Mathematical Finance, Vol. 20, No. 1 (January 2010), 1–33

MOMENT EXPLOSIONS AND STATIONARY DISTRIBUTIONS IN AFFINE DIFFUSION MODELS

PAUL GLASSERMAN

Columbia University

KYOUNG-KUK KIM

Korea Advanced Institute of Science and Technology

Many of the most widely used models in finance fall within the affine family of diffusion processes. The affine family combines modeling flexibility with substantial tractability, particularly through transform analysis; these models are used both for econometric modeling and for pricing and hedging of derivative securities. We analyze the tail behavior, the range of finite exponential moments, and the convergence to stationarity in affine models, focusing on the class of canonical models defined by Dai and Singleton (2000). We show that these models have limiting stationary distributions and characterize these limits. We show that the tails of both the transient and stationary distributions of these models are necessarily exponential or Gaussian; in the non-Gaussian case, we characterize the tail decay rate for any linear combination of factors. We also give necessary and sufficient conditions for a linear combination of factors to be Gaussian. Our results follow from an investigation into the stability properties of the systems of ordinary differential equations associated with affine diffusions.

KEY WORDS: affine diffusion, tail behavior, stationary distribution, moment explosion.

1. INTRODUCTION

The affine family of diffusion models includes many of the most widely used models in finance. The affine framework offers substantial modeling flexibility and a high degree of tractability, particularly through Laplace or Fourier transforms. Examples of affine diffusions include the Ornstein–Uhlenbeck (OU) process, the square-root diffusion associated with the Cox–Ingersoll–Ross (CIR) (1985) interest rate model, the Heston (1993) stochastic volatility model, the interest rate models of Brown and Schaefer (1994) and Longstaff and Schwartz (1992), and the Duffie–Kan (1996) family of term structure models. Affine models are used both for econometric modeling of time series data and for pricing and hedging of derivative securities.

Duffie, Pan, and Singleton (2000) develop a transform analysis for affine jump-diffusions in a very general setting. They derive generalized characteristic functions associated with these models and show that these are exponentials of affine functions of the state variables; the coefficients of these affine functions are characterized as solutions

The authors would like to thank the anonymous referees for their many detailed comments and corrections which have improved the paper.

Manuscript received October 2007; final revision received April 2008.

Address correspondence to Kyoung-Kuk Kim, Department of Industrial and Systems Engineering, KAIST, Daejeon, South Korea; e-mail: catenoid@kaist.edu.

to ordinary differential equations (ODEs). Duffie, Filipovic, and Schachermayer (2003) characterize regular affine processes and their associated differential equations. Dai and Singleton (2000) define equivalence classes of affine models that are invariant under certain affine transformations, and they define a canonical model within each class. See Singleton (2006) for an extensive discussion of the estimation of these models.

In this paper, we study the tail behavior of affine diffusions and their stationary distributions. We focus on canonical models and show that these models do indeed have limiting stationary distributions. We characterize the tail behavior of the transient and stationary distributions of these models, and we show that the tails are always exponential or Gaussian. This in turn allows us to characterize the range of finite moments for asset price processes constructed from affine diffusions.

We obtain our results through an analysis of the stability of the ODEs that determine the transforms associated with an affine model. To illustrate the connection between tail behavior and transforms, let X be a positive-valued random variable and let $\phi(\theta) = \mathbb{E}[\exp(\theta X)]$ denote its moment generating function (the mapping $\theta \to \phi(-\theta)$ is its Laplace transform). We can distinguish various types of tail behavior for X based on properties of $\phi(\theta)$ for $\theta \geq 0$: If $\phi(\theta) = \infty$ for all $\theta > 0$, then X is heavy-tailed; if $\phi(\theta)$ is finite all for $\theta \in [0, \theta_0)$, for some $\theta_0 > 0$, then the tail of X is exponentially bounded; if, in addition, $\phi(\theta) = \infty$ for all $\theta > \theta_0$, then the tail is exponentially bounded both above and below, so X has an exponential tail; and if $\phi(\theta) < \infty$ for all $\theta \geq 0$, then X is light-tailed. Similar statements apply to a two-sided random variable through consideration of both positive and negative values of θ. When we refer to the tails of a random vector $X \in \mathbb{R}^n$, we mean the tails of random variables of the form $u \cdot X,\ u \in \mathbb{R}^n$, with $u \cdot X$ denoting the scalar product of u and X.

Consider, now, an OU process

$$dY_t = a(b - Y_t)\,dt + \sigma\,dW_t, \tag{1.1}$$

with $a, \sigma > 0$ and $b \geq 0$, or a CIR process

$$dY_t = a(b - Y_t)\,dt + \sigma\sqrt{Y_t}\,dW_t \tag{1.2}$$

with, in addition, $2ab > \sigma^2$ and $Y_0 > 0$. In either case, take Y_0 fixed, for simplicity. Then, in the case of (1.1), Y_t has a Gaussian distribution for all $t > 0$ and a stationary Gaussian limit distribution as $t \to \infty$; in particular, Y_t has light tails for all t. In the case of (1.2), Y_t has a scaled noncentral chi-square distribution for all $t > 0$ and a stationary limit with a gamma distribution; thus, Y_t has an exponential tail for all t.

Our results extend this simple illustration to the full range of canonical affine models. We establish the existence of limiting stationary distributions, and we show that any linear combination of the state variables has either an exponential tail or a Gaussian distribution. The dynamics of a canonical affine model cannot produce heavy-tailed distributions, nor can they produce non-Gaussian light-tailed distributions; the same holds for any affine model obtained from a canonical model through an affine transformation. As a point of contrast, we note that GARCH models typically generate heavy-tailed marginal distributions, even when driven by light-tailed innovations; see Basrak, Davis, and Mikosch (2002).

The tail behavior of an affine process determines the maximal moments in an asset-price model constructed from the affine process. More explicitly, suppose the process

Y takes values in $\mathbb{R}^n$, and construct a price process $P_t = \exp(a_t + u_t \cdot Y_t)$, where a_t is a scalar function of time, and u_t is an $\mathbb{R}^n$-valued function of time. The points

$$\underline{\theta}_t = \inf\left\{\theta \in \mathbb{R} : \mathbb{E}[P_t^\theta] < \infty\right\} \quad \text{and} \quad \overline{\theta}_t = \sup\left\{\theta \in \mathbb{R} : \mathbb{E}[P_t^\theta] < \infty\right\}$$

coincide with the endpoints of the interval of convergence of the moment generating function of $u_t \cdot Y_t$. We use the structure of the transform of Y_t to characterize these points. It follows from our investigation that the interval $(\underline{\theta}_t, \overline{\theta}_t)$ shrinks (or, more precisely, does not expand) as t increases. Inverting the dependence on t leads to the smallest t at which $\mathbb{E}[P_t^\theta]$ becomes infinite, for fixed θ. This is the problem of finding the *moment explosion time* studied by Andersen and Piterbarg (2007) in the Heston model. Through results of Lee (2004), the extremal values $\underline{\theta}_t$, $\overline{\theta}_t$ determine the asymptotic slope of the implied volatility curve for options on P_t.

We derive our results through an analysis of the ODEs that arise in the transform analysis of affine models. We show that the moment generating function of $u \cdot Y_t$, $u \in \mathbb{R}^n$, is infinite at θ precisely if the solution to the ODE for Y explodes by time t from initial condition θu. It follows that Y_t has exponential tails if the solution remains finite on $[0, t]$ from all initial conditions in a neighborhood of the origin, and Y_t has light tails if this holds for all initial conditions in $\mathbb{R}^n$. The limiting behavior of the distribution of Y_t is determined by the behavior of the ODEs as $t \to \infty$. By characterizing the stability of the ODEs, we show that $\{Y_t, t \geq 0\}$ has a limiting distribution that does not depend on Y_0, and that this limiting distribution is, in fact, stationary for Y. The tails of this stationary distribution are determined by the stability region of the ODE for Y; properties of the stability region are themselves of some interest, as we illustrate through examples. Our final result shows that a linear combination of the components of Y_t is light-tailed only if it is Gaussian, and we characterize which linear combinations have this property through the model parameters defining Y.

The rest of this paper is organized as follows. Section 2 reviews the dynamics and parametric restrictions for canonical affine models and states our main results. Section 3 illustrates these results with examples. Sections 4–6 develop the analysis and proofs underlying our results. Section 4 includes relevant background on the theory of dynamical systems. We conclude in Section 7.

2. MAIN RESULTS

The canonical affine models introduced by Dai and Singleton (2000) follow equations of the form

$$dY_t = -A^\top(\Theta - Y_t)\,dt + \sqrt{diag(F_t)}\,dW_t, \tag{2.1}$$

evolving on $\mathbb{R}^n$ and driven by an n-dimensional standard Brownian motion W. Here, F_t is an affine function of Y_t, also taking values in $\mathbb{R}^n$, and $diag(F_t)$ denotes the $n \times n$ diagonal matrix whose diagonal entries are the components of F_t. The interpretation of the process Y depends on the application. For example, in some models, one defines a short rate process r_t by setting $r_t = u_0 + u_1 \cdot Y_t$, for some $u_0 \in \mathbb{R}$ and some $u_1 \in \mathbb{R}^n$; other models define an asset price process P_t by setting $\log(P_t) = a_t + b_t \cdot Y_t$, for some deterministic functions a and b.

The canonical specification of Dai and Singleton (2000) imposes additional restrictions on (2.1). To state these, we introduce some notational conventions to be used throughout the paper. For vectors or matrices a and b, we write $a \geq b$ if every entry of a is at least as large as the corresponding entry of b; we write $a > b$ if $a \geq b$ and $a \neq b$; and we write $a \gg b$ if every entry of a is strictly larger than the corresponding entry of b. We set $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x \geq 0\}$ and $\mathbb{R}^m_{++} = \{x \in \mathbb{R}^m : x \gg 0\}$, with the dimension of the zero vector determined by context. We write $|x|$ for the Euclidean norm of the vector x.

In the Dai–Singleton (2000) classification, the canonical model $\mathbb{A}_m(n)$ partitions the state vector Y as (Y^v, Y^d), with Y^v evolving on $\mathbb{R}^m_+$ and Y^d on $\mathbb{R}^{n-m}$, as a consequence of restrictions imposed on (2.1). The components of Y^v are called volatility factors, and the components of Y^d are called dependent factors. We use the superscripts v and d more generally to indicate partitions of vectors and matrices associated with the partioning of Y. Thus, we often write a vector $u \in \mathbb{R}^n$ as (u^v, u^d), with u^v having m components and u^d having $n-m$ components. The parameters of a canonical model $\mathbb{A}_m(n)$ are required to satisfy conditions (C1)–(C4), later. Dai and Singleton (2000) and Singleton (2006) explain the econometric identification issues that motivate these conditions.

(C1) The matrix A has the block form

$$A = \begin{pmatrix} A^v & A^c \\ 0 & A^d \end{pmatrix},$$

and it has real and strictly negative eigenvalues.

(C2) The off-diagonal entries of A^v are nonnegative.

(C3) The vector $\Theta = (\Theta^v, \Theta^d)$ has $\Theta^d = 0$, $\Theta^v \geq 0$, and $(-A^\top \Theta)^v \gg 0$.

(C4) The vector $F_t = (F_t^v, F_t^d)$ satisfies

$$F_t^v = Y_t^v, \quad F_t^d = e + (B^c)^\top Y_t^v,$$

where e is a vector of 1s and B^c is a matrix in $\mathbb{R}_+^{m\times(n-m)}$.

The eigenvalue condition in (C1) ensures mean reversion in Y. It implies (through, e.g., p. 62 of Horn and Johnson (1990)) that A^v and A^d also have strictly negative eigenvalues, in view of the block triangular form of A. Together, (C1) and (C2) imply that $-A^v$ is an M-matrix (as defined, e.g., in Berman and Plemmons (1994)). The vector Θ represents the long-run mean of Y. We could rewrite (2.1) in terms of

$$\Lambda = -A^\top \Theta. \tag{2.2}$$

Indeed, if we specify Λ^v rather than Θ, with $\Lambda^v \gg 0$, then the fact that $-A^v$ is an M-matrix guarantees (see p. 137 of Berman and Plemmons (1994): inverse-positivity of M-matrix) that we can find a $\Theta^v \geq 0$ for which $-A^{v\top}\Theta^v = \Lambda^v$; in fact, we can take $\Theta^v = -(A^{v\top})^{-1}\Lambda^v$. If we then set $\Theta^d = 0$ and $\Lambda^d = (-A^\top\Theta)^d = (A^{c\top})(A^{v\top})^{-1}\Lambda^v$, we complete the specification of Λ in a manner consistent with (C3) and (2.2). Thus, we can choose either Θ or Λ in specifying the model.

Condition (C4) requires that only the volatility factors Y^v appear inside the square root in (2.1), which is natural, given that the components of Y^d will be allowed to become negative. The form of F_t^v implies that the volatility factors are correlated only through the matrix A in the drift of Y. Cheridito, Filipović, and Kimmel (2006) show that the diffusion matrix of any affine diffusion on $\mathbb{R}^m_+ \times \mathbb{R}^{n-m}$ can be diagonalized through an affine transformation if $m \le 1$ or $m \ge n-1$ (in particular, if $n \le 3$); but they also provide examples for which no such transformation exists.

To illustrate this modeling framework, we formulate a stochastic volatility model in the class $\mathbb{A}_1(2)$—that is, a two-factor model with a single volatility factor. We write the state vector as $Y = (Y^v, Y^d)$, with dynamics

$$dY^v_t = \big(m_1 + pY^v_t\big)dt + \sqrt{Y^v_t}\, dW^1_t \tag{2.3}$$

$$dY^d_t = \big(m_2 + qY^v_t + rY^d_t\big)dt + \sqrt{1 + sY^v_t}\, dW^2_t, \tag{2.4}$$

for some constants $m_1,\ m_2,\ p,\ q,\ r$, and s. The restrictions of the general model $\mathbb{A}_m(n)$ require $m_1 > 0,\ p < 0,\ q \ge 0,\ r < 0,\ s \ge 0$, and $qm_1 = pm_2$. We can then construct an asset-price process P_t by setting

$$\log(P_t) = a_t + 2b_t Y^v_t + 2c_t Y^d_t, \tag{2.5}$$

for some deterministic functions $a_t,\ b_t$, and c_t. We will apply our general results to the moments of P_t in the next section and illustrate the qualitatively different behavior produced by different ranges of parameter values in the model.

The model (2.1) has associated with it a system of ODEs on $\mathbb{R}^n$ specified by

$$\begin{pmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{pmatrix} = \begin{pmatrix} A^v & A^c \\ 0 & A^d \end{pmatrix} \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} + \begin{pmatrix} I & B^c \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1^2(t) \\ \vdots \\ x_n^2(t) \end{pmatrix}. \tag{2.6}$$

We will write this system more compactly as

$$\dot{x} = f_o(x) = Ax + B\big(x_1^2, \dots, x_n^2\big), \quad x(0) = u, \tag{2.7}$$

with B the corresponding block matrix in (2.6), and the initial condition $u \in \mathbb{R}^n$ included here for future reference. We will see that, for any initial condition u, the system (2.7) admits a unique solution on a time interval $[0, t)$, for some $t > 0$. But, the solution may blow up in finite time and fail to exist beyond some finite time τ. We discuss this point in greater detail in Section 4.1.

The analysis in Duffie et al. (2000) leads to the representation

$$\mathbb{E}\left[\exp(2u \cdot Y_t)\right] = \exp\left(2\int_0^t \Lambda \cdot x(s)\, ds + 2\int_0^t |x^d(s)|^2\, ds + 2x(t) \cdot Y_0\right), \tag{2.8}$$

with x solving (2.7) and Λ as in (2.2), at least under some regularity conditions. Our first result asserts the validity of this formula (even in the infinite case) without further conditions and adds a stronger conclusion:

THEOREM 2.1. *The transform formula (2.8) holds in the sense that if either side is well-defined and finite, then the other is also finite and equality holds. Moreover, the right-hand side of (2.8) is well defined and finite if and only if the solution of (2.7) exists at time t. Consequently, for any $t \ge 0$, the right-hand side of (2.8) is finite for any vector u in a neighborhood of the origin.*

This result connects the stability of the ODE (2.7) with the tail behavior of Y_t:

COROLLARY 2.2. *Consider the system in (2.7) with initial condition* $x(0) = \theta u/2$, $\theta > 0$. *If the solution x exists at t, then*

$$\limsup_{y\to\infty} \frac{1}{y} \log \mathbb{P}(u \cdot Y_t > y) \leq -\theta.$$

If the solution explodes before t, then

$$\limsup_{y\to\infty} \frac{1}{y} \log \mathbb{P}(u \cdot Y_t > y) \geq -\theta.$$

For any $t \geq 0$, *the solution x exists at t for all sufficiently small* $|\theta| > 0$.

Corollary 2.2 describes the tail behavior of Y_t: the last statement of the corollary and the first limsup together imply that for any u and any $\epsilon > 0$, we have

$$\mathbb{P}(u \cdot Y_t > y) \leq e^{-(\theta-\epsilon)y},$$

for some $\theta > 0$ and all sufficiently large y. Thus, $u \cdot Y_t$ has an exponentially bounded right tail and, with an obvious modification to the argument, an exponentially bounded left tail as well.

A further consequence of Theorem 2.1 is a comparison of the tails of the volatility factors of two models. For processes Y^1 and Y^2 on $\mathbb{R}^m_+$, if $\mathbb{E}\exp(u \cdot Y^1_t) \geq \mathbb{E}\exp(u \cdot Y^2_t)$ for all $u \in \mathbb{R}^m_+$, then Y^1_t has heavier tails than Y^2_t. We give conditions for such a comparison for processes in $\mathbb{A}_m(m)$.

COROLLARY 2.3. *Let* Y^i *be a process in* $\mathbb{A}_m(m)$ *with parameters* A^i *and* Λ^i, $i = 1, 2$.

1. *Suppose* $A^1 = A^2$ *and* $Y^1_0 = Y^2_0$; *then* $\mathbb{E}\exp(2u \cdot Y^1_t) \geq \mathbb{E}\exp(2u \cdot Y^2_t)$ *for all* $u \in \mathbb{R}^m_+$ *and* $t \geq 0$ *if and only if* $\Lambda^1 \geq \Lambda^2$.
2. *Suppose* $\Lambda^1 = \Lambda^2$ *and* $Y^1_0 = Y^2_0 = Y_0$; *then* $\mathbb{E}\exp(2u \cdot Y^1_t) \geq \mathbb{E}\exp(2u \cdot Y^2_t)$ *for all* $(u, Y_0) \in \mathbb{R}^m_+ \times \mathbb{R}^m_+$ *and* $t \geq 0$ *if and only if* $A^1 \geq A^2$.

Our next result considers the limit as $t \to \infty$. Define the stability region S of the ODE (2.7) to be the set of initial conditions u for which the solution $x(t)$ exists for all $t \geq 0$ and $\lim_{t\to\infty} x(t) = 0$ if $x(0) = u$.

THEOREM 2.4. *The process Y has a unique stationary distribution, which is also the limiting distribution of* Y_t, *as* $t \to \infty$, *for any* Y_0. *Moreover, if* Y_∞ *has the stationary distribution of Y and we define*

$$\mathcal{S} = \{u \in \mathbb{R}^n : \mathbb{E}\exp(2u \cdot Y_\infty) < \infty\},$$

then $\mathcal{S}$ *coincides with S, the stability region of the system (2.7). This set contains a neighborhood of the origin.*

By arguing as in Corollary 2.2, we conclude that $u \cdot Y_\infty$ has exponentially bounded tails for all $u \in \mathbb{R}^n$. As a consequence of our analysis, we will identify the distribution of Y_∞ through its moment generating function.

Theorems 2.1 and 2.4 preclude the possibility of heavy tails for Y_t and Y_∞—any linear combination of the components of Y_t or Y_∞ has tails that are bounded by some exponential decay. We turn next to the possibility of light tails—tails that decay faster than any exponential. The Gaussian subfamily of canonical affine models (which corresponds to taking $m = 0$ and thus removing all volatility factors) demonstrates that such light-tailed models are indeed possible within the canonical affine framework. Our next result shows that the Gaussian case is the *only* light-tailed case among canonical models. More precisely, we show that if the moment generating function of $u \cdot Y_t$ is finite for all $\theta \in \mathbb{R}$, then the distribution of $u \cdot Y_t$ is Gaussian.

Before stating the theorem, we review some facts from linear algebra. By choosing an appropriate basis, we can transform A^d into a Jordan canonical form; in other words, there exists an invertible matrix P such that $P^{-1} A^d P = J$, and J is a block diagonal Jordan matrix. (The columns of P are eigenvectors or generalized eigenvectors of A^d.) Let $\lambda_1, \dots, \lambda_k$ denote the distinct eigenvalues of A^d, and let a_{λ_i} denote the algebraic multiplicity of λ_i, which is the multiplicity of $(x - \lambda_i)$ in the characteristic polynomial of A^d. The matrix J can then be chosen to have k diagonal blocks of the form $\lambda_i I_i + N_i$, $i = 1, \dots, k$, with I_i the identity matrix and N_i a nilpotent matrix, both of dimension $a_{\lambda_i} \times a_{\lambda_i}$. The entries of N_i immediately above its main diagonal take the values 0 or 1, and all other entries of N_i are equal to 0.

We introduce a special matrix W to state our last theorem. For this, we select the qth row of P if there exists some p with $B^c_{pq} \neq 0$, $q = 1, \dots, n - m$. Denoting the row vectors thus extracted from P by $w_1, \dots, w_l$, we define

$$W := \begin{pmatrix} w_1 \\ \vdots \\ w_l \\ \hline A^c P \end{pmatrix} = \left[\, W_1 \middle| \cdots \middle| W_k \,\right].$$

In the block decomposition on the right, W_1 consists of the first a_{λ_1} columns of W, W_2 consists of the next a_{λ_2} columns, and so on. Similarly, we define

$$\tilde{u} := P^{-1} u^d = \begin{pmatrix} \tilde{u}^1 \\ \vdots \\ \tilde{u}^k \end{pmatrix}, \quad \tilde{u}^i \in \mathbb{R}^{a_{\lambda_i}}.$$

THEOREM 2.5. *Assume that a Jordan canonical form J of A^d is given as aforementioned. Then for any given $t > 0$ and $u \in \mathbb{R}^n$, the following holds: $\mathbb{E}\exp(2\theta u \cdot Y_t) < \infty$ for all $\theta \in \mathbb{R}$ if and only if $u^v = 0$ and*

$$W_i N_i^l \tilde{u}^i = 0, \quad l = 0, \dots, a_{\lambda_i} - 1, \quad i = 1, \dots, k. \tag{2.9}$$

Moreover, $u \cdot Y_t$ has a Gaussian distribution if and only if these conditions hold.

Because the multiplicities of the roots of the characteristic polynomial of A^d are sensitive to the coefficients of the polynomial, small changes in the entries of A^d can make it diagonalizable. For diagonalizable A^d, (2.9) reduces to

$$W_i \tilde{u}^i = 0, \qquad i = 1, \dots, k. \tag{2.10}$$

Conditions (2.9) and (2.10) may seem surprisingly complicated, but we will illustrate their significance and application through examples in the next section. A more intuitive approach to checking whether a linear combination of factors has a Gaussian distribution would be to check if each of the factors is Gaussian; individual factors might then be checked recursively, as follows: no volatility factor is Gaussian, no dependent factor that has a volatility factor in its drift or diffusion coefficient is Gaussian, no dependent factor that has a non-Gaussian dependent factor in its drift is Gaussian, and so on. Our examples will show that this approach cannot cover all cases because of special cancellations that can occur; nevertheless, Theorem 2.5 does support sufficient conditions of this type, as we will show in the next corollary. These conditions become necessary when each eigenvalue of A^d has a geometric multiplicity of 1, a restriction that effectively rules out certain cancellations. The geometric multiplicity g_{λ_i} of an eigenvalue λ_i is the dimension of the eigenspace associated with λ_i.

We make precise the recursive procedure sketched above through a directed graph G on the coordinates of the dependent factors. Introduce an edge $(i, i+i)$ in G if $J_{i,i+1} = 1$. Call a node j of the graph *restricted* with respect to a matrix M if $M_{ij} \neq 0$ for some i. Extend this property to other nodes by saying that j is restricted if it is reachable from a restricted node through a directed path in G. For any matrix D, let $\mathbf{1}_D$ denote the matrix with $(\mathbf{1}_D)_{ij} = 1$ if $D_{ij} \neq 0$ and 0 otherwise.

COROLLARY 2.6. *A sufficient condition for (2.9) is that $\tilde{u}_j = 0$ for all j restricted with respect to $\mathbf{1}_{A^c P} + \mathbf{1}_{B^c}\mathbf{1}_P$. This condition becomes necessary if $g_{\lambda_i} = 1$ for all $i = 1, \ldots, k$.*

3. EXAMPLES AND APPLICATIONS

3.1. Stochastic Volatility: A Simple Case

To illustrate our results, we begin by considering the stochastic volatility model (2.3)–(2.5), based on the $\mathbb{A}_1(2)$ dynamics in (2.3)–(2.4). Through (2.8), moments of P_T are given by

$$\mathbb{E}\left[P_T^\theta\right] = \exp\left(a_T\theta + 2\int_0^T (m_1 x_1(t) + m_2 x_2(t))\,dt \right.$$
$$\left. + \ 2\int_0^T x_2(t)^2 dt + 2\left(x_1(T)Y_0^v + x_2(T)Y_0^d\right)\right), \tag{3.1}$$

where (x_1, x_2) solves the ODE

$$\dot{x}_1 = px_1 + qx_2 + x_1^2 + sx_2^2, \quad \dot{x}_2 = rx_2, \tag{3.2}$$

with initial condition $(x_1(0), x_2(0)) = (\theta b_T, \theta c_T)$.

We begin with the simple case $q = s = 0$, in which the ODE for x_1 reduces to a scalar quadratic differential equation. We digress briefly to record properties of this scalar system because it will be an important tool at several points in our analysis.

Consider, then, the scalar quadratic ODE $\dot{x} = \alpha x^2 + \beta x + \gamma$, with $\alpha > 0$. Let $D = \beta^2 - 4\alpha\gamma$, and denote by η_1 and η_2 the two solutions of $\alpha x^2 + \beta x + \gamma = 0$. The following properties of the solution x, which are easily derived from its closed form, are also used in Andersen and Piterbarg (2007). If $D > 0$ with $\eta_1 < \eta_2$, then

$$x(t) \to \eta_1 \text{ as } t \to \infty, \quad \text{if } x(0) < \eta_2;$$
$$x(t) \equiv \eta_1 \text{ or } \eta_2, \quad \text{if } x(0) = \eta_1 \text{ or } \eta_2, \text{ respectively;}$$
$$x(t) \to \infty \text{ as } t \to \tau, \quad \text{if } x(0) > \eta_2,$$

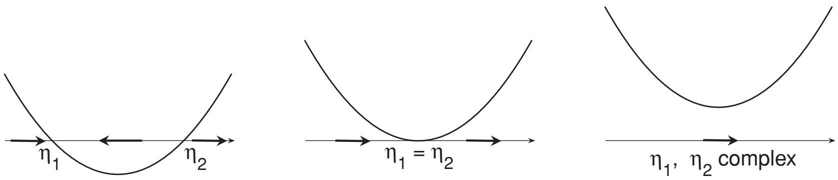

FIGURE 3.1. Qualitative behavior of $\dot{x} = \alpha x^2 + \beta x + \gamma$ with equilibria $\eta_1,\ \eta_2$

with

$$\tau = \frac{1}{\alpha(\eta_2 - \eta_1)} \log \frac{x(0) - \eta_1}{x(0) - \eta_2}. \tag{3.3}$$

If $D = 0$, then

$$\begin{aligned} &x(t) \to -\frac{\beta}{2\alpha} \text{ as } t \to \infty, \quad \text{if } x(0) < -\beta/2\alpha; \\ &x(t) \equiv -\frac{\beta}{2\alpha}, \quad \text{if } x(0) = -\beta/2\alpha; \\ &x(t) \to \infty \text{ as } t \to \tau, \quad \text{if } x(0) > -\beta/2\alpha, \end{aligned}$$

with

$$\tau = \frac{1}{x(0) - \beta/2\alpha}.$$

If $D < 0$, then

$$x(t) \to \infty \text{ as } t \to \tau = \frac{1}{\sqrt{-D}}\left(\pi - 2\tan^{-1}\frac{2\alpha x(0) + \beta}{\sqrt{-D}}\right).$$

These cases are illustrated in Figure 3.1. Consider, in particular, the first case, $D > 0$. The two roots are equilibrium points—points at which $\dot{x} = 0$. The root η_1 is a stable equilibrium for the ODE; $x(t)$ moves toward η_1 from any initial condition less than η_1 or between the two roots, so the stability region for the system is

$$S = \{x : x < \eta_2\}.$$

In contrast, η_2 is an unstable equilibrium, and x blows up in finite time τ if $x(0) > \eta_2$. The set S_T consists of all initial conditions from which x continues to exist throughout $[0, T)$. From the expression for the explosion time τ in (3.3), we find that

$$S_T = \left\{x : x \le \left(\eta_2 e^{\alpha T(\eta_2 - \eta_1)} - \eta_1\right) / \left(e^{\alpha T(\eta_2 - \eta_1)} - 1\right)\right\}.$$

We can now apply this to (3.1). In the case $q = s = 0$, the solution x_1 in (3.2) becomes infinite at $\tau = (\log(\theta b_T + p) - \log(\theta b_T))/p$, if $\theta b_T > -p$; otherwise, $x_1(t)$ is finite for all t and converges exponentially to zero. In other words, if $\theta b_T < -p/(1 - e^{pT})$, then the

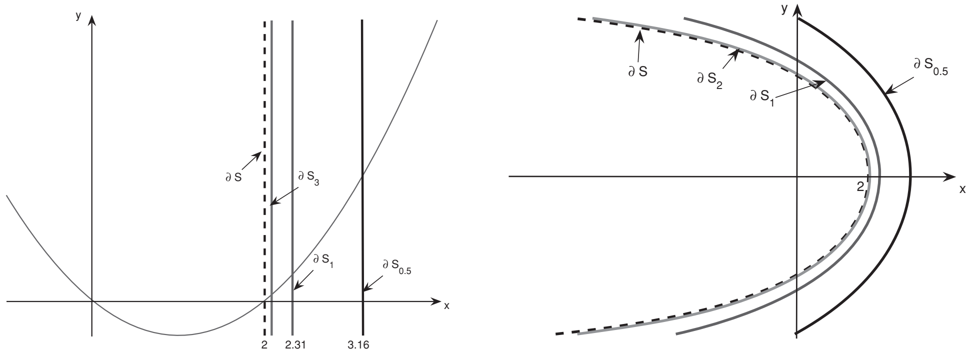

FIGURE 3.2. Boundaries of S and S_T for $\mathbb{A}_1(2)$ models. The left panel has parameters $p = -2,\ q = s = 0$; the right panel has $p = r = -2,\ q = 0,\ s = 1$.

right-hand side of (3.1) is finite; the second coordinate x_2 is always finite and integrable. We therefore conclude that

$$\sup\{\theta : \mathbb{E}\big[P_T^\theta\big] < \infty\} = \begin{cases} \dfrac{-p}{b_T(1-e^{pT})}, & \text{if } b_T > 0; \\ \infty, & \text{if } b_T \le 0; \end{cases}$$

$$\inf\{\theta : \mathbb{E}\big[P_T^\theta\big] < \infty\} = \begin{cases} -\infty, & \text{if } b_T \ge 0; \\ \dfrac{-p}{b_T(1-e^{pT})}, & \text{if } b_T < 0. \end{cases}$$

We can illustrate these properties through the following sets:

$$\begin{aligned} \mathcal{S} &= \big\{(x, y) : \lim_{t\to\infty} \mathbb{E}\exp\big(2xY_t^v + 2yY_t^d\big) < \infty\big\} \\ \mathcal{S}_T &= \big\{(x, y) : \mathbb{E}\exp\big(2xY_t^v + 2yY_t^d\big) < \infty,\ \forall t \in [0, T)\big\}. \end{aligned}$$

Theorems 2.1, 2.4 imply that these sets coincide, respectively, with the set S of initial conditions for which the solution to (3.2) exists for all time and converges to zero, and the set S_T for which the solution exists throughout $[0, T)$. Rewriting S and S_T above in terms of p and T, we get

$$\begin{aligned} S &= (-\infty, -p) \times \mathbb{R} \\ S_T &= (-\infty, -p/(1-e^{pT})] \times \mathbb{R}. \end{aligned}$$

If $(\theta b_T, \theta c_T) \in S_T^o$ (the interior of S_T), then (3.1) is finite; if $(\theta b_T, \theta c_T) \in S$, then (3.1) is finite for all T. The left panel of Figure 3.2 illustrates the boundaries of these sets. The parabola shows the values of $\dot{x}_1 = px_1 + x_1^2$ in (3.2) as a function of x_1. The larger of the two solutions to the equation $\dot{x}_1 = 0$ determines the upper limit of the stability region for x_1 (as in Figure 3.1), so ∂S passes through this point. As T decreases, ∂S_T shifts to the left.

We can also see from the figure that $(\theta b_T, \theta c_T)$ lies outside S_T for some (and then all) sufficiently large $\theta > 0$ or $\theta < 0$, unless (b_T, c_T) lies on the vertical axis. Thus, P_T^θ has infinite expectation for some θ unless $b_T = 0$. When $b_T = 0$, $\log(P_T) = a_T + 2c_T Y_T^d$ has a Gaussian distribution, and thus does indeed have finite moments of all orders. This is a simple graphical description of the conditions in Theorem 2.5 for this example.

3.2. Stochastic Volatility: Further Cases

We continue to work with the basic model (2.3)–(2.5), but now take $s > 0,\ q = 0$, and $p = r < 0$. In this case, the function $\xi(t) := e^{-pt}x_1(t)/\sqrt{sx_2(0)^2}$ solves $\dot{\xi}/(\xi^2+1) = \sqrt{sx_2(0)^2}e^{pt}$. Then, we have

$$\tan^{-1}(\xi(t)) - \tan^{-1}(\xi(0)) = \sqrt{sx_2^2(0)}(e^{pt}-1)/p.$$

Therefore,

$$x_1(t) = \sqrt{sx_2^2(0)}e^{pt}\tan\left(\sqrt{sx_2^2(0)}(e^{pt}-1)/p + \tan^{-1}\left(x_1(0)/\sqrt{sx_2^2(0)}\right)\right),$$
$$x_2(t) = x_2(0)e^{pt}.$$

Then,

$$S = \{(x, y) : x < \sqrt{sy^2}\tan(\pi/2 + \sqrt{sy^2}/p)\}$$

and

$$S_T = \{(x, y) : x \leq \sqrt{sy^2}\tan(\pi/2 + \sqrt{sy^2}(1-e^{pT})/p)\}.$$

These sets are illustrated in the right panel of Figure 3.2. For any nonzero point (b_T, c_T), the line defined by the points $(\theta b_T, \theta c_T)$ as θ ranges over $\mathbb{R}$ crosses the boundary of S_T twice, once with θ positive and once with θ negative. If (b_T, c_T) is in the interior of S_T, then these values of θ are the extremal moments $\overline{\theta}_T$ and $\underline{\theta}_T$ as a consequence of Theorem 2.1. In particular, $\mathbb{E}[P_T^\theta]$ becomes infinite for all sufficiently large positive or negative θ. The log price $\log(P_T)$ is never Gaussian.

We next consider the effect of varying $r < 0$, which is the coefficient on Y_t^d in the expression for dY_t^d in (2.4), while fixing $s > 0,\ q = 0$, and $p < 0$. We can represent $x_1(t)$ in terms of function $\psi(l)$ by setting

$$\psi'\left(-\frac{\sqrt{k}e^{rt}}{r}\right) = \frac{1}{\sqrt{k}e^{rt}}\left(x_1(t) + \frac{p}{2}\right)\psi\left(-\frac{\sqrt{k}e^{rt}}{r}\right), \tag{3.4}$$

with $k = sx_2(0)^2$. The function $\psi(l)$ solves a second-order ODE,

$$l^2\psi''(l) + l\psi'(l) + \left(l^2 - \left(\frac{p}{2r}\right)^2\right)\psi(l) = 0. \tag{3.5}$$

It follows that $\psi(l)$ is a linear combination of Bessel functions of the first and second kinds, respectively; see, for example, p. 748 of Polyanin and Zaitsev (2003) for properties of the solution. Because any multiple of $\psi(l)$ satisfies (3.4), we can set $\psi(l)$ as the solution to (3.5) for $l \in (0, -\sqrt{k}/r]$ with $\psi(-\sqrt{k}/r) = \sqrt{k}$, which then satisfies $\psi'(-\sqrt{k}/r) = x_1(0) + p/2$. Because $S = \{x(0) : \lim_{t\to\infty} x(t) = 0\}$, from (3.4) we get

$$S = \left\{x(0) : \lim_{l\downarrow 0}\frac{l\psi'(l)}{\psi(l)} = -\frac{p}{2r}\right\}.$$

A similar analysis can be carried out for $s = 0,\ q > 0$, and $p < 0$ case. Figure 3 shows the boundary of S for different values of r. The left panel has $q = 0$ and $s = 1$; the right panel has $q = 1$ and $s = 0$. In both cases, the stability region becomes smaller as r approaches zero, indicating that $Y_\infty = (Y_\infty^v, Y_\infty^d)$ has heavier (though still exponentially

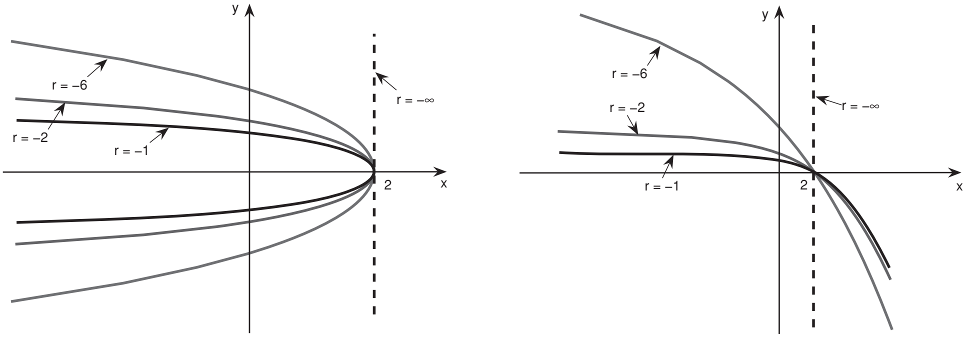

FIGURE 3.3. Stability boundaries for $\mathbb{A}_1(2)$ models. The left panel has parameters $p = -2,\ q = 0,\ s = 1$; the right panel has $p = -2,\ q = 1,\ s = 0$.

bounded) tails at smaller values of $|r|$. This is to be expected from the role of r in the dynamics (2.3)–(2.4) of the model.

The two panels of Figure 3.3 show an interesting contrast. In the right panel, we see that a line of the form $\{\theta u : \theta \in \mathbb{R}\}$, $u \in S \cap \mathbb{R}^2_{++}$, crosses the boundary of S just once, at some $\theta > 0$; in the left panel, such a line would cross the boundary of S at both a positive and negative value of θ, as noted in our discussion of Figure 3.2. This reflects an interesting distinction between two ways the volatility factor Y^v can influence the dependent factor Y^d. When Y^v appears in the diffusion coefficient of Y^d (the left panel, with $q = 0,\ s \neq 0$), it makes both the right and left tails of $u \cdot Y_\infty$ exponential, $u \in \mathbb{R}^2_{++}$; when Y^v appears only in the drift of Y^d (the right panel, with $q \neq 0,\ s = 0$), one tail of $u \cdot Y_\infty$ is exponential, but the other is light. The figure has $q > 0$, so the right tail is the exponential one; taking $q < 0$ would reflect the figure about the horizontal axis, corresponding to an exponential left tail.

3.3. Two Volatility Factors

Our next example is a model in $\mathbb{A}_2(2)$:

$$\begin{aligned} dY_t^1 &= \left(m_1 + pY_t^1 + rY_t^2\right)dt + \sqrt{Y_t^1}\,dW_t^1 \\ dY_t^2 &= \left(m_2 + qY_t^1 + sY_t^2\right)dt + \sqrt{Y_t^2}\,dW_t^2. \end{aligned}$$

This can be viewed as a two-factor CIR model; it also belongs to the family of continuous-state branching processes, as explained in Duffie et al. (2003). The associated system of ODEs is

$$\dot{x}_1 = px_1 + qx_2 + x_1^2 \tag{3.6}$$

$$\dot{x}_2 = rx_1 + sx_2 + x_2^2. \tag{3.7}$$

To satisfy the restrictions on the A matrix in (2.1), we require $p, s < 0,\ q, r \geq 0$, and $ps - qr > 0$.

The ODEs (3.6)–(3.7) do not admit a closed-form solution, but we can investigate the qualitative behavior of the system and illustrate this behavior graphically. (We review relevant background on dynamical systems in Section 4.1.) Figure 3.4 shows the vector

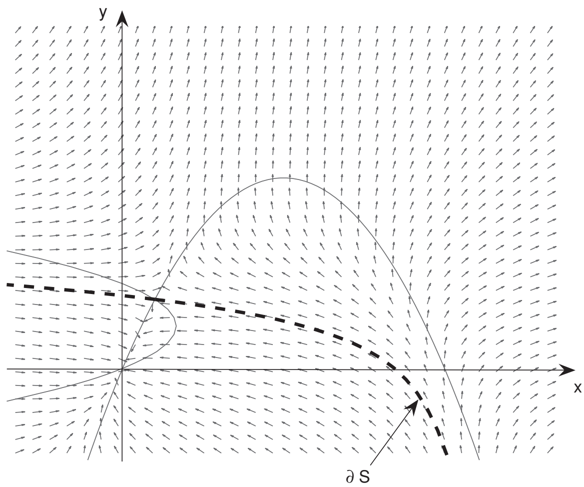

FIGURE 3.4. Vector field of an $\mathbb{A}_2(2)$ model and ∂S with, $p = -3,\ q = 1,\ r = 0.5$ and $s = -1$.

field defined by (3.6)–(3.7) with $p = -3,\ q = 1,\ r = 1/2$, and $s = -1$. The two parabolic curves are the points in the plane satisfying $\dot{x}_1 = 0$ in (3.6) and $\dot{x}_2 = 0$ in (3.7). At the intersections of the two parabolic curves we have $(\dot{x}_1, \dot{x}_2) = 0$, making these equilibrium points; there are two equilibrium points in the example of Figure 4, one of which is the origin. The origin is a stable equilibrium: the system approaches the origin from all initial conditions in a neighborhood of the origin. Indeed, the system approaches the origin from all initial conditions in the stability region S, whose boundary ∂S is indicated by a dashed line in the figure. If $x(0)$ lies outside of S, the system explodes, in the sense that $|x(t)| \to \infty$.

The other point of intersection of the two parabolas is an unstable equilibrium: there are initial conditions arbitrarily close to this point from which the system will approach either the origin or infinity. (In the language of dynamical systems, this is a hyperbolic equilibrium of type 1, and therefore unstable; see Section 4.1 and, e.g., Chiang, Hirsch, and Wu (1988) for background.) Associated with the unstable equilibrium is a stable manifold—a curve in the plane of initial conditions from which the system moves toward the unstable equilibrium. This curve is contained within ∂S.

From Theorem 2.4, we know that the points u in S are precisely the points for which $\mathbb{E}[\exp(2u \cdot Y_\infty)]$ is finite. Because S contains a neighborhood of the origin, any linear combination of the components of Y_∞ has exponentially bounded tails. For $u \in S \cap \mathbb{R}^2_{++}$, the line $\{\theta u : \theta \in \mathbb{R}\}$ crosses ∂S just once, at some $\theta > 0$, so $\mathbb{E}[\exp(\theta u \cdot Y_\infty)]$ becomes infinite at for all sufficiently large $\theta > 0$ but remains finite for all $\theta < 0$. In other words, $u \cdot Y_\infty$ has an exponential right tail and a light left tail (in fact, $u \cdot Y_\infty$ is nonnegative).

Figure 3.5 illustrates the behavior of this system for other parameter values. The left panel of the figure shows an example with three equilibrium points, and the right panel shows one with four equilibrium points. In both cases, the origin is the only stable equilibrium. Figure 3.6 shows a degenerate case with $q = r = 0$. Here, equations (3.6) and (3.7) decouple, and the stability of each reduces to the analysis of the scalar quadratic differential equation in Section 3.1.

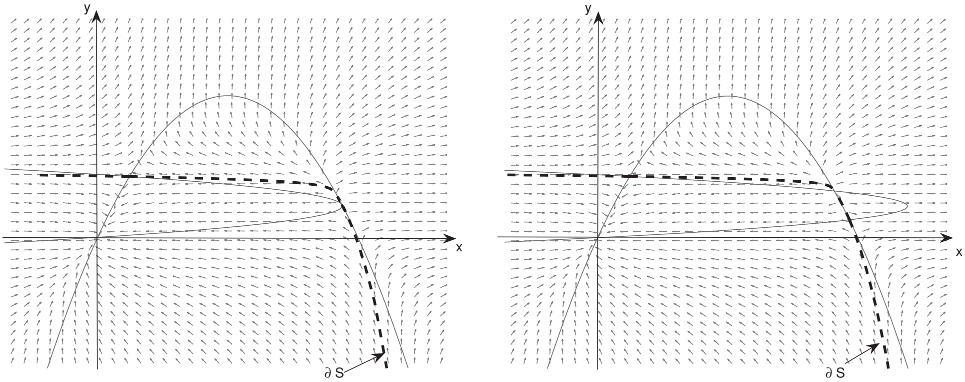

FIGURE 3.5. The stability boundary for $\mathbb{A}_2(2)$ models. The left panel has parameters $p = -3,\ q = 1,\ r = 0.089,\ s = -1$; the right panel has the same parameters, except with $r = 0.07$.

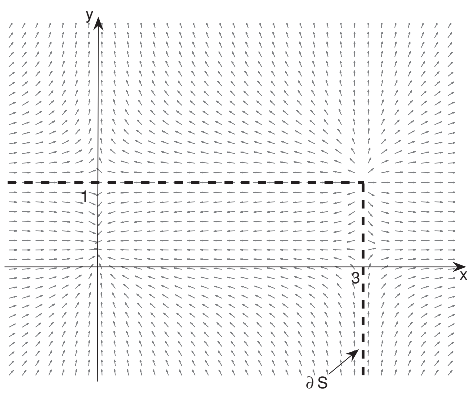

FIGURE 3.6. The stability boundary for $\mathbb{A}_2(2)$ with $p = -3,\ q = r = 0,\ s = -1$.

3.4. Gaussian Conditions

In Theorem 2.5, we gave conditions under which $u \cdot Y_t$ and $u \cdot Y_\infty$ have finite moments of all orders, and we noted that these conditions also determine when Y_t and Y_∞ are Gaussian. From the perspective of the associated ODEs, $u \cdot Y_\infty$ has finite moments of all orders precisely if the ODE solution exists for all $t \geq 0$, from all initial conditions $\theta u,\ \theta \in \mathbb{R}$; in other words, the stability region S includes all multiples u. We now illustrate these properties with examples.

Consider the following family of models in $\mathbb{A}_1(3)$:

$$dY_t^1 = \left(\Lambda_1 - Y_t^1\right) dt + \sqrt{Y_t^1}\, dW_t^1 \tag{3.8}$$

$$dY_t^2 = \left(\Lambda_2 + a\, Y_t^1 - Y_t^2\right) dt + dW_t^2 \tag{3.9}$$

$$dY_t^3 = \left(\Lambda_3 + b\, Y_t^1 + c\, Y_t^2 - Y_t^3\right) dt + dW_t^3 \tag{3.10}$$

The model has Y^1 as volatility factor and Y^2 and Y^3 as dependent factors. The matrix A has the form

$$A = \left(\begin{array}{c|c} A^v & A^c \\ \hline & A^d \end{array}\right) = \left(\begin{array}{c|cc} -1 & a & b \\ \hline 0 & -1 & c \\ 0 & 0 & -1 \end{array}\right),$$

and $B^c = 0$ because the volatility factor Y^1 does not appear in the diffusion coefficient of either Y^2 or Y^3.

Because A^d is already block diagonal, it is easy to check that

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1/c \end{pmatrix}$$

if $c \neq 0$, and $P = I_2$ if $c = 0$. Condition (2.9) becomes

$$a\,u_2 + b\,u_3 = 0, \quad a\,c\,u_3 = 0. \tag{3.11}$$

The case $c = 0$ reduces to (2.10). Theorem 2.5 requires $u^v = 0$, so we must have $u_1 = 0$.

We consider several cases for the parameters $a,\ b$, and c.

$a = 0$: We can satisfy (3.11) with any u that is a multiple of $(0, 1, 0)$; i.e., with $u \cdot Y_t = u_2 Y_t^2$. This is also evident from the fact that Y^2 is an OU process when $a = 0$. If we also have $b = 0$, then u_2 and u_3 are both free in (3.11) and, indeed, (Y^2, Y^3) is a Gaussian process.

$c = 0,\ a \neq 0,\ b \neq 0$: Condition (2.10) is satisfied by taking $u = (0, 1/a, -1/b)$, or any multiple thereof. From (3.8)–(3.10), we see that neither Y^2 nor Y^3 is Gaussian—each has the volatility factor Y^1 in its drift. Nevertheless, the linear combination $u^d \cdot Y^d$ is Gaussian. We can also see this by noting that

$$\begin{aligned} d\big(u^d \cdot Y_t^d\big) &= \left(m - \frac{1}{a} Y_t^2 + \frac{1}{b} Y_t^3\right) dt + \frac{1}{a} d W_t^2 - \frac{1}{b} d W_t^3 \\ &= -u^d \cdot Y_t^d\, dt + \frac{1}{a} d W_t^2 - \frac{1}{b} d W_t^3, \end{aligned}$$

with $m = (\Lambda_2/a) - (\Lambda_3/b) = 0$, in light of (2.2); thus $u^d \cdot Y^d$ is an OU process constructed from non-Gaussian processes. This example illustrates why Corollary 2.6 cannot cover all cases.

$c \neq 0,\ a \neq 0$: (3.11) requires $u_2 = u_3 = 0$; thus, no $u \cdot Y$ is Gaussian, except the degenerate case $u \equiv 0$. If $b = 0$, then the equation for Y^3 in (3.10) has no direct dependence on a volatility factor, but it fails to be Gaussian because it depends on Y^2 which depends on Y^1. This is also a consequence of Corollary 2.6; the first coordinate of $\tilde{u} = P^{-1} u^d = (u_2 \;\; c\,u_3)$ is restricted with respect to $\mathbf{1}_{A^c P}$ and the second coordinate has a directed path from the first coordinate.

In this example, the conclusion of the first case ($a = 0$) and that of the third case ($c \neq 0,\ a \neq 0$) coincide with what one would expect based on the intuitive approach to checking for Gaussian distributions outlined after (2.10) and formalized in Corollary 2.6. However, the second case ($c = 0,\ a \neq 0,\ b \neq 0$) shows that the intuitive approach cannot cover all cases. The necessary and sufficient conditions in Theorem 2.5 capture the possibility of a Gaussian distribution resulting from a cancellation of factors, as in this example.

4. ANALYSIS OF QUADRATIC DYNAMICAL SYSTEMS

4.1. Definitions and Terminology

In this section, we establish some properties of the ODE system (2.7), in particular viewing it as defining a mapping from the initial condition u to the solution $x(t)$ at time t. We begin by reviewing some definitions and basic properties from the theory of dynamical systems; additional background can be found in Hirsch and Smale (1974) and Chiang, Hirsch, and Wu (1988).

Consider, then, an equation

$$\dot{x} = f(x) \tag{4.1}$$

defined by a C^r function $f : W \to E$, with $W \subset E$ open and E a normed vector space. For each $u \in W$, there is a unique solution to (4.1), with $x(0) = u$, defined on a maximal open time interval $I(u) \subset \mathbb{R}$. For $t \in I(u)$, we denote this solution either by $x(t)$ or $\Phi_t(u)$; the notation $\Phi_t(u)$ makes explicit the dependence on the initial condition u. Also, the uniqueness of the solution allows us to write, for example,

$$\Phi_{s+t}(u) = \Phi_s(\Phi_t(u)),$$

for t and $s + t$ in $I(u)$. In particular, Φ_{-t} is the inverse of Φ_t.

Define

$$\Omega = \{(t, u) \in \mathbb{R} \times W : t \in I(u)\};$$

then Φ is a mapping from Ω to W. Standard properties of dynamical systems imply that Ω is open in $\mathbb{R} \times W$ and Φ is C^r if f is C^r, for $0 \leq r \leq \infty$. In fact, Φ is analytic in t and u as long as $\Phi_t(u)$ stays in the domain of analyticity of f.

Let τ denote the (possibly infinite) right endpoint of the interval $I(u)$. If $\tau < \infty$, then for any compact set $K \subset W$, there is a $t \in I(u)$ with $\Phi_t(u) \neq K$; in other words, the solution escapes the domain of definition in finite time, and τ is the "blow-up time" from u.

An *equilibrium point* of (4.1) is a point $\eta \in W$ at which $f(\eta) = 0$. An equilibrium point η is called *hyperbolic* if every eigenvalue of the Jacobian of f at η has a nonzero real part. The *type* of an equilibrium point is the number of eigenvalues (counted according to their multiplicity) with positive real parts. The stable manifold of a hyperbolic equilibrium is the set of points $u \in W$ for which $\Phi_t(u) \to \eta$ as $t \to \infty$; the unstable manifold is the set of $u \in W$ for which $\Phi_{-t}(u) \to \eta$ as $t \to \infty$. A hyperbolic equilibrium η_0 of type zero is a stable equilibrium; this means that its stable manifold contains a neighborhood of η_0 or, equivalently, that its unstable manifold consists solely of η_0. It is also a standard fact that this stable manifold of η_0 is an open set.

For the system (2.7) associated with a canonical affine model, the origin is a hyperbolic equilibrium of type zero and thus a stable equilibrium. The origin is, in fact, the unique stable equilibrium (see Kim (2008)). We denote its stable manifold by S and call this the *stability region* of the dynamical system. Part of the content of Theorem 2.4 is that the stable manifold of the origin determines the range of finite moments of the limiting stationary distribution of the model.

As an aside, we note that the unstable equilibrium in Figure 3.4 is of type 1; the equilibrium at the point of tangency of the two parabolic curves in the left panel of Figure 3.5 fails to be hyperbolic; and, in the right panel of Figure 3.5, the four equilibrium points defined by the four points of intersection of the two curves have types 0, 1, 2, and 1 when taken in clockwise order, starting from the origin. The type-2 equilibrium is a source: its stable manifold consists solely of the point itself.

4.2. Solution Properties

Our analysis of the dynamical system (2.7) makes extensive use of comparison theorems, and these in turn prove to be very useful in establishing some distributional properties of Y. The comparison results rely on a concept of quasi-monotonicity. Under the componentwise ordering of vectors introduced in Section 2, we call a function $f : \mathbb{R}^n \to \mathbb{R}^n$ *quasi-monotone increasing* if, whenever $x \leq y$ and $x_k = y_k$ for some k, then $f_k(x) \leq f_k(y)$. A mapping $x \mapsto Ax$ defined by a matrix A is thus quasi-monotone increasing if and only if $A_{ij} \geq 0$ whenever $i \neq j$. Suppose that f defined on $\mathbb{R}^n$ is quasi-monotone increasing and locally Lipschitz continuous. Let $x(t),\ y(t) : [a, b] \to \mathbb{R}^n$ be differentiable functions such that

$$\dot{x}(t) - f(x(t)) \leq \dot{y}(t) - f(y(t)), \quad \forall\, t \in [a, b];$$

then it follows from Volkmann (1972) that

$$x(a) \leq y(a) \Rightarrow x(t) \leq y(t) \quad \forall t \in [a, b]. \tag{4.2}$$

When $n = 1$, this reduces to a standard comparison result for scalar differential equations.

The relevance of this result to our setting comes from property (C2), which makes A^y quasi-monotone, and the fact that the mapping $(x_1, \ldots, x_n) \mapsto (x_1^2, \ldots, x_n^2)$ is also quasi-monotone. Through (4.2), we arrive at the following comparison property for the solution Φ to (2.7):

LEMMA 4.1. *For any $u \in \mathbb{R}^n$ and $\theta > 1$, we have*

$$\theta \Phi_t(u) \leq \Phi_t(\theta u),$$

for all $t \geq 0$ at which both sides are well defined.

The proofs of this result and the next two lemmas are deferred to the Appendix.

For later reference, we also record the following results on the decay of solutions. See, for example, Chapter 7 of Verhulst (1996). For the system (2.7), there exist positive constants $C,\ \delta$, and μ such that

$$|\Phi_t(u)| \leq C|u|e^{-\mu t} \tag{4.3}$$

for all $|u| \leq \delta$, and

$$\left|\Phi_t^d(u)\right| \leq C|u^d|e^{-\mu t}, \tag{4.4}$$

for all $u \in \mathbb{R}^n$. The constant $-\mu$ can be chosen to be the eigenvalue of A of smallest magnitude.

LEMMA 4.2. *For each $u \in \mathbb{R}^n$, the trajectory $\{\Phi_t(u) : t \in [0, \tau)\}$ of (2.7) is bounded later.*

LEMMA 4.3. *Suppose $|\Phi_t(u)| \to \infty$ as $t \to \tau$, for some $\tau \le \infty$. Then $\int_0^t \Lambda \cdot \Phi_s(u)\, ds \to \infty$ as $t \to \tau$.*

4.3. Proof of Theorem 2.1 and its Corollaries

In light of the expression that appears in the exponent of (2.8), it is natural to introduce the notation

$$\Psi_t(u) = \int_0^t \Lambda \cdot \Phi_s(u)\, ds + \int_0^t \left|\Phi_s^d(u)\right|^2 ds + \Phi_t(u) \cdot Y_0.$$

For $(t, u) \in \Omega$, $\Phi_s(u)$ is bounded for $s \in [0, t]$, so $\Psi_t(u)$ is well defined and finite. As part of the proof of Theorem 2.1, we will show that $\Psi_t(u)$ blows up at τ precisely if $\Phi_t(u)$ does.

Proof of Theorem 2.1. We first show that the finiteness of $\Phi_t(u)$ is equivalent to that of $\Psi_t(u)$. One direction is trivial: if $(t, u) \in \Omega$, then $\Phi_s(u)$ is bounded for $s \in [0, t]$ and thus $\Psi_t(u)$ is finite. To show the converse, observe that $|\Phi_t^d(u)|$ is bounded on $t \in \mathbb{R}_+$ (by (4.4)) and $\Phi_t(u)$ is bounded below for its entire life span $t \in [0, \tau)$ (by Lemma 4.2). It follows that $\Phi_t(u) \cdot Y_0 = \Phi_t^v(u) \cdot Y_0^v + \Phi_t^d(u) \cdot Y_0^d$ is also bounded below because $Y_0^v \ge 0$. It thus follows from Lemma 4.3 and the continuity of $\Phi_t(u)$ (as a function of t) that

$$\Phi_t(u) \text{ blows up at time } \tau \Leftrightarrow \Psi_t(u) \text{ blows up at } \tau. \tag{4.5}$$

Next, we show that if $\Psi_t(u)$ is finite, then $\mathbb{E}\exp(2u \cdot Y_t)$ is also finite and equality holds in (2.8). Duffie, Filipović, and Schachermayer (2003) define regular affine Markov processes and show that there are necessary and sufficient conditions for parameters of an affine model to ensure regularity, namely, admissibility. They also show that the transform formula holds true for all $(t, u) \in \mathbb{R}_+ \times \mathbb{C}_-^m \times i\mathbb{R}^{n-m}$ for affine models with admissible parameters. It is not hard to check that canonical affine models satisfy the admissibility condition. And the processes generated by them are conservative, as defined in Duffie et al. (2003). This follows easily from Proposition 9.1 in (2003); we note that the generalized Riccati equation (2.14) with (2.15) in Duffie et al. (2003) is (2.7) in the canonical case.

Now suppose $\Psi_t(u)$ is finite. Because the process Y is conservative regular affine, by Lemma A.2 we can invoke Theorem 2.16 in (2003) and conclude that $\mathbb{E}\exp(2u \cdot Y_t)$ is finite and the transform formula holds.

We now prove the converse of the main statement of the theorem. Suppose, then, that $\mathbb{E}\exp(2u \cdot Y_t) < \infty$ for some $t > 0$ and $u \in \mathbb{R}^n$. Because the origin is a stable equilibrium and its stability region S is open (see Section 4.1), there is a $\theta_0 \in (0, 1)$ such that $\theta_0 u \in S$. But if $\theta_0 u \in S$, then $\lim_{s\to\infty} \Phi_s(\theta_0 u) = 0$, and it follows that $\sup_s |\Phi_s(\theta_0 u)| < \infty$. We may then define a positive θ^* by setting

$$\theta^* = \sup\left\{\theta > 0 : \frac{1}{\theta}\int_0^t \Lambda \cdot \Phi_s(\theta u)\, ds < \infty\right\}, \tag{4.6}$$

the supremum taken over those $\theta > 0$ for which $\Phi_t(\theta u)$ is well defined—i.e., those for which $t \in I(\theta u)$. (If $\Phi_s(\theta u)$ blows up before t, then the integral in (4.6) is infinite.)

If $\theta^* > 1$, then $\Phi_t(u)$ is finite, and we have already shown that this implies that $\Psi_t(u)$ is finite, and we have also shown that (2.8) holds in this case. To complete the proof, we will show that $\theta^* \leq 1$ leads to a contradiction.

Suppose, then, that $\theta^* \leq 1$. Because $\Lambda^v \gg 0$ and $\Phi_s^d(u)$ is linear in the initial condition u, Lemma 4.1 implies that the function

$$\theta \mapsto \frac{1}{\theta}\int_0^t \Lambda \cdot \Phi_s(\theta u)\, ds, \quad \theta \in [\theta_0, \theta^*)$$

is increasing. This implies that

$$\lim_{\theta \uparrow \theta^*} \int_0^t \Lambda \cdot \Phi_s(\theta u)\, ds = \infty.$$

Also by Lemma 4.1, we have

$$\frac{1}{\theta_0}\Phi_s(\theta_0 u) \leq \frac{1}{\theta}\Phi_s(\theta u)$$

for all $(\theta, s) \in R \equiv [\theta_0, \theta^*) \times [0, t]$. Because $\Phi_s(\theta_0 u)$ is bounded below (by Lemma 4.2), $\Phi_s(\theta u)$ is bounded below uniformly on R. Moreover, the solution $\Phi_s^d(\theta u)$ to the linear part of (2.7) is uniformly bounded above as well on R, as is easily deduced from (4.4). Thus,

$$\Psi_t(\theta u) \geq \int_0^t \Lambda \cdot \Phi_s(\theta u)\, ds + K,$$

for some constant K and all $\theta \in [\theta_0, \theta^*)$. It follows that $\lim_{\theta \uparrow \theta^*} \Psi_t(\theta u) = \infty$.

However, for any $\theta \in (0, \theta^*)$, we have $\Psi_t(\theta u) < \infty$, which we already know implies that (2.8) holds at θu, so

$$\exp(2\Psi_t(\theta u)) = \mathbb{E}\exp(2\theta u \cdot Y_t) \leq \left(\mathbb{E}\exp(2u \cdot Y_t)\right)^\theta < \infty,$$

by Jensen's inequality. This implies that $\limsup_{\theta \uparrow \theta^*} \Psi_t(\theta u) < \infty$. But this is a contradiction, so we must in fact have $\theta^* > 1$.

The last assertion of the theorem now follows directly from the fact that the stability region S of the origin is open. □

Proof of Corollary 2.2. The indicated tail properties are standard consequences of finite moment generating functions, but we include a brief proof for completeness. From the inequality $1\{z > y\} \leq \exp(\theta(z - y))$, $\theta \geq 0$, we get $\mathbb{P}(u \cdot Y_t > y) \leq \exp(-\theta y)\mathbb{E}\exp(\theta u \cdot Y_t)$, from which the first limsup follows. Suppose now that

$$\limsup_{y \to \infty} \frac{1}{y} \log \mathbb{P}(u \cdot Y_t > y) \leq -\theta - \epsilon,$$

for some $\epsilon > 0$. Then $\mathbb{P}(u \cdot Y_t > y) \leq \exp(-(\theta + \epsilon)y)$ for all sufficiently large y, and so

$$\theta \int_{-\infty}^{\infty} e^{\theta y}\mathbb{P}(u \cdot Y_t > y)\, dy < \infty.$$

With the change of variables $x = \exp(\theta y)$, this becomes

$$\int_0^\infty \mathbb{P}(\exp(\theta u \cdot Y_t) > x)\, dx = \mathbb{E} \exp(\theta u \cdot Y_t).$$

The last statement in the corollary is an easy consequence of the fact that the stability region of (2.7) contains a neighborhood of the origin. □

Proof of Corollary 2.3. In the case of $\mathbb{A}_m(m)$, the vector field $f_o(x)$ of (2.7) is quasi-monotone increasing. We may therefore apply the comparison result in (4.2) with the trivial solution $x \equiv 0$ to conclude that $\Phi_t(u) \geq 0$, for all $t \geq 0$, for any $u \geq 0$.

Fix a $u \geq 0$. If $\Phi_s(u)$ blows up at or before t, then there is nothing to prove because both expectations are infinite. If $\Phi_t(u)$ is finite, then the transform formula (2.8) holds due to Theorem 2.1. It follows from (2.8) and the nonnegativity of $\Phi_t(u)$ that the ordering of Λ^1 and Λ^2 implies the ordering of the $\mathbb{E}\exp(2u \cdot Y_t^i)$, $i = 1, 2$. Conversely, if $\mathbb{E}\exp(2u \cdot Y_t^1) \geq \mathbb{E}\exp(2u \cdot Y_t^2)$ for all $u \in \mathbb{R}_+^m$ and $t \geq 0$, then we get

$$\Lambda^1 \cdot u = \lim_{t \downarrow 0} \frac{1}{t} \Lambda^1 \cdot \int_0^t \Phi_s(u)\, ds \geq \Lambda^2 \cdot u = \lim_{t \downarrow 0} \frac{1}{t} \Lambda^2 \cdot \int_0^t \Phi_s(u)\, ds.$$

Because this holds for any $u \geq 0$, $\Lambda^1 \geq \Lambda^2$.

For the second statement of the corollary, we write $x(t)$ for a solution to (2.7) with A^1 for A, and $y(t)$ for a solution with A^2. Suppose $u \geq 0$ and $Y_0 \geq 0$ are given. Then, if $A^1 \geq A^2$, we have

$$\dot{x} - \left(A^2 x + \left(x_1^2, \ldots, x_m^2\right)\right) = (A^1 - A^2)x \geq 0 = \dot{y} - \left(A^2 y + \left(y_1^2, \ldots, y_m^2\right)\right),$$

the inequality following from the fact that $x(t) \geq 0$ for $x(0) = u \geq 0$. Thus, $x(t) \geq y(t)$ and so the inequality for exponential moments follows from (2.8), because x, y, Λ, and Y_0 are nonnegative. Conversely, if the inequality holds for all nonnegative u and Y_0, then

$$\lim_{t \downarrow 0} \frac{1}{t} \left(\frac{1}{2} \log \mathbb{E} \exp\left(2u \cdot Y_t^1\right) - u \cdot Y_0 \right) \geq \lim_{t \downarrow 0} \frac{1}{t} \left(\frac{1}{2} \log \mathbb{E} \exp\left(2u \cdot Y_t^2\right) - u \cdot Y_0 \right)$$

yields $((A^1 - A^2)u) \cdot Y_0 \geq 0$. Because Y_0 is an arbitrary vector in $\mathbb{R}_+^m$, $(A^1 - A^2)u \geq 0$, and this in turn implies $A^1 \geq A^2$. □

5. CONVERGENCE TO STATIONARITY

In this section, we use the transform formula (2.8) and our analysis of the ODE (2.7) to prove that a canonical affine model has a unique limiting distribution, that this limiting distribution is stationary, and that the domain of the moment generating function of this limiting stationary distribution coincides with the stability region of the associated dynamical system.

As a first step in our analysis, we show that the moment generating function of Y_t converges, as $t \to \infty$, precisely on the stability region.

LEMMA 5.1. *Let S be the stability region of the system (2.7). Then,*

$$S = \left\{u \in \mathbb{R}^n : \lim_{t \to \infty} \mathbb{E} \exp(2u \cdot Y_t) < \infty\right\}.$$

Proof. Suppose $u \in S$. Then, as in (4.3), $\Phi_t(u)$ converges to the origin exponentially as $t \to \infty$; we may therefore define

$$t_\delta = \inf\{t : |\Phi_t(u)| \le \delta\} < \infty.$$

Let μ and C be as in (4.3). Then, for $t \ge t_\delta$,

$$\begin{aligned}\int_0^t |\Lambda \cdot \Phi_s(u)|\, ds &\le \int_0^t |\Lambda| \cdot |\Phi_s(u)|\, ds \\ &\le \int_0^{t_\delta} |\Lambda| \cdot |\Phi_s(u)|\, ds + C\delta|\Lambda| \int_{t_\delta}^t e^{-\mu(s-t_\delta)}\, ds.\end{aligned}$$

The last integral converges to a finite value as $t \to \infty$. The integrability of $|\Phi_t^d(u)|^2$ as a function of t follows similarly from (4.4). Therefore, $\lim_{t\to\infty} |\Psi_t(u)| < \infty$, and thus Theorem 2.1 implies

$$\lim_{t\to\infty} \mathbb{E}\exp(2u \cdot Y_t) = \lim_{t\to\infty} \exp(2\Psi_t(u)) = \exp(2\Psi_\infty(u)) < \infty. \tag{5.1}$$

For the converse, suppose $u \notin S$. If $\Phi_t(u)$ blows up in finite time τ, then $\lim_{t\to\tau} \exp(2\Psi_t(u)) = \infty$, as shown in (4.5), so no further argument is required in this case. Assume that $\Phi_t(u)$ exists for all $t \ge 0$. Because S is open and it contains the origin, we can choose $k > 1$ sufficiently large that $u/k \in S$. Then Lemma 4.1 implies $k\Phi_t(u/k) \le \Phi_t(u)$ for all t. This implies that

$$\liminf_{t\to\infty} \int_0^t \Phi_{s,i}(u)\, ds \ge c_i := \int_0^\infty k\Phi_{s,i}(u/k)\, ds,$$

for some real number c_i, for each $i \in \{1, \dots, m\}$. We also have

$$\liminf_{t\to\infty} \Phi_t(u) \ge \liminf_{t\to\infty} k\Phi_t(u/k) = 0.$$

But this lim inf cannot be the zero vector; if it were, $\Phi_t(u)$ would reach S in finite time and then converge to 0, which would contradict the fact that $u \notin S$. Thus some component i of $\Phi_t(u)$ has a positive lim inf, and i must be in $\{1, \dots, m\}$ because $\Phi_t^d(u)$ converges to zero. As a consequence,

$$\liminf_{t\to\infty} \int_0^t \Lambda^v \cdot \Phi_s^v(u)\, ds \ge \sum_{j\ne i} \Lambda_j c_j + \liminf_{t\to\infty} \int_0^t \Lambda_i \Phi_{s,i}(u)\, ds = \infty.$$

It follows that $\liminf_{t\to\infty} \Psi_t(u) = \infty$ and thus $\liminf_{t\to\infty} \mathbb{E}\exp(2u \cdot Y_t) = \infty$. □

Proof of Theorem 2.4. We start by showing that the sequence $\{Y_t\}$ is tight (as defined, for example, in Chung (2001), p. 90). For this, we need to show $\lim_{r\to\infty} \sup_t \mathbb{P}(|Y_t| > r) = 0$. But, we have

$$\begin{aligned}
\mathbb{P}(|Y_t| > r) &\le \mathbb{P}\Big(\bigcup_i \{|Y_{t,i}| > r/\sqrt{n}\}\Big) \\
&\le \sum_i \mathbb{P}(|Y_{t,i}| > r/\sqrt{n}) \\
&= \sum_i \{\mathbb{P}(Y_{t,i} > r/\sqrt{n}) + \mathbb{P}(-Y_{t,i} > r/\sqrt{n})\} \\
&= \sum_i \{\mathbb{P}(e^{2\delta Y_{t,i}} > e^{2\delta r/\sqrt{n}}) + \mathbb{P}(e^{-2\delta Y_{t,i}} > e^{2\delta r/\sqrt{n}})\} \\
&\le \sum_i \left\{ \frac{\mathbb{E}e^{2\delta Y_{t,i}}}{e^{2\delta r/\sqrt{n}}} + \frac{\mathbb{E}e^{-2\delta Y_{t,i}}}{e^{2\delta r/\sqrt{n}}} \right\},
\end{aligned}$$

where δ is a positive constant such that $B_\delta(0) \subset S$. From Lemma 5.1, we get $\sup_t \mathbb{E}\exp(\pm 2\delta Y_{t,i}) \le M_i < \infty$, for some M_i, for each i. Therefore,

$$\sup_t \mathbb{P}(|Y_t| > r) \le 2 \sum_i M_i \exp(-2\delta r/\sqrt{n})$$

which converges to zero as $r \to \infty$.

Because the sequence $\{Y_t\}$ is tight, it is relatively compact (Chung (2001), p. 90), so each subsequence $\{Y_{t'}\}$ contains a further subsequence $\{Y_{t''}\}$ converging weakly to some limiting random vector Y^a. Because we have $\sup_{t''} \mathbb{E}\exp(2u \cdot Y_{t''}) < \infty$, for any $u \in B_\delta(0)$ (by Lemma 5.1) and because $Y_{t''} \Rightarrow Y^a$, Theorem 4.5.2 in Chung (2001) implies that

$$\lim_{t''\to\infty} \mathbb{E}\exp(2\theta u \cdot Y_{t''}) = \mathbb{E}\exp(2\theta u \cdot Y^a), \quad \forall \theta \in (0, 1). \tag{5.2}$$

Equality continues to hold if we replace θu by u because $B_\delta(0)$ is open: we can find $u' \in B_\delta(0)$ such that $u = \theta u'$ for some $\theta \in (0, 1)$ and then apply (5.2) at u'. From (5.1) we know that the original sequence $\{Y_t\}$ satisfies $\lim_{t\to\infty} \mathbb{E}\exp(2u \cdot Y_t) = \exp(2\Psi_\infty(u))$ for $u \in B_\delta(0)$, so the same limit applies to $\{Y_{t''}\}$. Applying the same argument to any other weakly convergent subsequence of $\{Y_t\}$, say with limit Y^b, we find that

$$\mathbb{E}\exp(2u \cdot Y^a) = \exp(2\Psi_\infty(u)) = \mathbb{E}\exp(2u \cdot Y^b), \quad \forall u \in B_\delta(0).$$

But the distribution of a random vector is uniquely determined by its moment generating function in a neighborhood of the origin, so $Y^a \sim Y^b$. Because every convergent subsequence has the same limiting distribution, the original sequence $\{Y_t\}$ also converges to Y^a in distribution, so we now denote Y^a by Y_∞. We have shown that $\mathbb{E}\exp(2u \cdot Y_t) \to \mathbb{E}\exp(2u \cdot Y_\infty)$ for all $u \in B_\delta(0)$. Our next step will be to show that this holds for all $u \in S$, and to show that $\mathbb{E}\exp(2u \cdot Y_\infty) = \infty$ if $u \notin S$.

For any $u \in S$, we can find $u' \in S$ and $\theta \in (0, 1)$ with $u = \theta u'$, because S is an open set containing the origin. We know that $Y_t \Rightarrow Y_\infty$ and, by Lemma 5.1, that $\sup_t \mathbb{E}\exp(2u' \cdot Y_t)$ is finite. It follows from Theorem 4.5.2 of Chung (2001) that $\mathbb{E}\exp(2u \cdot Y_t) \to \mathbb{E}\exp(2u \cdot Y_\infty)$, so we conclude that $S \subseteq \{u : \mathbb{E}\exp(2u \cdot Y_\infty) < \infty\}$.

We prove the opposite inclusion by contradiction. For this, suppose that $u \notin S$ and that $\mathbb{E}\exp(2u \cdot Y_\infty) < \infty$. Define

$$\theta^* = \sup\{\theta \in [0, 1] : \theta u \in S\};$$

then $\theta^* > 0$ and $\theta^* u$ is on ∂S, the topological boundary of S, because S is open and $u \notin S$. Fix a $\theta_0 \in (0, \theta^*)$, so that $\theta_0 u \in S$, and set $g(t) = \Phi_t(\theta_0 u)/\theta_0$. Lemma 4.1 implies that $\Phi_t(\theta u) \geq \theta g(t)$, for all $t \geq 0$ and all $\theta \in [\theta_0, \theta^*)$. Consider the trajectory of $\Phi_t(\theta^* u)$. We claim that $\tau = \infty$. To see this, choose a $\theta \in (\theta_0, \theta^*)$. Then, for each $i \in \{1, \ldots, m\}$,

$$\begin{aligned} x_i^2 + \sum_j A_{ij} x_j + \sum_j B_{ij} x_j^2 &\geq x_i^2 + A_{ii} x_i + \theta \sum_{j \neq i} A_{ij} g_j(t) \\ &\geq x_i^2 + A_{ii} x_i + \theta M \end{aligned}$$

where $x(t) = \Phi_t(\theta u)$ and M is a lower bound of the summation. Next, we define a new function y starting at t_0 by

$$\dot{y} = y^2 + A_{ii} y + \theta M, \quad y(t_0) = x_i(t_0).$$

If $y(t_0)$ is sufficiently large, then $y(t)$ blows up in finite time (see Section 3.1) and so does $x_i(t)$. Suppose $\tau < \infty$. Then, it is possible to choose θ close to θ^* and $t_0 < \tau$ such that some $x_i(t_0)$ becomes large enough to make $y(t)$ blow up in finite time. This is a contradiction to $\theta u \in S$.

Therefore, we have $\lim_{t \to \infty} \Psi_t(\theta^* u) = \infty$ as shown in the proof of Lemma 5.1. On the other hand, we have

$$\begin{aligned} \int_0^\infty \Lambda^v \cdot (\Phi_t^v(\theta^* u) - \theta^* g^v(t))\, dt &= \int_0^\infty \lim_{\theta \uparrow \theta^*} \Lambda^v \cdot (\Phi_t^v(\theta u) - \theta g^v(t))\, dt \\ &\leq \liminf_{\theta \uparrow \theta^*} \int_0^\infty \Lambda^v \cdot \Phi_t^v(\theta u)\, dt - \theta^* \int_0^\infty \Lambda^v \cdot g^v(t)\, dt \end{aligned} \tag{5.3}$$

where the equality comes from the continuity of the flow Φ and the inequality is from Fatou's lemma. Because $\Lambda^v \cdot g^v(t)$ and $\Phi_t^d(\theta^* u)$ are integrable, $\lim_{t \to \infty} \Psi_t(\theta^* u) = \infty$ implies that the left-hand side of (5.3) is infinite. Therefore, $\liminf_{\theta \uparrow \theta^*} \int_0^\tau \Lambda^v \cdot \Phi_t^v(\theta u)\, dt = \infty$. But for $\theta \in (0, \theta^*)$, $\theta u \in S$ and utilizing Jensen's inequality,

$$\exp(2\Psi_\infty(\theta u)) = \mathbb{E} \exp(2\theta u \cdot Y_\infty) \leq (\mathbb{E} \exp(2u \cdot Y_\infty))^\theta < \infty.$$

Therefore, $\limsup_{\theta \uparrow \theta^*} \Psi_\infty(\theta u) < \infty$ and this is a contradiction.

To conclude the proof, we need to show that the limiting distribution is a stationary distribution. Suppose, therefore, that $Y_0 \sim Y_\infty$. Then for any $u \in S$, by taking a conditional expectation,

$$\begin{aligned} \mathbb{E} \exp(2u \cdot Y_t) &= \mathbb{E} \exp\left(2 \int_0^t \Lambda \cdot \Phi_s(u)\, ds + 2 \int_0^t \left|\Phi_s^d(u)\right|^2 ds + 2\Phi_t(u) \cdot Y_0\right) \\ &= \exp\left(2 \int_0^t \Lambda \cdot \Phi_s(u)\, ds + 2 \int_0^t \left|\Phi_s^d(u)\right|^2 ds\right) \mathbb{E} \exp(2\Phi_t(u) \cdot Y_0) \\ &= \exp\left(2 \int_0^t \Lambda \cdot \Phi_s(u)\, ds + 2 \int_0^t |\Phi_s^d(u)|^2\, ds\right) \\ &\quad \times \exp\left(2 \int_0^\infty \Lambda \cdot \Phi_s(\Phi_t(u))\, ds + 2 \int_0^\infty \left|\Phi_s^d(\Phi_t(u))\right|^2 ds\right) \\ &= \exp\left(2 \int_0^\infty \Lambda \cdot \Phi_t(u)\, dt + 2 \int_0^\infty \left|\Phi_t^d(u)\right|^2 dt\right) \\ &= \mathbb{E} \exp(2u \cdot Y_\infty). \end{aligned} \tag{5.4}$$

Because the distribution of a random vector is determined by the values of its moment generating function in a neighborhood of the origin, we conclude that Y_t has the distribution of Y_∞ whenever Y_0 does. $\square$

Observe that (5.4) gives the moment generating function of Y_∞ and thus characterizes the stationary distribution of Y_t.

From the preceding proof, we see that the distribution of Y_∞ is determined by the behavior of the dynamical system (2.7) on the stable manifold S of the stable equilibrium at the origin: the fact that $\Phi_t(u) \to 0$ for $u \in S$ is crucial to the convergence of $\Psi_t(u)$ and thus the moment generating function of $u \cdot Y_t$. This raises the question of whether other, unstable equilibria play any role in the stochastic behavior of the basic model (2.1). Our next result illustrates a setting in which they do.

PROPOSITION 5.2. *Suppose that η is a hyperbolic equilibrium of system (2.7) of type less than n. Then for any u in the stable manifold of η, we have*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp(2u \cdot Y_t) = 2\Lambda \cdot \eta. \tag{5.5}$$

Proof. If u lies on the stable manifold of η, then $\lim_{t\to\infty} \Phi_t(u) = \eta$, so $\Psi_t(u)$ is well defined for all $t \geq 0$. The limit on the left side of (5.5) is given by the limit of $2\Psi_t(u)$ as $t \to \infty$; i.e., by

$$\lim_{t\to\infty} \frac{2}{t} \int_0^t \Lambda \cdot \Phi_s(u)\, ds + \lim_{t\to\infty} \frac{2}{t} \int_0^t \left|\Phi_s^d(u)\right|^2 ds + \lim_{t\to\infty} \frac{2}{t} \Phi_t(u) \cdot Y_0.$$

The last term is clearly zero, and the second term also vanishes because

$$\frac{1}{t} \int_0^t \left|\Phi_s^d(u)\right|^2 ds \leq \frac{1}{t} \int_0^t C^2 |u^d|^2 \exp(-2\mu s)\, ds = \frac{1}{t} C^2 |u^d|^2 \frac{1 - \exp(-2\mu t)}{2\mu} \to 0,$$

in light of (4.4). The first limit is $2\Lambda \cdot \eta$. $\square$

The condition in the proposition on the equilibrium's type ensures the existence of a stable manifold. An equilibrium of type n is a source, an example of which appears in the right panel of Figure 3.5, at the upper right intersection of the two curves. The limit in (5.5) arises in the definition of the rate function used in the Gärtner–Ellis Theorem (see, e.g., Dembo and Zeitouni (1998)). The behavior in (5.5) is somewhat pathological because the limit, viewed as a function of u, fails to be a closed convex function. As a consequence, the Gärtner–Ellis Theorem does not apply to the sequence $\{Y_t/t\}$.

Theorem 2.4 characterizes the set of u for which $\mathbb{E} \exp(2u \cdot Y_\infty)$ is finite and identifies this set with the stability region S of (2.7). The problem of describing the boundary of S has attracted considerable attention. Genesio et al. (1985) survey methods using a Lyapunov approach; Chiang et al. (1988) characterize ∂S in terms of stable submanifolds of unstable equilibria. Kim (2008) establishes a similar result for the quadratic system (2.7).

Theorem 2.4 raises the question of characterizing the region in which Y_t has finite exponential moments, for finite t; that is, characterizing

$$S_t = \{u \in \mathbb{R}^n : \mathbb{E} \exp(2u \cdot Y_s) < \infty,\ \forall\, s \in [0, t)\}.$$

This set coincides with the set of initial conditions u for which the solution $\Phi_s(u)$ exists throughout $[0, t)$. Directly from the definition of S_t, we see that S_t shrinks as t increases; that S_t is convex follows from Hölder's inequality. Beyond these basic properties, it is generally more difficult to characterize S_t than S, at least from the perspective of the dynamical system (2.7). Theorem 2.5 and the analysis in the next section give some results in this direction.

6. GAUSSIAN CONDITIONS

LEMMA 6.1. *For any $t > 0$ and $u \in \mathbb{R}^n$, $\mathbb{E}\exp(2\theta u \cdot Y_t) < \infty$ for all $\theta \in \mathbb{R}$ if and only if*

$$u^v = 0, \quad A^c x^d(s) = 0, \quad B^c\big(x_{m+1}^2(s), \ldots, x_n^2(s)\big) = 0$$

for all $s \geq 0$, where x^d is the solution to $\dot{x} = A^d x$ with $x(0) = u^d$. Moreover, in this case, $u \cdot Y_t$ has a Gaussian distribution.

Proof. See the Appendix.

Proof of Theorem 2.5. In writing $P^{-1} A^d P = J$, we may assume P is chosen to give J the specific form described before the statement of the theorem. We further assume that the k distinct eigenvalues of A^d are numbered in decreasing order, $\lambda_k < \cdots < \lambda_1 < 0$.

Define $y(t) = P^{-1}x(t)$, where x is the solution to $\dot{x} = A^d x$ with $x(0) = u^d$. Then y satisfies $\dot{y} = Jy$ with $y(0) = \tilde{u}$ and $\tilde{u} = P^{-1}u^d$. Let y^i denote the block of y corresponding to the ith block $J_i = \lambda_i I_i + N_i$ of J. We use this notation similarly for other vectors. In other words, if the $a_{\lambda_i} \times a_{\lambda_i}$ matrix J_i runs through coordinates $(p+1, p+1), \ldots, (p + a_{\lambda_i}, p + a_{\lambda_i})$ of J, then v^i of $v \in \mathbb{R}^n$ is $(v_{p+1}, \ldots, v_{p+a_{\lambda_i}})$. Because we have $\dot{y}^i = J_i y^i$, $y^i(0) = \tilde{u}^i$, the solution is expressed as follows:

$$y^i(t) = \exp(\lambda_i t)\left[I_i + \sum_{l=1}^{a_{\lambda_i}-1} \frac{t^l}{l!} N_i^l \right] \tilde{u}^i.$$

Suppose that $w^\top y \equiv 0$ for some $w \in \mathbb{R}^n$. Then $\sum_{i=1}^{k} {w^i}^\top y^i \equiv 0$. If we divide this by $\exp(\lambda_1 t)$, which has the smallest magnitude among eigenvalues, and send $t \to \infty$, then $\exp(-\lambda_1 t){w^1}^\top y^1 \equiv 0$; otherwise, we equate one exponentially decreasing function with a polynomial, which is absurd. By applying the same procedure with other λ_i's, we conclude that ${w^i}^\top y^i \equiv 0$ for each i. Consequently, $w^\top y \equiv 0$ is equivalent to

$$\text{(6.1)} \qquad {w^i}^\top N_i^l \tilde{u}^i = 0, \quad i = 1, \ldots, k, \quad l = 0, \ldots, a_{\lambda_i} - 1.$$

This observation implies that the first two conditions in Lemma 6.1 are equivalent to requiring that $u^v = 0$ and that (6.1) holds for all ${w^i}^\top$ that are rows of $A^c P$. As for the third condition in Lemma 6.1, we note that $x_q = \sum_l P_{ql} y_l \equiv 0$ if there exists some p such that $B^c_{pq} \neq 0$. Therefore, (2.9) follows. □

Proof of Corollary 2.6. Choose any block J_i of J and $\tilde{u}^i$. By construction, J_i is itself a block diagonal matrix consisting of Jordan blocks associated with λ_i; each Jordan block has a 1 in every entry immediately above the main diagonal. Let Q be any Jordan block of J_i and $\tilde{u}^Q$ the corresponding block of $\tilde{u}^i$ with dimension d, say. Then, the following condition becomes a sufficient condition that induces (6.1):

$$w^{Q\top} N^l \tilde{u}^Q = 0, \quad l = 0, \dots, d-1, \quad \forall\, Q,$$

where N is Q less the diagonal part. But, then, this is just

$$w^{Q\top}\begin{pmatrix} \tilde{u}_1^Q \\ \vdots \\ \tilde{u}_{d-1}^Q \\ \tilde{u}_d^Q \end{pmatrix} = 0, \quad w^{Q\top}\begin{pmatrix} \tilde{u}_2^Q \\ \vdots \\ \tilde{u}_d^Q \\ 0 \end{pmatrix} = 0, \quad \dots, w^{Q\top}\begin{pmatrix} \tilde{u}_d^Q \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0.$$

Therefore, an equivalent statement is that if j is a coordinate with $w_j^Q \neq 0$, then $\tilde{u}_j^Q = \tilde{u}_{j+1}^Q = \cdots = \tilde{u}_d^Q = 0$.

The directed graph G in this case consists of paths such as $n \to n+1 \to \cdots \to n + d - 1$ if Q starts at the coordinate (n, n). If j is restricted with respect to $\mathbf{1}_{A^c P} + \mathbf{1}_{B^c}\mathbf{1}_P$, then $(A^c P)_{ij} \neq 0$ or $B^c_{iq} P_{qj} \neq 0$ for some $i,\ q$. This in turn means that $w_j \neq 0$ where w is the ith row of $A^c P$ or the qth row of P, and thus $\tilde{u}_j = 0$. In this case, the observation in the previous paragraph requires that any other components of $\tilde{u}$ that have a directed path from $\tilde{u}_j$ in G are also zero.

If $g_{\lambda_i} = 1$ for all i, then there is only one Jordan block for each λ_i and thus Q coincides with J_i. Therefore, the condition above becomes necessary, too. □

Corollary 2.6 essentially means that we achieve a non-Gaussian distribution for $u \cdot Y_t$ as long as it has some dependence on one or some of volatility driving factors by including them in the dynamics or by including a factor that depends on volatility factors. Of course, u has to be outside the closed set specified by (2.9). The vectors in this set cancel out the effects of the volatility factors in $u \cdot Y_t$. The next examples illustrate this feature in more detail.

EXAMPLE. $\mathbb{A}_m(n)$ with diagonal A^d. In this case, we have

$$dY_j^d(t) = \left(\Lambda_j^d + \sum_k A^c_{kj} Y_k^v + A^d_{jj} Y_j^d(t)\right) dt + \sqrt{1 + \sum_k B^c_{kj} Y_k^v}\, dW_j^d(t)$$

and

$$d\left(u^d \cdot Y^d(t)\right) = \left(u^d \cdot \Lambda^d + \sum_k \left(\sum_j u_j^d A^c_{kj}\right) Y_k^v + \sum_j u_j^d A^d_{jj} Y_j^d(t)\right) dt$$
$$+ \sum_j u_j^d \sqrt{1 + \sum_k B^c_{kj} Y_k^v}\, dW_j^d(t). \tag{6.2}$$

For $u \cdot Y$ not to have any dependence on Y^v, we must have $u^v = 0$,

$$\sum_j u_j^d A^c_{kj} = 0, \quad k = 1, \dots, m$$

and $u^d_j = 0$ whenever there exists k such that $B^c_{kj} \neq 0$. However, these conditions are not enough to remove all the dependence on Y^v. For example, suppose A^d is given by

$$A^d = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}.$$

Then, (6.2) becomes

$$d\left(u^d \cdot Y^d(t)\right) = \left(u^d \cdot \Lambda^d + \lambda_1\left(u^d \cdot Y^d(t)\right) + (\lambda_2 - \lambda_1)u^d_3 Y^d_3(t)\right) dt + \sum_{j \notin \mathcal{J}} u^d_j dW^d_j(t)$$

where $\mathcal{J}$ is a set of coordinates that are restricted with respect to $\mathbf{1}_{B^c}$. Therefore, if Y^d_3 has a volatility factor in its drift or diffusion, then $u^d \cdot Y^d$ is not free of Y^v effects. This kind of additional dependency is captured by (2.9).

EXAMPLE. $\mathbb{A}_m(m+2)$. This class of models has two dependent factors. We consider the case in which A^d has only one eigenvalue λ with $g_\lambda = 1$. The other possible case is diagonal and is similar to the example above but with a lower dimension. Let $P = (v_1 \quad v_2)$ be the non-singular matrix of an eigenvector and a generalized eigenvector in a Jordan canonical form of A^d and let $P^{-1}u = (a, b)$. We write

$$A^d P = P\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad L = \begin{pmatrix} I^v & 0 \\ 0 & P^\top \end{pmatrix}.$$

Next we apply an invariant affine transformation as defined in Dai and Singleton (2000), $Y \mapsto LY$. Then the dynamics of Y^v are the same as the original and that of $\tilde{Y} = P^\top Y^d$ becomes

$$d\tilde{Y}_t = \left(P^\top \Lambda^d + (A^c P)^\top Y^v_t + J^\top \tilde{Y}_t\right) dt + P^\top \sqrt{diag\left(F^d_t\right)} dW^d_t.$$

Denoting $\tilde{Y}$ by $(\tilde{Y}_1, \tilde{Y}_2)$,

$$d\begin{pmatrix} \tilde{Y}_1(t) \\ \tilde{Y}_2(t) \end{pmatrix} = P^\top \Lambda^d dt + (A^c P)^\top Y^v_t dt + \begin{pmatrix} \lambda \tilde{Y}_1(t) \\ \tilde{Y}_1(t) + \lambda \tilde{Y}_2(t) \end{pmatrix} dt$$
$$+ \begin{pmatrix} (v_1)_1 \sqrt{1 + \sum B^c_{k1} Y^v_k} dW_2(t) + (v_1)_2 \sqrt{1 + \sum B^c_{k2} Y^v} dW_3(t) \\ (v_2)_1 \sqrt{1 + \sum B^c_{k1} Y^v_k} dW_2(t) + (v_2)_2 \sqrt{1 + \sum B^c_{k2} Y^v_k} dW_3(t) \end{pmatrix}$$

Note that $u \cdot Y_t = \tilde{u} \cdot \tilde{Y}_t = a\tilde{Y}_1(t) + b\tilde{Y}_2(t)$ (we assume $u^v = 0$). Now suppose $a \neq 0$. Then, $u \cdot Y_t$ has a dependence on Y^v unless $(A^c P)_{k1} = 0$ and $B^c_{ki} = 0$ for all k whenever $(v_1)_i \neq 0$. This is the same as asking whether coordinate 1 is restricted with respect to $\mathbf{1}_{A^c P} + \mathbf{1}_{B^c}\mathbf{1}_P$. A similar argument applies to the case $b \neq 0$ regarding the second coordinate.

If $a = 0$ but $b \neq 0$, then we still have to consider the dependence of $\tilde{Y}_1$ on Y^v because $\tilde{Y}_2$ is correlated with $\tilde{Y}_1$ through the drift term. This means that $u \cdot Y_t$ has dependence on Y^v if coordinate 1 is restricted. It is clear from the dynamics of $\tilde{Y}$ that the final dynamics induce a Gaussian distribution after we remove the dependence on Y^v.

7. CONCLUSION

We have established three general results for affine models. Our first result confirms the validity of the transform representation without further conditions and shows that the range of exponents for which the transform is finite at time t coincides with the set of initial conditions from which the ODE solution exists on $[0, t]$. Based on this result, we are able to investigate the properties of affine models by analyzing the associated differential equations. As an example, we gave two comparison criteria for processes in $\mathbb{A}_m(m)$.

Our second result establishes the existence of a limiting stationary distribution and characterizes this limit through its transform; the tail behavior of the limiting distribution is determined by the stability region of the associated dynamical system.

Our last result gives necessary and sufficient conditions for a linear combination of factors to have a Gaussian distribution and shows that any non-Gaussian linear combination has exponential tails. Essentially, a Gaussian distribution is obtained by removing from a linear combination all the dependence on the volatility factors, but the precise conditions that achieve this turn out to be subtle.

APPENDIX A: PROOFS

Proof of Lemma 4.1. Define $x(t) = \Phi_t(u)$ and $y(t) = \Phi_t(\theta u)/\theta$; then

$$\begin{aligned}\dot{x} &= Ax + B\left(x_1^2, \ldots, x_n^2\right)\\ \dot{y} &= Ay + \theta B\left(y_1^2, \ldots, y_n^2\right)\end{aligned}$$

with $x(0) = y(0) = u$. It is immediate that $x^d \equiv y^d$ because they satisfy the same linear ODE with the same initial condition. So, we concentrate on x^v and y^v, for which the corresponding ODEs are

$$\begin{aligned}\dot{x}^v &= A^v x^v + \left(x_1^2, \ldots, x_m^2\right) + c(t) + d(t)\\ \dot{y}^v &= A^v y^v + \theta\left(y_1^2, \ldots, y_m^2\right) + c(t) + \theta d(t)\end{aligned}$$

where $c(t) = A^c x^d(t)$ and $d(t) = B^c(x_{m+1}^2, \ldots, x_n^2)$. Now define

$$f(x^v) = A^v x^v + \left(x_1^2, \ldots, x_m^2\right).$$

By condition **(C2)** (see the discussion preceding Lemma 4.1), the mapping $x^v \mapsto A^v x^v$ is quasi-monotone increasing, as is the mapping $x^v \mapsto (x_1^2, \ldots, x_m^2)$, and thus also f. Recalling that B^c has nonnegative entries and $\theta > 1$, we get

$$\begin{aligned}\dot{x}^v - f(x^v) &= c(t) + d(t)\\ &\le (\theta - 1)(y_1^2, \ldots, y_m^2) + c(t) + \theta d(t)\\ &= \dot{y}^v - f(y^v).\end{aligned}$$

It now follows from the comparison result (4.2) that $x(t) \le y(t)$. □

For the proof of Lemma 4.2, we need a preliminary result that limits the crossing of coordinates of the solution to (2.7).

LEMMA A.1. *For the system (2.7), suppose* $(t, u) \in \Omega$ *and let* $x(t) = \Phi_t(u)$. *For* $i, j \in \{1, \ldots, n\}$, *the set* $\{s \in [0, t] : x_i(s) = x_j(s)\}$ *has only finitely many isolated points.*

Proof. As noted in Section 4.1, $\Phi_t(u)$ is analytic in (t, u) so long as it lies within the domain of analyticity of f_o in (2.7); but this function is analytic in the entire domain. It follows that $x_i(s) - x_j(s)$ is analytic in s. An analytic function can have only a finite number of isolated zeros on a compact interval. □

Proof of Lemma 4.2. Fix $u \in \mathbb{R}^n$ and let us denote $\Phi_t(u)$ by $x(t)$ to simplify notation. We define a piecewise differentiable function $\gamma(t) = \min_{i=1,\dots,m} x_i(t)$. Because $x^d(t)$ converges to zero (as implied by (4.4)), we can find $M > 0$ such that $\sup_t |x^d(t)| < M$. The value of M depends on $x^d(0)$. Lemma A.1 implies that in any bounded interval $[0, t]$ with $x(t)$ finite, the set of s at which $x_i(s) = x_j(s)$ is either finite or an interval. Therefore, we can define a sequence of closed intervals of $\mathbb{R}_+$, $\{I_j\}$, such that $I(u) \cap \mathbb{R}_+ = \bigcup_{j=1}^{\infty} I_j$ and

$$\gamma(t) = x_{i(j)}, \quad \forall t \in I_j^o,$$

for some $i(j) \in \{1, \dots, m\}$, where I_j^o denotes the interior of I_j.

In an interval I^o throughout which $\gamma(t) = x_i(t)$, we have

$$\begin{aligned}
\dot{\gamma}(t) &= \gamma^2 + \sum_{k=1}^{n} A_{ik} x_k + \sum_{k=m+1}^{n} B_{ik} x_k^2 \\
&\geq \gamma^2 + \sum_{k=1}^{m} A_{ik} \gamma + \sum_{k=m+1}^{n} A_{ik} x_k \\
&\geq \gamma^2 + \sum_{k=1}^{m} A_{ik} \gamma - M \max_{j=1,\dots,m} \sum_{k=m+1}^{n} |A_{jk}|. \qquad \text{(A.1)}
\end{aligned}$$

In the first inequality, we used the assumption that A^v has nonnegative off-diagonal entries and $B^c \geq 0$. Next, we define a continuous, piecewise differentiable function v by

$$\dot{v} = L(v), \quad L(v) := v^2 + \sum_{k=1}^{m} A_{ik} v - K$$

whenever $\gamma(t) = x_i(t)$, with $K = M \max_j \sum_{k=m+1}^{n} |A_{jk}|$ and $\gamma(0) \geq v(0)$. Then, because γ and v satisfy

$$\dot{\gamma} - L(\gamma) \geq \dot{v} - L(v),$$

we get $\gamma \geq v$ by applying the standard comparison result repeatedly on the intervals I_j. If we show that v is bounded below, then γ is also bounded below and the statement follows.

To see that v is indeed bounded below, we observe that M can be set large enough to make $L(x) = 0$ have two real solutions, $\eta_1^i < \eta_2^i$, for each i; in this case $\dot{v}(t) \geq 0$ or $v(t) \geq \eta_1^i$ (as is evident in Figure 3.1) when $\gamma(t) = x_i(t)$. □

Proof of Lemma 4.3. We write $x(t)$ for $\Phi_t(u)$ to simplify notation. Define a piecewise differentiable function $\gamma = \max_{i=1,\dots,m} x_i$, similarly as in the proof of Lemma 4.2. We saw there that we can define a sequence of intervals $\{I_j\}$ until τ with $\gamma(t) = x_{i(j)}(t)$ in I_j^o. In an interval I on which $\gamma = x_i$, γ satisfies (A.1). Because the trajectory of $x(t)$ is bounded below (by Lemma 4.2), $\gamma \to \infty$. So, at some time $t_0 < \tau$, $\gamma(t_0)$ is sufficiently large that the right-hand side of (A.1) becomes positive for all $i = 1, \dots, m$, and then γ never decreases. We can then divide both sides of (A.1) by γ to get

$$\begin{aligned}\frac{\dot{\gamma}}{\gamma} &= \sum_{k=1}^{m} A_{ik}\frac{x_k}{\gamma} + \gamma + \frac{1}{\gamma}\sum_{k=m+1}^{n} A_{ik}x_k + \frac{1}{\gamma}\sum_{k=m+1}^{n} B_{ik}x_k^2 \\ &\le \sum_{k=1}^{m} A_{ik} + \gamma + M\end{aligned}$$

on (t_0, τ), for some sufficiently large M. Here we have used the fact that $x_k \le \gamma$, $k \in \{1, \ldots, m\}$, and the nonnegativity of the off-diagonal entries of A^v. The existence of M is guaranteed by the fact that $|x^d|$ is bounded and γ never decreases after t_0. Then,

$$\int_{t_0}^{\tau} \frac{\dot{\gamma}}{\gamma} dt \le \int_{t_0}^{\tau} \gamma\, dt + \left(\max_i \sum_{k=1}^{m} A_{ik} + M\right)(\tau - t_0).$$

However, the left-hand side is infinite, so $\int_{t_0}^{\tau} \gamma\, dt = \infty$ as well. We can pick a constant C such that $|\Lambda^d x^d(t)| \le C$ for all $t \ge 0$; then, because $\Lambda^v \gg 0$, we have

$$\int_0^{\tau} \Lambda \cdot x(t)\, dt \ge \left(\min_i \Lambda_i^v\right) \int_0^{\tau} \gamma\, dt - C\tau = \infty.$$

$\square$

The system (2.7) can be thought of as a system of equations defined in $\mathbb{C}^n$ by setting $x(t) = \operatorname{Re} x(t) + i \operatorname{Im} x(t)$. Based on the analyticity of f_o, the solution $x(t)$ also has a nice analytic property which is used in the proof of Theorem 2.1.

LEMMA A.2. *For the system (2.7), suppose* $(t, u) \in \Omega$. *Then we can find an open convex subset of* $\mathbb{C}^n$, *containing the line segment* $L = \{\lambda u \in \mathbb{R}^n : \lambda \in [0, 1]\}$, *in which* $\Phi_t(\cdot)$ *is analytic.*

Proof. Because $\Phi_t(u)$ is finite, $\Phi_t(\lambda u)$ is finite for all $\lambda \in [0, 1]$. This is because, first, $\Phi_t(\lambda u) \le \lambda \Phi_t(u)$ by Lemma 4.1 (take $\theta = 1/\lambda$, for $\lambda \in (0, 1]$) and, second, it is bounded below by Lemma 4.2.

For each $\lambda u \in L$, there is an open ball B_λ in $\mathbb{C}^n$ centered at λu in which $\Phi_t(\cdot)$ is analytic, because of the analyticity of f_o. Because L is compact, we can cover L by a finite number of such balls. We can then find an open convex set U that contains L and is contained within the cover; for example, we can define U to be the set of points less than a distance ϵ from L, for sufficiently small $\epsilon > 0$. Then $\Phi_t(\cdot)$ is analytic in U. $\square$

Proof of Lemma 6.1. The proof uses an approach of Getz and Jacobson (1977). We write the ODE for x^v in (2.7) as

$$\dot{x}^v = \begin{pmatrix} x_1^2 \\ \vdots \\ x_m^2 \end{pmatrix} + A^v x^v + A^c x^d + B^c \begin{pmatrix} x_{m+1}^2 \\ \vdots \\ x_n^2 \end{pmatrix}, \quad x^v(0) = u^v.$$

Choose any $w \in \mathbb{R}_{++}^m$ and let $\rho = \min_i w_i$. Multiplying both sides of the ODE by $w^\top$, we get

$$w^\top \dot{x}^v = x^{v\top} diag(w) x^v + (w^\top A^v) x^v + w^\top A^c x^d + x^d diag(w^\top B^c) x^d.$$

Define $b = A^{v\top} w/2$ and $\tilde{x} = x^v + diag(w)^{-1}b$. Then,

$$
\begin{aligned}
w^\top \dot{\tilde{x}} &= \tilde{x}^\top diag(w)\tilde{x} - b^\top diag(w)^{-1}b + w^\top A^c x^d + x^d diag(w^\top B^c)x^d \\
&\geq \rho \tilde{x}^\top \tilde{x} - b^\top diag(w)^{-1}b + w^\top A^c x^d + x^d diag(w^\top B^c)x^d \\
&\geq \frac{\rho}{|w|^2}(w^\top \tilde{x})^2 - b^\top diag(w)^{-1}b + w^\top A^c x^d + x^d diag(w^\top B^c)x^d. \qquad \text{(A.2)}
\end{aligned}
$$

Let $g(w) = b^\top diag(w)^{-1}b,\ y = w^\top \tilde{x}$ and $y(0) = w^\top \tilde{x}(0)$.

We want to determine whether there is a real number θ such that $x(s)$ blows up as $s \to t$ for the scaled initial condition $x(0) = \theta u$. We divide the rest of the proof into four cases.

Case (i): Suppose $u^v \neq 0$. From (A.2) we get

$$
\begin{aligned}
\dot{y} &\geq \frac{\rho}{|w|^2} y^2 - g(w) + w^\top A^c x^d \\
&\geq \frac{\rho}{|w|^2} y^2 - g(w) - C|\theta| \cdot |w^\top A^c| \cdot |u^d| \qquad \text{(A.3)}
\end{aligned}
$$

with $y(0) = \theta w^\top u^v + e^\top b$, using (4.4) in the second inequality. Now choose w so that $w^\top u^v \neq 0$. Define a new function z by setting

$$
\dot{z} = \frac{\rho}{|w|^2} z^2 - g(w) - |\theta| M, \qquad \text{(A.4)}
$$

with $z(0) = y(0)$ and $M = C|w^\top A^c| \cdot |u^d|$; then $y \geq z$ on their common interval of existence. Let $\eta_2 = \sqrt{(g(w) + |\theta| M)|w|^2/\rho}$ and $\eta_1 = -\eta_2$, the two equilibria of the ODE (A.4). Because $w \in \mathbb{R}_{++},\ g(w) > 0$ so $\eta_2 \neq 0$. By increasing θ (if $w^\top u^v > 0$) or increasing $-\theta$ (if $w^\top u^v < 0$), we can make $z(0) > \eta_2$. Then, as in (3.3), z has a finite blow-up time

$$
\tau = \frac{|w|^2}{\rho(\eta_2 - \eta_1)} \log \frac{z(0) - \eta_1}{z(0) - \eta_2}.
$$

Because we always have $y \geq z,\ \tau$ is an upper bound on the blow-up time of y. Moreover, this upper bound can be made arbitrarily small because $\tau \downarrow 0$ as $\theta \to \infty$ or $\theta \to -\infty$, depending on the sign of $w^\top u^v$. Thus, by taking θ of sufficiently large magnitude and with the sign of $w^\top u^v$, we ensure that x blows up by time t.

Case (ii): Next, suppose $u^v = 0$ but $A^c x^d(s)$ is not identically zero, x^d having initial condition $x^d(0) = u^d$. The solution $x^d(t)$ is given by $\exp(A^d t)u^d$. So, there is some $t_0 < t$ for which $\int_0^{t_0} A^c \exp(A^d t)u^d ds \neq 0$; otherwise, $A^c x^d(s) = 0$ for all $s \in [0, t)$ and this implies $A^c x^d \equiv 0$ because $A^c x^d$ is analytic. Now consider the scaled initial condition $x(0) = \theta u$, and let y be the function defined above by $y = w^\top \tilde{x}$. Then, the initial condition becomes $y(0) = e^\top b$. For $s \leq t_0$, (A.3) yields

$$
\dot{y} \geq \frac{\rho}{|w|^2} y^2 - g(w) + w^\top A^c x^d \geq -g(w) + w^\top A^c x^d,
$$

and so

$$
y(t_0) \geq e^\top b - g(w)t_0 + \theta w^\top \int_0^{t_0} A^c \exp(A^d s)u^d ds.
$$

The integral in this expression is nonzero, so the last term is nonzero for some $w \in \mathbb{R}^m_{++}$. On the other hand, for $t \geq t_0$, we use

$$\dot{y} \geq \frac{\rho}{|w|^2} y^2 - g(w) - |\theta| M,$$

with M as before. We can make $y(t_0)$ greater than η_2 by increasing θ or $-\theta$. Applying the same argument we applied to z following (A.4), we conclude that y blows up in time t, and then x does too.

Case (iii): Suppose that $u^v = 0$ and $A^c x^d \equiv 0$, but $B^c(x^2_{m+1}(s), \dots, x^2_n(s))$ is not identically zero. We can pick $t_0 < t$ such that

$$N \equiv \int_0^{t_0} (\exp(A^d s)u^d)^\top diag(w^\top B^c) \exp(A^d s)u^d ds \neq 0.$$

Now consider x with $x(0) = \theta u$ and take $y = w^\top \tilde{x}$. Then (A.2) yields

$$\begin{aligned} \dot{y} &\geq \frac{\rho}{|w|^2} y^2 - g(w) + w^\top A^c x^d + x^d diag(w^\top B^c) x^d \\ &\geq -g(w) + x^d diag(w^\top B^c) x^d \end{aligned}$$

and so $y(t_0) \geq e^\top b - g(w)t_0 + \theta^2 N$. And we use the following inequality for $t \geq t_0$ (by (A.2)):

$$\dot{y} \geq \frac{\rho}{|w|^2} y^2 - g(w).$$

By the argument in Cases (i)–(ii), we conclude that x blows up by time t for sufficiently large $|\theta|$.

Case (iv): Suppose $u^v = 0$, $A^c x^d \equiv 0$ and $B^c(x^2_{m+1}, \dots, x^2_n) \equiv 0$. This means that x^v is a solution of

$$\dot{x}^v = \begin{pmatrix} x_1^2 \\ \vdots \\ x_m^2 \end{pmatrix} + A^v x^v, \quad x^v(0) = 0.$$

This makes $x^v \equiv 0$ and thus

$$\mathbb{E} \exp(2\theta u \cdot Y_t) = \exp\left(2\theta^2 \int_0^t |x^d(s)|^2 ds + 2\theta \left(\int_0^t \Lambda^d \cdot x^d(s)\, ds + x^d(t) \cdot Y_0^d\right)\right)$$

where x^d is the solution from the original (unscaled) initial condition, $x^d(0) = u^d$. Because the moment generating function of $u \cdot Y_t$ is the exponential of a quadratic function of θ, we conclude that $u \cdot Y_t$ is Gaussian. □

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