\tilde{W}_{\tau_b} - \frac{\mu^2-\theta^2}{2\sigma^2}\tau_b\right\}\mathbf{1}\{\tau_b \le T\}\right], \tag{C1}$$

where $\mathbf{E}_\theta$ indicates expectation with the drift of $\tilde{W}$ changed to θ. This identity holds for any real θ; if we choose $\theta = \theta_1$, with $\theta_1 = \sqrt{2\sigma^2 r + \mu^2}$, then, recalling that $\tilde{W}_{\tau_b} = \log(b/V_0)$, the expectation in (C1) becomes

$$\left(\frac{b}{V_0}\right)^{(\mu-\theta_1)/\sigma^2}\mathbb{P}_{\theta_1}(\tau_b \le T). \tag{C2}$$

Observing that $\mathbb{P}_{\theta_1}(\tau_b \le T) = \mathbb{P}_{\theta_1}(\tilde{m}_T \le \log(b/V_0))$ and applying formula (B1) with m replaced by $\log(b/V_0)$ and μ replaced by θ_1, it follows that (C2) is equal to

$$\left(\frac{b}{V_0}\right)^{(\mu-\theta_1)/\sigma^2}\Phi\left(\frac{\log(b/V_0)-\theta_1 T}{\sigma\sqrt{T}}\right) + \left(\frac{b}{V_0}\right)^{(\mu+\theta_1)/\sigma^2}\Phi\left(\frac{\log(b/V_0)+\theta_1 T}{\sigma\sqrt{T}}\right). \tag{C3}$$

Thus we have shown that

$$\begin{aligned}&\mathbf{E}[e^{-r\tau_b}(1-\pi_{\tau_b})X_1\alpha V_{\tau_b}\mathbf{1}_{\{\tau_b\le T\}}]\\ &= R_1\alpha b\left(1-\left(\frac{b}{a}\right)^{(q((1-\alpha)/\alpha))}\right)\\ &\quad\cdot\left[\left(\frac{b}{V_0}\right)^{(\mu-\theta_1)/\sigma^2}\Phi\left(\frac{\log(b/V_0)-\theta_1 T}{\sigma\sqrt{T}}\right)\right.\\ &\quad\left.+\left(\frac{b}{V_0}\right)^{(\mu+\theta_1)/\sigma^2}\Phi\left(\frac{\log(b/V_0)+\theta_1 T}{\sigma\sqrt{T}}\right)\right].\end{aligned}$$

A further application of the Girsanov theorem yields

$$\begin{aligned}\mathbb{P}_{\theta_1}(\tau_b \le T) &= \mathbf{E}\left[\exp\left\{\frac{\theta_1-\mu}{\sigma^2}\tilde{W}_T + \frac{\mu^2-\theta_1^2}{2\sigma^2}T\right\}\mathbf{1}\{\tau_b\le T\}\right]\\ &= e^{-rT}H(T,(\theta_1-\mu)/\sigma^2,0,\log(b/V_0)),\end{aligned}$$

and thus the expression in (38).

Next we turn to (37). On the event that the firm survives until the debt matures, the present value of the equity held by the contingent capital investors is given by the first term in (23). We can replace the indicator $\mathbf{1}\{\tau_b > T\}$ in this expression with $\mathbf{1}\{\tilde{m}_T > \log(b/V_0)\}$, and if $\tilde{m}_T > \log(a/V_0)$, then no debt was converted and $(1-\pi_T) = 0$, so we may restrict the expectation to the event that $\tilde{m}_T$ lies between $\log(b/V_0)$ and $\log(a/V_0)$. Moreover, for $\tilde{m}_T$ in this interval, $L_T = a - V_0\exp(\tilde{m}_T)$. On this event, we therefore get (from (17))

$$\pi_T = \left(\frac{V_0}{a}\right)^{q((1-\alpha)/\alpha)} e^{(q((1-\alpha)/\alpha))\tilde{m}_T},$$

and $B-(1-\alpha)L_T + D = a(1-\alpha)-(1-\alpha)L_T = (1-\alpha)V_0 e^{\tilde{m}_T}$. Making these substitutions, the first term in (23) becomes $\exp(-rT)$ times

$$\left(1-\left(\frac{V_0}{a}\right)^{q(1-\alpha)/\alpha} e^{(q((1-\alpha)/\alpha))\tilde{m}_T}\right)\left(V_0 e^{\tilde{W}_T} - (1-\alpha)V_0 e^{\tilde{m}_T}\right) \cdot\mathbf{1}\left\{\log\left(\frac{b}{V_0}\right) < \tilde{m}_T \le \log\left(\frac{a}{V_0}\right)\right\}.$$

By expanding the product and taking the expectation, we get four terms, each of the type that defines the function ΔH, and this yields (37).

C.3. Proof of Proposition 6.4

If $\tilde{m}_t \le \log(b/V_0)$, then $\tau_b \le t$, and if $\tilde{m}_t > \log(a/V_0)$, then $\pi_t = 1$. In addition, for $\tilde{m}_t$ in the interval $[\log(b/V_0), \log(a/V_0)]$, L_t and π_t are respectively equal to $a - V_0 e^{\tilde{m}_t}$ and $(V_0 e^{\tilde{m}_t}/a)^{(q((1-\alpha)/\alpha))}$. It follows that $(1-\pi_t)(\delta V_t - (1-\kappa)[c_1(B-(1-\alpha)L_t)+c_2 D])\mathbf{1}\{\tau_b > t\}$ equals

$$\begin{aligned}&\left(1-\left(\frac{V_0 e^{\tilde{m}_t}}{a}\right)^{(q((1-\alpha)/\alpha))}\right)\\ &\cdot\left(\delta V_0 e^{\tilde{W}_t} - (1-\kappa)(1-\alpha)\left[(c_2-c_1)b + c_1 V_0 e^{\tilde{m}_t}\right]\right)\\ &\cdot\mathbf{1}\left\{\log\left(\frac{b}{V_0}\right) < 1\tilde{m}_t \le \log\left(\frac{a}{V_0}\right)\right\}.\end{aligned}$$

Here again the expectation is a linear combination of values of ΔH, as given in (40).

C.4. Proof of Proposition 6.5

We have $\mathbb{P}(\tau_b > T) = 1 - \mathbb{P}(\tilde{m}_T \le \log(b/V_0)) = 1 - H(T, 0, 0, \log(b/V_0))$, and we showed that

$$\mathbf{E}\left[e^{-r\tau_b}\mathbf{1}\{\tau_b \le T\}\right] = e^{-rT}H\left(T,(\theta_1-\mu)/\sigma^2, 0, \log(b/V_0)\right)$$

in the proof of Proposition 6.3. The result follows from making these substitutions in (41)–(42).

the RHS in above should be multiplied by (b / V_0)^((\mu - \theta_1) / \sigma^2)

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