

## Design of Risk Weights

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# Design of Risk Weights

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## Abstract

Banking regulations set minimum levels of capital for banks. These requirements are generally formulated through a ratio of capital to *risk-weighted* assets. A risk-weighting scheme assigns a weight to each asset or category of assets and effectively functions as a linear constraint on a bank's portfolio choice; it also changes the incentives for banks to hold various kinds of assets. In this paper, we investigate the design of risk weights to align regulatory and private objectives in a simple mean-variance framework for portfolio selection. By setting risk weights proportional to profitability rather than risk, the regulator can induce a bank to reduce its overall level of risk without distorting its asset mix. Because the regulator is unlikely to know the true profitability of assets, we introduce an adaptive formulation in which the regulator sets weights by observing a bank's portfolio. The adaptive scheme converges to the same combination of weights and portfolio choice that would hold if the regulator knew the asset profitability. We also investigate other objectives, including steering banks to a target mix of assets, adding robustness, mitigating procyclicality, and reducing system-wide risk in a setting with multiple heterogeneous banks.

## 1 Introduction

Capital requirements for banks are intended to ensure that banks have adequate capital to withstand large losses in the assets they hold. The simplest type of capital requirement limits a bank's overall leverage by putting an upper bound on the ratio of a bank's total assets to its equity. Since the 1980s, most regulatory capital requirements have instead been formulated as a percentage of *risk-weighted* assets, with the objective of aligning a bank's capital cushion with the riskiness of its assets.

A risk-weighting scheme assigns a risk weight to each type of asset or group of assets, such as residential mortgages, corporate loans, securities, and so on. Risk-weighted assets, in their simplest form, are then calculated by taking a linear combination of a bank's investments across categories, using the risk weights as coefficients. Required capital is then set at a fixed fraction (e.g., 8 percent) of the risk-weighted sum. See Section 2 for details and background.

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This is a rather curious scheme, if we consider that risk is not ordinarily considered additive. One might construe the additive formulation as conservative, but capital requirements affect which assets a bank chooses to hold, so the choice of risk weights affects a bank’s asset mix and not just the overall risk of its portfolio. A risk-weighting scheme may be conservative in its effect on overall risk and yet introduce unintended distortions in the levels of different kinds of lending activities.

This paper undertakes a theoretical investigation into the design of risk weights. Our goal is to understand what types of objectives can be achieved by imposing linear risk-weight constraints. We work within a simple model of portfolio selection based on various mean-variance objectives. This simple setting provides a high degree of tractability, which makes the implications of the results easier to interpret.

The theoretical underpinnings of risk weights have not received a great deal of attention. Pyle [28] and Hart and Jaffee [20] provide early formulations of a financial institution’s portfolio problem using mean-variance optimization; Hart and Jaffee [20] associate a reserve requirement with each asset that functions much like a risk weight, but they take these requirements as given. Koehn and Santomero [24] and Kim and Santomero [23] use the mean-variance framework to argue that a simple leverage limit can actually increase risk, and Kim and Santomero [23] go on to derive risk weights that preclude this outcome. Their formulation is critiqued by Keeley and Furlong [22] for assuming that a bank can buy and sell its own equity the way it trades in any other asset. In this respect, our formulation is closer to Rochet [31, Chapter 8], in that we treat the level of capital available to a bank as fixed. Using a mean-variance analysis, Rochet [31, p.244] proposes setting the risk weight for each asset proportional to its systematic risk, as defined through the capital asset pricing model. Calomiris [11] and Morgan and Ashcraft [27] propose tying capital requirements to the interest rates banks charge on loans because higher rates should reflect higher risk. This approach implicitly takes the view that the risk weight for an asset should reflect only the risk in that asset, without consideration of the effect on portfolio mix. See Santos [32] and VanHoose [36] for surveys of research on capital requirements and many additional references.

Gordy [17] analyzes the connection between a linear risk-weighting scheme and a value-at-risk measure for portfolio credit risk, based on the internal ratings based approach introduced under Basel II capital requirements (BCBS [7]). He shows that in an “asymptotic single risk factor” version of the portfolio model, a value-at-risk based capital requirement is equivalent to a linear risk-weighting scheme. Repullo and Suarez [30] analyze implications of this framework for bank portfolio choice. Shin [33] interprets value-at-risk based capital constraints from the perspective of mean-variance optimization.

The empirical literature on bank capital requirements is extensive, but most of it focuses on the

level of bank capital rather than the validity of risk weights. An exception is Cordell and King [12] who compare regulatory risk weights with market-based risk weights derived from the performance of bank stocks. Recent studies comparing risk-weighted assets with the market risk of banks include Acharya, Engle, and Pierret [2], Das and Sy [13] and Vallascas and Hagendorff [35].

For our investigation, we take a risk-weighting scheme to have two primary interlinked objectives: to limit the overall risk in a bank portfolio and to do so without an unintended distortion of the mix of assets held by the bank. The first of these objectives is common to all capital regulation, but the second is specific to a risk-weighting scheme because risk weights implicitly assign prices (in terms of additional capital) to asset categories and thus inevitably create incentives for banks to choose some assets over others. As a starting point, we suppose that the regulator would prefer not to change the mix of assets — just the overall levels — before we consider the more general case in which the regulator seeks to steer banks toward a different mix.

Our first main result (in Section 3) shows that this objective can be achieved — surprisingly, the ideal risk-weights turn out to have little to do with risk and are instead proportional to the profitability (expected excess return) on each asset. With these weights, the regulator can limit the bank’s overall risk; Kim and Santomero [23] arrive at a similar conclusion in their formulation, but their result does not appear to be well known. Moreover, we show that this choice of weights leaves the relative mix of assets in the bank’s portfolio unchanged from the relative mix the bank would choose in the absence of a risk-weight constraint. If the regulator does want to change the asset mix as well as the overall risk level, we identify the set of target portfolios the regulator can induce the bank to hold through suitable choice of risk weights.

Setting risk weights proportional to asset profitability has attractive theoretical properties but is difficult in practice because the regulator is unlikely to have good information on expected returns. We therefore analyze an *adaptive* implementation in which the regulator sets weights based on observing a bank’s portfolio. Changing the weights changes the bank’s choice of portfolio which leads to a further change in weights. The result is an iterative process. We show that the process converges to an equilibrium in which the risk weights and the bank’s portfolio coincide with the values they would have if the regulator knew the profitability of each asset. The details of this adaptive process are specific to our model, but we view the main insight from this analysis as more broadly applicable: to compensate for imperfect knowledge about bank assets, the regulator should increase the risk weight for an asset category as banks increase their positions in that category. These results are in Section 4.

Banks face multiple capital constraints, including an overall leverage ratio and, more recently, constraints based on stress tests. In Section 5, we extend our analysis to consider multiple con-

straints, and we connect the extension with a robustness interpretation.

A significant concern with risk-based capital requirements is that they are *procyclical*. In an economic downturn, defaults become more likely, so loans become riskier, forcing banks to hold more capital and thus reduce lending, aggravating the economic downturn. In Section 6, we show that our basic approach to the design of risk weights can be modified to mitigate procyclicality. In the simplest case, these “macroprudential” risk weights are still proportional to asset profitability, but the constant of proportionality changes to mitigate procyclicality.

Finally, in Section 7 we consider a system-wide objective for the regulator in a model with multiple heterogeneous banks. Banks differ in the set of assets to which they have access. We show that a single set of risk weights can ensure that all banks meet a regulatory risk limit. Using a common set of risk weights implicitly imposes a capital surcharge on banks that participate in a wider range of activities, which is consistent with heightened capital requirements for global banks. This effect can be offset through a simple multiplier based on portfolio concentration.

## 2 Background on Risk Weights

Nearly all of the various capital adequacy measures defined in the international standards set by the Basel Committee on Banking Supervision (BCBS [6, 7, 9]) are based on a bank’s risk-weighted assets. These measures include, in particular, ratios for Tier 1 and Tier 2 capital, total capital, common equity Tier 1 capital, and core Tier 1 capital. These ratios differ in the scope of capital they include in the numerator, but they all take risk-weighted assets as the denominator.

A capital standard based on risk-weighted assets was introduced in the 1988 Basel Accord [6], now generally referred to as Basel I. The accord sought to harmonize capital requirements internationally and set a minimum capital standard of 8 percent of risk-weighted assets. It put forward three reasons for risk-weighting that remain relevant today: ensuring comparability across banking systems with different structures, incorporating off-balance-sheet exposures, and not deterring banks from holding assets that carry low risk. The accord did not, however, lay out any principles by which the risk weights would be set.

The Basel I rules allow just five weights: 0 percent, 10 percent, 20 percent, 50 percent, and 100 percent, in increasing order of “riskiness.” For example, sovereign exposures have a 0 percent weight for OECD countries and 100 percent for non-OECD countries; short-term loans to other banks carry a 20 percent risk weight; first-lien residential mortgages for owner-occupied housing carry a 50 percent risk weight, and all corporate loans carry a 100 percent risk weight. The weight for off-balance-sheet exposures is determined by the type of counterparty — sovereign, bank, or corporate. The 1988 accord includes general discussion of risk categories, but, again, it does not

provide principles to support the relative magnitudes of the weights.

In response to the growing importance of trading activities in large banks, the 1996 amendment to Basel I expanded capital requirements to include capital charges for market risk. The amendment's standardized approach assigned risk weights to various categories of assets, covering specific risk associated with, for example, a particular issuer, and general risks from interest rates, exchange rates, and similar broad market factors. As one would expect, the risk weights introduced suggest an effort to align the weights with perceived risk — the risk weights for debt securities increase with maturity, for example. However, the amendment does not provide underlying principles that would imply a particular relative weighting across risk categories like interest rates, equities, currencies, and commodities.

For banks with advanced internal risk management procedures, the amendment offered an alternative approach, and this alternative would appear to be the first attempt to make a rigorous connection between risk and regulatory capital in banks. Under the internal models approach, a bank estimates a value-at-risk (VaR) for its trading activities, which is simply the first percentile of the profit and loss distribution over a ten-day horizon. The VaR is scaled and then added to the risk-weighted assets of the banking book (calculated from the original Basel I weights); the overall capital requirement is then 8 percent of the total. The net effect of this procedure is to assign a risk weight to the trading book that is proportional to its VaR. If we go a step further and assume that VaR is roughly proportional to portfolio standard deviation,<sup>1</sup> then the risk weight for the trading book is proportional to its standard deviation.

The internal models approach marks a departure from earlier schemes in that the regulator does not explicitly assign a risk weight for each asset. Instead, the regulator specifies the rules for calculating the risk weight for a set of assets and leaves it to the bank to carry out the calculation.

This perspective is also important for Basel II [7], which revisited the calculation of risk-weighted assets in the banking book. The standardized approach under Basel II is, for the most part, based on tables of fixed risk weights for various types of exposure, though with greater differentiation of risk categories than Basel I. But the internal ratings based approach instead defines procedures by which banks themselves are to calculate risk weights for various types of lending activities. The procedures are complicated but, at their core, they take the risk weight for a category of loans to be proportional to a VaR figure for the portfolio of those loans. A significant current concern is the wide disparity in the application of the internal ratings based approach across large banks; see, for example, Le Leslé and Avramova [25], European Banking Authority [14], and many accounts in the financial press.

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<sup>1</sup>This does not require a normal distribution; rather, it assumes that the shape of the distribution remains stable so that the first percentile remains at a fixed number of standard deviations from zero.

The Basel 2.5 rules (BCBS [8]) add capital charges for certain types of trading activities; we may view these provisions as changing the calculation of risk weights for the trading book. Basel III imposes stricter capital ratios primarily by narrowing the definition of capital in the numerator while leaving the denominator — risk-weighted assets — largely unchanged. Thus, although the risk weights themselves have become much more complex since the introduction of Basel I, the basic framework — setting minimum capital requirements as a fraction of risk-weighted assets, with risk weights assigned to asset categories or subportfolios — remains the same.

### 3 Optimal Risk Weights

#### 3.1 Bank and Regulator Objectives

We begin by considering a bank’s portfolio selection problem in the absence of regulatory constraints. The bank can undertake various types of lending — mortgages, commercial and industrial loans, and credit cards, for example — and we think of a loan category as an asset. Although our model is generic, we primarily have in mind the bank’s loan portfolio — its banking book — rather than its trading book. We do not address the practical problem of determining the appropriate level of granularity at which risk weights should be imposed but return to this point after Proposition 1. The bank’s portfolio problem is to decide how much of each asset to acquire. At this point, we assume the bank is able to fund whatever portfolio it selects. The returns to the various asset categories are summarized through a vector  $\boldsymbol{\mu} \geq 0$  of expected excess returns above the bank’s funding cost and a positive definite covariance matrix  $\boldsymbol{\Sigma}$ . A two-moment summary would not be a very informative description of the return on a single loan, but it becomes useful when we take an asset to be a portfolio of loans of a single category — a collection of hundreds of mortgages, say, rather than a single mortgage. In the absence of regulatory constraints, the bank would choose a portfolio vector  $\mathbf{x}$  by solving

$$(BU) \quad \max_{\mathbf{x}} \boldsymbol{\mu}^\top \mathbf{x} - \frac{\gamma}{2} \mathbf{x}^\top \boldsymbol{\Sigma} \mathbf{x} \tag{1}$$

for some  $\gamma > 0$ . The optimal solution is  $\mathbf{x}^\circ = \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} / \gamma$ .

The vector  $\mathbf{x}$  records the dollars invested in each asset, not the proportion invested. We do not model the funding side of the bank’s problem but instead assume that the bank has a fixed amount of equity which it can leverage through deposits and other types of debt. The parameter  $\gamma$  reflects an assumption that the unregulated bank has a concern for risk, and this parameter ensures that (1) is maximized at a finite level of leverage. For tractability, we assume that the expected excess returns  $\boldsymbol{\mu}$  do not depend on the portfolio choice  $\mathbf{x}$ .

The formulation in (1) is a stylized model of the bank’s portfolio selection problem. In the appendix, we extend the main result of this section to a more general class of objectives in which

the portfolio variance is replaced by a more general measure of risk. One may question the inclusion of any concern for risk in the objective function: because of the limited liability of equity, the bank's shareholders face a limited downside and an unlimited upside, creating an incentive to increase risk. In practice, this incentive is mitigated by the obstacles to operating a firm perceived to be financially unsound. A profit-maximizing bank would incorporate risk into its portfolio selection if, for example, risk affects its funding costs and access to certain markets; the parameter  $\gamma$  incorporates these considerations in reduced form. Thus, we view the bank as sensitive to risk, but perhaps not as sensitive as the regulator deems necessary.

The regulator's objective is to minimize the risk in the bank's portfolio  $\mathbf{y}$  while allowing the bank an adequate rate of return:

$$\begin{aligned} \min_{\mathbf{y}} \quad & \mathbf{y}^\top \Sigma \mathbf{y} \\ \text{s.t.} \quad & \boldsymbol{\mu}^\top \mathbf{y} \geq \ell, \end{aligned}$$

for some  $\ell \geq 0$ . It will be convenient to work with a dual formulation

$$\begin{aligned} \text{(RR)} \quad \max_{\mathbf{y}} \quad & \boldsymbol{\mu}^\top \mathbf{y} \\ \text{s.t.} \quad & \sqrt{\mathbf{y}^\top \Sigma \mathbf{y}} \leq \eta; \end{aligned}$$

the problems are made equivalent by setting  $\ell = \eta \sqrt{\boldsymbol{\mu}^\top \Sigma^{-1} \boldsymbol{\mu}}$ . In (RR), the regulator's primary concern is with satisfying the risk constraint. The constraint is an upper bound on the portfolio standard deviation and could also be viewed as approximating a value-at-risk constraint.<sup>2</sup> Among all combinations of lending activities that satisfy the risk constraint, the regulator would choose the one that maximizes the market return  $\boldsymbol{\mu}^\top \mathbf{y}$ . In doing so, the regulator implicitly leaves it to the market to determine the relative value of the various categories of lending in which the bank might participate. In Section 3.3, we consider variants in which the regulator targets other combinations.

To try to get the bank to operate in accordance with (RR), the regulator announces a vector of risk weights  $\mathbf{w}$ . Given a portfolio vector  $\mathbf{x}$ , the bank's risk-weighted assets total  $\mathbf{w}^\top \mathbf{x}$ . The regulator limits the bank's risk-weighted assets to some level  $\kappa > 0$ , which we think of as a multiple of the bank's equity. (If a bank needs at least 8 percent of its risk-weighted assets in equity, then  $\kappa$  is 12.5 times the bank's equity.) The bank now faces the following constrained portfolio selection problem:

$$\begin{aligned} \text{(BC)} \quad \max_{\mathbf{x}} \quad & \boldsymbol{\mu}^\top \mathbf{x} - \frac{\gamma}{2} \mathbf{x}^\top \Sigma \mathbf{x} \\ \text{s.t.} \quad & \mathbf{w}^\top \mathbf{x} \leq \kappa. \end{aligned}$$

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<sup>2</sup>In practice, a risk limit would more commonly be formulated as a percentage of the bank's capital rather than as a dollar amount. We could replace  $\eta$  with the product of some capital-independent constant  $\eta'$  and the bank's capital. Because we are taking the bank's capital as fixed, the two formulations are equivalent.

### 3.2 Optimal Weights

How should the regulator choose  $\mathbf{w}$ ? We offer two perspectives on this question. First, the regulator would like to select the risk weights  $\mathbf{w}$  to induce the bank to choose the portfolio  $\mathbf{y}^*$  that optimizes (RR). Second, the regulator would like to enforce the risk constraint in (RR) without distorting the relative mix of assets the bank would select in optimizing the unconstrained objective (1). In other words, the regulator would like to scale down the bank's overall risk without pushing the bank to shift investment from one type activity to another. Using risk weights

$$\mathbf{w}^* = \alpha \boldsymbol{\mu} \equiv \frac{\kappa}{\eta \sqrt{\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}}} \boldsymbol{\mu}, \quad (2)$$

achieves both objectives:

**Proposition 1** *Suppose the regulator chooses the risk weights in (2). Then the optimal solution  $\mathbf{x}^*$  to (BC) satisfies the risk constraint in (RR). Moreover,  $\mathbf{x}^*$  is a scalar multiple of  $\mathbf{x}^\circ$ , the optimal solution to (1), so the two portfolios have the same relative mix of assets. If the risk-weight constraint in (BC) is binding, then  $\mathbf{x}^*$  coincides with  $\mathbf{y}^*$ , the optimal solution to (RR).*

*Proof.* The optimal solution to (RR) is given by

$$\mathbf{y}^* = \frac{\eta}{\sqrt{\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}}} \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}. \quad (3)$$

For any risk-weight vector  $\mathbf{w}$ , the optimal solution to (BC) is given by

$$\begin{aligned} \mathbf{x}^* &= \frac{1}{\gamma} \boldsymbol{\Sigma}^{-1} \left( \boldsymbol{\mu} - \frac{(\mathbf{w}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} - \gamma \kappa)^+}{\mathbf{w}^\top \boldsymbol{\Sigma}^{-1} \mathbf{w}} \mathbf{w} \right) \\ &= \frac{1}{\gamma} \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} - \frac{(\mathbf{w}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} - \gamma \kappa)^+}{\mathbf{w}^\top \boldsymbol{\Sigma}^{-1} \mathbf{w}} \frac{1}{\gamma} \boldsymbol{\Sigma}^{-1} \mathbf{w}. \end{aligned} \quad (4)$$

If we set  $\mathbf{w} = \alpha \boldsymbol{\mu}$  and if the corresponding linear risk-weight constraint in (BC) is binding, then

$$\begin{aligned} \mathbf{x}^* &= \frac{1}{\gamma} \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} - \frac{\alpha \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} - \gamma \kappa}{\alpha \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}} \frac{1}{\gamma} \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \\ &= \frac{1}{\gamma} \left( 1 - \frac{\alpha \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} - \gamma \kappa}{\alpha \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}} \right) \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \\ &= \frac{1}{\gamma} \left( \frac{\gamma \kappa}{\alpha \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}} \right) \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \\ &= \frac{\kappa}{\alpha \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}} \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}. \end{aligned} \quad (5)$$

In particular, taking

$$\alpha = \frac{\kappa}{\eta \sqrt{\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}}} \quad (6)$$

gives  $\mathbf{x}^* = \mathbf{y}^*$ , and then  $\mathbf{x}^*$  satisfies the risk constraint in (RR). Moreover, it is immediate that  $\mathbf{y}^*$  is a scalar multiple of  $\mathbf{x}^\circ$ .

If the linear risk-weight constraint in (BC) is not binding, then  $\mathbf{x}^* = \mathbf{x}^\circ$ . By substituting the definitions of  $\mathbf{x}^\circ$  and  $\mathbf{w}^*$ , we find that  $\mathbf{w}^{*\top} \mathbf{x}^\circ < \kappa$  implies  $\sqrt{\boldsymbol{\mu} \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}} < \eta \gamma$ . But then  $\sqrt{\mathbf{x}^{\circ\top} \boldsymbol{\Sigma} \mathbf{x}^\circ} = \sqrt{\boldsymbol{\mu} \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}} / \gamma < \eta$ , so the risk constraint is satisfied.  $\square$

We make some observations on this result:

- Remarkably, this result shows that the ideal risk weights (2) have almost nothing to do with risk — at least not explicitly — and are instead proportional to  $\boldsymbol{\mu}$ . A similar conclusion is reached by a different argument in Kim and Santomero [23], but their result does not appear to be well known.<sup>3</sup> In fact, risk considerations enter through the bank’s objective in (1) in (BC), and it is essential to our result that the bank have *some* concern for risk so that  $\gamma > 0$ . Whatever constraint the regulator imposes, the bank will choose an optimal risk-return tradeoff. The tool available to the regulator is a linear constraint. By choosing risk weights proportional to  $\boldsymbol{\mu}$ , the regulator limits the bank’s portfolio return and thus induces the bank to limit its risk.
- To gain some insight into why this approach leaves the bank’s relative mix of assets unchanged, observe that imposing a risk weight on an asset is similar to reducing its expected return, and choosing  $\mathbf{w}$  proportional to  $\boldsymbol{\mu}$  effectively reduces  $\boldsymbol{\mu}$  by the same factor across all assets. In the unconstrained optimum  $\mathbf{x}^\circ = \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} / \gamma$ , scaling down  $\boldsymbol{\mu}$  is equivalent to increasing  $\gamma$ . Thus, imposing the risk weights  $\mathbf{w}^*$  has the same effect as increasing the bank’s risk aversion.
- A more conventional formulation would take  $\mathbf{w}$  proportional to  $\boldsymbol{\sigma}$ , the vector of standard deviations. Indeed, under Basel II, banks using internal ratings to calculate a value-at-risk (VaR) for a loan portfolio, use a multiplier to convert the VaR to a level of risk-weighted assets, and then take a percentage of risk-weighted assets to get required capital. This is equivalent to using a risk weight proportional to the VaR per dollar invested. From (4) we see that taking  $\mathbf{w}$  proportional to  $\boldsymbol{\sigma}$  in (BC) leads the bank to choose a portfolio  $\mathbf{x} = \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - k \boldsymbol{\sigma}) / \gamma$ , for some scalar  $k > 0$ . This is the solution to a mean-variance optimization problem with the expected returns  $\boldsymbol{\mu}$  replaced by “stressed” values  $\boldsymbol{\mu} - k \boldsymbol{\sigma}$ . Viewed from our framework, choosing risk weights proportional to  $\boldsymbol{\sigma}$  seems hard to justify. This approach is often taken to be conservative because it ignores correlations between loan portfolios (though correlations within each portfolio are captured in each VaR calculation); however, that perspective over-

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<sup>3</sup>Most references to Kim and Santomero [23] refer to the negative result in Section I.C of their paper, showing that a leverage constraint can increase the probability of insolvency. We refer to Section II.B of their paper.

looks the fact that risk weights that are not proportional to  $\boldsymbol{\mu}$  change the relative mix of assets.

- In the case of a single risky asset, the expected return cancels from (2) leaving a risk weight proportional to the asset’s standard deviation. Taking  $\mathbf{w}$  proportional to  $\boldsymbol{\sigma}$  may thus be viewed as an incorrect generalization from the case of a single asset to the case of multiple assets.
- Importantly, the optimal weight vector  $\mathbf{w}^*$  in (2) does not depend on  $\gamma$ . The same weights work for banks with different risk appetites. Also,  $\mathbf{w}^*$  depends on the relative magnitudes of the expected returns  $\boldsymbol{\mu}$ , but not their absolute levels:  $\mathbf{w}^*$  is invariant to the norm of  $\boldsymbol{\mu}$ .
- Through a similar argument, Rochet [31] proposes setting an asset’s risk weight proportional to its systematic risk. Under the capital asset pricing model (CAPM), the expected excess return on each asset is proportional to the asset’s beta with respect to the market portfolio. Choosing risk weights proportional to  $\boldsymbol{\mu}$  is then equivalent to choosing risk weights proportional to asset betas.
- A referee has observed that risk weights proportional to mean excess returns also result from a Gaussian counterpart of the asymptotic single risk factor model used in Basel II. This analogy connects the weight vector  $\mathbf{w}^*$  with risk weights as calculated in Basel II, under special conditions on  $\boldsymbol{\Sigma}$ . The details of this case are discussed in the appendix.

We acknowledge that there are practical obstacles to implementing the approach suggested by Proposition 1, including a precise interpretation of the set of assets on which our formulation relies. We think of the different assets as corresponding to different lending categories (or “risk buckets”), but pinning down the risk and return for an asset requires a more granular specification — for example, mortgage loans to borrowers of a given credit quality with similar loan-to-value ratios, and similar contractual features. Basel II leaves it to banks to calculate their risk-weighted assets in each risk bucket, recognizing that different banks may make very different kinds of loans within the same loan category. Applying a single risk weight to a category regardless of differences in loan characteristics could create an incentive for a bank to make riskier loans.

To illustrate Proposition 1, we consider a simple two-asset example. We set  $\boldsymbol{\mu} = (1, 1)^\top$  and

$$\boldsymbol{\Sigma} = \begin{pmatrix} 1 & 2\rho \\ 2\rho & 4 \end{pmatrix}$$

and vary the parameter  $\rho$ . We set  $\gamma = 1$  and  $\kappa = \eta = 1$ . Figure 1 shows the optimal portfolios for  $\rho = -0.45, -0.4, \dots, 0.4, 0.45$  in three cases. The open circles show the optimal solutions to

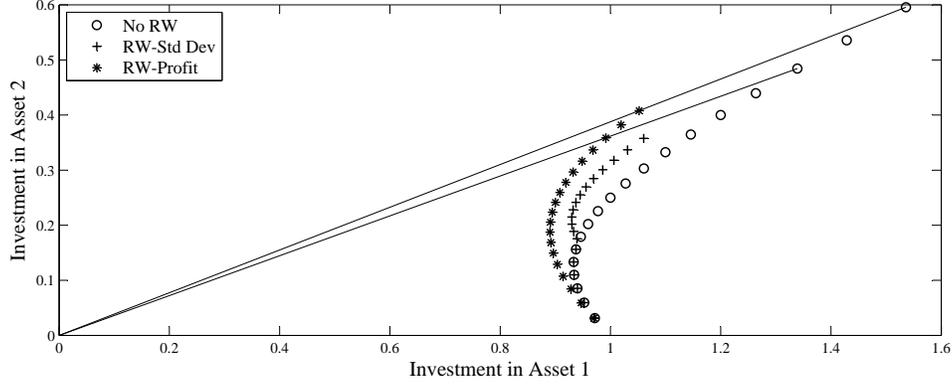


Figure 1: Optimal solutions to (BU) (circles), (BC) with  $\mathbf{w}^*$  (asterisks), and (BC) with risk weights proportional to standard deviations (crosses). As illustrated by the two lines through the origin, choosing risk weights  $\mathbf{w}^*$  scales down the unconstrained portfolios without changing the relative mix of assets.

(BU), the portfolios the bank would choose without a risk-weight constraint; the asterisks show the optimal solutions to (BC) with risk weights  $\mathbf{w}^*$  as in (2); and the crosses show solutions to (BC) with the risk weight for each asset proportional to its standard deviation and the proportionality constant chosen to bring these portfolios close to the others to facilitate comparison. The risk weights  $\mathbf{w}^*$  have the effect of scaling down the unconstrained portfolios — a line from each circle to the corresponding asterisk passes through the origin. In contrast, risk weights proportional to standard deviations will in general change the relative mix of assets. The solutions to (BC) coincide with solutions to (BU) when the risk-weight constraint is nonbinding.

Proposition 1 assumes that the bank optimizes the risk-adjusted objective in (BC) in selecting its portfolio. But the risk weights  $\mathbf{w}^*$  remain effective and cannot be gamed, regardless of how the bank chooses its portfolio, provided we restrict attention to portfolios taking only long positions:

**Proposition 2** *Suppose the unconstrained optimum satisfies  $\mathbf{x}^\circ \geq 0$ . Then any nonnegative portfolio vector  $\mathbf{x}$  satisfying the linear constraint  $\mathbf{x}^\top \mathbf{w}^* \leq \kappa$  satisfies the risk constraint  $\sqrt{\mathbf{x}^\top \Sigma \mathbf{x}} \leq \eta$ .*

*Proof.* We can bound the portfolio risk by maximizing  $\mathbf{x}^\top \Sigma \mathbf{x}$  subject to  $\mathbf{x}^\top \mathbf{w}^* \leq \kappa$  and  $\mathbf{x} \geq 0$ . Applying the KKT conditions, the maximum is attained at a vector  $\mathbf{x} = \lambda_0 \Sigma^{-1} \mathbf{w}^* - \Sigma^{-1} \boldsymbol{\lambda}$ , for some vector  $\boldsymbol{\lambda} \geq 0$  satisfying  $\boldsymbol{\lambda}^\top \mathbf{x} = 0$ , and a scalar  $\lambda_0 \geq 0$ , with  $\lambda_0 = 0$  if  $\mathbf{x}^\top \mathbf{w}^* < \kappa$ . These conditions are satisfied by taking  $\mathbf{x} = \lambda_0 \Sigma^{-1} \boldsymbol{\mu}$ ,  $\lambda_0 = \eta^2 / \kappa$ , and  $\boldsymbol{\lambda} = 0$ : nonnegativity holds because  $\mathbf{x} = \lambda_0 \gamma \mathbf{x}^\circ \geq 0$ , and the risk-weight constraint holds because  $\mathbf{x}^\top \mathbf{w}^* = \lambda_0 \mathbf{w}^{*\top} \Sigma^{-1} \mathbf{w}^* = \lambda_0 \kappa^2 / \eta^2 = \kappa$ . At this  $\mathbf{x}$ , we have  $\mathbf{x}^\top \Sigma \mathbf{x} = \lambda_0 \mathbf{x}^\top \mathbf{w}^* = \lambda_0 \kappa = \eta^2$ .  $\square$

Taken together, Propositions 1 and 2 show that with weights  $\mathbf{w}^*$ , a bank that chooses its asset mix prudently will not have that mix changed, and regardless of how the bank chooses its portfolio the regulator's risk constraint in (RR) will be enforced.

The only important qualification to this statement is that we require  $\mathbf{x}^\circ \geq 0$  and  $\mathbf{x} \geq 0$  in Proposition 2. Bank portfolios are, for the most part, long-only. Banks make loans of various types and cannot directly take short positions in these assets. Short positions are possible in a bank's trading book, but for most banks this is a small fraction of total assets, and within our stylized framework we can think of the entire trading portfolio as a single asset. Proposition 1 carries over to the case of long-only portfolios. Let  $\boldsymbol{\mu}_1$  and  $\boldsymbol{\Sigma}_{11}$  be the mean vector and covariance matrix for the set of assets the bank chooses to hold when it solves (BU) with a nonnegativity constraint, and set  $\alpha_1 = \kappa/\eta\sqrt{\boldsymbol{\mu}_1^\top \boldsymbol{\Sigma}_{11}^{-1} \boldsymbol{\mu}_1}$ .

**Proposition 3** *Proposition 1 continues to hold if we add nonnegativity constraints  $\mathbf{x} \geq 0$  and  $\mathbf{y} \geq 0$  in problems (BU), (RR), and (BC) and set  $\mathbf{w}^* = \alpha_1 \boldsymbol{\mu}$ .*

Once we require  $\mathbf{x} \geq 0$ , it is possible that the set of assets in which the bank takes strictly positive positions will differ in (BU) and (BC) because of the risk-weight constraint. A key implication of Proposition 3 is that this does not happen with the weights  $\mathbf{w}^*$ , and, moreover, the same assets are active in the solution to (RR). Based on this result, we can often ignore the nonnegativity constraint and instead assume that the set of assets available to the bank is restricted to the set of assets in which the bank would choose to invest if it solved (BU) under the constraint  $\mathbf{x} \geq 0$ . We defer the proof of Proposition 3 to the appendix.

### 3.3 Weights for Other Targets

Proposition 1 shows how the regulator can select  $\mathbf{w}$  to drive the bank to choose its portfolio  $\mathbf{x}^*$  to match the solution to the regulator's objective (RR). As already noted, this choice reduces the bank's overall risk without changing its relative asset mix. If, however, the regulator does want to change the asset mix, it can do so through a suitable choice of weights. More explicitly, we suppose the bank will choose its portfolio by solving (BC). The regulator would like the bank to choose a particular portfolio  $\mathbf{y}$  and seeks a weight vector  $\mathbf{w}$  that delivers this outcome. As before,  $\mathbf{x}^\circ$  denotes the optimal solution to the unconstrained problem (1).

**Proposition 4** *Suppose the regulator's target portfolio  $\mathbf{y}$  satisfies  $\mathbf{y}^\top(\boldsymbol{\mu} - \gamma\boldsymbol{\Sigma}\mathbf{y}) > 0$ . Setting*

$$\mathbf{w} = \frac{\kappa}{\mathbf{y}^\top \boldsymbol{\Sigma}(\mathbf{x}^\circ - \mathbf{y})} \boldsymbol{\Sigma}(\mathbf{x}^\circ - \mathbf{y}) \quad (7)$$

*makes  $\mathbf{y}$  the optimal solution to the bank's problem (BC).*

The form of the weight vector (7) is easiest to interpret when the assets are uncorrelated and have a common variance, so that  $\Sigma$  is a diagonal matrix with all entries on the diagonal equal. In this case, we see that the weight for each asset is proportional to the difference between the bank's unconstrained investment in that asset and the regulator's target level of investment. Thus, the weights lean against the bank's preference to deviate from the target. The greater the deviation  $\mathbf{x}_i^\circ - \mathbf{y}_i$  in a particular asset  $i$ , the greater the weight attached to that asset.

To interpret the condition in the proposition, write  $u(t) = t\boldsymbol{\mu}^\top \mathbf{y} - \gamma t^2 \mathbf{y}^\top \Sigma \mathbf{y} / 2$  for the bank's objective function along the ray  $t\mathbf{y}$ ,  $t \geq 0$ . The condition requires  $u'(1) > 0$ , meaning that the objective function is increasing along this ray at the point  $\mathbf{y}$ . If we had  $u'(1) < 0$ , the bank would prefer to shrink  $\mathbf{y}$  toward the origin and could do so while satisfying the risk-weight constraint defined by (7); thus, risk weights would not compel the bank to choose  $\mathbf{y}$ . With  $u'(1) = 0$ , the denominator of (7) would become zero.

*Proof.* The bank's optimal choice is given by (4), so the regulator's problem is to choose  $\mathbf{w}$  so that

$$\frac{1}{\gamma} \Sigma^{-1} \boldsymbol{\mu} - \rho(\mathbf{w}) \frac{1}{\gamma} \Sigma^{-1} \mathbf{w} = \mathbf{y} \quad (8)$$

with

$$\mathbf{w}^\top \Sigma^{-1} \boldsymbol{\mu} - \gamma \kappa > 0 \quad (9)$$

and hence

$$\rho(\mathbf{w}) = \frac{\mathbf{w}^\top \Sigma^{-1} \boldsymbol{\mu} - \gamma \kappa}{\mathbf{w}^\top \Sigma^{-1} \mathbf{w}} > 0.$$

The condition in the proposition implies that  $\mathbf{y} \neq \mathbf{x}^\circ$ , so with  $\mathbf{w}$  as in (7),

$$\mathbf{w}^\top \mathbf{x}^\circ = \kappa \frac{\mathbf{x}^{\circ\top} \Sigma (\mathbf{x}^\circ - \mathbf{y})}{\mathbf{y}^\top \Sigma (\mathbf{x}^\circ - \mathbf{y})}.$$

This is greater than  $\kappa$  because  $0 < (\mathbf{x}^\circ - \mathbf{y})^\top \Sigma (\mathbf{x}^\circ - \mathbf{y}) = \mathbf{x}^{\circ\top} \Sigma (\mathbf{x}^\circ - \mathbf{y}) - \mathbf{y}^\top \Sigma (\mathbf{x}^\circ - \mathbf{y})$  and  $\mathbf{y}^\top \Sigma (\mathbf{x}^\circ - \mathbf{y}) > 0$  by the condition in the proposition, so (9) holds. Rearranging (8) shows that we need to verify that (7) satisfies

$$\mathbf{w} = \frac{1}{\rho(\mathbf{w})} (\boldsymbol{\mu} - \gamma \Sigma \mathbf{y}) \equiv a (\boldsymbol{\mu} - \gamma \Sigma \mathbf{y}). \quad (10)$$

For any  $\mathbf{w}$  of this form, the definition of  $\rho$  yields

$$\rho(\mathbf{w}) \equiv \frac{a(\boldsymbol{\mu}^\top \Sigma^{-1} \boldsymbol{\mu} - \gamma \boldsymbol{\mu}^\top \mathbf{y}) - \gamma \kappa}{a^2(\boldsymbol{\mu}^\top \Sigma^{-1} \boldsymbol{\mu} - 2\gamma \boldsymbol{\mu}^\top \mathbf{y} + \gamma^2 \mathbf{y}^\top \Sigma \mathbf{y})}.$$

But we also have  $a = 1/\rho(\mathbf{w})$ , and this allows us to solve for  $a$  to get  $a = \kappa/\mathbf{y}^\top (\boldsymbol{\mu} - \gamma \Sigma \mathbf{y})$ . Using this  $a$  in (10) yields (7) because  $\mathbf{x}^\circ = \Sigma^{-1} \boldsymbol{\mu} / \gamma$ .  $\square$

A simple corollary to Proposition 4 allows the regulator to choose effective risk weights even if the bank and the regulator disagree about the risk and return characteristics of the assets. Suppose the regulator’s view is summarized by the pair  $(\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\Sigma}})$  and the bank’s view by  $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ . The regulator’s target, as defined by the solution to (RR) given in (3) is

$$\tilde{\mathbf{y}} = \frac{\eta}{\sqrt{\tilde{\boldsymbol{\mu}}^\top \tilde{\boldsymbol{\Sigma}}^{-1} \tilde{\boldsymbol{\mu}}}} \tilde{\boldsymbol{\Sigma}}^{-1} \tilde{\boldsymbol{\mu}}. \quad (11)$$

Inserting this target in (7) yields a risk-weight vector  $\tilde{\mathbf{w}}$ . If the regulator imposes the risk weights  $\tilde{\mathbf{w}}$ , then a bank optimizing (BC) based on its own view  $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$  will choose the portfolio vector  $\tilde{\mathbf{y}}$  in (11). We revisit this application from a different perspective in Section 6.

## 4 Adaptive Risk Weights

Within the framework of Section 3, we have seen that the regulator’s ideal risk-weight vector  $\mathbf{w}^*$  is proportional to the expected return vector  $\boldsymbol{\mu}$ . In practice,  $\boldsymbol{\mu}$  is difficult to measure. A bank may have better information about the profitability of its investment opportunities, but the regulator has limited ability to extract this information from the bank.

With the substitution  $\boldsymbol{\mu} = \gamma \boldsymbol{\Sigma} \mathbf{x}^\circ$ , we can rewrite (2) as

$$\mathbf{w}^* = \frac{\kappa \boldsymbol{\Sigma}}{\eta \sqrt{\mathbf{x}^\circ{}^\top \boldsymbol{\Sigma} \mathbf{x}^\circ}} \mathbf{x}^\circ, \quad (12)$$

which provides an alternative interpretation: the optimal risk weights are proportional to the bank’s ideal portfolio  $\mathbf{x}^\circ$ , as reshaped by multiplication by  $\boldsymbol{\Sigma}$ . The interpretation is clearest when  $\boldsymbol{\Sigma}$  is diagonal; in that case, the risk weight for an asset is the product of the variance of the asset’s return and the size of the bank’s preferred investment in that asset. The expression in (12) is invariant to the norm of  $\mathbf{x}^\circ$ , so the weights depend only on the relative sizes of the bank’s preferred investments.

We can similarly apply (2)–(6) to write

$$\mathbf{w}^* = \frac{\kappa}{\eta^2} \boldsymbol{\Sigma} \mathbf{x}^*. \quad (13)$$

This is an equilibrium relationship in the sense that the weights  $\mathbf{w}^*$  generated by  $\mathbf{x}^*$  in (13) make  $\mathbf{x}^*$  the bank’s optimal constrained portfolio in (BC). The representations in (12) in (13) show that the regulator need not know  $\boldsymbol{\mu}$  explicitly. Instead, (12) and (13) suggest that the regulator can set the weights by observing the bank’s portfolio. For small changes, (13) yields

$$\Delta \mathbf{w}^* = \frac{\kappa}{\eta^2} \boldsymbol{\Sigma} \Delta \mathbf{x}^*.$$

Of course, once new weights are imposed, the bank’s portfolio changes in response to the new constraint. If the regulator and the bank make iterative adjustments, we get a sequence of weight vectors and portfolios. We analyze this process.

## 4.1 Iteration to Equilibrium

In more detail, the regulator will impose a sequence of weight vectors  $\mathbf{w}_k$ ,  $k = 0, 1, \dots$ , of the form

$$\mathbf{w}_k = \beta_k \mathbf{v}_k \quad \text{with} \quad \beta_k = \frac{\kappa}{\eta \sqrt{\mathbf{v}_k^\top \boldsymbol{\Sigma}^{-1} \mathbf{v}_k}}. \quad (14)$$

The multiplier  $\beta_k$  scales the candidate weight vector to be consistent with the capital constraint  $\kappa$ , whereas the vector  $\mathbf{v}_k$  determines the relative weights assigned to the assets. Given the weight vector  $\mathbf{w}_k$ , the bank solves (BC) to get portfolio vector  $\mathbf{x}_k^*$ ; the regulator observes the bank's portfolio and responds by adjusting the weight vector to  $\mathbf{w}_{k+1} = \beta_{k+1} \mathbf{v}_{k+1}$ , with  $\mathbf{v}_{k+1}$  updated according to the rule

$$\mathbf{v}_{k+1} = \begin{cases} \boldsymbol{\Sigma} \mathbf{x}_k^* + \frac{\mathbf{w}_k^\top \boldsymbol{\Sigma}^{-1} \mathbf{v}_{k-\kappa}}{\mathbf{w}_k^\top \boldsymbol{\Sigma}^{-1} \mathbf{w}_k} \mathbf{w}_k, & \mathbf{w}_k^\top \mathbf{x}_k^* \geq \kappa; \\ \boldsymbol{\Sigma} \mathbf{x}_k^*, & \mathbf{w}_k^\top \mathbf{x}_k^* < \kappa. \end{cases} \quad (15)$$

The updating rule in (15) distinguishes two cases depending on whether the capital constraint was binding under the previous set of risk weights.

**Proposition 5** *For any  $\mathbf{v}_0$ , the sequence of risk-weight vectors  $\mathbf{w}_k$ ,  $k = 0, 1, 2, \dots$ , converges to the optimal value  $\mathbf{w}^*$  in (2), and the portfolios  $\mathbf{x}_k^*$ ,  $k = 0, 1, 2, \dots$  converge to  $\mathbf{x}^*$ . The limit  $(\mathbf{w}^*, \mathbf{x}^*)$  defines an equilibrium between the regulator and the bank in the sense that it is a fixed point of the updating process.*

Thus, the regulator can arrive at the optimal risk weights iteratively by responding to changes in the bank's portfolio, *without knowing  $\boldsymbol{\mu}$* . At each update through (15), the regulator looks at the difference between the bank's current portfolio multiplied by  $\boldsymbol{\Sigma}$  and a current target to adjust the weights. If  $\boldsymbol{\Sigma}$  is diagonal, then, other things being equal, a higher investment by the bank in an asset will lead to a higher risk weight for that asset.

We have assumed that each time the risk weights change, the bank chooses a new portfolio by solving (BC) and does not choose a suboptimal portfolio to throw off the regulator's updating scheme. Our view is that the bank would be reluctant to choose a bad portfolio in the present in the hope of reaping future gains from more advantageous risk weights if it were uncertain about the timing and procedure for the regulator's updates.

In practice, risk weights cannot be changed very often, accounting information is available at most quarterly, and banks need time to adjust their portfolios. But making updates every one or two years does not seem implausible. Many important practical considerations are outside the scope of our model, including the possibility that  $\boldsymbol{\mu}$  itself changes over time. In the discussion

following Proposition 1, we noted difficulties in determining the appropriate level of granularity at which to map actual lending categories to the assets in our model. This problem becomes more acute in an adaptive setting because individual loans may move from one asset category to another as their credit quality changes, altering a bank’s asset mix. Banks may be unable to offset this effect and rebalance to a target portfolio because they cannot easily sell existing loans.

Setting aside these significant practical obstacles, the key implication of Proposition 5 is that weights should be updated in the direction of the risk-scaled portfolio  $\Sigma \mathbf{x}_k^*$ . In particular, a large buildup in any asset category should be met with an increase in the risk weight for that category.<sup>4</sup> Even if growth in a lending category is justified by a technological or economic change, a bank in our framework would tend to take on too much risk in adjusting its portfolio, relative to the regulator’s objective. The adaptive scheme limits the bank’s risk to the regulatory constraint without, in the limit, changing the relative mix of assets that the bank would choose, unconstrained, in response to the technological or economic change.

## 4.2 A Perspective Through Recent History

To shed some light on how adaptive risk weights might operate in practice, we examine historical data drawn from the Report of Condition and Income (“Call Report”) filed quarterly by all banks in the United States. We use the data to examine changes in bank portfolios. We use the average over all U.S. commercial banks as reported in the Uniform Bank Performance Reports on the Federal Financial Institutions Examination Council website, which are available from 2002. Our objective is to examine which lending categories grew during this period; these are the categories for which risk weights would likely have been increased under our adaptive method. It is impossible to determine what would have happened in the past under different rules for setting bank capital. There are many reasons why the sizes of different lending categories might change over time, and any retrospective examination of lending levels is subject to selection bias. With these caveats, it is nevertheless informative to look at trends in recent history.

Figure 2 shows the change in size of various loan categories, as a percentage of total assets, relative to the end of 2002. The values are normalized to start at 1 in 2002. The left panel shows, in particular, growth in the commercial real estate and total real estate categories through the end of 2008. The right panel focuses on subcategories of real estate lending and shows that the

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<sup>4</sup>Such an approach has been adopted by the Reserve Bank of India. Sinha [34] includes tables showing changes in risk weights under this policy and states (p.12) “Noticing the steep increase in bank credit to the commercial real estate sector in conjunction with that in the prices of real estate, risk weights for banks’ exposure to commercial real estate were increased from 100 percent to 125 percent in July 2005, and further to 150 percent in May 2006... . When there was a boom in consumer credit and equities, risk weights for consumer credit and capital market exposures were increased from 100 percent to 125 percent.” We thank Viral Acharya for pointing out this precedent.

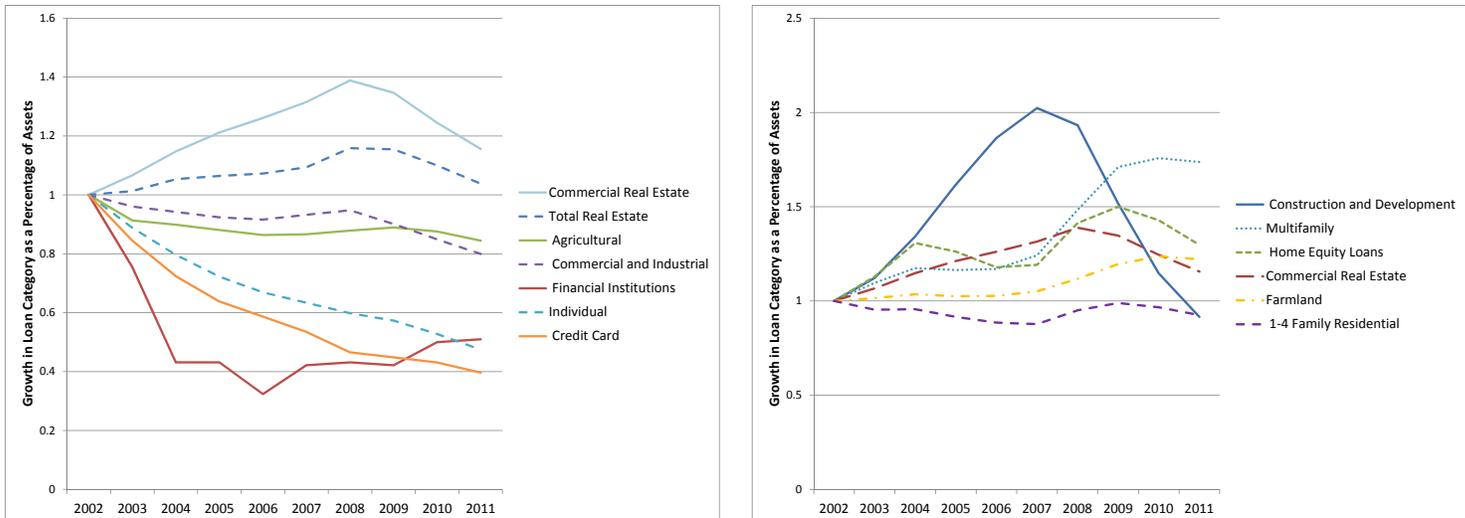


Figure 2: Growth in loan categories as a percentage of bank assets, relative to 2002.

growth in construction and development was particularly rapid leading up to the financial crisis. Indeed, Acharya [1] argues that the risk-weighting framework contributed to the growth in real estate lending by setting the risk weights for mortgages too low. An adaptive scheme would have increased risk weights as the amount of real estate lending grew.

Figure 3 compares bank holdings of sovereign debt issued by Greece, Italy, Portugal, and Spain (GIPS) with debt issued by Germany, based on data from Merler and Pisani-Ferry [26]. The yields of the GIPS debt increased significantly compared to that of Germany starting in 2008, but debt issued by any of these countries carries a risk weight of zero; this combination of factors explains the pattern in the figure. See Acharya and Steffen [3] for a more extensive investigation of the incentives faced by European banks in holding sovereign debt. Under our adaptive scheme, the risk weights on the GIPS debt would increase as bank holdings of this debt increased if the political objections could be overcome.

## 5 Additional Constraints

### 5.1 Leverage Ratios and Stress Tests

Banks in the United States have long been subject to an overall leverage constraint as well as a constraint on risk-weighted assets, and such a constraint is included in Basel III capital requirements. A leverage constraint limits total (unweighted) assets relative to capital; in our framework, this reduces to setting  $\mathbf{w} = (1, 1, \dots, 1)^\top$ .

More recently, stress test results have become a binding constraint on capital for the largest

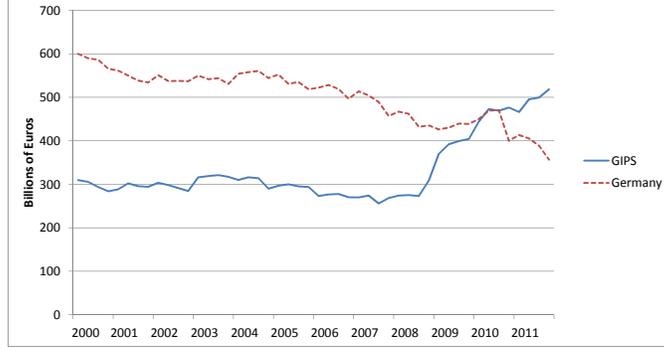


Figure 3: Bank holdings of sovereign debt issued by Greece, Italy, Portugal, and Spain (GIPS) and by Germany.

U.S. bank holding companies through the Federal Reserve’s Comprehensive Capital Analysis and Review [10]. In its simplest form, a stress test scenario can be viewed as a vector of negative returns  $\mathbf{r}$  for the assets held by banks. (More precisely, the stress scenario posits values for broad economic and financial variables, and banks map these variables to losses in each asset category.) A bank with portfolio holdings  $\mathbf{x}$  incurs a total loss of  $-\mathbf{r}^\top \mathbf{x}$  in the stress scenario, and it passes the stress test if the loss is less than an upper bound tied to the bank’s capital. Within our framework, a stress test constraint is therefore a special case of a constraint on risk-weighted assets with risk weights  $\mathbf{w} = -\mathbf{r}$ .

These examples lead us to consider the effect of adding a further constraint in (BC) of the form

$$\tilde{\mathbf{w}}^\top \mathbf{x} \leq \tilde{\kappa}, \quad (16)$$

for some  $\tilde{\mathbf{w}}$  and  $\tilde{\kappa}$ . We think of the constraint in (16) as supplemental to the primary risk-weight constraint in (BC). We therefore first examine if (16) is consistent with the original constraint in the following sense:

**Proposition 6** *Suppose the optimal solution to (BC) with weights  $\mathbf{w}^*$  satisfies  $\mathbf{x}^* \geq 0$ , and suppose  $\Sigma^{-1}\tilde{\mathbf{w}} \geq 0$ . Suppose  $\tilde{\kappa}/\sqrt{\tilde{\mathbf{w}}^\top \Sigma^{-1}\tilde{\mathbf{w}}} = \kappa/\sqrt{\mathbf{w}^{*\top} \Sigma^{-1}\mathbf{w}^*}$ . Then the optimal solution  $\mathbf{x}^*$  to (BC) satisfies (16), and any portfolio vector  $\mathbf{x} \geq 0$  that satisfies (16) satisfies the risk constraint in (RR), meaning that  $\sqrt{\mathbf{x}^\top \Sigma \mathbf{x}} \leq \eta$ .*

*Proof.* If the risk-weight constraint  $\mathbf{w}^{*\top} \mathbf{x}^* = \kappa$  is binding, then  $\mathbf{x}^* = \kappa \Sigma^{-1} \mathbf{w}^* / (\mathbf{w}^{*\top} \Sigma^{-1} \mathbf{w}^*)$  and

$$\tilde{\mathbf{w}}^\top \mathbf{x}^* = \frac{\kappa \tilde{\mathbf{w}}^\top \Sigma^{-1} \mathbf{w}^*}{\mathbf{w}^{*\top} \Sigma^{-1} \mathbf{w}^*} = \frac{\tilde{\kappa} \tilde{\mathbf{w}}^\top \Sigma^{-1} \mathbf{w}^*}{\sqrt{\tilde{\mathbf{w}}^\top \Sigma^{-1} \tilde{\mathbf{w}}} \sqrt{\mathbf{w}^{*\top} \Sigma^{-1} \mathbf{w}^*}} \leq \tilde{\kappa},$$

the last step following from the Cauchy-Schwarz inequality. If the risk-weight constraint is not binding, then  $\mathbf{x}^* = \mathbf{x}^\circ$  and the first step holds as an inequality.

For the second statement, consider the problem of maximizing  $\mathbf{x}^\top \Sigma \mathbf{x}$  subject to the constraint (16) and  $\mathbf{x} \geq 0$ , which is similar to the problem in Proposition 2. Under the condition  $\Sigma^{-1} \tilde{\mathbf{w}} \geq 0$ , the maximum is attained at  $\tilde{\kappa} \tilde{\mathbf{w}} / (\tilde{\mathbf{w}}^\top \Sigma^{-1} \tilde{\mathbf{w}})$  where it takes the value  $\tilde{\kappa}^2 / (\tilde{\mathbf{w}}^\top \Sigma^{-1} \tilde{\mathbf{w}})$ . By assumption, this is the same as  $\kappa^2 / (\mathbf{w}^{*\top} \Sigma^{-1} \mathbf{w}^*)$ , which equals  $\eta^2$ .  $\square$

This result provides a way to set the upper bound  $\tilde{\kappa}$  in a way that is consistent with the original bound  $\kappa$  on risk-weighted assets; that is,

$$\frac{\tilde{\kappa}}{\kappa} = \sqrt{\frac{\tilde{\mathbf{w}}^\top \Sigma^{-1} \tilde{\mathbf{w}}}{\mathbf{w}^{*\top} \Sigma^{-1} \mathbf{w}^*}}. \quad (17)$$

For example, with a capital requirement of 8 percent of risk-weighted assets and 3 percent of total assets, the ratio on the left is  $8/3 \approx 1.63$ .

## 5.2 Adding Robustness

The framework of Section 3 leaves little room for additional constraints because a single set of risk weights suffices to achieve the regulator’s objective. In practice, additional constraints provide a “belts and suspenders” to capital regulation<sup>5</sup> and serve as a backstop against measurement errors and gaming by banks. An overall leverage ratio guards against misspecification of risk weights, and a stress test focuses on extreme scenarios where modeling assumptions breakdown. In short, the objective of introducing additional constraints is *robustness*.

If we start from this objective, we can draw on the literature on robust portfolio selection to see how robustness can be incorporated into the design of risk weights directly. We focus on uncertainty in the covariance matrix  $\Sigma$ , our principal measure of risk. The regulator’s best estimate of the covariance matrix determines the risk constraint in (RR); the regulator’s uncertainty is described by a set  $\mathcal{S}$  of alternative covariance matrices, and problem (RR) is made robust through an additional constraint of the form

$$\sup_{\tilde{\Sigma} \in \mathcal{S}} \sqrt{\mathbf{y}^\top \tilde{\Sigma} \mathbf{y}} \leq \tilde{\eta}, \quad (18)$$

for some  $\tilde{\eta} > 0$ . We may have  $\tilde{\eta} > \eta$  because (18) serves as a backup to the original constraint in (RR).

The set  $\mathcal{S}$  could be specified in many ways; see, for example, the alternatives studied in Goldfarb and Iyengar [16]. If we assume a normal distribution for asset returns, then describing uncertainty through relative entropy, as in Hansen and Sargent [19] is particularly convenient. In this setting,

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<sup>5</sup>As described by Stefan Ingves, chairman of the Basel Committee on Banking Supervision, in “Basel III is Simpler and Stronger,” *Wall Street Journal*, October 12, 2012.

Glasserman and Xu [15] show that given a portfolio vector  $\mathbf{y}$ , the worst-case covariance matrix — which achieves the supremum in (18) for the corresponding set  $\mathcal{S}$  — is given by

$$\tilde{\Sigma} = (\Sigma^{-1} - \theta \mathbf{y} \mathbf{y}^\top)^{-1},$$

where  $\theta \geq 0$  controls the degree of robustness or uncertainty. In other words, the worst-case mis-measurement of risk occurs at a covariance matrix whose inverse is shrunk in a direction determined by  $\mathbf{y}$ . From the regulator's perspective, the worst-case occurs at the target portfolio  $\mathbf{y}^*$  defined by (RR). We show below that the worst-case covariance is then of the form

$$\tilde{\Sigma}^* \equiv (\Sigma^{-1} - \theta \mathbf{y}^* \mathbf{y}^{*\top})^{-1} = \Sigma + \delta \boldsymbol{\mu} \boldsymbol{\mu}^\top, \quad (19)$$

with  $\delta \geq 0$ , if  $\theta \eta^2 < 1$ . Thus, the worst-case covariance for a target portfolio of  $\mathbf{y}^*$  inflates the baseline covariance estimate  $\Sigma$  along a direction determined by  $\boldsymbol{\mu}$ .

To achieve robustness, the regulator would like to limit risk as measured by the worst-case covariance  $\tilde{\Sigma}^*$  as well as the risk measured by the baseline  $\Sigma$ . We have seen that the regulator can get the bank to meet the risk constraint in (RR) through appropriate choice of risk weights. The next result shows that the regulator can enforce the robust risk constraint as well:

**Proposition 7** *Suppose the regulator selects the risk-weight vector  $\mathbf{w}^*$  in (2), and suppose the resulting capital constraint is binding in (BC). Suppose the optimal solution  $\mathbf{x}^*$  to (BC) is nonnegative. Then any nonnegative feasible solution  $\mathbf{x}$  to the bank's problem (BC) satisfies  $\mathbf{x}^\top \Sigma \mathbf{x} \leq \eta^2$  and*

$$\mathbf{x}^\top \tilde{\Sigma}^* \mathbf{x} \leq \frac{\eta^2}{1 - \theta \eta^2}, \quad (20)$$

provided  $\theta \eta^2 < 1$ .

*Proof.* The first inequality restates part of Proposition 1 and is included for comparison. Using the expression for  $\mathbf{y}^*$  in (3), we can evaluate  $\tilde{\Sigma}^*$  as

$$\begin{aligned} \tilde{\Sigma}^* &= (\Sigma^{-1} - \theta \mathbf{y}^* \mathbf{y}^{*\top})^{-1} \\ &= \Sigma + \frac{\theta}{1 - \theta \mathbf{y}^{*\top} \Sigma \mathbf{y}^*} \Sigma \mathbf{y}^* \mathbf{y}^{*\top} \Sigma \\ &= \Sigma + \frac{\theta}{1 - \theta \eta^2} \frac{\eta^2}{\boldsymbol{\mu}^\top \Sigma^{-1} \boldsymbol{\mu}} \boldsymbol{\mu} \boldsymbol{\mu}^\top \\ &\equiv \Sigma + \delta \boldsymbol{\mu} \boldsymbol{\mu}^\top, \end{aligned}$$

where the expression for the inverse follows from the the Sherman-Morrison formula and can also be verified directly by matrix multiplication. To bound  $\mathbf{x}^\top \tilde{\Sigma}^* \mathbf{x}$ , observe that if  $\mathbf{x}$  is feasible for

(BC) and  $\mathbf{x}^* = \mathbf{y}^*$  is optimal for (BC), then

$$\boldsymbol{\mu}^\top \mathbf{x} - \frac{\gamma}{2} \mathbf{x}^\top \boldsymbol{\Sigma} \mathbf{x} \leq \boldsymbol{\mu}^\top \mathbf{x}^* - \frac{\gamma}{2} \mathbf{x}^{*\top} \boldsymbol{\Sigma} \mathbf{x}^* = \boldsymbol{\mu}^\top \mathbf{x}^* - \frac{\gamma}{2} \eta^2,$$

and therefore

$$\boldsymbol{\mu}^\top \mathbf{x} = \boldsymbol{\mu}^\top \mathbf{x}^* - \frac{\gamma}{2} \left( \eta^2 - \mathbf{x}^\top \boldsymbol{\Sigma} \mathbf{x} \right) \leq \boldsymbol{\mu}^\top \mathbf{x}^*.$$

For any feasible  $\mathbf{x} \geq 0$ , we have  $\boldsymbol{\mu}^\top \mathbf{x} \geq 0$  (under our standing assumption that  $\boldsymbol{\mu} \geq 0$ ) and thus

$$\mathbf{x}^\top \tilde{\boldsymbol{\Sigma}}^* \mathbf{x} = \mathbf{x}^\top \boldsymbol{\Sigma} \mathbf{x} + \delta (\boldsymbol{\mu}^\top \mathbf{x})^2 \leq \eta^2 + \delta (\boldsymbol{\mu}^\top \mathbf{x}^*)^2.$$

Substituting for  $\delta$  and  $\mathbf{x}^*$  and simplifying, we get

$$\eta^2 + \delta (\boldsymbol{\mu}^\top \mathbf{x}^*)^2 = \eta^2 + \left( \frac{\theta}{1 - \theta \eta^2} \frac{\eta^2}{\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}} \right) \left( \eta^2 \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \right) = \frac{\eta^2}{1 - \theta \eta^2}.$$

□

Proposition 7 shows that the risk weights  $\mathbf{w}^*$  that we introduced in Section 3 have some built-in robustness. Imposing these weights on the bank in (BC) automatically ensures that any feasible portfolio  $\mathbf{x}$  satisfies the robust risk constraint (20) as well as the baseline constraint  $\mathbf{x}^\top \boldsymbol{\Sigma} \mathbf{x} \leq \eta^2$ , under the conditions stated in the result. The bound in (20) is larger than the baseline bound  $\eta^2$ , but some slack is appropriate because (20) is a backup to the baseline constraint; if the baseline estimate  $\boldsymbol{\Sigma}$  is accurate, the robust constraint should not bind. If, however, the regulator would like to limit  $\mathbf{x}^\top \tilde{\boldsymbol{\Sigma}}^* \mathbf{x}$  to some level  $\tilde{\eta}^2$ , then, as a second interpretation, the proposition shows how  $\eta$  needs to be adjusted in setting the risk weights  $\mathbf{w}^*$  in (2) to achieve the desired constraint; specifically,  $\eta$  needs to be set so that  $\eta^2 / (1 - \theta \eta^2) = \tilde{\eta}^2$ .

## 6 Mitigating Procyclicality

Risk-based capital requirements are generally procyclical, meaning that they tend to require more capital during economic downturns and less capital in periods of growth. This pattern amplifies the booms and busts of the business cycle by limiting the supply of bank credit when it is needed most and by potentially fueling an asset bubble with cheap credit during an upswing. Procyclicality results from the tendency of near-term risk (as measured by market volatility and default probabilities, for example) to be higher during downturns. A general concern about Basel II capital requirements is that the greater the sensitivity to near-term risk, the greater the procyclicality. The procyclicality of risk-based capital has been investigated extensively; see, for example, Adrian and Shin [4], Gordy and Howells [18], Repullo, Saurina, and Trucharte [29], and Shin [33].

The weights  $\mathbf{w}^*$  in (2) derived in a single-period framework are potentially procyclical when  $\Sigma$  (and possibly  $\boldsymbol{\mu}$ ) changes with the business cycle. For example, increasing  $\Sigma$  by a factor of  $\delta > 1$  increases  $\mathbf{w}^*$  by a factor of  $\sqrt{\delta}$ , and this is equivalent to reducing the bound on risk-weighted assets from  $\kappa$  to  $\kappa/\sqrt{\delta}$ . Thus, an increase in volatility would force a bank to shrink the size of its balance sheet.

To mitigate procyclicality, the regulator would like to use “through-the-cycle” measures of risk rather than “point-in-time” measures. To reflect this objective, we reformulate the constraint in (RR) to read

$$\sqrt{\mathbf{y}\bar{\Sigma}\mathbf{y}} \leq \eta, \quad (21)$$

where  $\bar{\Sigma}$  is a longer-term measure of risk than the near-term value  $\Sigma$ . We do not assume that the bank takes a similar perspective; on the contrary, we assume that the bank continues to use  $\Sigma$ , and the regulator needs to account for this discrepancy in designing risk weights. The through-the-cycle risk constraint (21) leads the regulator to a target of

$$\bar{\mathbf{y}} = \frac{\eta}{\sqrt{\boldsymbol{\mu}^\top \bar{\Sigma}^{-1} \boldsymbol{\mu}}} \bar{\Sigma}^{-1} \boldsymbol{\mu};$$

the regulator seeks risk weights that induce the bank to select  $\bar{\mathbf{y}}$  in solving (BC), even though the bank continues to use the short-term covariance  $\Sigma$ .

**Proposition 8** *If the regulator selects risk weights*

$$\bar{\mathbf{w}} = \frac{\kappa}{\lambda \boldsymbol{\mu}^\top \bar{\Sigma}^{-1} (\boldsymbol{\mu} - \gamma \lambda \Sigma \bar{\Sigma}^{-1} \boldsymbol{\mu})} (\boldsymbol{\mu} - \gamma \lambda \Sigma \bar{\Sigma}^{-1} \boldsymbol{\mu}), \quad (22)$$

with  $\lambda = \eta / \sqrt{\boldsymbol{\mu}^\top \bar{\Sigma}^{-1} \boldsymbol{\mu}}$ , then the bank’s optimal solution to (BC) using the short-term covariance matrix  $\Sigma$  coincides with the portfolio vector  $\bar{\mathbf{y}}$  based on the long-term covariance matrix. If the two covariance matrices are related through an expression of the form  $\Sigma = \delta_1 \bar{\Sigma} + \delta_2 \boldsymbol{\mu} \boldsymbol{\mu}^\top$ , for some  $\delta_1, \delta_2$ , then the weights simplify to

$$\bar{\mathbf{w}} = \frac{\kappa}{\eta \sqrt{\boldsymbol{\mu}^\top \bar{\Sigma}^{-1} \boldsymbol{\mu}}} \boldsymbol{\mu}. \quad (23)$$

The expression in (22) follows from (7) for the particular choice of target  $\bar{\mathbf{y}}$ , and it reduces to (23) under the stated condition on the covariance matrix. The special case (23) is remarkable because it shows that the regulator can ignore the bank’s internal choice of  $\Sigma$  (and  $\gamma$ ) and simply replace  $\Sigma$  in the original prescription for  $\mathbf{w}^*$  with  $\bar{\Sigma}$  to get through-the-cycle risk weights. The most natural case leading to (23) holds when  $\Sigma$  is just a multiple of  $\bar{\Sigma}$ , but the possibility that  $\delta_2 \neq 0$  adds flexibility to the result.

This solution can also be implemented adaptively:

**Corollary 1** *Suppose the long-term and short-term covariance matrices satisfy  $\Sigma = \delta_1 \bar{\Sigma} + \delta_2 \mu \mu^\top$ , for some  $\delta_1, \delta_2$ . Suppose the regulator chooses  $\mathbf{w}_k$  adaptively as in (14), but now with  $\beta_k = \kappa/\eta \sqrt{\mathbf{v}_k^\top \bar{\Sigma}^{-1} \mathbf{v}_k}$ . Then the risk-weights  $\mathbf{w}_k$  and optimal portfolios  $\mathbf{x}_k^*$  converge to  $(\bar{\mathbf{w}}, \bar{\mathbf{y}})$ , and this limit is a fixed-point of the updating process.*

## 7 Multiple Banks

To this point, we have limited our discussion to the case of a single bank. In this section, we explore the design of risk weights in a setting with multiple heterogeneous banks.

To introduce heterogeneity, we assume that different banks have access to different (but overlapping) sets of assets. For example, a regional bank may effectively be unable to make loans outside its region, and smaller banks may be unable to participate in some lines of business available to larger banks. We denote by  $\mathbf{x}_i$  the portfolio vector for bank  $i$ . Without explicitly labeling the assets to which each bank has access, we denote by  $\boldsymbol{\mu}_i$  the subvector of  $\boldsymbol{\mu}$  defined by the components in which bank  $i$  may invest, and we use  $\Sigma_i$  for the submatrix of  $\Sigma$  defined by the same components. In particular, the mean and variance for an asset do not depend on which bank holds the asset, leaving out the possibility of differences in skill or monopoly rents across banks. Paralleling the notation used in Section 3, each bank  $i$  has a risk-aversion parameter  $\gamma_i$ , a limit  $\kappa_i$  on risk-weighted assets, and a regulatory risk bound of  $\eta_i$ .

We first consider the consequences of having the regulator select a universal set of risk weights  $\mathbf{w} = \alpha \boldsymbol{\mu}$ , for some  $\alpha > 0$ , to which each bank responds by solving (BC). Through appropriate choice of  $\alpha$ , the regulator can indeed enforce the risk bound  $\eta_i$  on each bank; moreover, the risk weights do not distort the bank portfolios in the sense that each bank chooses the same relative mix of assets as it would without a risk-weight constraint.

**Proposition 9** *Suppose the regulator sets risk weights  $\mathbf{w} = \alpha \boldsymbol{\mu}$ , with*

$$\alpha \geq \max_i \frac{\kappa_i}{\eta_i \sqrt{\boldsymbol{\mu}_i^\top \Sigma_i^{-1} \boldsymbol{\mu}_i}}.$$

*Then each bank's optimal portfolio  $\mathbf{x}_i^*$  satisfies  $\sqrt{\mathbf{x}_i^{*\top} \Sigma_i \mathbf{x}_i^*} \leq \eta_i$ . Moreover, each  $\mathbf{x}_i^*$  is a scalar multiple of the corresponding unconstrained optimal portfolio  $\mathbf{x}_i^\circ$  for the bank.*

*Proof.* Each bank  $i$  faces the portfolio selection problem (BC) with weight vector  $\alpha \boldsymbol{\mu}_i$ . Let  $\alpha_i = \kappa_i/\eta_i \sqrt{\boldsymbol{\mu}_i^\top \Sigma_i^{-1} \boldsymbol{\mu}_i}$ . If  $\alpha = \alpha_i$ , then the bank  $i$ 's problem reduces to the one solved in Proposition 1 because the risk-weight vector reduces to (2), and the result follows. If  $\alpha_i < \alpha$ , there are two possibilities. If the risk-weight constraint is not binding, then bank  $i$  chooses  $\mathbf{x}_i^* = \mathbf{x}_i^\circ$ , its

unconstrained optimum and then  $\sqrt{\mathbf{x}_i^{\circ\top} \boldsymbol{\Sigma} \mathbf{x}_i^{\circ}} < \eta_i$  by the argument used at the end of the proof of Proposition 1. If the risk-weight constraint is binding, then it follows from (5) that bank  $i$ 's optimal portfolio is given by  $\mathbf{x}_i^* = (\alpha_i/\alpha) \mathbf{y}_i^*$ , where  $\mathbf{y}_i^*$  is the portfolio vector in (3) but with all variables on the right subscripted by  $i$ . With this choice we again have  $\sqrt{\mathbf{x}_i^{*\top} \boldsymbol{\Sigma} \mathbf{x}_i^*} < \eta_i$ .  $\square$

This result confirms that the regulator can achieve the desired risk constraints with a single risk-weight vector, and that it can do so without distorting the mix of assets held by each bank, even when banks have access to different sets of assets. A consequence of imposing a single set of risk weights is that banks with  $\alpha_i < \alpha$  are more tightly constrained by common risk weights than they would be under the bank-specific weights (2). We now argue that the solution generated through Proposition 9 is the best possible, given the tightened risk limits. Consider the regulator's system-wide problem for banks  $i = 1, \dots, n$ ,

$$\begin{aligned} \text{(RRS)} \quad & \max_{\{\mathbf{y}_i\}} \quad \boldsymbol{\mu}^\top (\mathbf{y}_1 + \dots + \mathbf{y}_n) \\ \text{s.t.} \quad & \forall i \quad \sqrt{\mathbf{y}_i^\top \boldsymbol{\Sigma} \mathbf{y}_i} \leq \tilde{\eta}_i, \end{aligned}$$

with

$$\tilde{\eta}_i = \min\{\alpha_i \eta_i / \alpha, \sqrt{\boldsymbol{\mu}_i^\top \boldsymbol{\Sigma}_i^{-1} \boldsymbol{\mu}_i / \gamma_i}\}.$$

In (RRS), we think of  $\mathbf{y}_i$  as the portfolio vector for bank  $i$  expanded to the full dimension of all assets through the addition of zeros in those coordinates representing assets to which bank  $i$  does not have access. Thus, problem (RRS) extends the objective in (RR) to the system-wide portfolio  $\sum_{i=1}^n \mathbf{y}_i$  while imposing a separate risk constraint on each bank.

**Proposition 10** *If the regulator chooses the risk-weight vector  $\mathbf{w} = \alpha \boldsymbol{\mu}$  as in Proposition 9, then the portfolios chosen by the individual banks solve the system-wide optimization problem (RRS).*

This result follows from combining the argument used in Proposition 9 with the argument used in Proposition 1 and the fact that the objective in (RRS) is separable. The details are straightforward so we omit them.

We can gain further insight by making the reasonable assumption that the ratio  $\kappa_i/\eta_i$  appearing in  $\alpha_i$  is common across all banks: we interpret  $\kappa_i$  as a multiple of the bank's capital, and  $\eta_i$  is the regulator's choice of risk limit for the bank, so the added assumption is that the risk limit is proportional to the bank's capital. Under this condition, the banks with smaller risk-weight multipliers  $\alpha_i < \alpha$  are the banks with larger values of

$$R_i \equiv \sqrt{\boldsymbol{\mu}_i^\top \boldsymbol{\Sigma}_i^{-1} \boldsymbol{\mu}_i}.$$

We show below (Proposition 11) that this quantity is an increasing function of the set of assets to which a bank has access. Thus, imposing a single multiplier  $\alpha$  as in Proposition 9 means imposing a capital surcharge on banks with access to a wider set of assets. We may interpret these to be large international banks, as opposed to community banks. Under Basel III, global systemically important financial institutions (“G-SIFIs”) are indeed subject to additional capital requirements.<sup>6</sup>

There is an alternative interpretation of these results under which the regulator would instead want to compensate for the disparities in Proposition 9: the regulator may prefer to impose stricter capital requirements on banks with more concentrated portfolios. This can be accomplished as follows. The regulator selects the risk-weight vector  $\mathbf{w} = \alpha\boldsymbol{\mu}$ , with

$$\alpha = \frac{\kappa_j/\eta_j}{\sqrt{\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}}} \equiv \frac{\kappa_j/\eta_j}{R}$$

using the common ratio of  $\kappa_j/\eta_j$  across banks and the full vector  $\boldsymbol{\mu}$  and matrix  $\boldsymbol{\Sigma}$ . The regulator then imposes a *concentration penalty* on bank  $i$  by assigning it risk weights  $(R/R_i)\mathbf{w}$ . As previously noted,  $R_i$  is determined by the set of assets to which bank  $i$  has access. The net effect of this procedure is to put bank  $i$  in exactly the position it would face in isolation, facing the same risk weights as in Section 3 with no change in the constraint  $\eta_i$  and with only modest tailoring of a common risk-weight vector. Our interpretation of  $R_i$  is justified by the following result:

**Proposition 11** *For any pair  $(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)$  extracted from  $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$  by deleting those entries to which bank  $i$  does not have access,*

$$\boldsymbol{\mu}_i^\top \boldsymbol{\Sigma}_i^{-1} \boldsymbol{\mu}_i \leq \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}.$$

*Thus,  $R_i \leq R$  and  $\alpha_i \geq \alpha$ .*

*Proof.* Without loss of generality, we may suppose that the mean return vector can be decomposed as  $\boldsymbol{\mu} = (\boldsymbol{\mu}_1; \boldsymbol{\mu}_0)$ , for some  $\boldsymbol{\mu}_0$ , and then  $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_{11}$  in the partitioning

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{10} \\ \boldsymbol{\Sigma}_{10}^\top & \boldsymbol{\Sigma}_{00} \end{bmatrix}.$$

It now follows from Corollary A.3.2 of Anderson [5] that  $\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \geq \boldsymbol{\mu}_1^\top \boldsymbol{\Sigma}_{11}^{-1} \boldsymbol{\mu}_1$ .  $\square$

In addition to justifying our interpretation of the concentration penalty  $R/R_i$ , this result rules out a certain kind of regulatory arbitrage. Once we scale risk-weight vectors differently for different portfolios, we need to consider the possibility of a bank artificially splitting itself into two banks holding different sets of assets, with the objective of facing less stringent risk-weight constraints.

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<sup>6</sup>International coordination presents many political challenges to our scheme because tying risk weights to profitability could result in different risk weights for similar categories of loans in different countries.

Proposition 11 ensures that there can be no advantage to doing so. It further ensures that a bank has nothing to gain by pretending not to have access to assets in which it can in fact invest.

Observe that if  $\Sigma_{10}^\top \Sigma_{11}^{-1} \boldsymbol{\mu}_1 = \boldsymbol{\mu}_0$ , then equality holds in Proposition 11. To further support our interpretation of  $R_i$ , we will show that under this condition the assets not contained in the set corresponding to  $\boldsymbol{\mu}_1$  add no diversification. To see this, let  $(\mathbf{X}_1; \mathbf{X}_0)$  be the partitioned vector of returns. We can always write

$$\mathbf{X}_0 = \boldsymbol{\alpha} + \boldsymbol{\beta} \mathbf{X}_1 + \boldsymbol{\epsilon},$$

with  $\boldsymbol{\alpha}$  constant,  $\boldsymbol{\beta} = \Sigma_{10}^\top \Sigma_{11}^{-1}$ , and the error  $\boldsymbol{\epsilon}$  uncorrelated with  $\mathbf{X}_1$  and having mean zero. Taking expectations, we get  $\boldsymbol{\mu}_0 = \boldsymbol{\alpha} + \Sigma_{10}^\top \Sigma_{11}^{-1} \boldsymbol{\mu}_1$ , so the condition reduces to  $\boldsymbol{\alpha} = 0$ .

Now consider a portfolio  $\mathbf{x} = (\mathbf{x}_1; \mathbf{x}_0)$  of holdings in  $(\mathbf{X}_1; \mathbf{X}_0)$ . The portfolio's return is

$$\begin{aligned} P &= \mathbf{X}_1^\top \mathbf{x}_1 + \mathbf{X}_0^\top \mathbf{x}_0 &= \mathbf{X}_1^\top \mathbf{x}_1 + \mathbf{X}_1^\top \Sigma_{11}^{-1} \Sigma_{10} \mathbf{x}_0 + \boldsymbol{\epsilon}^\top \mathbf{x}_0 \\ & &= \mathbf{X}_1^\top (\mathbf{x}_1 + \Sigma_{11}^{-1} \Sigma_{10} \mathbf{x}_0) + \boldsymbol{\epsilon}^\top \mathbf{x}_0. \end{aligned}$$

Then,

$$\mathbb{E}[P] = \mathbb{E}[\mathbf{X}_1^\top (\mathbf{x}_1 + \Sigma_{11}^{-1} \Sigma_{10} \mathbf{x}_0)].$$

But because  $\mathbf{X}_1$  and  $\boldsymbol{\epsilon}$  are uncorrelated,  $P$  has greater variance than  $\mathbf{X}_1^\top (\mathbf{x}_1 + \Sigma_{11}^{-1} \Sigma_{10} \mathbf{x}_0)$ . Hence, compared with any investments  $\mathbf{x}_1$  and  $\mathbf{x}_0$  in  $\mathbf{X}_1$  and  $\mathbf{X}_0$ , an investment of  $\mathbf{x}_1 + \Sigma_{11}^{-1} \Sigma_{10} \mathbf{x}_0$  in  $\mathbf{X}_1$  produces lower variance with the same expected value. In other words, the assets in  $\mathbf{x}_0$  are not needed to construct an optimal portfolio.

## 8 Concluding Remarks

We have investigated the design of risk weights to meet a variety of regulatory objectives in setting bank capital requirements. We have derived our results in a simple model of portfolio selection. We see the insights from this analysis as more broadly applicable:

- tying risk weights to profitability controls risk;
- adapting risk weights to changes in bank portfolios controls risk with less information about bank assets;
- multiple constraints can be applied for robustness;
- risk weights can be modified to mitigate procyclicality, even if banks use short-term measures of risk;

- applying a common set of risk weights to multiple heterogeneous banks imposes a capital surcharge on banks participating in a wider range of activities, and this effect can be offset through a simple multiplier based on portfolio concentration.

Our analysis is based on a stylized model and thus leaves open many questions for future work. Among these is the important practical problem of determining the right level of detail at which to map lending categories to assets. If the categories are too coarse, risk weights create bad incentives; if they are too detailed, then loans can move from one category to another over time, creating an unintended portfolio shift.

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## A Appendix

### A.1 Long-Only Portfolios

We begin with a proof of Proposition 3.

*Proof.* Write (BU<sub>+</sub>) for problem (BU) with the nonnegativity constraint  $\mathbf{x} \geq \mathbf{0}$ , and define problems (RR<sub>+</sub>) and (BC<sub>+</sub>) accordingly. The Lagrangian for (BU<sub>+</sub>) is  $\boldsymbol{\mu}^\top \mathbf{x} - (\gamma/2)\mathbf{x}^\top \boldsymbol{\Sigma} \mathbf{x} + \boldsymbol{\lambda}^\top \mathbf{x}$ . By relabeling assets if necessary, we may assume that the solution to (BU<sub>+</sub>) takes the form  $\bar{\mathbf{x}} = (\bar{\mathbf{x}}_1; \bar{\mathbf{x}}_2)$ ,  $\bar{\mathbf{x}}_1 > \mathbf{0}$ ,  $\bar{\mathbf{x}}_2 = \mathbf{0}$  with a corresponding partition of  $\boldsymbol{\mu} = (\boldsymbol{\mu}_1; \boldsymbol{\mu}_2)$ ,  $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_1; \boldsymbol{\lambda}_2)$ , and

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}.$$

The resulting KKT conditions for optimality are

$$\begin{aligned} \text{(KKT-BU}_+) \quad & \boldsymbol{\lambda}_1 = \mathbf{0}, \boldsymbol{\lambda}_2 \geq \mathbf{0}, \\ & -\boldsymbol{\mu}_1 + \gamma \boldsymbol{\Sigma}_{11} \bar{\mathbf{x}}_1 = \mathbf{0}, \\ & -\boldsymbol{\mu}_2 + \gamma \boldsymbol{\Sigma}_{21} \bar{\mathbf{x}}_1 - \boldsymbol{\lambda}_2 = \mathbf{0}. \end{aligned}$$

Problems (RR<sub>+</sub>) and (BC<sub>+</sub>) also admit partitioned solutions. The respective partitions into active and inactive assets are not a priori the same as those of (BU<sub>+</sub>), but we use the same notation. In problem (RR<sub>+</sub>), we may replace the decision variable  $\mathbf{y}$  with  $\mathbf{x}$  to simplify the comparison and then write the Lagrangian as  $\boldsymbol{\mu}^\top \mathbf{x} - (\gamma/2)\mathbf{x}^\top \boldsymbol{\Sigma} \mathbf{x} + \boldsymbol{\lambda}^\top \mathbf{x} - \lambda_0 \mathbf{x}^\top \boldsymbol{\Sigma} \mathbf{x}$ . The KKT conditions become

$$\begin{aligned} \text{(KKT-RR}_+) \quad & \lambda_0 \geq 0, \boldsymbol{\lambda}_1 = \mathbf{0}, \boldsymbol{\lambda}_2 \geq \mathbf{0}, \\ & \lambda_0(\eta^2 - \bar{\mathbf{x}}_1^\top \boldsymbol{\Sigma}_{11} \bar{\mathbf{x}}_1) = 0, \\ & -\boldsymbol{\mu}_1 + 2\lambda_0 \boldsymbol{\Sigma}_{11} \bar{\mathbf{x}}_1 = \mathbf{0}, \\ & -\boldsymbol{\mu}_2 + 2\lambda_0 \boldsymbol{\Sigma}_{21} \bar{\mathbf{x}}_1 - \boldsymbol{\lambda}_2 = \mathbf{0}. \end{aligned}$$

For problem (BC<sub>+</sub>), we may write the risk-weight constraint  $\mathbf{w}^{*\top} \mathbf{x} \leq \kappa$  as  $\boldsymbol{\mu}^\top \mathbf{x} \leq b$  by setting  $b = \kappa/\alpha_1$ . The Lagrangian is then  $\boldsymbol{\mu}^\top \mathbf{x} - (\gamma/2)\mathbf{x}^\top \boldsymbol{\Sigma} \mathbf{x} + \boldsymbol{\lambda}^\top \mathbf{x} - \lambda_0 \boldsymbol{\mu}^\top \mathbf{x}$  and the KKT conditions are

$$\begin{aligned}
(\text{KKT-BC}_+) \quad & \lambda_0 \geq 0, \boldsymbol{\lambda}_1 = \mathbf{0}, \boldsymbol{\lambda}_2 \geq \mathbf{0}, \\
& \lambda_0(b - \boldsymbol{\mu}_1^\top \bar{\mathbf{x}}_1) = 0, \\
& -\boldsymbol{\mu}_1 + \gamma \boldsymbol{\Sigma}_{11} \bar{\mathbf{x}}_1 + \lambda_0 \boldsymbol{\mu}_1 = \mathbf{0}, \\
& -\boldsymbol{\mu}_2 + \gamma \boldsymbol{\Sigma}_{21} \bar{\mathbf{x}}_1 + \lambda_0 \boldsymbol{\mu}_2 - \boldsymbol{\lambda}_2 = \mathbf{0},
\end{aligned}$$

We now claim that a solution to any of these problems determines a solution to the other two with the same set of active assets. In particular, a solution to (BU<sub>+</sub>) has  $\bar{\mathbf{x}}_1 = \boldsymbol{\Sigma}_{11}^{-1} \boldsymbol{\mu}_1 / \gamma$  and  $\boldsymbol{\lambda}_2$  proportional to  $\boldsymbol{\Sigma}_{21} \boldsymbol{\Sigma}_{11}^{-1} \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2 \geq \mathbf{0}$ . From this solution we can construct a solution to (RR<sub>+</sub>) by setting  $\lambda_0 = \sqrt{\boldsymbol{\mu}_1^\top \boldsymbol{\Sigma}_{11}^{-1} \boldsymbol{\mu}_1 / 2\eta}$ ; the resulting solution has the same set of active assets because it simply scales  $\bar{\mathbf{x}}_1$ . For (BC<sub>+</sub>), we take  $\lambda_0 = [1 - b\gamma / \boldsymbol{\mu}_1^\top \boldsymbol{\Sigma}_{11}^{-1} \boldsymbol{\mu}_1]^+$  and again arrive at a solution with the same set of active assets.  $\square$

This argument shows that the partition into active and inactive assets in the optimal solution to each of the three problems is fully characterized by the conditions

$$\boldsymbol{\Sigma}_{11}^{-1} \boldsymbol{\mu}_1 > \mathbf{0} \tag{24}$$

and

$$\boldsymbol{\Sigma}_{21} \boldsymbol{\Sigma}_{11}^{-1} \boldsymbol{\mu}_1 \geq \boldsymbol{\mu}_2. \tag{25}$$

Now consider the original problems (BU), (RR), and (BC) without nonnegativity constraints but restricted to the subportfolio  $\mathbf{x}_1$  under conditions (24) and (25). By writing out the KKT conditions for these reduced problems, it is easy to see that the optimal solution  $\mathbf{x}_1$  to each reduced problem coincides with the optimal levels for the active assets in the corresponding problems with nonnegativity constraints. Thus, once we have identified the set of active assets through the conditions (24) and (25), we may restrict attention to just the active assets and drop the nonnegativity constraints.

From the discussion that follows Proposition 11, we see that condition (25) has a simple interpretation: it implies that for any investment in the assets in  $\mathbf{x}_2$ , one can find a portfolio holding only assets in  $\mathbf{x}_1$  with higher expected returns and/or lower variance. In other words, condition (25) implies that the assets in  $\mathbf{x}_2$  are dominated.

## A.2 More General Risk Measures

Proposition 1 continues to hold under portfolio risk measures beyond variance. Consider any differentiable function  $\rho(\cdot)$  mapping portfolio vectors to real numbers such that

- $\nabla\rho(\mathbf{x}) = \mathbf{y}$  is solvable for any  $\mathbf{y} \geq \mathbf{0}$  with a unique solution, say  $\mathbf{x} = (\nabla\rho)^{-1}(\mathbf{y})$ ;
- if  $0 < \theta_1 \leq \theta_2$ ,  $\theta_i \in \mathbb{R}$ , then  $\rho((\nabla\rho)^{-1}(\theta_1\boldsymbol{\mu})) \leq \rho((\nabla\rho)^{-1}(\theta_2\boldsymbol{\mu}))$  holds. (Recall that  $\boldsymbol{\mu} > \mathbf{0}$ .)

Replace the optimization problems in Section 3 with the following:

$$\begin{aligned} \text{(BU)} \quad & \max \quad \boldsymbol{\mu}^\top \mathbf{x} - \gamma\rho(\mathbf{x}), \\ \text{(RR)} \quad & \max \quad \boldsymbol{\mu}^\top \mathbf{y} \quad \text{s.t.} \quad \rho(\mathbf{y}) \leq \eta^2, \\ \text{(BC)} \quad & \max \quad \boldsymbol{\mu}^\top \mathbf{x} - \gamma\rho(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{w}^\top \mathbf{x} \leq \kappa. \end{aligned}$$

The optimal solutions take the respective forms  $\mathbf{x}^\circ = (\nabla\rho)^{-1}(\frac{1}{\gamma}\boldsymbol{\mu})$ ,

$$\mathbf{y}^* = (\nabla\rho)^{-1}\left(\frac{1}{\lambda_R}\boldsymbol{\mu}\right),$$

and

$$\mathbf{x}^* = (\nabla\rho)^{-1}\left(\frac{1}{\gamma}(\boldsymbol{\mu} - \lambda_B\mathbf{w})\right),$$

where  $\lambda_R$  and  $\lambda_B$  are Lagrange multipliers for (RR) and (BC). Proposition 1 holds in this more general setting with appropriate modification of the proportionality constant  $\alpha$  in (2). If  $\gamma < \lambda_R$ , then

$$\alpha = \frac{\kappa}{\boldsymbol{\mu}^\top \mathbf{y}^*} = \frac{\kappa}{\boldsymbol{\mu}^\top (\nabla\rho)^{-1}\left(\frac{1}{\lambda_R}\boldsymbol{\mu}\right)};$$

otherwise,  $\alpha = \kappa/\boldsymbol{\mu}^\top \mathbf{x}^\circ$ . These assertions follow from the KKT conditions; we omit the details.

## B Convergence to Equilibrium

In this section, we prove Proposition 5 and Corollary 1. To lighten the notation, we take  $\gamma = 1$ ; equivalently, we may replace  $\boldsymbol{\mu}$  with  $\bar{\boldsymbol{\mu}} = \boldsymbol{\mu}/\gamma$  everywhere in the proof. From (4), the bank's rule for setting its portfolio under (BC) becomes

$$\mathbf{x}_k^* = \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \frac{(\mathbf{w}_k^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \kappa)^+}{\mathbf{w}_k^\top \boldsymbol{\Sigma}^{-1}\mathbf{w}_k} \boldsymbol{\Sigma}^{-1}\mathbf{w}_k. \quad (26)$$

We can rearrange this expression to get

$$\boldsymbol{\mu} = \begin{cases} \boldsymbol{\Sigma}\mathbf{x}_k^* + \frac{(\mathbf{w}_k^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \kappa)}{\mathbf{w}_k^\top \boldsymbol{\Sigma}^{-1}\mathbf{w}_k} \mathbf{w}_k, & \text{if } \mathbf{w}_k^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} \geq \kappa; \\ \boldsymbol{\Sigma}\mathbf{x}_k^*, & \text{if } \mathbf{w}_k^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} < \kappa. \end{cases} \quad (27)$$

We claim that the condition  $\mathbf{w}_k^\top \mathbf{x}_k^* < \kappa$  used in the update rule (15) is equivalent to the condition  $\mathbf{w}_k^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} < \kappa$  in (27). To see this, observe from (26) that if  $\mathbf{w}_k^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} < \kappa$  then  $\mathbf{w}_k^\top \mathbf{x}_k^* = \mathbf{w}_k^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$ ,

and if  $\mathbf{w}_k^\top \Sigma^{-1} \boldsymbol{\mu} \geq \kappa$  then  $\mathbf{w}_k^\top \mathbf{x}_k^* = \kappa$ . It follows that if  $\mathbf{w}_k^\top \mathbf{x}_k^* < \kappa$  for some  $k$ , then  $\mathbf{v}_{k+1} = \boldsymbol{\mu}$ . Comparison of (15) and (27) further shows that then  $\mathbf{v}_{k+n} \equiv \boldsymbol{\mu}$  for all  $n \geq 1$ . The definition of  $\beta_k$  then shows that  $\mathbf{w}_{k+n} = \mathbf{w}^*$  for all  $n \geq 1$ ; in other words, the limit  $\mathbf{w}^*$  is reached in a finite number of steps.

We may therefore suppose that  $\mathbf{w}_k^\top \mathbf{x}_k^* \geq \kappa$  for all  $k$  and thus restrict attention to the first case in each of (15) and (27). Taking the difference between these two expressions, we get

$$\begin{aligned} \mathbf{v}_{k+1} - \boldsymbol{\mu} &= \frac{\mathbf{w}_k^\top \Sigma^{-1} (\mathbf{v}_k - \boldsymbol{\mu})}{\mathbf{w}_k^\top \Sigma^{-1} \mathbf{w}_k} \mathbf{w}_k \\ &= \frac{\mathbf{v}_k^\top \Sigma^{-1} (\mathbf{v}_k - \boldsymbol{\mu})}{\mathbf{v}_k^\top \Sigma^{-1} \mathbf{v}_k} \mathbf{v}_k. \end{aligned} \quad (28)$$

We may represent  $\mathbf{v}_k$  in the form  $\mathbf{v}_k = a_k \boldsymbol{\mu} + b_k \boldsymbol{\xi}$  where  $\boldsymbol{\mu}^\top \Sigma^{-1} \boldsymbol{\mu} = c$ ,  $\boldsymbol{\mu}^\top \Sigma^{-1} \boldsymbol{\xi} = 0$  and  $\boldsymbol{\xi}^\top \Sigma^{-1} \boldsymbol{\xi} = c$ . To see why, write  $\Sigma^{-1/2} \mathbf{v}_k$  as the sum of a projection onto  $\Sigma^{-1/2} \boldsymbol{\mu}$  and a residual to get  $\Sigma^{-1/2} \mathbf{v}_k = a \Sigma^{-1/2} \boldsymbol{\mu} + \boldsymbol{\xi}'$  and  $\boldsymbol{\mu}^\top \Sigma^{-1/2} \boldsymbol{\xi}' = 0$ , for some  $a$  and  $\boldsymbol{\xi}'$ . This gives  $\mathbf{v}_k = a \boldsymbol{\mu} + \Sigma^{1/2} \boldsymbol{\xi}'$  and  $\boldsymbol{\mu}^\top \Sigma^{-1} \Sigma^{1/2} \boldsymbol{\xi}' = \boldsymbol{\mu}^\top \Sigma^{-1/2} \boldsymbol{\xi}' = 0$ , and scaling of  $\Sigma^{1/2} \boldsymbol{\xi}'$  gives  $\boldsymbol{\xi}$ . Making this substitution on the right side of (28) yields.

$$\begin{aligned} \mathbf{v}_{k+1} &= \boldsymbol{\mu} + \frac{ca_k(a_k - 1) + cb_k^2}{ca_k^2 + cb_k^2} (a_k \boldsymbol{\mu} + b_k \boldsymbol{\xi}) \\ &= \boldsymbol{\mu} + \left(1 - \frac{ca_k}{ca_k^2 + cb_k^2}\right) (a_k \boldsymbol{\mu} + b_k \boldsymbol{\xi}) \\ &= \left(1 + a_k - \frac{a_k^2}{a_k^2 + b_k^2}\right) \boldsymbol{\mu} + \left(1 - \frac{a_k}{a_k^2 + b_k^2}\right) b_k \boldsymbol{\xi} \\ &= \left(a_k + \frac{b_k^2}{a_k^2 + b_k^2}\right) \boldsymbol{\mu} + \left(1 - \frac{a_k}{a_k^2 + b_k^2}\right) b_k \boldsymbol{\xi} \\ &\equiv a_{k+1} \boldsymbol{\mu} + b_{k+1} \boldsymbol{\xi}. \end{aligned}$$

In particular,  $\boldsymbol{\xi}$  remains fixed and only the coefficients change. The coefficients satisfy

$$a_{k+1} = a_k + \frac{b_k^2}{a_k^2 + b_k^2}, \quad (29)$$

$$b_{k+1} = \left(1 - \frac{a_k}{a_k^2 + b_k^2}\right) b_k. \quad (30)$$

Since  $a_{k+1} \geq a_k$ ,  $a = \lim_k a_k = \sup_k a_k$  exists, though it is potentially infinite. We consider the possible cases.

- Case  $a < \infty$ : If  $\limsup_k b_k^2 = \epsilon > 0$ , then from (29)

$$a_{k+1} - a_k = \frac{b_k^2}{a_k^2 + b_k^2} > \frac{\epsilon/2}{(|a| + 1)^2 + \epsilon/2}$$

for infinitely many  $k$ 's, which is impossible if  $a < \infty$ . Hence  $\lim_k b_k = 0$ .

- Case  $a = \infty$ : We will show that this is impossible.

- (a) If  $a = \infty$ , then for all sufficiently large  $k$  we have  $a_k > 1$  and  $0 < 1 - \frac{a_k}{a_k^2 + b_k^2} < 1$ . Then (30) implies that  $|b_k|$  is decreasing for all sufficiently large  $k$ , hence  $|b_k|$  is bounded.
- (b) Since  $a_k$  diverges to  $\infty$ ,  $\frac{b_k^2}{a_k} < 1$  for large  $k$  because of (a).
- (c) We have  $a_{k+1} - a_k = \frac{b_k^2}{a_k^2 + b_k^2} \leq 1$  for all  $k$ . For any index  $j$ , if  $a_j \leq m$  for some integer  $m$ , then  $a_{j+k} \leq m + k \leq m + j + k$ ; i.e.  $a_k \leq m + k$  for all large  $k$ .
- (d) Using (b) and (c), for some sufficiently large  $k^*$  and all  $k \geq k^*$ ,

$$1 - \frac{a_k}{a_k^2 + b_k^2} = 1 - \frac{1}{a_k + \frac{b_k^2}{a_k}} \leq 1 - \frac{1}{m + k + 1} = \frac{m + k}{m + k + 1}$$

and if we plug this bound in (30), then

$$|b_{k^*+\ell}| \leq \frac{m + k^*}{m + k^* + \ell + 1} |b_{k^*}|. \quad (31)$$

- (e) From (29) and (31), for any  $p = 1, 2, \dots$ ,

$$\begin{aligned} a_{k^*+p} &= a_{k^*} + \sum_{\ell=0}^{p-1} \frac{b_{k^*+\ell}^2}{a_{k^*+\ell}^2 + b_{k^*+\ell}^2} \\ &\leq a_{k^*} + \frac{1}{a_{k^*}^2} \sum_{\ell=0}^{p-1} b_{k^*+\ell}^2 \\ &\leq a_{k^*} + \frac{(m + k^*)^2 b_{k^*}^2}{a_{k^*}^2} \sum_{\ell=0}^{p-1} \frac{1}{(m + k^* + \ell + 1)^2} \\ &\leq a_{k^*} + \frac{(m + k^*)^2 b_{k^*}^2}{a_{k^*}^2} \sum_{\ell=1}^{\infty} \frac{1}{\ell^2} \\ &< \infty. \end{aligned}$$

Hence  $a_k$  is a bounded sequence, contradicting the assumption  $a = \infty$ .

We conclude from this analysis that the  $a_k$  increase to a finite limit  $a$  and the  $b_k$  converge to zero. In other words,  $\mathbf{v}_k$  converges to a multiple of  $\boldsymbol{\mu}$ , and then  $\mathbf{w}_k$  converges to  $\mathbf{w}^*$ . Convergence of  $\mathbf{x}_k^*$  to  $\mathbf{x}^*$  follows from (26). That  $(\mathbf{w}^*, \mathbf{x}^*)$  is a fixed point follows from (15) and (26).  $\square$

The same argument works for Corollary 1. Equations (26) and (27) hold as before with  $\mathbf{w}_k$  defined using the modified value of  $\beta_k$ . In (28),  $\beta_k$  cancels from the numerator and denominator, so the convergence proof in Proposition 5 applies directly.

As a further consequence of the argument in Proposition 5, we may conclude that the iteration produces only strictly positive weight vectors under our assumption that the entries of  $\boldsymbol{\mu}$  are strictly

positive. More precisely, we claim that if  $\mathbf{v}_k > \mathbf{0}$ ,  $a_k > 1$  and  $b_k > 0$  for some  $k$ , then  $\mathbf{v}_n > \mathbf{0}$  for all  $n \geq k$ . The argument is by induction. With  $a_k > 1$ , we have  $0 < 1 - a_k/(a_k^2 + b_k^2) < 1$ , which implies  $0 < b_{k+1} < b_k$ . We also have  $a_{k+1} \geq a_k$ . Hence

$$\mathbf{v}_{k+1} = a_{k+1}\boldsymbol{\mu} + b_{k+1}\boldsymbol{\xi} \geq a_k\boldsymbol{\mu} - b_k\boldsymbol{\xi}^- = \min\{a_k\boldsymbol{\mu}, \mathbf{v}_k\} > \mathbf{0},$$

where the min is applied componentwise. Hence we have  $\mathbf{v}_{k+1} > \mathbf{0}$ ,  $a_{k+1} > 1$ , and  $b_{k+1} > 0$ .

## C An Asymptotic Single-Factor Gaussian Setting

In this section, we develop a connection suggested by a referee between the risk weights in Proposition 1 and those that follow from tying risk weights to marginal value-at-risk (VaR) contributions in a single-factor Gaussian model. This model offers a rough analogy with the asymptotic single risk factor model that underpins the Basel II internal ratings based approach for capital requirements; see Gordy [17].

Suppose that the asset returns have a multivariate normal distribution,  $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ . The portfolio return for any portfolio vector  $\mathbf{x}$  is then normally distributed. In measuring portfolio VaR, it is customary to take the expected return to be zero, which is conservative if the expected return is positive. Thus,  $\text{VaR}(\mathbf{x}) = k(\mathbf{x}^\top \boldsymbol{\Sigma} \mathbf{x})^{1/2}$ , where  $k$  is a quantile from the standard normal distribution and does not depend on  $\mathbf{x}$ .

The marginal contribution to VaR of the  $i$ th asset in the portfolio vector  $\mathbf{x} = (x_1, \dots, x_d)^\top$  is given by

$$\frac{\partial \text{VaR}(\mathbf{x})}{\partial x_i} = k \frac{\sum_j x_j \sigma_{ij}}{\sqrt{\sum_{k,j} x_k x_j \sigma_{kj}}},$$

where we have written  $\sigma_{ij}$  for the  $ij$ -entry of  $\boldsymbol{\Sigma}$ . If we chose a risk weight vector  $\mathbf{w} = (w_1, \dots, w_d)^\top$  in which  $w_i$  is proportional to this marginal VaR contribution, then  $\mathbf{w}$  would in general depend on the choice of portfolio.

In a single-factor model, such as the CAPM, the excess return on the  $i$ th asset takes the form

$$R_i = \mu_i + \beta_i R_M + \epsilon_i,$$

where  $R_M$  is the excess return on the common factor and  $\epsilon_i$  and  $\epsilon_j$  are uncorrelated for  $i \neq j$  and uncorrelated with  $R_M$ . Thus, writing  $\sigma_{\epsilon_i}^2$  for the variance of  $\epsilon_i$ , we have

$$\sigma_{ij} = \begin{cases} \beta_i \beta_j \sigma_M^2, & \text{if } i \neq j; \\ \beta_i^2 \sigma_M^2 + \sigma_{\epsilon_i}^2, & \text{otherwise.} \end{cases}$$

In the Basel II asymptotic single risk factor model, idiosyncratic risk terms drop out leaving only the effect of a common factor. By analogy, here we consider the simplification  $\sigma_{ii} = \beta_i^2 \sigma_M^2$ . In this

case, if we choose risk weights proportional to marginal VaR contributions we get

$$w_i \propto \frac{\partial \text{VaR}(\mathbf{x})}{\partial x_i} = k \frac{\sum_j x_j \beta_i \beta_j \sigma_M^2}{\sqrt{\sum_{k,j} x_k x_j \beta_k \beta_j \sigma_M^2}} = k \beta_i \frac{\mathbf{x}^\top \boldsymbol{\beta}}{\sqrt{(\mathbf{x}^\top \boldsymbol{\beta})(\mathbf{x}^\top \boldsymbol{\beta})}} \sigma_M = k \beta_i \sigma_M.$$

Thus, in this setting, the resulting risk weights are portfolio independent, as they are in Gordy [17], and they are proportional to the expected excess returns, as they are in Proposition 1. This setting therefore provides a link between our result and Basel II risk weights, albeit under very special conditions.

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