Addendum to
"Numerical Solution of Jump-Diffusion LIBOR Market Models"

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Abstract

This addendum provides a detailed proof of Theorem 6.1 in Glasserman and Merener [1], establishing the convergence order of a discretization scheme.

1 Introduction

The purpose of this addendum is to present a detailed proof of Theorem 6.1 in Glasserman and Merener [1], establishing the convergence order of a discretization scheme for a class of processes constructed from Brownian motion and Poisson random measure. The scheme is based on a technique of Mikulevicius and Platen [3], but is analyzed under a different set of assumptions.

For clarity, we recount here the notation and framework of Section 5 and Section 6 in [1]. We consider an M-dimensional process $X(t)$, $t \in [0,T]$ with dynamics

$$dX(t) = \tilde{a}(X(t))\,dt + b(X(t))\,dW(t) + \int_{R^d} c(X(t),z)p(dz,dt), \tag{1}$$

where $p(dz,dt)$ is a Poisson random measure on $R^d \times [0,T]$ with intensity $\lambda_0\,h(z)$, h a probability density on R^d. Define

$$a(y) = \tilde{a}(y) + \int_{R^d} c(y,z)\,h(z)\,\lambda_0\,dz,$$

so the dynamics (1) can be written as

$$dX(t) = a(X(t))\,dt + b(X(t))\,dW(t) + \int_{R^d} c(X(t),z)\,q(dz,dt)$$

where $q(dz,dt) = p(dz,dt) - h(z)\,\lambda_0\,dz$ is a Poisson martingale measure on $R^d \times [0,T]$. In the application that motivates our analysis, the support of h is $[0,\infty) \times [0,1]$.

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Let $B^\xi(C)$ be the class of $2(\xi+1)$-times continuously differentiable real-valued functions for which the function itself and its partial derivatives up to order $2(\xi+1)$ are uniformly bounded by a constant C.

For bounded $\psi : \Re^M \to \Re$ let

$$\phi(x) = \int_{R^d} \psi(x + c(x,z))\, h(z)\, dz \tag{2}$$

and let

$$\bar{\phi}(x) = \int_{R^d} (x + c(x,z))\, h(z)\, dz. \tag{3}$$

The coefficients of the first order, or Euler, scheme introduced in Mikulevicius and Platen [3] and used in Section 5 of Glasserman and Merener [1] are

$$f_0(x) = \tilde{a}(x) \text{ and } f_1(x) = b(x). \tag{4}$$

The coefficients of the second order, or Milstein, scheme are f_0 and f_1 in (4), and

$$\begin{aligned} f_{00}(x) &= \sum_{j=1}^{M} \tilde{a}_j(x)\partial_j \tilde{a}(x) + \frac{1}{2}\sum_{j=1}^{M}\sum_{k=1}^{M} b_j(x)b_k(x)\partial_{jk}\tilde{a}(x), \quad f_{11}(x) = \sum_{j=1}^{M} b_j(x)\partial_j b(x), \\ f_{10}(x) &= \sum_{j=1}^{M} b_j(x)\partial_j \tilde{a}(x), \quad f_{01}(x) = \sum_{j=1}^{M} \tilde{a}_j(x)\partial_j b(x) + \frac{1}{2}\sum_{j=1}^{M}\sum_{k=1}^{M} b_j(x)b_k(x)\partial_{jk} b(x). \end{aligned} \tag{5}$$

where ∂_j denotes partial derivative with respect to the jth component of x.

We are concerned with the numerical calculation of $E[g(X(T))]$ for some *payoff* function $g : \Re^M \to \Re$. Theorem 6.1 in Glasserman and Merener [1] reads:

THEOREM 1.1 (Theorem 6.1 in [1])

Fix $\xi \in \{1,2\}$. Let the payoff function $g : \Re^M \to \Re$ be in $B^\xi(G)$ for some G and let $\{X(t), t \in [0,T]\}$ be as in (1). We assume:

(i) $\bar{\phi}(x)$ is $2(\xi+1)$-times continuously differentiable with uniformly bounded derivatives;

(ii) there is a constant K such that if $\psi \in B^\xi(\Psi)$ for some Ψ then $\phi(x) \in B^\xi(K\Psi)$ in (2);

(iii) a and b are $2(\xi+1)$-times continuously differentiable with uniformly bounded derivatives;

(iv) there is a constant K_2 such that any $f \in S_\xi$ satisfies $|f(y)| \leq K_2(1 + \|y\|)$, with $S_1 = \{f_0, f_1\}$ as in (4) and $S_2 = \{f_0, f_1, f_{00}, f_{10}, f_{01}, f_{11}\}$ as in (5).

Then the approximation defined by (28), (29), and (30) in [1] has weak convergence order one and the approximation defined by (31) and (32) in [1] has weak convergence order two.

Before proceeding with the proof of Theorem 1.1 we briefly discuss its connection with Theorem 3.3 of Mikulevicius and Platen [3] which is also a weak convergence result. As mentioned in Section 6 of [1], Mikulevicius and Platen [3] introduced a hierarchy of schemes which, under regularity conditions on a, b, c and the payoff function g, are shown to have arbitrarily high order of weak convergence. In particular, the scheme defined by (28), (29), and (30) in [1] converges weakly with order one and the scheme defined by (31) and (32) in [1] converges weakly with order two.

More precisely, Mikulevicius and Platen [3] assume that the payoff g is $2(\xi+1)$-times continuously differentiable and with polynomial growth, and that the coefficients a, b, and c satisfy:

(a) a, b and c are $2(\xi+1)$-times continuously differentiable with uniformly bounded derivatives;

(b) there is a constant K_2 such that the functions $f(y)$ in S_ξ satisfy $|f(y)| \leq K_2(1+\|y\|)$, with $S_1 = \{f_0, f_1\}$ as in (4) and $S_2 = \{f_0, f_1, f_{00}, f_{10}, f_{01}, f_{11}\}$ as in (5).

We show below (in the proof of Theorem 1.1) that assumptions (i) and (iii) of Theorem 1.1 guarantee that the functions $f \in S_\xi$ are well defined. This fact, and assumption (iv) in Theorem 1.1, are equivalent to assumption (b) of Mikulevicius and Platen. Also, it is clear that regularity conditions for a and b are identical for both convergence results. The results differ in their requirements for c and the payoff g. Theorem 1.1 allows for discontinuous c, though it imposes stronger requirements on g. Next, we will show explicitly that the requirements for c in Mikulevicius and Platen indeed imply assumptions (i) and (ii) in Theorem 1.1. Though not necessary for the proof of Theorem 1.1, this observation helps clarify the relationship between this result and the one of Mikulevicius and Platen.

PROPOSITION 1.1 *If the function c in (1) is $2(\gamma+1)$-times continuously differentiable with derivatives uniformly bounded by a constant C, then assumptions (i) and (ii) in Theorem 1.1 are satisfied.*

Proof. We denote by $1_k \in \Re^M$ the vector with kth component equal to one, and the rest equal to zero. For (i) in Theorem 1.1 observe that

$$\frac{\partial \bar{\phi}(x)}{\partial x_k} = \frac{\partial}{\partial x_k} \int_{R^d} (x + c(x,z))\, h(z)\, dz = \int_{R^d} (1_k + \frac{\partial}{\partial x_k} c(x,z))\, h(z)\, dz, \quad k = 1 \ldots, M \tag{6}$$

where the boundedness of $\frac{\partial}{\partial x_k} c(x,z)$ has allowed us to invoke the Bounded Convergence Theorem and exchange differentiation and integration to show that $\frac{\partial \bar{\phi}(x)}{\partial x_k}$ exists, and can be written as in the rightmost expression in (6). Furthermore, since $\frac{\partial}{\partial x_k} c(x,z)$ is uniformly bounded, and $h(z)$ is a probability density, then $\frac{\partial \bar{\phi}(x)}{\partial x_k}$ is also uniformly bounded. The same argument applies to show that derivatives of $\bar{\phi}$ up to order $2(\gamma+1)$ exist and are uniformly bounded. Therefore, assumption (i) in Theorem 1.1 holds.

For assumption (ii) in Theorem 1.1 we take $\psi \in B^{\xi}(\Psi)$. That ϕ in (2) is uniformly bounded follows from the fact that ψ is uniformly bounded. Next, we consider the derivatives of ϕ. Differentiating, we have that

$$\left|\frac{\partial}{\partial x_k}\psi(x+c(x,z))\right| = \left|\nabla\psi \cdot (1_k + \frac{\partial c(x,z)}{\partial x_k})\right|$$

which is bounded by $M\Psi(1+C)$. Therefore, we can invoke the Bounded Convergence Theorem to write

$$\frac{\partial \phi(x)}{\partial x_k} = \frac{\partial}{\partial x_k}\int_{R^d}\psi(x+c(x,z))\,h(z)\,dz = \int_{R^d}\frac{\partial}{\partial x_k}\psi(x+c(x,z))\,h(z)\,dz$$

Furthermore, $|\frac{\partial\phi(x)}{\partial x_k}| < M\Psi(1+C)$. Computation of the bound for higher order derivatives is straightforward. Therefore, assumption (ii) in Theorem 1.1 holds. $\Box$

This verifies that hypotheses (i) and (ii) in Theorem 1.1 are implied by the regularity condition for c in Theorem 3.3 of Mikulevicius and Platen [3]. But, as mentioned above, Theorem 1.1 is obtained under more restrictive conditions for g than in the result of Mikulevicius and Platen [3]. Thus, neither complete set of conditions implies the other.

Next we present a proof of Theorem 1.1. The proof holds in fact for the entire hierarchy of schemes proposed in Mikulevicius and Platen [3], which have arbitrarily high orders of convergence. Schemes of order higher than two are constructed using the funcions f in S_ξ which are defined in a recursive way below.

2 Proof of Theorem 1.1

The result will follow from the proof of Theorem 3.3 of Mikulevicius and Platen [3] once we establish that two key properties used in their proof hold in our setting as well: the existence of a stochastic Taylor formula, and smoothness of the solution v of a backward Kolmogorov equation and associated functionals.

The stochastic Taylor formula of Mikulevicius and Platen [3] requires the existence of a set of coefficient functions $f \in S_{\xi+1}$. The functions in $S_{\xi+1}$ are defined through up to $\xi+1$ recursive applications of the differential operators

$$\Pi^0 f(x) \equiv \sum_{j=1}^{M}\tilde{a}_j(x)\partial_j f(x) + 1/2\sum_{i,r=1}^{M} b_j(x)b_r(x)\partial_{jr}f(x), \quad \text{and} \quad \Pi^1 \equiv \sum_{j=1}^{M} b_j(x)\partial_j f(x), \qquad (7)$$

and involve derivatives up to order 2ξ. The functions in S_1 and S_2 are displayed in (4) and (5), and appear in condition (iv) of Theorem 1.1.

The solution of a Kolmogorov equation used in Lemma 4.3 of Mikulevicius and Platen [3] is

$$v(s,x) \equiv E\Big[g(X(T))|X(s)=x\Big],$$

and we will also use the functional

$$\Pi v(s,x) \equiv \int_{R^d} \Big[v(s,x+c(x,z)) - v(s,x)\Big]\, h(z)\ \lambda_0\, dz. \tag{8}$$

The following lemma presents sufficient conditions on $g, v, \Pi v$ and the coefficients functions $f \in S_{\xi+1}$ in order to prove Theorem 1.1 using the approach of Mikulevicius and Platen [3].

LEMMA 2.1 *Fix $\xi \in \{1,2\}$. Let the payoff function $g : \Re^M \to \Re$ be in $B^\xi(G)$ and let $\{X(t), t \in [0,T]\}$ be as in (1). We assume:*

(i) $v(s,x)$ and $\Pi v(s,x)$ are $2(\xi+1)$-times continuously differentiable on the initial condition x;

(ii) a and b are $2(\xi+1)$-times continuously differentiable with uniformly bounded derivatives;

(iii) the functions $f \in S_{\xi+1}$ are well defined;

(iv) there is a constant K_2 such that any $f \in S_\xi$ satisfies $|f(y)| \le K_2(1+\|y\|)$, with $S_1 = \{f_0, f_1\}$ as in (4) and $S_2 = \{f_0, f_1, f_{00}, f_{10}, f_{01}, f_{11}\}$ as in (5).

Then the approximation defined by (28), (29), and (30) in [1] has weak convergence order one and the approximation defined by (31) and (32) in [1] has weak convergence order two.

This follows from the proof of Theorem 3.3 of Mikulevicius and Platen [3] because its conditions imply the conditions needed for the proof. In more detail, $g \in B^\xi(G)$ implies that g is $2(\xi+1)$-times continuously differentiable and with polynomial growth, as assumed in Theorem 3.3 of [3]. Assumption (i) ensures the conclusions of Lemma 4.3 and Lemma 4.8 in [3]. Assumptions (ii) and (iv) are explicit in Theorem 3.3 of [3]. Assumption (iii) is needed for the existence of a stochastic Taylor formula.

Armed with Lemma 2.1, it is clear that to prove Theorem 1.1 it will suffice to show that the assumptions of Theorem 1.1 imply the assumptions of Lemma 2.1. In particular, we need to check that the functions f in $S_{\xi+1}$ are well defined and that smoothness properties of v and Πv hold. Verification of these two properties divides the proof into two parts.

Part 1 of proof. As mentioned above, the stochastic Taylor formula requires that the coefficient functions $f \in S_{\xi+1}$, computed through the repeated application of Π^0 and Π^1 (7), are well defined. In order to be able to apply Π^0 and Π^1 it will suffice to show 2ξ-times continuous differentiability of

$$\tilde{a}(x) = a(x) - \int_{R^d} c(x,z)\, h(z)\, \lambda_0\, dz.$$

From (3) we have that

$$\int_{R^d} c(x,z)\,h(z)\,\lambda_0\,dz = \lambda_0\,\bar{\phi}(x) - \lambda_0\,x$$

leading to

$$\tilde{a}(x) = a(x) - \lambda_0\,\bar{\phi}(x) + \lambda_0\,x\,. \tag{9}$$

Therefore, both $\tilde{a}$ and the right hand side of the equality are $2(\xi+1)$ -times continuously differentiable, the latter by assumptions (i) and (iii) in Theorem 1.1.

This ends the first part of the proof. Next we introduce an auxiliary result that we will later use to prove the $2(\xi+1)$-fold continuous differentiability of the solution v of a Kolmogorov equation and associated functionals (8).

We have assumed in Theorem 1.1 that a, b and $\bar{\phi}$ are $2(\xi+1)$-times continuously differentiable with bounded derivatives, so (9) implies that the derivatives of $\tilde{a}$ are also bounded. We have also assumed in Theorem 1.1 that both $f_0 = \tilde{a}(y)$ and $f_1 = b(y)$ are of linear growth. Therefore, the assumptions of the following auxiliary lemma are implicit in the hypotheses of Theorem 1.1.

LEMMA 2.2 *Let $Z(t) \in \Re^M$ be an Ito process, $t \in [0,T], Z(0) = x$ a.s., with*

$$dZ(t) = \tilde{a}(Z(t))\,dt + b(Z(t))\,dW(t) \tag{10}$$

where the functions $\tilde{a}$, b are $2(\xi+1)$-times continuously differentiable with uniformly bounded partial derivatives. Let a, b be of linear growth; i.e., $\|\tilde{a}(y)\| + \|b(y)\| \leq K_2\,(1+\|y\|)$ for some constant K_2. Let $g : \Re^M \to \Re$ be in $B^\xi(G)$ and define $\phi^D(s,t,x) = E[g(Z(t))|Z(s)=x]$. Then $\phi^D(s,t,\cdot) \in B^\xi(D(\xi)\,G)$ where $D(\xi)$ is independent of g

Proof of lemma. We need to prove that ϕ^D is bounded and that $\phi^D(s,t,\cdot)$ has continuous bounded partial derivatives up to order $2(\xi+1)$. That $|\phi^D| \leq G$ follows from the fact that $|g| \leq G$.

We analyze the derivatives of ϕ^D within the framework of Chapter V of Krylov [2] in which, under technical conditions, it is possible to exchange differentiation and expectation. We introduce the following notation. The vector $\eta^{k_1 \ldots k_n}(t) \in \Re^M$ is obtained by formally differentiating $Z(t)$ with respect to the components $x_{k_1}, \ldots, x_{k_n}$ of x, $n \in \{1, \ldots, 2(\xi+1)\}$. These processes will be derivatives of $Z(t)$ in probability though we will not need to check this explicitly. We need to show that these processes are *solvable by Euler's method in the mean* (SEM) as defined in V.3 of Krylov [2]. The dynamics of the i-th component of the first order derivative process is

$$d\eta_i^{k_1} = (\nabla\tilde{a}_i \cdot \eta^{k_1})\,dt + (\nabla b_i \cdot \eta^{k_1})\,dW \tag{11}$$

with $\eta_{k_1}^{k_1}(0) = 1, \eta_j^{k_1}(0) = 0$ for $j \neq k_1$, $\nabla\tilde{a}_i$, $\nabla b_i \in \Re^M$. The components of the second order derivative processes evolve as

$$d\eta_i^{k_1 k_2} = (\eta^{k_2^\top} \cdot (\nabla\nabla\tilde{a}_i) \cdot \eta^{k_1}) + \nabla\tilde{a}_i \cdot \eta^{k_1 k_2})dt + (\eta^{k_2^\top} \cdot (\nabla\nabla b_i) \cdot \eta^{k_1} + \nabla b_i \cdot \eta^{k_1 k_2})\,dW$$

with $\eta^{k_1 k_2}(0) = 0$ (and $\nabla\nabla$ denoting the Hessian). In general, for each $k_1, \ldots, k_n$ through repeated differentiation we define a^* and b^* and then

$$d\eta_i^{k_1 k_2 \ldots k_n} = \tilde{a}_i^*\, dt + b_i^*\, dW$$

with $\eta^{k_1 k_2 \ldots k_n}(0) = 0$ for derivatives of order higher than two. It is easy to check, as is clear for the lowest two derivatives, that for a derivative of order n, both $\tilde{a}^*$ and b^* are polynomials in the derivative processes of order less than n and affine functions of $\eta^{k_1 k_2 \ldots k_n}$. Viewing a^* and b^* as functions of $\eta^{k_1 k_2 \ldots k_n}$ with all other derivative processes held fixed, we therefore have

$$\begin{aligned}
&\|\tilde{a}^*(\eta^{k_1 k_2 \ldots k_n})\| + \|b^*(\eta^{k_1 k_2 \ldots k_n})\| \le K_1(1 + \|\eta^{k_1 k_2 \ldots k_n}\|), \\
&\|\tilde{a}^*(\eta^{k_1 k_2 \ldots k_n}) - \tilde{a}^*(\eta^{k_1 k_2 \ldots k_n} + h)\| + \|b^*(\eta^{k_1 k_2 \ldots k_n}) - b^*(\eta^{k_1 k_2 \ldots k_n} + h)\| \le K_2\|h\| \qquad (12)
\end{aligned}$$

where $K_1,\ K_2$ are independent of $\eta^{k_1 k_2 \ldots k_n}$. These are Lipschitz and linear growth conditions. Furthermore, $\tilde{a}^*$ and b^* are continuously differentiable in $\eta^{k_1 k_2 \ldots k_n}$. Remark V.7.3 in Krylov [2] and the hypotheses of Lemma 2.2 imply that the system of equations formed by (10) and (11) is SEM. Then, as in Remark V.3.5 of [2], we can inductively add higher order derivatives to the system. These derivatives satisfy the regularity conditions (12) so the expanded system is SEM. Let η be the vector formed by *all* derivatives processes up to order $2(\xi + 1)$. It follows from Remark V.3.2 in [2] that for any positive p there exist positive constants q, M^* such that $E[\|\eta(t)\|^p] \le M^*(1 + \|\eta(0)\|^q)$. The norm of each derivative process at time 0 is bounded by one, so M^* may be chosen to satisfy $E[\|\eta^{k_1 \ldots k_n}(t)\|^p] \le M^*$.

We consider now $\partial\phi^D/\partial x_{k_1}$. By Lemma V.7.1 of [2], the hypotheses of Lemma 2.2 and the fact that the derivative processes are SEM, we have that ϕ^D is continuously differentiable and we may exchange differentiation and expectation to get

$$\frac{\partial \phi^D}{\partial x_{k_1}}(t) \;=\; E[\nabla g \cdot \eta^{k_1}(t)] \le E[\|\nabla g\|\|\eta^{k_1}(t)\|] \le (E[\|\nabla g\|^2])^{\frac{1}{2}} (E[\|\eta^{k_1}(t)\|^2])^{\frac{1}{2}}$$

where last step is the Cauchy-Schwarz inequality. Also, $(E\|\nabla g\|^2)^{\frac{1}{2}} \le G d^{\frac{1}{2}}$ because $\nabla g \in \Re^M$, $g \in B^\xi(G)$. We also have that $E\|\eta^{k_1}(t)\|^2$ is bounded as shown above. Therefore $|\partial\phi^D/\partial x_{k_1}(t)| \le GD$ with D independent of g.

Next we consider second order derivatives. These are

$$\frac{\partial^2 \phi^D}{\partial^2 x_{k_1} x_{k_2}} = \frac{\partial}{\partial x_{k_2}} E[\nabla g \cdot \eta^{k_1}]. \qquad (13)$$

The quantity between brackets in (13) is of polynomial growth, and the derivative processes $\eta^{k_1}, \eta^{k_1 k_2}$ satisfy the hypotheses of Lemma V.7.1 in [2]. This, again, ensures continuous differentiability of $\partial\phi^D/\partial x_{k_1}$ and allows us to interchange differentiation and expectation to get

$$\begin{aligned}
E[\eta^{k_2^\top} \cdot (\nabla\nabla g) \cdot \eta^{k_1} + \nabla g \cdot \eta^{k_1 k_2}] &\le E[\|\eta^{k_2^\top} \cdot (\nabla\nabla g) \cdot \eta^{k_1} + \nabla g \cdot \eta^{k_1 k_2}\|] \\
&\le E[\|\eta^{k_2^\top} \cdot (\nabla\nabla g) \cdot \eta^{k_1}\|] + E[\|\nabla g \cdot \eta^{k_1, k_2}\|].
\end{aligned}$$

All partial derivatives of g have been assumed bounded by G and we have shown before that the norm of the derivative processes have bounded moments for finite time t. Thus, Hölder's inequality and some algebra lead to $|\partial^2\phi^D/\partial^2 x_{k_1}x_{k_2}(t)| \leq GD$ with D independent of g.

We avoid presenting here the cumbersome but straightforward computations that generalize the result to higher derivatives. The proof repeatedly uses the regularity of the derivative processes to apply Lemma V.7.1 in [2]. These computations are analogous to those made explicit above and prove that, for finite ξ, partial derivatives of ϕ^D up to order $2(\xi+1)$ exist, are continuous, and bounded in absolute value by $D(\xi, G) = D(\xi)G$. □

Part 2 of proof. We continue now with the proof of Theorem 1.1. The remaining step is the $2(\xi+1)$-fold continuous differentiability of the solution v of a Kolmogorov equation and associated functionals (8). That is, we need to show that

$$\begin{aligned} v(s,x) &= E\Big[g(X(T))|X(s)=x\Big], \\ \Pi v(s,x) &= \int_{R^d} \Big[v(s,x+c(x,z)) - v(s,x)\Big]\, h(z)\,\lambda_0\,dz \end{aligned}$$

are $2(\xi+1)$-times continuously differentiable in the initial condition x.

We begin with $v(s,x)$. Let N be the number of points in $[s,T]$ of the Poisson random measure in (1), with strictly increasing jump times $\{\tau_1,\ldots,\tau_N\}$, $N<\infty$ a.s. We take the paths of X to be right-continuous and write $X(\tau_j-)$ for $\lim_{t\to\tau_j-} X(t)$. Conditioning on the jump times we define

$$v_n(s,s_1,\ldots,s_n,x) = E\left[g(X(T))|X(s)=x, N=n, \tau_i=s_i, i=1,\ldots,n\right]$$

with $s \leq s_1 < \ldots < s_n$. We show by induction in the number of jumps that $v_n(s,s_1,\ldots,s_n,\cdot)$ is in $B^\xi(K^nD^{n+1}G)$ for all $s,s_1,\ldots,s_n$, with G and K as in Theorem 1.1. For $n=1$ we have

$$\begin{aligned} &v_1(s,s_1,x) \\ &\quad = E\left[E\left[E[g(X(T))|X(s_1),N=1,\tau_1=s_1]\Big|X(s_1-),N=1,\tau_1=s_1\right]\Bigg|X(s)=x,N=1,\tau_1=s_1\right]. \end{aligned}$$

The innermost expectation is computed conditional on no jumps in $(s_1,T]$; in the notation of Lemma 2.2, it is $\phi^D(s_1,T,X(s_1))$, which is in $B^\xi(D\,G)$. Thus,

$$v_1(s,s_1,x) = E\Big[E\Big[\phi^D(s_1,T,X(s_1))|X(s_1-),N=1,\tau_1=s_1\Big]\,|X(s)=x,N=1,\tau_1=s_1\Big].$$

Since $\phi^D(s_1,T,\cdot)$ is in $B^\xi(DG)$, by hypothesis (ii) of Theorem 1.1 the inner conditional expectation is in $B^\xi(KDG)$. The outer conditional expectation is computed conditional on no jumps in $[s,s_1)$ so again applying Lemma 2.2 we conclude that $v_1(s,s_1,\cdot)\in B^\xi(KD^2G)$.

For the inductive step define $S_n = \{s_1,\ldots,s_n\}$ and $\Theta_n = \{\tau_1,\ldots,\tau_n\}$. Take as induction hypothesis that

$$v_{n-1}(s,s_1,\ldots,s_{n-1},x) = E\left[g^*(X(t))|X(s)=x,N=n-1,\Theta_{n-1}=S_{n-1}\right]$$

belongs to $B^\xi(K^{n-1}D^nG^*)$ for any fixed $t \le T$ and $g^* \in B^\xi(G^*)$. Now

$$v_n(s,s_1,\ldots,s_n,x) = E\left[g(X(T))|X(s)=x, N=n, \Theta_n = S_n\right]$$
$$= E\left[E\left[E[g(X(T))|X(s_n), N=n, \tau_n = s_n]\Big|X(s_n-), N=n, \tau_n = s_n\right]\Bigg|X(s)=x, N=n, \Theta_n = S_n\right].$$

The same argument as in the case of one jump applies for the two innermost expectations, allowing us to write

$$v_n(s,s_1,\ldots,s_n,x) = E\left[\phi(s_n, X(s_n-))|X(s)=x, \Theta_{n-1} = S_{n-1}\right]$$

for some $\phi(s_n,\cdot)$ in $B^\xi(KDG)$. For the last expectation we apply the induction hypothesis with $G^* = KDG$ to conclude that $v_n(s,s_1,\ldots,s_n,x) \in B^\xi(K^nD^{n+1}G)$.

Next we integrate over the jump times and write

$$v(s,x,n) = \int \ldots \int q_n(s_1,\ldots,s_n) v_n(s,s_1,\ldots,s_n,x)\, ds_1,\ldots,ds_n,$$

where q_n is the joint density of the jump times in $[s,T]$ of the Poisson random measure, conditional on $N=n$. Because $v(s,s_1,\ldots,s_n,x) \in B^\xi(K^nD^{n+1}G)$, the Bounded Convergence Theorem allows us to interchange differentiation (in x) and integration and conclude that the derivatives of $v(s,x,n)$ up to order $2(\xi+1)$ exist and are continuous. Furthermore, $v_n(s,s_1,\ldots,s_n,\cdot) \in B^\xi(K^nD^{n+1}G)$ implies that $v(s,x,n) \in B^\xi(K^nD^{n+1}G)$ too.

Finally we treat $v(s,x) = E[g(X(T))|X(s)=x]$. This can be written as

$$v(s,x) = \sum_{n=0}^{\infty} P(N=n)v(s,x,n), \quad \text{with} \quad P(N=n) = \frac{e^{-\lambda_0(T-s)}(\lambda_0(T-s))^n}{n!}.$$

Any series of the form $\sum_{n=0}^{\infty} P(N=n)f_n$ with $|f_n| \le C^n$ for some constant C is absolutely convergent. Therefore $v(s,x)$ is bounded. Notice that $\frac{\partial^m v(s,x,n)}{\partial x_{k_1}\ldots\partial x_{k_m}}$ is continuous and that

$$\sum_{n=0}^{\infty} P(N=n)\frac{\partial^m v(s,x,n)}{\partial x_{k_1}\ldots\partial x_{k_m}} \le \sum_{n=0}^{\infty} |P(N=n)\frac{\partial^m v(s,x,n)}{\partial x_{k_1}\ldots\partial x_{k_m}}| \le \sum_{n=0}^{\infty} P(N=n)K^nD^{n+1}G = C < \infty.$$

Then $\sum_{n=0}^{\infty} P(N=n)\frac{\partial^m v(s,x,n)}{\partial x_{k_1}\ldots\partial x_{k_m}}$ converges uniformly and is continuous. Therefore

$$\frac{\partial^m v(s,x)}{\partial x_{k_1}\ldots\partial x_{k_m}} = \sum_{n=0}^{\infty} P(N=n)\frac{\partial^m v(s,x,n)}{\partial x_{k_1}\ldots\partial x_{k_m}}$$

which implies that $\frac{\partial^m v(s,x)}{\partial x_{k_1}\ldots\partial x_{k_m}}$ exists and is continuous. Furthermore, $v(s,x) \in B^\xi(C)$.

To complete the proof of the theorem we need to show that

$$\Pi v(s,x) = \lambda_0 \int_{R^d} v(s, x + c(x,z))\, h(z)\, dz - \lambda_0\, v(s,x)$$

is $2(\xi+1)$-times continuously differentiable in x. We only need to consider the first term. The function $v(s,\cdot)$ is in $B^\xi(C)$ so by hypothesis (ii) of Theorem 1.1 the integral (viewed as a function of x) is in $B^\xi(KC)$ and in particular is $2(\xi+1)$-times continuously differentiable. $\square$

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