Saddlepoint Approximations for Affine Jump-Diffusion Models

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Abstract

Affine jump-diffusion (AJD) processes constitute a large and widely used class of continuous-time asset pricing models that balance tractability and flexibility in matching market data. The prices of e.g., bonds, options, and other assets in AJD models are given by extended pricing transforms that have an exponential-affine form; these transforms have been characterized in great generality by Duffie, Pan and Singleton [28]. Calculating model prices requires inversion of these transforms, and this has limited the application of AJD models to the comparatively small subclass for which the transforms are available in closed form. This article seeks to widen the scope of AJD models amenable to practical application through approximate transform inversion techniques. More specifically, we develop the use of saddlepoint approximations for AJD models. These approximations facilitate the calculation of prices in AJD models whose transforms are not available explicitly. We derive and test several alternative saddlepoint approximations and find that they produce accurate prices over a wide range of parameters.

JEL classification: C13; C32; C49

Keywords: Transform inversion; Characteristic function; Option prices; Numerical approximations

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1 Introduction

Affine jump-diffusion (AJD) processes constitute a large class of continuous-time asset pricing models that balance tractability and flexibility in matching market data. In an AJD model, the drift vector, the diffusion matrix and the jump intensity all have affine dependence on the state vector. As shown by Duffie, Pan and Singleton [28], this restriction leads to considerable tractability in term structure modeling and option pricing, while at the same time allowing model features like state-dependent conditional variances and flexible correlations between state variables that are absent from simpler models. The objective of this article is to further expand the scope of tractable AJD models through the use of approximate transform inversion techniques.

The AJD family of models includes many widely used special cases, such as the Gaussian model of Vasicek [49], the square-root diffusion of Cox, Ingersoll and Ross [20], the Heston [36] stochastic volatility model, and extensions of these models to include jumps. AJD processes have been used extensively in empirical work, including, for example, Bakshi, Cao and Chen [3], Bates [6, 7], Broadie, Chernov and Johannes [11], Chernov [15], Duffie, Pedersen and Singleton [29], Duffie and Singleton [30], Eraker [32], Eraker, Johannes and Polson [33] and Pan [44]. The yield factor models of Dai and Singleton [21] and Duffie and Kan [26] fall within the AJD family. Duffie, Filipović and Schachermayer [25] develop the theoretical foundations of AJD processes. A detailed account of the econometric aspects of AJD models is given in Singleton [48].

As demonstrated in Duffie, Pan and Singleton [28] (henceforth DPS), the tractability of AJD models lies in the special form taken by a wide class of transforms, including various Fourier and Laplace transforms as special cases. These transforms have an exponential-affine form, meaning that they are exponentials of affine functions of the state vector; the coefficients of these affine functions are in some cases available explicitly and, more generally, can be characterized through solutions of ordinary differential equations. Through their transform analysis, DPS derive what could be viewed as a far-reaching generalization of the Black-Scholes formula for option prices. This makes the AJD family of models particularly attractive for empirical studies that combine option prices with time series data on underlying prices or rates. Studies of this type include Andersen, Benzoni and Lund [2], Bakshi, Cao and Chen [3], Bates [6, 7, 8], Broadie, Chernov and Johannes [11], Chen and Scott [13], Chernov [15], Chernov and Ghysels [16], Eraker [32], Eraker, Johannes and Polson [33] and Pan [44].

Despite the many examples of studies using AJD models, the models used in empirical work have remained limited to a relatively small subclass for which the pricing transforms are available in closed form. This restriction appears to be driven more by convenience of implementation than by considerations of empirical validity. In the general framework of DPS, the pricing transforms are characterized in terms of solutions of ordinary differential equations (ODEs). The AJD models used in practice (such as those of Cox, Ingersoll and Ross [20] and Heston [36]) are those for which these ODEs can be solved explicitly, thus providing explicit expressions for the pricing transforms. In this setting, each model-price calculation requires the numerical inversion of a closed-form transform, which can be accomplished with relatively modest computational effort.

For more general AJD models — those for which the pricing transforms are not available in closed form — each price calculation requires, in principle, embedding the numerical solution of a system of ODEs within a numerical inversion routine. Numerical transform inversion is a numerical integration problem that typically uses hundreds or thousands of evaluations of the transform, and each such function evaluation requires the solution of a system of ODEs. It is the impracticality of this combination that has limited the application of AJD models to the most tractable cases.

In this article, we develop the use of saddlepoint approximations as alternatives to numerical transform inversion in order to widen the scope of practical AJD models. The saddlepoint method is rooted in asymptotic expansions for evaluating contour integrals in the complex plane. It was introduced in statistics by Daniels [22] to approximate the probability density function of the sum of independent random variables. Lugannani and Rice [42] derive a saddlepoint approximation for the distribution function. See Daniels [23] and Jensen [37] for overviews of applications in statistics. Rogers and Zane [45] apply saddlepoint approximations to option pricing; applications in credit risk include Dembo, Deuschel and Duffie [24], Gordy [35], Martin, Thompson and Browne [43], and Yang, Hurd and Zhang [52]. Aït-Sahalia and Yu [1] derive saddlepoint approximations for transition densities of continuous-time Markov processes with applications to statistical inference. In the affine framework, Collin-Dufresne and Goldstein [18] use Edgeworth expansions for swaption pricing. Saddlepoint approximations also have potential applicability to risk management in the setting of Duffie and Pan [27].

Saddlepoint approximations rely on the solution to an equation defined by the derivative of the transform to be inverted; this solution is the saddlepoint. We investigate various ways of computing or approximating the saddlepoint in the setting of AJD models. We also compare alternative versions of saddlepoint approximations for price calculations. We find that saddlepoint approximations do indeed provide an effective way to calculate prices in AJD models whose ODEs do not admit explicit solutions.

This paper consists of five sections. After this introductory section, in Section 2 we present the extended transforms of AJD models that are necessary in calculating the derivatives used in the approximations. In Section 3, we review the saddlepoint method and associated approximations, and we explain how the saddlepoint method applies to AJD models. We test the approximations numerically in Section 4, and find that saddlepoint techniques yield surprisingly small relative errors over a wide range of parameters. We conclude the paper in Section 5.

2 Affine Jump-diffusion Model and Extended Transforms

We start by reviewing basic facts about AJD processes. Following the notation in DPS, an AJD process $X \in \mathbb{R}^n$ is defined as a solution of the stochastic differential equation (SDE)

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + dZ_t$$

where W is an $(\mathcal{F}_t)$-adapted Brownian motion in $\mathbb{R}^n$, $\mathcal{F}_t$ stands for the σ-field of information sets available up to time t, and Z is a pure jump process whose jumps have a fixed probability distribution ν on $\mathbb{R}^n$ and arrive with intensity $\lambda(X_t)$. The asset price of interest, S_t, at time t is assumed to be $(\overline{a}_t + \overline{b}_t \cdot X_t)\exp(a_t + b_t \cdot X_t)$ for deterministic $\overline{a}_t$, $\overline{b}_t$, a_t and b_t; for simplicity we assume $S_t = e^{d \cdot X_t}$. The more general case can be reduced to this case at the expense of introducing time-dependency in the characteristics of X defined below. The dynamics of other assets, stochastic interest rates or stochastic volatility can be included as coordinates of the vector-valued process X. The functional forms of $\mu(X_t)$, $\sigma(X_t)$, $\lambda(X_t)$ and the interest rate $r(X_t)$ are specified as follows:

$$\begin{aligned}
&\mu(x) = K_0 + K_1 x, \quad K_0 \in \mathbb{R}^n,\ K_1 \in \mathbb{R}^{n\times n},\ x \in \mathbb{R}^n \\
&(\sigma(x)\sigma(x)^\top)_{ij} = H_{0ij} + H_{1ij} \cdot x, \quad H_{0ij} \in \mathbb{R},\ H_{1ij} \in \mathbb{R}^n \\
&\lambda(x) = l_0 + l_1 \cdot x, \quad l = (l_0, l_1) \in \mathbb{R} \times \mathbb{R}^n \\
&r(x) = \rho_0 + \rho_1 \cdot x, \quad \rho = (\rho_0, \rho_1) \in \mathbb{R} \times \mathbb{R}^n \\
&\theta(c) = \int_{\mathbb{R}^n} \exp(c \cdot z) d\nu(z) \quad \text{for } c \in \mathbb{C}^n,\ \text{“jump transform”}.
\end{aligned}$$

The process X is said to have the characteristic (K, H, l, θ, ρ).

The state variable X_t at time t takes values in a domain $D \subset \mathbb{R}^n$ on which the process is defined. For instance, $(\sigma(X_t)\sigma(X_t)^\top)_{ii}$ should be non-negative for each i. A discussion of the state space D and constraints on the charateristic of X can be found in Chapter 5 of Singleton [48], and Duffie, Filipović and Schachermayer [25] deal with this issue in a more general framework. The definition above implies that the process X is Markovian and that when a jump occurs, its jump size is independent of the jump arrival rate or the past history of X.

In DPS, the authors prove that certain Fourier-type transforms of an AJD process can be found by solving the following set of ODEs:

$$\dot\beta(t) = -\rho_1 + K_1^\top \beta(t) + \frac{1}{2}\beta(t)^\top H_1 \beta(t) + l_1(\theta(\beta(t)) - 1) \tag{1}$$

$$\dot\alpha(t) = -\rho_0 + K_0 \cdot \beta(t) + \frac{1}{2}\beta(t)^\top H_0 \beta(t) + l_0(\theta(\beta(t)) - 1) \tag{2}$$

$$\dot B(t) = K_1^\top B(t) + \beta(t)^\top H_1 B(t) + l_1 \nabla\theta(\beta(t)) B(t) \tag{3}$$

$$\dot A(t) = K_0 \cdot B(t) + \beta(t)^\top H_0 B(t) + l_0 \nabla\theta(\beta(t)) B(t) \tag{4}$$

with $\beta(0) = u$, $\alpha(0) = 0$, $B(0) = v$, $A(0) = 0$ for some $u \in \mathbb{C}^n$, $v \in \mathbb{R}^n$, with $\nabla\theta(c)$ a row vector. These transforms facilitate the pricing of many financial derivatives such as European calls or puts, quanto options, Asian options and others using Fourier inversion. To apply saddepoint techniques, we will need ODEs that characterize cumulant generating functions (CGFs) and their derivatives. See DPS for the proof of the next theorem.

Theorem 2.1 (DPS) *Suppose the system of ODEs (1)–(4) has a unique solution and the other technical conditions in [28], p.1351, hold. Then*

$$\begin{aligned}\psi_0(u, X_t, t, T) &= \mathbb{E}\left[\exp\left(-\int_t^T r(X_s)ds\right) e^{u\cdot X_T}\Big|\mathcal{F}_t\right] \\ &= e^{\alpha(T-t)+\beta(T-t)\cdot X_t} \\ \psi_1(v, u, X_t, t, T) &= \mathbb{E}\left[\exp\left(-\int_t^T r(X_s)ds\right)(v\cdot X_T)e^{u\cdot X_T}\Big|\mathcal{F}_t\right] \\ &= \psi_0(u, X_t, t, T)\Big(A(T-t) + B(T-t)\cdot X_t\Big)\end{aligned}$$

where $u \in \mathbb{C}^n$, $v \in \mathbb{R}^n$, $t \le T$ *and the process* X *has the characteristic* (K, H, l, θ, ρ).

The integral that we shall consider in later sections is $\mathbb{E}[\exp(-\int_t^T r(X_s)ds)(b\cdot X_T)^k e^{(a+zb)\cdot X_T} \mid \mathcal{F}_t]$ for some $a, b \in \mathbb{R}^n$ and $z \in \mathbb{R}$. When $k = 0$ and $t = 0$, it becomes $\psi_0(a + zb, X_0, 0, T) = \exp(\alpha(T, z) + \beta(T, z)\cdot X_0)$. Note that here we include z to express the dependence of α, β on z through the initial conditions $\alpha(0, z) = 0$, $\beta(0, z) = a + zb$. If $k = 1$, $t = 0$, then by Theorem 2.1 we get $\psi_1(b, a+zb, X_0, 0, T) = (A(T, z)+B(T, z)\cdot X_0)\exp(\alpha(T, z)+\beta(T, z)\cdot X_0)$ with initial conditions $A(0, z) = 0$, $B(0, z) = b$. Provided we can interchange differentiation and expectation in

$$\frac{\partial\psi_0(a + zb, X_t, t, T)}{\partial z} = \frac{\partial}{\partial z}\mathbb{E}\left[\exp\left(-\int_t^T r(X_s)ds\right) e^{(a+zb)\cdot X_T}\Big|\mathcal{F}_t\right],$$

viewing ψ_0 as a function of two variables z and t, we get

$$\frac{\partial\alpha(T-t, z)}{\partial z} + \frac{\partial\beta(T-t, z)}{\partial z}\cdot X_t = A(T-t, z) + B(T-t, z)\cdot X_t$$

for all t and X_t, so we conclude $\partial\alpha(t, z)/\partial z = A(t, z)$, $\partial\beta(t, z)/\partial z = B(t, z)$. One condition that justifies the interchange of differentiation and integration is the finiteness of ψ_0 for some interval $z \in (-l, l)$ containing 0 as an interior point. This can be proved by the Dominated Convergence Theorem and the Mean Value Theorem; see, e.g., page 43 of Shreve [47]. By repeating the same argument, one can calculate the k-th partial derivative of ψ_0, $\partial^k\psi_0/\partial z^k$, by interchanging the order of differentiation and integration without changing the interval in which $\partial^k\psi_0/\partial z^k$ becomes finite.

Through this line of reasoning, we arrive at Theorem 2.3, below, and the following new set of ODEs:

$$\begin{aligned}\dot{D}(t) &= K_1^\top D(t) + \beta(t)^\top H_1 D(t) + l_1\nabla\theta(\beta(t))D(t) + B(t)^\top H_1 B(t) + l_1 B(t)^\top\nabla^2\theta(\beta(t))B(t) \quad (5)\\ \dot{C}(t) &= K_0\cdot D(t) + \beta(t)^\top H_0 D(t) + l_0\nabla\theta(\beta(t))D(t) + B(t)^\top H_0 B(t) + l_0 B(t)^\top\nabla^2\theta(\beta(t))B(t) \quad (6)\end{aligned}$$

with $\alpha(t)$, $\beta(t)$, $A(t)$, $B(t)$, $\nabla(\theta(c))$ as before, $C(0) = 0$, $D(0) = 0$, and $(\nabla^2\theta(c)_{i,j}) = (\int e^{c\cdot z}z_i z_j d\nu(z))$ the Hessian of $\theta(c)$. We also need the following technical conditions, which extend conditions in

DPS. The proof of Theorem 2.3 is based on showing that a certain process is a martingale; these conditions are useful in verifying the martingale property.

Definition 2.2 *(K,H,l,θ,ρ) is well-behaved at (v,u,T) if ODEs (1)–(6) are solved uniquely* [1]*, if θ is twice differentiable at $\beta(t)$ for all $t \le T$, and if the following conditions are satisfied:*

$$
\begin{aligned}
&(i) \qquad \mathbb{E}\Big[\int_0^T |\gamma(t)\lambda(X_t)|dt\Big] < \infty, \ \textit{where } \gamma(t) = \Big(\Phi_t'(\theta(\beta_t)-1) + 2\Phi_t \nabla\theta(\beta_t)B_t + \Psi_t B_t^\top \nabla^2\theta(\beta_t)B_t \\
&\qquad\qquad + \Psi_t'(\theta(\beta_t)-1) + \Psi_t\nabla\theta(\beta_t)D_t\Big) \\
&(ii) \qquad \mathbb{E}\Big[\Big(\int_0^T \eta(t)\cdot\eta(t)dt\Big)^{1/2}\Big] < \infty, \ \textit{where } \eta(t) = \Big(\Phi_t'\beta_t^\top + 2\Phi_t B_t^\top + \Psi_t'\beta_t^\top + \Psi_t D_t^\top\Big)\,\sigma(X_t) \\
&(iii) \qquad \mathbb{E}\Big[|\Phi_T' + \Psi_T'|\Big] < \infty
\end{aligned}
$$

Here Φ_t, Φ_t', Ψ_t, Ψ_t' are processes defined in the appendix and $\beta_t = \beta(T-t)$, $B_t = B(T-t)$, $D_t = D(T-t)$ for notational convenience. The next theorem is a natural extension of Theorem 2.1 and will play a key role in later sections.

Theorem 2.3 *Suppose (K,H,l,θ,ρ) is well-behaved at (v,u,T). Then*

$$
\begin{aligned}
\psi_2(v,u,X_t,t,T) &= \mathbb{E}\left[\exp\left(-\int_t^T r(X_s)ds\right)(v\cdot X_T)^2 e^{u\cdot X_T} \mid \mathcal{F}_t\right] \\
&= \psi_0(u,X_t,t,T)\Big((A(T-t)+B(T-t)\cdot X_t)^2 + (C(T-t)+D(T-t)\cdot X_t)\Big)
\end{aligned}
$$

where $v \in \mathbb{R}^n$, $u \in \mathbb{C}^n$, $t \le T$, the process X has the characteristic (K,H,l,θ,ρ).

Proof See the appendix. ∎

Again assuming that we can interchange the order of differentiation and expectation (for example, supposing $|\psi_0| < \infty$ for all $z \in (-l,l)$ for some l and treating ψ_0 as a function of z and t), we have

$$
\frac{\partial^2\psi_0(a+zb,X_t,t,T)}{\partial z^2} = \mathbb{E}\left[\exp\left(-\int_t^T r(X_s)ds\right)(b\cdot X_T)^2\, e^{(a+zb)\cdot X_T}\Big|\mathcal{F}_t\right] = \psi_2(b,a+zb,X_t,t,T);
$$

and from this we conclude

$$
\frac{\partial^2\alpha(t,z)}{\partial z^2} = C(t,z), \qquad \frac{\partial^2\beta(t,z)}{\partial z^2} = D(t,z).
$$

These transforms can be continued as long as we are working with a sufficiently well behaved AJD process. Indeed, it is easy to find a pattern in the related ODEs. From the relationships above between α, β, A, B, C and D and the corresponding ODEs (1)–(6), we observe that if we have a set of ODEs for the k-th derivative of ψ_0, then we get a new set of ODEs for the $(k+1)$-th derivative

[1] Conditions that ensure this are presented in Duffie et al. [25] in a more general framework.

just by differentiating the previous ODEs with respect to the variable z.[2] For a rigorous proof we would need to define suitable processes as in Theorem 2.3 and give some extended conditions to make the Brownian part and the jump part martingales. We write the next set of ODEs for later use.

Theorem 2.4 *Under the conditions in the appendix we have*

$$\begin{aligned}\psi_3(v,u,X_t,t,T) &= \mathbb{E}\left[\exp\left(-\int_t^T r(X_s)ds\right)(v\cdot X_T)^3 e^{u\cdot X_T}\Big|\mathcal{F}_t\right]\\ &= \psi_0(u,X_t,t,T)\Big((A(T-t)+B(T-t)\cdot X_t)^3+3(A(T-t)+B(T-t)\cdot X_t)\\ &\quad\times(C(T-t)+D(T-t)\cdot X_t)+(E(T-t)+F(T-t)\cdot X_t)\Big)\end{aligned}$$

where $v\in\mathbb{R}^n$, $u\in\mathbb{C}^n$, $t\le T$, *the process* X *has the characteristic* (K,H,l,θ,ρ) *and*

$$\begin{aligned}\dot F(t) &= K_1^\top F(t)+\beta(t)^\top H_1 F(t)+l_1\nabla\theta(\beta(t))F(t)\\ &\quad +3B(t)^\top H_1 D(t)+3l_1 B(t)^\top\nabla^2\theta(\beta(t))D(t)+l_1\int_{\mathbb{R}^n} e^{z\cdot\beta(t)}(z\cdot B(t))^3 d\nu(z) \qquad (7)\\ \dot E(t) &= K_0\cdot F(t)+\beta(t)^\top H_0 F(t)+l_0\nabla\theta(\beta(t))F(t)\\ &\quad +3B(t)^\top H_0 D(t)+3l_0 B(t)^\top\nabla^2\theta(\beta(t))D(t)+l_0\int_{\mathbb{R}^n} e^{z\cdot\beta(t)}(z\cdot B(t))^3 d\nu(z) \qquad (8)\end{aligned}$$

with $\alpha(t)$, $\beta(t)$, $A(t)$, $B(t)$, $C(t)$, $D(t)$, $\nabla(\theta(c))$, $\nabla^2(\theta(c))$ *as before, and* $E(0)=0$, $F(0)=0$.

Proof See the appendix. ∎

3 Saddlepoint Approximation and Option Pricing

3.1 Option Pricing

When we price options with the log of underlying asset following an AJD process, $S_t=e^{d\cdot X_t}$, the basic building block is

$$G_{a,b}(y;X_0,T)=\mathbb{E}\left[\exp\left(-\int_0^T r(X_s)ds\right)e^{a\cdot X_T}\mathbf{1}_{\{b\cdot X_T\le y\}}\right]$$

so that, as shown in DPS, a European call option price, for example, can be calculated as follows:

$$\begin{aligned}C(T,c) &= \mathbb{E}\left[\exp\left(-\int_0^T r(X_s)ds\right)(e^{d\cdot X_T}-c)^+\right]\\ &= \mathbb{E}\left[\exp\left(-\int_0^T r(X_s)ds\right)(e^{d\cdot X_T}-c)\mathbf{1}_{\{d\cdot X_T\ge \ln c\}}\right]\\ &= G_{d,-d}(-\ln c\,;X_0,T)-c\,G_{0,-d}(-\ln c\,;X_0,T).\end{aligned}$$

[2]This leads us to conjecture the functional form of $\mathbb{E}[\exp(-\int_t^T r(X_s)ds)(b\cdot X_T)^N e^{(a+zb)\cdot X_T}\mid\mathcal{F}_t]$ should be

$$\sum_{(m_1,\ldots,m_N):\sum km_k=N}\frac{N!}{m_1!m_2!\cdots m_N!}\psi_0(a+zb,X_t,t,T)\prod_{j:m_j\ne 0}\left(\frac{\partial^j\alpha}{j!\partial z^j}(T-t,z)+\frac{\partial^j\beta}{j!\partial z^j}(T-t,z)\cdot X_t\right)^{m_j}$$

from the Faà di Bruno's formula and the ODEs satisfied by $\partial^j\alpha/\partial z^j$, $\partial^j\beta/\partial z^j$ can be derived by applying the same formula to the ODEs (1), (2).

To facilitate the application of saddlepoint approximations, we will express this as a difference of two probabilities, after some possible scaling and change of measure. This will reduce the calculation of the option price to the task of calculating those probabilities. To this end, first suppose the characteristic (K, H, l, θ, ρ) of the AJD process X is well-behaved at (b, a, T). Then there exist $\tilde{\alpha}(t)$, $\tilde{\beta}(t)$ solving the ODEs (1), (2) in Theorem 2.1 with the boundary conditions $\tilde{\alpha}(0) = 0$, $\tilde{\beta}(0) = a$. On the other hand, it is easy to show, as noted in DPS, that

$$\xi_t = \exp\left(-\int_0^t r(X_s)ds\right) e^{\tilde{\alpha}(T-t)+\tilde{\beta}(T-t)\cdot X_t}$$

is a positive martingale, using Itô's formula and (1), (2). So an equivalent probability measure $\mathbb{Q}$ given by $d\mathbb{Q}/d\mathbb{P} = \xi_T/\xi_0$ is well defined. Also note that from the definition of ψ_0 in Section 2, $\psi_0(a, X_0, 0, T) = \mathbb{E}[\exp(-\int_0^T r(X_s)ds)e^{a\cdot X_T}] = \xi_0$. Thus the random variable $Y := b \cdot X_t$ has a moment generating function under $\mathbb{Q}$ given by

$$\begin{aligned} e^{\mathcal{K}(z)} &= \mathbb{E}^Q\left[e^{zY}\right] = \frac{1}{\xi_0}\mathbb{E}\left[\exp\left(-\int_0^T r(X_s)ds\right) e^{(a+zb)\cdot X_T}\right] \\ &= \frac{\psi_0(a+zb, X_0, 0, T)}{\psi_0(a, X_0, 0, T)} = \exp\Big(\alpha(T,z) - \alpha(T,0) + (\beta(T,z) - \beta(T,0))\cdot X_0\Big) \end{aligned}$$

where $\alpha(t,z)$, $\beta(t,z)$ denote the solutions of (1), (2) with $\alpha(0,z) = 0$, $\beta(0,z) = a + zb$ so that $\tilde{\alpha}(t) = \alpha(t,0)$, $\tilde{\beta}(t) = \beta(t,0)$.

The CGF of Y is $\mathcal{K}(z)$ under $\mathbb{Q}$. Unless Y is a constant almost surely, Y has a positive variance and so $\mathcal{K}(z)$ is strictly convex in z. Proposition 5 in DPS implies that X is again an AJD process under $\mathbb{Q}$ with the characteristic (K^Q, H, l^Q, θ^Q) where

$$\begin{aligned} &K_0^Q(t) = K_0 + H_0\tilde{\beta}(T-t), \quad K_1^Q(t) = K_1 + H_1\tilde{\beta}(T-t), \\ &l_0^Q(t) = l_0\theta(\tilde{\beta}(T-t)), \quad l_1^Q(t) = l_1\theta(\tilde{\beta}(T-t)), \\ &\theta^Q(c,t) = \theta(c+\tilde{\beta}(T-t))/\theta(\tilde{\beta}(T-t)). \end{aligned}$$

Finally we note that $G_{a,b}(y; X_0, T) = \mathbb{E}[\exp(-\int_0^T r(X_s)ds)e^{a\cdot X_T}1_{\{b\cdot X_T\le y\}}] = \xi_0\mathbb{Q}(Y \le y)$. So the option-pricing problem is reduced to the calculation of the cumulative distribution function (CDF) $\mathbb{Q}(Y \le y)$ or its complement $\mathbb{Q}(Y > y)$.

In the AJD setting, this tail probability can be represented through the Fourier inversion formula,

$$\mathbb{Q}(Y > y) = \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} e^{(\mathcal{K}(z)-zy)}\frac{dz}{z}, \quad \tau > 0.$$ [3]

Numerical calculation of this integral requires evaluation of the integrand at hundreds or thousands of points. Unless $\mathcal{K}(z)$ is available in closed form, we would need to solve the ODEs (1), (2) numerically at each evaluation point. This computational burden limits the scope of AJD models

[3] This can be shown using the Plancherel Theorem and the Dominated Convergence Theorem (see the appendix of Rogers and Zane [45]).

amenable to practical application and motivates our investigation of approximations. In the next subsection, we review the saddlepoint method and explain how we apply this method to option pricing in AJD models.

Remark For European call options, a simpler calculation is possible. To simplify the measure transform, suppose the short rate is a constant r. Then the option price is given by

$$\begin{aligned} C(T,c) &= \mathbb{E}\left[e^{-rT}(S_T-c)^+\right] \\ &= e^{-rT}\Big\{\mathbb{E}e^{X_T} - \mathbb{E}\left[e^{X_T}\wedge c\right]\Big\} \\ &= e^{-rT}\Big\{e^{\mathcal{K}(1)} - c\,\mathbb{P}(X_T+Y>\ln c)\Big\} \end{aligned}$$

where $S_T = e^{X_T}$, Y is exponentially distributed with unit mean, independent of X_T, and $e^{\mathcal{K}(z)} = \mathbb{E}[e^{zX_T}]$. So

$$\mathbb{E}\left[e^{z(X_T+Y)}\right] = e^{\mathcal{K}(z)}\frac{1}{1-z} = e^{\mathcal{K}(z)-\ln(1-z)}, \quad z<1.$$

This means we need to calculate only one tail probability. If we want to use the Fourier inversion formula, this reduces the workload by almost a half. A similar but different use of exponential density functions was made in Butler and Wood [12] to approximate the moment generating functions of truncated random variables.

3.2 Saddlepoint Approximation

Daniels [22] introduced the saddlepoint method to statistics in order to approximate the probability density function (PDF) of the mean of i.i.d. random variables X_i's. Assuming we know the CGF $\mathcal{K}(z)$ where $e^{\mathcal{K}(z)} = \mathbb{E}[e^{zX_1}]$, the PDF $f_n(\overline{x})$ of $\overline{X} = \sum_1^n X_i/n$ is given by

$$f_n(\overline{x}) = \frac{n}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} e^{n(\mathcal{K}(z)-z\overline{x})}dz, \qquad \text{for any } \tau\in\{x\in\mathbb{R}:|\mathcal{K}(x)|<\infty\}.$$

Daniels [22] used the method of steepest descent to expand this contour integral. The saddlepoint $\hat{z}$ is defined by the saddlepoint equation $\mathcal{K}'(\hat{z}) = \overline{x}$; the modulus of the integrand is minimized along the real axis at $\hat{z}$ and maximized at $\hat{z}$ along the contour parallel to the imaginary axis passing through $\hat{z}$. So, the region outside a neighborhood of the saddlepoint contributes little to the integration, and we get Daniels' formula through a Taylor expansion of the exponent $\mathcal{K}(z) - z\overline{x}$ around $\hat{z}$. (The method of steepest descent is explained in Chapter 7 of Bleistein and Handelsman [9].)

Lugannani and Rice [42] approximated tail probabilities rather than densities. The following form of the Lugannani-Rice (LR) formula can be found in Daniels [23]:

$$\mathbb{P}(\overline{X}>\overline{x}) = 1-\Phi(\sqrt{n}\hat{w}) + \phi(\sqrt{n}\hat{w})\left\{\frac{b_0}{n^{1/2}} + \frac{b_1}{n^{3/2}} + o(n^{-3/2})\right\} \tag{9}$$

where $b_0 = 1/\hat{u} - 1/\hat{w}$, $b_1 = (\lambda_4/8 - 5\lambda_3^2/24)/\hat{u} - \lambda_3/(2\hat{u}^2) - 1/\hat{u}^3 + 1/\hat{w}^3$ and $\hat{w} = \mathrm{sgn}(\hat{z})\sqrt{2(\hat{z}y-\mathcal{K}(\hat{z}))}$, $\hat{u} = \hat{z}\sqrt{\mathcal{K}''(\hat{z})}$, $\lambda_3 = \mathcal{K}^{(3)}(\hat{z})/\mathcal{K}''(\hat{z})^{3/2}$, $\lambda_4 = \mathcal{K}^{(4)}(\hat{z})/\mathcal{K}''(\hat{z})^{4/2}$. When $\overline{x} = \mathbb{E}[X_1] = \mathcal{K}'(0)$, the formula reduces to

$$\mathbb{P}(\overline{X}>\mathcal{K}'(0)) = \frac{1}{2} - \frac{\lambda_3(0)}{6\sqrt{2\pi n}} + O(n^{-3/2}). \tag{10}$$

Here Φ, ϕ are the CDF and the PDF of the standard normal distribution, respectively. We will use this formula with $n = 1$ and b_0 in test cases. The accuracy of the approximation (9) for small n depends on the proximity of the underlying distribution to the normal distribution. Wood, Booth and Butler [51] study the saddlepoint approximation with a non-normal distribution replacing Φ and ϕ for a better approximation. We will test such a variant with a stochastic volatility jump-diffusion model using a gamma distribution as the base distribution in the approximation.

To apply the LR formula (9), we need to find the solution $\hat{z}$ of the saddlepoint equation $\mathcal{K}'(z) = y$ for some given real number y and compute $\mathcal{K}(\hat{z})$ and its derivatives. In an AJD setting, from Section 2 we have

$$\begin{aligned}
\mathcal{K}(z) &= \alpha(T,z) - \alpha(T,0) + (\beta(T,z) - \beta(T,0)) \cdot X_0 \\
\mathcal{K}'(z) &= A(T,z) + B(T,z) \cdot X_0 \\
\mathcal{K}''(z) &= C(T,z) + D(T,z) \cdot X_0, \quad \text{etc.},
\end{aligned}$$

and these functions can be evaluated by solving a set of ODEs, the size of which depends on the order of derivatives one wants to compute. Once $\hat{z}$ is found, each system of ODEs need only be solved once. The total number of ODE solutions required depends on the approximation chosen through the number of derivatives of $\mathcal{K}(\hat{z})$ used. In contrast, numerical inversion of the characteristic function requires the solution of ODEs (1), (2) for each evaluation point in the numerical integration. Finding $\hat{z}$ is therefore critical to the method.

Under rather mild conditions, the saddlepoint equation $\mathcal{K}'(z) = y$ has a unique root. We will, in particular, impose the following two conditions on the AJD process X, option maturity T and real vectors a, b.

Assumption 1 *There exists an $l > 0$ such that $|\psi_0(a + zb, X_0, 0, T)| < \infty$ for all $z \in (-l, l)$.*

Assumption 2 *The CGF $\mathcal{K}(z)$ of $b \cdot X_T$ is strictly convex and steep at the boundary of $\mathcal{D} = \{z \in \mathbb{R} : |\mathcal{K}(z)| < \infty\}$.*

Unless $b \cdot X_T$ is constant almost surely, $\mathcal{K}(z)$ is strictly convex and the convexity of $\mathcal{K}(z)$ implies that $\mathcal{D}$ is an interval. Steepness means $\lim_{z\to v} \mathcal{K}'(z) = -\infty$ and $\lim_{z\to u} \mathcal{K}'(z) = \infty$ where $v = \inf \mathcal{D}$ and $u = \sup \mathcal{D}$ (see Barndorff-Nielsen [5] for more details). These assumptions are conditions on the tails of the random variable $b \cdot X_T$. Assumption 1 allows us to interchange the order of differentiation and integration as discussed in Section 2. Assumption 2 ensures the existence of a unique solution of the saddlepoint equation for any given $y \in \mathbb{R}$ and is not restrictive in practice.

Remark Although we focus on AJD models, the same approximations can be applied to quadratic term structure models (see, e.g., Leippold and Wu [40] or Cheng and Scaillet [14]) where extended transforms are again given by systems of ODEs. We also note that such systems of equations can be derived by re-writing quadratic term structure models as AJD models as observed in [14], Proposition 3.

3.3 Approximating the Saddlepoint

As already noted, solving the saddlepoint equation is a key step in applying the saddlepoint method. Numerical solution of the equation might require many iterations, each iteration requiring evaluation of the derivative of the CGF. This could be problematic in high-dimensional models without a closed-form CGF. The approximations to the saddlepoint $\hat{z}$ discussed in this section address this difficulty.

Several authors have addressed the problem of analytically intractable CGFs. Easton and Ronchetti [31] approximate $\mathcal{K}(z)$ by

$$\tilde{\mathcal{K}}(z) = \mu z + \frac{1}{2}\sigma^2 z^2 + \frac{1}{6}\kappa_3 z^3 + \frac{1}{24}\kappa_4 z^4$$

using the first four cumulants, and use $\tilde{z}$ for which $\tilde{\mathcal{K}}'(\tilde{z}) = y$ instead of the true saddlepoint $\hat{z}$. This approximate saddlepoint equation for $\tilde{\mathcal{K}}$ might have multiple roots, so Wang [50] modifies this method and uses

$$\tilde{\mathcal{K}}(z;b) = \mu z + \frac{1}{2}\sigma^2 z^2 + \left(\frac{1}{6}\kappa_3 z^3 + \frac{1}{24}\kappa_4 z^4\right) g_b(z)$$

where $g_b(z) = \exp(-\kappa_2 b^2 z^2/2)$ with a properly chosen constant $b > 0$.

Starting from a Taylor expansion of $\mathcal{K}'(z)$ around $z = 0$, Lieberman [41] presents a series reversion of the saddlepoint equation $\mathcal{K}'(\hat{z}) = y$ as a power series in $(y-\mu)/\sigma^2$. When expanded to third order, this yields

$$\hat{z}_3 = \frac{y-\mu}{\sigma^2} - \frac{\kappa_3}{2\sigma^2}\left(\frac{y-\mu}{\sigma^2}\right)^2 + \left(\frac{\kappa_3^2}{2\sigma^4} - \frac{\kappa_4}{6\sigma^2}\right)\left(\frac{y-\mu}{\sigma^2}\right)^3 \tag{11}$$

as an approximation to the exact saddlepoint $\hat{z}$. Here, $(y-\mu)/\sigma^2$ is the first iteration of a Newton-Raphson algorithm starting from $z_0 = 0$. Lieberman [41] then derives a saddlepoint approximation based on $\hat{z}_3$. With $\hat{v}_3 = \hat{z}_3\sqrt{n\mathcal{K}''(\hat{z}_3)}$, $\hat{\lambda}_3 = \mathcal{K}^{(3)}(\hat{z}_3)/\mathcal{K}''(\hat{z}_3)^{3/2}$, $\hat{\lambda}_4 = \mathcal{K}^{(4)}(\hat{z}_3)/\mathcal{K}''(\hat{z}_3)^{4/2}$ and $H(x) = \mathbf{1}_{\{x>0\}} + \frac{1}{2}\mathbf{1}_{\{x=0\}}$, Lieberman's approximation is

$$\begin{aligned}
\mathbb{P}(\overline{X} > y) \;=\; & H(-\hat{v}_3) + \exp\left(n(\mathcal{K}(\hat{z}_3) - y\hat{z}_3) + \frac{\hat{v}_3^2}{2}\right) \\
& \times\Bigg[(H(\hat{v}_3) - \Phi(\hat{v}_3))\left(1 - \frac{\hat{\lambda}_3\hat{v}_3^3}{6\sqrt{n}} + \frac{1}{n}\left(\frac{\hat{\lambda}_4\hat{v}_3^4}{24} + \frac{\hat{\lambda}_3^2\hat{v}_3^6}{72}\right)\right) \\
& +\phi(\hat{v}_3)\left(\frac{\hat{\lambda}_3(\hat{v}_3^2-1)}{6\sqrt{n}} - \frac{1}{n}\left(\frac{\hat{\lambda}_4(\hat{v}_3^3-\hat{v}_3)}{24} + \hat{\lambda}_3^2\frac{\hat{v}_3^5 - \hat{v}_3^3 + 3\hat{v}_3}{72}\right)\right)\Bigg]\left(1 + O(n^{-3/2})\right).
\end{aligned} \tag{12}$$

We will test this approximation in the next section. We will see that Lieberman's method is not uniformly accurate over a large range of strikes because the error in Lieberman's approximate saddlepoint, $\hat{z}_3$ which is an expansion in terms of $(y-\mu)/\sigma^2$, becomes large as y increases.

We propose an improvement that proceeds one more step. We expand $\mathcal{K}'(z)$ around $z = \hat{z}_3$ (rather than $z = 0$) to third order to get

$$\tilde{z}_3 = \hat{z}_3 + \frac{y - \mathcal{K}'(\hat{z}_3)}{\mathcal{K}''(\hat{z}_3)} - \frac{\mathcal{K}'''(\hat{z}_3)}{2\mathcal{K}''(\hat{z}_3)} \left(\frac{y - \mathcal{K}'(\hat{z}_3)}{\mathcal{K}''(\hat{z}_3)} \right)^2 + \left(\frac{\mathcal{K}^{(3)}(\hat{z}_3)^2}{2\mathcal{K}''(\hat{z}_3)^2} - \frac{\mathcal{K}^{(4)}(\hat{z}_3)}{6\mathcal{K}''(\hat{z}_3)} \right) \left(\frac{y - \mathcal{K}'(\hat{z}_3)}{\mathcal{K}''(\hat{z}_3)} \right)^3. \quad (13)$$

Note that (13) reduces to (11) if $\hat{z}_3$ is replaced by zero. Evaluation of $\tilde{z}_3$ uses the same set of ODEs which are used to get $\hat{z}_3$; we do not need higher order derivatives of $\mathcal{K}(z)$ or any extra set of ODEs for (13). To evaluate (13), we solve one set of ODEs associated with $\mathcal{K}(z)$ through $\mathcal{K}^{(4)}(z)$ to get $\hat{z}_3$, and then solve the same set of ODEs to get $\tilde{z}_3$.

In our numerical tests, we will test the effectiveness of using the approximate saddlepoints $\hat{z}_3$ and $\tilde{z}_3$ in the LR formula (9) in place of the exact value $\hat{z}$. The approximations $\hat{z}_3$ and $\tilde{z}_3$ can also be used to initialize the root-finding procedure to solve for $\hat{z}$, and we will test this idea with $\hat{z}_3$.

Remark The problem of solving the saddlepoint equation can be transformed from a root-finding problem into a matter of function evaluation through a duality relation. If we define

$$\mathcal{H}(t, x, z) = \alpha(t, z) + \beta(t, z) \cdot x,$$

then $\mathbb{E}[e^{-\int_t^T r_s ds} e^{(a+zb)\cdot X_T} \mid \mathcal{F}_t, X_t = x] = e^{\mathcal{H}(T-t,x,z)}$ and thus $\mathcal{H}(T, X_0, z) = \mathcal{K}(z) + \alpha(T, 0) + \beta(T, 0) \cdot X_0$. The function $\mathcal{H}(t, x, z)$ is convex in z, and strictly convex as long as $b \cdot X_t$ is not constant almost surely. This allows us to apply a technique developed by Jonsson and Sircar [38] in their analysis of a partial hedging strategy. We define the convex dual

$$\mathcal{H}^*(t, x, y) = \sup_{z \in \mathcal{D}(t)} \{yz - \mathcal{H}(t, x, z)\}, \quad \mathcal{D}(t) = \{z \in \mathbb{R} : |\mathcal{H}(t, x, z)| < \infty\}.$$

Then, it can be shown that there exists a continuously differentiable function $\hat{z}(t, x, y)$ for $(t, x, y) \in (0, T] \times \mathbb{R}^n \times \mathbb{R}$ such that $(\partial \mathcal{H}^*/\partial y)(t, x, y) = \hat{z}(t, x, y)$ and $\mathcal{H}^*$, $\hat{z}$ satisfy some partial differential equations (PDEs). An approximate saddlepoint is then obtained by solving these PDEs numerically. However, in a numerical test using the Heston stochastic volatility model, an explicit finite different method applied to associated PDEs does not perform uniformly better than the methods considered in this paper. A further investigation in this direction using other numerical methods for PDEs remains as a future research. For more details, see Kim [39].

4 Test Cases

In this section we test the performance of saddlepoint approximation technique, for the Heston model, a stochastic volatility jump-diffusion (SVJ) model and the Scott model. Particularly, we look at the following methods:

LR method	equation (9) with numerical calculation of the saddlepoint $\hat{z}$
Lieberman method	equation (12)
L-LR method	equation (9) with $\hat{z}$ approximated using $\hat{z}_3$ in (11)
App-LR method	equation (9) with $\hat{z}$ approximated using $\tilde{z}_3$ in (13)

In applying the LR method, we exclude b_1 and higher order terms as their inclusion does not consistently improve the results. The motivation for testing the last three methods lies in avoiding potentially time-consuming calculation of $\hat{z}$.

4.1 Heston Model

In the Heston model [36], the pricing transforms are available in closed form, so no approximations are necessary. We use this as a test case for the approximations precisely because the tractability of the model allows us to compare the approximations with values computed through transform inversion.

The stock price and the volatility in the Heston model [36] under a risk-neutral measure are assumed to follow

$$
\begin{aligned}
dS_t &= rS_t dt + \sqrt{v_t} S_t dW_t^1 \\
dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^2
\end{aligned}
$$

where r is the constant interest rate and (dW_t^1, dW_t^2) is a 2-dimensional Brownian motion with $< dW_t^1, dW_t^2 >= \rho dt$. We define $X_t = \log S_t$ and apply Itô's formula to X_t to get an AJD process (X, v) with

$$
dX_t = \left(r - \frac{1}{2}v_t\right) dt + \sqrt{v_t} dW_t^1
$$

and v is as above. See the appendix for the characteristic of this process. The price of a European call option is then given by

$$
C(T, c) = \mathbb{E}\left[e^{-rT}(S_T - c)^+\right] = S_0 \mathbb{Q}(X_T > \ln c) - c\, e^{-rT} \mathbb{P}(X_T > \ln c),
$$

where $\mathbb{Q}$ is defined by the measure transform $d\mathbb{Q}/d\mathbb{P} = e^{-rT} e^{X_T - X_0}$, which corresponds to taking S_T as numeraire asset. The dynamics of (X, v) can be written as

$$
\begin{aligned}
dX_t &= (r + v_t/2)dt + \sqrt{v_t} dW_t^{1,Q} \\
dv_t &= (\kappa\theta - (\kappa - \rho\sigma)v_t)dt + \sigma\sqrt{v_t} dW_t^{2,Q},
\end{aligned}
$$

where $W^{1,Q}$ and $W^{2,Q}$ are standard Brownian motions under $\mathbb{Q}$ with correlation parameter ρ. The CGF of X_T under $\mathbb{P}$ is defined by $e^{\mathcal{K}(z)} = \mathbb{E}[e^{zX_T}] = \exp(\alpha(T) + \beta(T) \cdot (X_0, v_0))$ where $\beta(0) = (z, 0)$, $\alpha(0) = 0$. Through Heston [36], we have an explicit solution for the CGF of X_T given by

$$
\begin{aligned}
\mathcal{K}(z) &= C + Dv_0 + zX_0 \\
C &= rzT + \frac{\kappa\theta}{\sigma^2}\left\{(\kappa - \rho\sigma z + d)T - 2\ln\left[\frac{1 - ge^{dT}}{1 - g}\right]\right\} \\
D &= \frac{\kappa - \rho\sigma z + d}{\sigma^2}\left[\frac{1 - e^{dT}}{1 - ge^{dT}}\right] \\
g &= \frac{\kappa - \rho\sigma z + d}{\kappa - \rho\sigma z - d} \\
d &= \sqrt{(\rho\sigma z - \kappa)^2 - \sigma^2(-z + z^2)}.
\end{aligned}
$$

60	70	80	90	100	110	120	130	140
48	45	42	36	20	35	39	43	45
13	9	7	7	5	6	8	12	15

Table 1: Average number of function evaluations in the numerical solution of the saddlepoint equation in the Heston model, by strike price. The first row corresponds to initializing the root-finding procedure at zero; the second row corresponds to starting at Lieberman's approximate saddlepoint.

Again the idea is that when this kind of analytic solution is not available, we use the associated ODEs to find the saddlepoint and apply the saddlepoint method. How many calculations does this require? Let us suppose, for simplicity, that the computation times in solving ODEs for (α, β), (A, B) or (C, D) are approximately the same, say τ. Although the dimensions of the ODEs will grow exponentially as we differentiate repeatedly, we are interested in ODEs associated with $\mathcal{K}^{(j)}$ for $j = 4$ at most. Also some special structure of the models helps to simplify the equations. For example, $B_1(t) = 1$, $D_1(t) = F_1(t) = H_1(t) = 0$ in the Heston model. With the assumption of constant τ, we can compare the computational loads of different saddlepoint approximations. The computing time to approximate $G_{a,b}(y; X_0, T) = \xi_0 \mathbb{Q}(Y \leq y)$ using the LR method is about $\tau + 2k\tau + 3\tau$, where k is the number of iterations to solve the saddlepoint equation numerically. Here the first term is for $e^{\mathcal{K}(0)} = \xi_0$ and the last term is for $\mathcal{K}(\hat{z})$, $\mathcal{K}''(\hat{z})$. On the other hand, the time needed to apply the Lieberman method is then about $5\tau + 5\tau$ because we have to find $\mathcal{K}(0), \ldots, \mathcal{K}^{(4)}(0)$ and evaluate $\mathcal{K}(\hat{z}_3), \ldots, \mathcal{K}^{(4)}(\hat{z}_3)$, while the L-LR method would require approximately $5\tau + 3\tau$ because we evaluate only up to $\mathcal{K}''(\hat{z}_3)$. The time for the App-LR method is $10\tau + 3\tau$. In each case, the most time-consuming step is getting an accurate or approximate saddlepoint, and the computational load of this step determines the efficiency of the approximation. It will become clear in our examples that the cost of this step depends on option moneyness and maturity.

4.1.1 Numerical Results

The LR method. In our numerical tests, the initial asset price S_0 is set equal to 100, the strike c varies from 60 to 140 and the option maturity T is in the range of 0.1 to 2 years. We solve the saddlepoint equation numerically by using the `fzero` function in MATLAB (which uses a bisection and interpolation algorithm) and solving the ODEs (1)–(4) at each iteration. Table 1 shows the average number of iterations in this step for each strike. Initializing `fzero` at the approximate saddlepoint $\hat{z}_3$ in (11) reduces the number of iterations by 66%–84%. The lower half of Table 4 shows the relative errors of the LR method with respect to the accurate prices shown in the upper half.[4] The relative errors are less than 0.1% over the whole range considered.

The Lieberman method and the L-LR method. Tables 5 and 6 show the relative errors of

[4] The analytic prices presented here for the Heston model and the SVJ model are produced using the program SecPrcV2.7 by Mark Broadie, Ozgur Kaya and Guy Shahar. They employed a modified trapezoidal-type routine for transform inversion. We thank Mark Broadie for providing us with a copy of this program.

the Lieberman method and the L-LR method, respectively. As mentioned earlier, the approximate saddlepoint $\hat{z}_3$ incurs large errors as y (log of strike) moves away from the mean μ. So the Lieberman method works best for at-the-money (ATM) options while the L-LR method yields the smallest errors for deep in-the-money (ITM) calls. Also, we find that relative errors are enormous in the upper right part of the tables, but the out-of-the-money (OTM) call prices in that section are very small, so even small absolute errors become very large relative errors.

The App-LR method. In Table 7, we use the App-LR method. This method solves the ODEs for $\partial\alpha^j/\partial z^j$, $\partial\beta^j/\partial z^j$, $j = 0, \ldots, 4$, one more time, but it reduces the relative errors a lot compared to the Lieberman method and the L-LR method. An important advantage of this method is that, while keeping the errors small, we solve the ODEs a fixed number of times. Using a root-finding iteration like `fzero` requires solving ODEs an unpredictable number of times.

In light of the greater accuracy of the App-LR method compared with the Lieberman method and the L-LR method, in the subsequent examples we restrict attention to the LR method and the App-LR method.

Dependence of Approximation on Saddlepoint. The results above have the implication that the accuracy of saddlepoint approximations largely depends on how well we approximate the saddlepoint itself. To illustrate this more clearly, we display the shapes of the curves $\mathcal{K}(z)$ and $\mathcal{K}'(z)$ in Figure 1.[5] The shape of $\mathcal{K}'(z)$ looks approximately cubic. This suggests the following approach: solve ODEs (1)–(4) for some fixed values of z and for a fixed maturity, and apply a cubic spline interpolation to get an approximation for $\mathcal{K}(z)$.[6] The results are reported in Table 8. In most cases, the relative errors are close to the values from the LR method in Table 4 except in the upper right section of the table where we have small option prices. However, this approximation has an exceptionally large relative error at $T = 1.9$, $c = 110$. This again shows the importance of accurate evaluation of the saddlepoint. Any user who wants to adopt this approach should be very careful regarding this matter. One advantage of this spline approach is, first, the time for computation is relatively small (in the example, it resolves ODEs (1)–(4) 30 times for each maturity) and, second, a single approximation can be used for options with the same maturity but different strikes.

4.2 SVJ Model

As in Bates [6], the asset price and volatility processes in the SVJ model under a risk-neutral measure $\mathbb{P}$ are as follows:

$$\frac{dS_t}{S_{t-}} = (r - \lambda k)dt + \sqrt{v_t}dW_t^1 + (\xi_{N_{t-}} - 1)dN_t$$
$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t^2$$

where N is a Poisson process with rate λ and the ξ_i's are i.i.d. lognormal random variables with mean μ_J and variance σ_J^2. Since $\{e^{-rt}S_t\}$ is a martingale under the risk-neutral measure, this

[5] The graph of $\mathcal{K}(z)$ shows the moment generating function explodes around $20 + \epsilon$ and $-25 - \epsilon$.

[6] `interp1` in MATLAB

60	70	80	90	100	110	120	130	140
47	43	41	35	18	35	40	43	44
27	16	9	7	6	6	6	7	8

Table 2: Average number of function evaluations used in the numerical solution of the saddlepoint equation for each strike in the SVJ model. The first row initiates the root-finding at zero and the second row initiates it at Lieberman's approximate saddlepoint.

condition gives the relation $k = e^{\mu_J + \sigma_J^2/2} - 1$. Also, W^1 and W^2 are standard Brownian motions with correlation parameter ρ as in the Heston model. We define $X_t = \log S_t$ as usual and then Itô's formula yields

$$dX_t = (r - \lambda k - v_t/2)dt + \sqrt{v_t}dW_t^1 + \eta_{N_{t-}}dN_t,$$

where $\eta_i \sim N(\mu_J, \sigma_J^2)$. The characteristic of this AJD process (X, v) is given in the appendix. Its CGF $\mathcal{K}(z)$ under $\mathbb{P}$ is defined by $e^{\mathcal{K}(z)} = \mathbb{E}[e^{zX_T}]$.

A European call option price is, with a new probability measure $\mathbb{Q}$ defined by $d\mathbb{Q}/d\mathbb{P} = e^{X_T - \mathcal{K}(1)}$,

$$C(T, c) \quad = \quad e^{-rT}\left\{e^{\mathcal{K}(1)}\mathbb{Q}(X_T > \ln c) - c\,\mathbb{P}(X_T > \ln c)\right\}.$$

And $e^{\mathcal{K}_Q(z)} = \mathbb{E}^Q[e^{zX_T}] = e^{\mathcal{K}(1+z)-\mathcal{K}(1)}$, $\mathcal{K}_Q(z)$ denoting the CGF of X_T under $\mathbb{Q}$. From this relation between $\mathcal{K}_Q(z)$ and $\mathcal{K}(z)$, the solution $\tilde{z}$ of $\mathcal{K}_Q'(z) = y$ is given by $\hat{z} - 1$ with $\mathcal{K}'(\hat{z}) = y$.

4.2.1 Numerical Results

The LR method. As in Section 5.1.1, we test the LR method and compare the results with analytical option prices. Table 2 shows the effectiveness of using the approximate saddlepoint $\hat{z}_3$ in (11) as a starting point for the root-finding routine for the saddlepoint equation. The average number of function evaluations for each strike is reduced considerably, as we noted in the Heston model. We use the parameters $r = 3\%$, $\kappa = 2$, $\theta = 4\%$ (long run mean volatility = 20%), $v_0 = 4\%$ (initial volatility = 20%), $\sigma = 20\%$, $\rho = -20\%$, $\mu_J = -3\%$, $\sigma_J = 2\%$, $\lambda = 100\%$, $S_0 = 100$. The second part of Table 9 shows that the relative errors of the LR method are less than 0.4% in the whole region.

The App-LR method. With the same parameters, the App-LR method produces small relative errors close to those of Table 9, as reported in Table 10, except the one fairly extreme case of $T = 0.1$ and $c = 60$. The reason that the method fails for this case is that the approximate saddlepoint, $\hat{z}_3 = 23.9788$, from (11) is too far from the true saddlepoint, $\hat{z} = -64.4843$, resulting in the huge error of the modified approximate saddlepoint, $\tilde{z}_3 = 63.4224$, from (13). In fact, this error makes $\hat{w}$ in the LR formula (9) imaginary. More precisely, $\hat{z}y - \mathcal{K}(\hat{z})$ becomes negative, as illustrated in Figure 2. (One could address this problem by checking if $\hat{z}y - \mathcal{K}(\hat{z})$ is positive and reverting to a root-finding iteration if it is not.) This indicates the potential limitation of the application of the App-LR method when a call option is deep ITM with a short maturity. We will see a similar pattern in the Scott model.

Sensitivity of Approximation. With the option strike 100, $T = 0.1$ and $c = 100$, the effects of λ, μ_J, and σ_J are shown in Figures 3, 4 and 5. As the jump arrival rate λ increases from 0 to 200%, relative errors increase linearly up to 0.085%. As the mean of the jump size μ_J decreases from 0 to -20%, relative errors make a smooth curve with a peak of 1.4% at $\mu_J = -16\%$. The volatility of the jump size has the biggest effect, making the relative error more than 10% as σ_J grows.[7] However, empirical values found in the literature stay small enough for the LR method to produce small relative errors. More specifically, as Broadie et al. [11] summarize in their paper, Eraker et al. [33], Andersen et al. [2], Chernov et al. [17] and Eraker [32] report 4.07%, 1.95%, 0.7% and 6.63% for σ_J, respectively. Broadie et al. [11] report σ_J between 9% and 10% when a risk premium for σ_J is assumed to exist.

Nonnormal-based Approximation. The added skewness due to the jump component in the SVJ model makes the saddlepoint approximation using a gamma distribution for the base distribution attractive. We test this method for two strikes in Table 11.[8] The gamma-based approximation is better for $c = 90$, but not for $c = 100$. This result reasserts the conclusion of Wood et al. [51], "…any gains are likely to be small when the normal-based approximation does well."

4.3 Scott Model

As the last test case, we apply the methods to Scott [46]'s jump-diffusion model with stochastic volatility and stochastic interest rates. Under a risk-neutral measure $\mathbb{P}$, the dynamics of the state variables are given by

$$
\begin{aligned}
dX_t &= (r_t - \lambda k - \sigma^2 y_t^1/2)dt + \sigma\sqrt{y_t^1}dW_t + \eta_{N_{t-}}dN_t \\
dy_t^1 &= \kappa_1(\theta_1 - y_t^1)dt + \sigma_1\sqrt{y_t^1}dW_t^1 \\
dy_t^2 &= \kappa_2(\theta_2 - y_t^2)dt + \sigma_2\sqrt{y_t^2}dW_t^2
\end{aligned}
$$

where W_t, W_t^1, W_t^2 are Brownian motions with $< dW_t, dW_t^1 >= \rho dt$, $< dW_t, dW_t^2 >= 0$, $r_t = y_t^1 + y_t^2$, $\eta_i \overset{iid}{\sim} N(\mu_J, \sigma_J^2)$ and $k = e^{\mu_J + \sigma_J^2/2} - 1$. The stock price S_t is $\exp(X_t)$.

The characteristics for this model are given in the appendix. A function $\mathcal{K}(z)$ is defined by $e^{\mathcal{K}(z)} = \mathbb{E}[e^{-\int_0^T r_s ds}e^{zX_T}]$. Note that $\mathcal{K}(z)$ is not the CGF of X_T under $\mathbb{P}$. The European call option price is

$$C(T, c) = e^{\mathcal{K}(1)}\mathbb{Q}_1(X_T > \ln c) - c\, e^{\mathcal{K}(0)}\mathbb{Q}_2(X_T > \ln c),$$

[7] Figure 5 shows the relative errors grow as σ_J becomes larger. Numerical values are obtained from (9) with $n = 1$ and b_0 only or (10) if $\hat{z}$ is close to zero (in our case, (10) is used if $\hat{z} < 10^{-4}$). Indeed, when $\sigma_J = 14\%$, we have $\hat{z} = 4.38 \times 10^{-5}$ and (9) yields a 184.86% relative error while (10) gives a relative error of 6.97%.

[8] The PDF of a gamma distribution Gamma(k, θ) is expressed as $f(x, k, \theta) = x^{k-1}e^{-(x/\theta)}/(\Gamma(k)\theta^k)$ for $x > 0$, the shape parameter k and the scale parameter θ. We use a chi-square distribution $\chi^2(\nu)$ of which PDF is that of Gamma$(\nu/2, 2)$ where ν is the degree of freedom. In Table 11, ν is set equal to 4. Other values for ν have similar results.

60	70	80	90	100	110	120	130	140
47	45	43	38	29	34	42	43	46
21	17	10	7	7	5	7	8	10

Table 3: Average number of function evaluations used in the numerical solution of the saddlepoint equation for each strike in the Scott model. The first row initiates the root-finding at zero and the second row initiates it at Lieberman's approximate saddlepoint.

where the probability measures $\mathbb{Q}_i$, $i = 1, 2$, are defined by $d\mathbb{Q}_1/d\mathbb{P} = e^{-\int_0^T r_s ds} e^{X_T - \mathcal{K}(1)}$ and $d\mathbb{Q}_2/d\mathbb{P} = e^{-\int_0^T r_s ds - \mathcal{K}(0)}$, so that

$$e^{\mathcal{K}_{Q_1}(z)} = \mathbb{E}^{Q_1}\left[e^{zX_T}\right] = e^{\mathcal{K}(1+z)-\mathcal{K}(1)}, \quad e^{\mathcal{K}_{Q_2}(z)} = \mathbb{E}^{Q_2}\left[e^{zX_T}\right] = e^{\mathcal{K}(z)-\mathcal{K}(0)}$$

and $\mathcal{K}_{Q_i}$ is the CGF of X_T under $\mathbb{Q}_i$. The saddlepoint equation is given by $\mathcal{K}'_{Q_1}(\tilde{z}) = \mathcal{K}'(1+\tilde{z}) = y$ for $\mathbb{Q}_1$ and $\mathcal{K}'_{Q_2}(\hat{z}) = \mathcal{K}'(\hat{z}) = y$ for $\mathbb{Q}_2$. So implementing the LR method requires solving $\mathcal{K}'(z) = y$ only.

There are two ways to use the App-LR method. One is to use this method for each of $\mathbb{Q}_i(X_T > \ln c)$, $i = 1, 2$, trying to approximate the corresponding saddlepoints separately. The other is to set the approximation of $\tilde{z}$ equal to the approximation of $\hat{z}$ minus one, based on the relation $\tilde{z} = \hat{z} - 1$. Using this consistent approximation requires solving half as many ODEs. In more detail, the first method solves the ODEs for $\partial\alpha^j/\partial z^j$, $\partial\beta^j/\partial z^j$, $j = 0, \ldots, 4$, four times to get two approximate saddlepoints $\hat{z}$ and $\tilde{z}$, while the latter one solves the same ODEs just twice. In our tests, the second method produces smaller errors, particularly at short maturities.

4.3.1 Numerical Results

The LR method. The analytical values in the upper section of Table 12 were computed using Fourier inversion, using the `quad` function in MATLAB with a large interval for the numerical integration. Different integration intervals give different values, but we find the errors to be very small. Again in Table 3, we find that initiating `fzero` at the approximate saddlepoint $\hat{z}_3$ in (11) helps to reduce the computation time for solving the saddlepoint equation, and in Table 12 we observe small relative errors (less than 0.1% in most cases) for the LR method with respect to the analytical valuation.

The App-LR method. The first part of Table 13 is the result of the App-LR method. As noted in Section 5.2.1, we see that the method is not applicable to some deep ITM calls with short maturities. There are also two big errors in the upper right part of the table that do not have counterparts in the SVJ model. These errors, however, disappear when we use the second implementation, setting the approximation of $\tilde{z}$ equal to the approximation of $\hat{z}$ minus one. We find that this method dominates the first method throughout the whole region considered. Even though this second method still cannot be applied to some deep ITM calls with short maturities, it produces relative errors very close to those of the LR method.

5 Conclusion

When a closed-form solution for the characteristic function in an affine jump-diffusion model is not available, transform inversion combining numerical integration with hundreds or thousands of ODE solutions can be very time consuming. We have seen that saddlepoint approximations can be an effective alternative computational tool for calculating prices in affine jump-diffusion models.

In saddlepoint approximations, we find that accurate calculation of the saddlepoint is the most critical and often the most challenging task. We can address this issue either by solving the saddlepoint equation numerically or by obtaining an approximate saddlepoint. Results in this paper can be summarized as follows:

- The LR method (the Lugannani-Rice formula with a numerical solution of the saddlepoint equation) yields the smallest relative errors, ranging from 0.0% – 0.3% in most cases for the models considered here.
- Initiating a root-finding iteration at the approximate saddlepoint $\hat{z}_3$ of Lieberman substantially reduces the number of iterations.
- The App-LR method (the LR formula with an improved series approximation to the saddlepoint) gives small relative errors close to those of the LR method. However, it gives poor results for some deep ITM options with short maturities.
- For ATM options, the LR method dominates. For OTM or ITM options, the App-LR method is better, considering speed and accuracy together.
- If speed is of greater concern than accuracy, then it is best to use the Lieberman method for ATM options and to use the LR method for ITM options.

In our numerical tests, we have considered a wide range of strikes and maturities. Empirical work with AJD models generally focuses on a much more limited range, and this further supports the use of saddlepoint approximations.

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Appendix

Proof of Theorem 2.3. We follow the approach used in Theorem 1 in DPS. Throughout the proof, let us denote $\alpha(T-t)$, $\beta(T-t)$, $A(T-t)$, $B(T-t)$, $C(T-t)$ and $D(T-t)$ by α_t, β_t, A_t, B_t, C_t,

D_t, respectively, for notational convenience. We also write $r(X_t)$, $\mu(X_t)$, $\sigma(X_t)$, $\lambda(X_t)$ as r_t, μ_t, σ_t and λ_t. We use a dot, as in $\dot{f}$, to denote a time derivative df/dt. Next we define

$$\Psi_t = \exp\left(-\int_0^t r(X_s)ds\right) e^{\alpha_t+\beta_t\cdot X_t}$$

and $\Phi_t = \Psi_t(A_t + B_t \cdot X_t)$. In addition, we set $\Phi'_t = \Psi_t(A_t + B_t \cdot X_t)^2$ and $\Psi'_t = \Psi_t(C_t + D_t \cdot X_t)$. If we show that $\Phi'_t + \Psi'_t$ is a martingale, then $\Phi'_t + \Psi'_t = \mathbb{E}[\Phi'_T + \Psi'_T \mid \mathcal{F}_t]$ leads to the desired result.

Itô's formula for jump-diffusion processes (as in Cont and Tankov [19]) yields

$$\begin{aligned} d\Phi'_t &= \Phi'_t\left((-r_t + \dot{\alpha}_t + \dot{\beta}_t \cdot X_t)dt + (\beta_t \cdot \mu_t dt + \beta_t^\top \sigma_t dW_t) + \frac{1}{2}\beta^\top(\sigma_t\sigma_t^\top)\beta_t dt\right) \\ &\quad +2\Phi_t\left((\dot{A}_t + \dot{B}_t \cdot X_t)dt + (B_t \cdot \mu_t dt + B_t^\top \sigma_t dW_t) + \beta_t^\top(\sigma_t\sigma_t^\top)B_t dt\right) \\ &\quad +\Psi_t B_t^\top(\sigma_t\sigma_t^\top)B_t dt + dJ_t \\ &= \Pi_t dt + \Upsilon_t dW_t + dJ_t \end{aligned}$$

for appropriate drift and volatility coefficients Π_t, Υ_t and $J_t = \sum_{0<\tau(i)\le t}(\Phi'_{\tau(i)} - \Phi'_{\tau(i)-})$ with $\tau(i) = \inf\{t : N_t = i\}$. Here N_t is the counting process with intensity λ_t. Letting $\mathbb{E}_t$ be the $\mathcal{F}_t$-conditional expectation under $\mathbb{P}$ for $0 \le t \le s \le T$, and writing $\triangle X_i$ for the increment in X at $\tau(i)$, we have

$$\begin{aligned} &\mathbb{E}_t\Big[\sum_{t<\tau(i)\le s}(\Phi'_{\tau(i)} - \Phi'_{\tau(i)-})\Big] \\ &= \mathbb{E}_t\Big[\sum_{t<\tau(i)\le s}\mathbb{E}[\Phi'_{\tau(i)} - \Phi'_{\tau(i)-} \mid X_{\tau(i)-}, \tau(i)]\Big] \\ &= \mathbb{E}_t\Big[\sum_{t<\tau(i)\le s}\Big\{\Phi'_{\tau(i)-}\left(\mathbb{E}_{\tau(i)-}e^{\beta_{\tau(i)}\cdot\triangle X_i} - 1\right) + 2\Phi_{\tau(i)-}\mathbb{E}_{\tau(i)-}[e^{\beta_{\tau(i)}\cdot\triangle X_i}B_{\tau(i)}\cdot\triangle X_i] \\ &\quad +\Psi_{\tau(i)-}\mathbb{E}_{\tau(i)-}[e^{\beta_{\tau(i)}\cdot\triangle X_i}(B_{\tau(i)}\cdot\triangle X_i)^2]\Big\}\Big] \\ &= \mathbb{E}_t\Big[\int_{t+}^s\Big\{\Phi'_{u-}(\theta(\beta_u) - 1) + 2\Phi_{u-}\nabla\theta(\beta_u)B_u + \Psi_{u-}B_u^\top\nabla^2\theta(\beta_u)B_u\Big\}dN_u\Big]. \end{aligned}$$

Proceeding similarly,

$$d\Psi'_t = \tilde{\Pi}_t dt + \tilde{\Upsilon}_t dW_t + d\tilde{J}_t$$

for suitable coefficients $\tilde{\Pi}_t$, $\tilde{\Upsilon}_t$ (they are straightforward to compute, but omitted to save some space) and $\tilde{J}_t = \sum_{0<\tau(i)\le t}(\Psi'_{\tau(i)} - \Psi'_{\tau(i)-})$. The last term satisfies

$$\mathbb{E}_t\Big[\sum_{t<\tau(i)\le s}(\Psi'_{\tau(i)} - \Psi'_{\tau(i)-})\Big] = \mathbb{E}_t\Big[\int_{t+}^s\Big\{\Psi'_{u-}(\theta(\beta_u) - 1) + \Psi_{u-}\nabla\theta(\beta_u)D_u\Big\}dN_u\Big].$$

Now, we observe that if the condition (i) of Definition 2.2 is satisfied, then

$$\mathbb{E}_t\Big[J_s + \tilde{J}_s - J_t - \tilde{J}_t\Big] = \mathbb{E}_t\Big[\int_{t+}^s \gamma(u-)dN_u\Big] = \mathbb{E}_t\Big[\int_t^s \gamma(u)\lambda_u du\Big]$$

and $J_t + \tilde{J}_t - \int_0^t \gamma(u)\lambda_u du$ becomes a martingale thanks to the Integration theorem in p.27 of Brémaud [10].

From these observations, by adding and subtracting $\gamma(t)\lambda_t dt$ we get

$$\begin{aligned}
d(\Phi'_t + \Psi'_t) &= d(J_t + \tilde{J}_t) - \gamma(t)\lambda_t dt + (\Upsilon_t + \tilde{\Upsilon}_t)dW_t \\
&\quad + \Phi'_t \left(-r_t + \dot{\alpha}_t + \dot{\beta}_t \cdot X_t + \beta_t \cdot \mu_t + \frac{1}{2}\beta^\top(\sigma_t\sigma_t^\top)\beta_t + (\theta(\beta_t) - 1)\lambda_t \right) dt \\
&\quad + 2\Phi_t \left(\dot{A}_t + \dot{B}_t \cdot X_t + B_t \cdot \mu_t + \beta_t^\top(\sigma_t\sigma_t^\top)B_t + \nabla\theta(\beta_t)B_t\lambda_t \right) dt \\
&\quad + \Psi_t \left(B_t^\top(\sigma_t\sigma_t^\top)B_t + B_t^\top\nabla^2\theta(\beta_t)B_t\lambda_t \right) dt \\
&\quad + \Psi'_t \left(-r_t + \dot{\alpha}_t + \dot{\beta}_t \cdot X_t + \beta_t \cdot \mu_t + \frac{1}{2}\beta^\top(\sigma_t\sigma_t^\top)\beta_t + (\theta(\beta_t) - 1)\lambda_t \right) dt \\
&\quad + \Psi_t \left(\dot{C}_t + \dot{D}_t \cdot X_t + D_t \cdot \mu_t + \beta_t^\top(\sigma_t\sigma_t^\top)D_t + \nabla\theta(\beta_t)D_t\lambda_t \right) dt \\
&= d(J_t + \tilde{J}_t) - \gamma(t)\lambda_t dt + (\Upsilon_t + \tilde{\Upsilon}_t)dW_t \qquad (14)
\end{aligned}$$

as α_t, β_t, A_t, B_t, C_t and D_t are solutions to (1)–(6). The condition (ii) of Definition 2.2 ensures that $\int_0^t (\Upsilon_u + \tilde{\Upsilon}_u)dW_u$ is a martingale. Therefore, $\Phi'_t + \Psi'_t$ is a martingale and the proof is complete. $\blacksquare$

Theorem 2.3 can also be established as a consequence of Proposition 2 in Cheng and Scaillet [14]; for higher-order derivatives we need to consider higher powers of $b \cdot X_T$, and these require separate treatment.

Conditions for Theorem 2.4. The characteristics (K, H, l, θ, ρ) are well-behaved at (v, u, T), if all ODEs in Theorems 2.1, 2.3, 2.4 are solved uniquely, if θ is three times differentiable at $\beta(t)$ for all $t \leq T$, and if the following conditions are satisfied:

$$\begin{aligned}
(i) \quad & \mathbb{E}\Big[\int_0^T |\gamma(t)\lambda(X_t)|dt \Big] < \infty, \\
& \text{where } \gamma(t) = f_1(t) + f_2(t) + f_3(t), \\
& f_1(t) := \Phi_t^1(\theta(\beta_t) - 1) + 3\Psi_t\Big\{ (A_t + B_t \cdot X_t)^2 \nabla\theta(\beta_t)B_t \\
& \quad + (A_t + B_t \cdot X_t)B_t^\top \nabla^2\theta(\beta_t)B_t \Big\} + \Psi_t \int_{\mathbb{R}^n} e^{z\cdot\beta_t}(z \cdot B_t)^3 d\nu(z) \\
& f_2(t) := \Phi_t^2(\theta(\beta_t) - 1) + 3\Psi_t\Big\{ (A_t + B_t \cdot X_t)\nabla\theta(\beta_t)D_t \\
& \quad + (C_t + D_t \cdot X_t)\nabla\theta(\beta_t)B_t \Big\} + \Psi_t B_t^\top \nabla^2\theta(\beta_t)D_t \\
& f_3(t) := \Phi_t^3(\theta(\beta_t) - 1) + \Psi_t\nabla\theta(\beta_t)F_t \\
(ii) \quad & \mathbb{E}\Big[\Big(\int_0^T \eta(t) \cdot \eta(t)dt \Big)^{1/2} \Big] < \infty, \ \text{where } \eta(t) = (g_1(t) + g_2(t) + g_3(t))\,\sigma(X_t) \\
& g_1(t) := \Phi_t^1\beta_t^\top + 3\Psi_t(A_t + B_t \cdot X_t)^2 B_t^\top \\
& g_2(t) := \Phi_t^2\beta_t^\top + 3\Psi_t\Big\{ (C_t + D_t \cdot X_t)B_t^\top + (A_t + B_t \cdot X_t)D_t^\top \Big\} \\
& g_3(t) := \Phi_t^3\beta_t^\top + \Psi_t F_t^\top \\
(iii) \quad & \mathbb{E}\Big[|\Phi_T^1 + \Phi_T^2 + \Phi_T^3| \Big] < \infty
\end{aligned}$$

where Ψ_t, Φ_t^i for $i = 1, 2, 3$ are defined in the proof of Theorem 2.4 and $\alpha_t, \ldots, F_t$ stand for $\alpha(T-t), \ldots, F(T-t)$ which are the solutions to (1)–(8). ∎

Proof of Theorem 2.4. This can be proved by defining appropriate functions, as in the previous theorems. We set $\Psi_t = \exp(-\int_0^t r(X_s)ds)e^{\alpha(T-t)+\beta(T-t)\cdot X_t}$ as before and

$$\begin{aligned}
\Phi_t^1 &= (A(T-t) + B(T-t)\cdot X_t)^3 \Psi_t \\
\Phi_t^2 &= 3(A(T-t) + B(T-t)\cdot X_t)(C(T-t) + D(T-t)\cdot X_t)\Psi_t \\
\Phi_t^3 &= (E(T-t) + F(T-t)\cdot X_t)\Psi_t
\end{aligned}$$

and apply Itô's formula. Under the assumed conditions, $\Phi_t^1 + \Phi_t^2 + \Phi_t^3$ becomes a martingale. ∎

Characteristic of the model dynamics in the Heston model:

$$\begin{aligned}
&K_0 = \begin{pmatrix} r \\ \kappa\theta \end{pmatrix},\ K_1 = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & -\kappa \end{pmatrix},\ H_0 = 0 \\
&H_{1,11} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ H_{1,12} = H_{1,21} = \begin{pmatrix} 0 \\ \sigma\rho \end{pmatrix},\ H_{1,22} = \begin{pmatrix} 0 \\ \sigma^2 \end{pmatrix}
\end{aligned}$$

Characteristic of the model dynamics in the SVJ model:

$$\begin{aligned}
&K_0 = \begin{pmatrix} r - \lambda k \\ \kappa\theta \end{pmatrix},\ K_1 = \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & -\kappa \end{pmatrix},\ H_0 = 0 \\
&H_{1,11} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ H_{1,12} = H_{1,21} = \begin{pmatrix} 0 \\ \sigma\rho \end{pmatrix},\ H_{1,22} = \begin{pmatrix} 0 \\ \sigma^2 \end{pmatrix} \\
&\theta(c) = \int_{\mathbb{R}^2} \exp(c\cdot z)d\nu(z) = \exp(c_1\mu_J + c_1^2\sigma_J^2/2),\ l_0 = \lambda,\ l_1 = 0
\end{aligned}$$

Characteristic of the model dynamics in the Scott model:

$$\begin{aligned}
&K_0 = \begin{pmatrix} -\lambda k \\ \kappa_1\theta_1 \\ \kappa_2\theta_2 \end{pmatrix},\ K_1 = \begin{pmatrix} 0 & 1 - \frac{1}{2}\sigma^2 & 1 \\ 0 & -\kappa_1 & 0 \\ 0 & 0 & -\kappa_2 \end{pmatrix},\ H_0 = 0 \\
&H_{1,11} = \begin{pmatrix} 0 \\ \sigma^2 \\ 0 \end{pmatrix},\ H_{1,12} = H_{1,21} = \begin{pmatrix} 0 \\ \rho\sigma\sigma_1 \\ 0 \end{pmatrix},\ H_{1,22} = \begin{pmatrix} 0 \\ \sigma_1^2 \\ 0 \end{pmatrix},\ H_{1,33} = \begin{pmatrix} 0 \\ 0 \\ \sigma_2^2 \end{pmatrix} \\
&H_{1,13} = H_{1,31} = H_{1,23} = H_{1,32} = 0 \\
&\theta(c) = \int_{\mathbb{R}^2} \exp(c\cdot z)d\nu(z) = \exp(c_1\mu_J + c_1^2\sigma_J^2/2),\ l_0 = \lambda,\ l_1 = 0
\end{aligned}$$

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Table 4: Heston Model: (1) Analytic values (2) Relative errors of the LR method. Parameters: $S_0 = 100$, $v_0 = 4\%$, $\kappa = 2$, $\theta = 4\%$, $\sigma = 0.2$, $\rho = 20\%$, $r = 3\%$, maturities (column), strikes (row)

(1)	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**
0.1	40.180	30.210	20.240	10.366	2.662	0.236	0.008	1.7E-04	2.4E-06
0.2	40.359	30.419	20.490	10.940	3.840	0.826	0.121	0.014	0.001
0.3	40.538	30.628	20.769	11.546	4.773	1.455	0.353	0.075	0.015
0.4	40.716	30.840	21.077	12.141	5.581	2.068	0.656	0.190	0.052
0.5	40.894	31.057	21.403	12.714	6.308	2.658	0.997	0.349	0.119
0.6	41.072	31.279	21.743	13.266	6.979	3.224	1.360	0.544	0.212
0.7	41.250	31.507	22.089	13.799	7.608	3.770	1.734	0.764	0.330
0.8	41.429	31.739	22.440	14.313	8.202	4.297	2.115	1.004	0.469
0.9	41.608	31.976	22.793	14.812	8.769	4.807	2.499	1.260	0.626
1.0	41.789	32.217	23.145	15.296	9.313	5.304	2.884	1.527	0.799
1.1	41.970	32.460	23.497	15.767	9.837	5.787	3.269	1.803	0.985
1.2	42.153	32.706	23.848	16.225	10.343	6.258	3.653	2.087	1.182
1.3	42.336	32.954	24.196	16.673	10.834	6.719	4.035	2.377	1.388
1.4	42.521	33.204	24.542	17.111	11.312	7.170	4.414	2.671	1.604
1.5	42.706	33.455	24.885	17.539	11.777	7.612	4.792	2.969	1.827
1.6	42.893	33.707	25.225	17.959	12.231	8.046	5.167	3.270	2.056
1.7	43.079	33.959	25.562	18.370	12.675	8.472	5.539	3.574	2.291
1.8	43.267	34.211	25.896	18.774	13.109	8.890	5.908	3.879	2.532
1.9	43.455	34.463	26.227	19.171	13.535	9.302	6.275	4.185	2.776
2.0	43.644	34.716	26.554	19.561	13.953	9.707	6.638	4.492	3.025

(2)	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**
0.1	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.001%	0.002%
0.2	0.000%	0.000%	0.000%	0.000%	-0.001%	-0.001%	0.000%	0.002%	0.005%
0.3	0.000%	0.000%	0.000%	-0.001%	-0.004%	-0.005%	-0.003%	0.002%	0.008%
0.4	0.000%	0.000%	0.000%	-0.003%	-0.008%	-0.011%	-0.008%	-0.001%	0.008%
0.5	0.000%	0.000%	-0.001%	-0.005%	-0.013%	-0.018%	-0.015%	-0.006%	0.004%
0.6	0.000%	0.000%	-0.001%	-0.007%	-0.017%	-0.025%	-0.023%	-0.014%	-0.002%
0.7	0.000%	0.000%	-0.002%	-0.009%	-0.022%	-0.031%	-0.032%	-0.024%	-0.012%
0.8	0.000%	0.000%	-0.003%	-0.011%	-0.025%	-0.037%	-0.040%	-0.035%	-0.024%
0.9	0.000%	-0.001%	-0.004%	-0.013%	-0.028%	-0.042%	-0.048%	-0.045%	-0.036%
1.0	0.000%	-0.001%	-0.004%	-0.014%	-0.030%	-0.045%	-0.054%	-0.055%	-0.048%
1.1	0.000%	-0.001%	-0.005%	-0.016%	-0.031%	-0.048%	-0.059%	-0.063%	-0.060%
1.2	0.000%	-0.001%	-0.006%	-0.016%	-0.032%	-0.049%	-0.063%	-0.069%	-0.069%
1.3	0.000%	-0.002%	-0.006%	-0.017%	-0.033%	-0.050%	-0.065%	-0.074%	-0.077%
1.4	0.000%	-0.002%	-0.007%	-0.017%	-0.033%	-0.050%	-0.066%	-0.077%	-0.083%
1.5	0.000%	-0.002%	-0.007%	-0.018%	-0.033%	-0.050%	-0.066%	-0.079%	-0.088%
1.6	0.000%	-0.002%	-0.007%	-0.018%	-0.032%	-0.049%	-0.066%	-0.080%	-0.090%
1.7	0.000%	-0.002%	-0.008%	-0.017%	-0.031%	-0.048%	-0.065%	-0.080%	-0.092%
1.8	-0.001%	-0.003%	-0.008%	-0.017%	-0.031%	-0.047%	-0.063%	-0.079%	-0.092%
1.9	-0.001%	-0.003%	-0.008%	-0.017%	-0.030%	-0.045%	-0.062%	-0.077%	-0.091%
2.0	-0.001%	-0.003%	-0.008%	-0.017%	-0.029%	-0.044%	-0.059%	-0.075%	-0.090%

Table 5: Heston Model: Relative errors of the Lieberman method

	60	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**
0.1	0.00%	0.00%	0.12%	0.23%	0.00%	-19.46%	-602.61%	-2.1E+3%	-1.9E+4%
0.2	0.00%	0.23%	1.13%	0.27%	0.00%	-4.29%	-116.01%	-428.92%	-1.3E+3%
0.3	0.05%	1.65%	1.79%	0.28%	0.00%	-2.04%	-45.91%	-200.64%	-466.73%
0.4	0.55%	3.58%	1.90%	0.28%	0.00%	-1.26%	-24.07%	-117.31%	-265.96%
0.5	2.03%	4.93%	1.78%	0.28%	0.00%	-0.88%	-14.60%	-75.62%	-181.11%
0.6	4.32%	5.46%	1.58%	0.27%	0.00%	-0.66%	-9.67%	-51.56%	-132.77%
0.7	6.66%	5.38%	1.38%	0.26%	0.00%	-0.51%	-6.78%	-36.52%	-100.62%
0.8	8.37%	4.98%	1.20%	0.24%	0.00%	-0.40%	-4.96%	-26.63%	-77.58%
0.9	9.21%	4.44%	1.04%	0.23%	0.00%	-0.33%	-3.73%	-19.87%	-60.47%
1.0	9.29%	3.89%	0.91%	0.22%	0.00%	-0.27%	-2.88%	-15.11%	-47.55%
1.1	8.84%	3.37%	0.80%	0.20%	0.00%	-0.23%	-2.27%	-11.69%	-37.68%
1.2	8.10%	2.91%	0.70%	0.19%	0.00%	-0.20%	-1.81%	-9.17%	-30.08%
1.3	7.26%	2.51%	0.62%	0.18%	0.00%	-0.17%	-1.47%	-7.29%	-24.19%
1.4	6.40%	2.17%	0.56%	0.17%	0.00%	-0.15%	-1.21%	-5.86%	-19.59%
1.5	5.61%	1.89%	0.50%	0.16%	0.00%	-0.13%	-1.00%	-4.76%	-15.97%
1.6	4.90%	1.65%	0.45%	0.15%	0.00%	-0.12%	-0.84%	-3.91%	-13.11%
1.7	4.27%	1.45%	0.41%	0.14%	0.01%	-0.10%	-0.71%	-3.23%	-10.83%
1.8	3.74%	1.28%	0.37%	0.13%	0.01%	-0.09%	-0.60%	-2.70%	-9.01%
1.9	3.28%	1.13%	0.34%	0.12%	0.01%	-0.08%	-0.52%	-2.27%	-7.53%
2.0	2.88%	1.01%	0.31%	0.12%	0.01%	-0.08%	-0.44%	-1.92%	-6.34%

Table 6: Heston Model: Relative errors of the L-LR method

	60	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**
0.1	0.00%	0.00%	0.00%	0.04%	0.02%	-2.61%	-25.92%	-337.50%	-1.2E+4%
0.2	0.00%	0.00%	0.02%	0.19%	0.05%	-2.66%	-17.37%	-121.26%	-1.2E+3%
0.3	0.00%	0.01%	0.09%	0.34%	0.09%	-2.52%	-13.55%	-67.76%	-422.93%
0.4	0.00%	0.02%	0.18%	0.45%	0.13%	-2.29%	-11.03%	-44.65%	-207.63%
0.5	0.01%	0.05%	0.27%	0.54%	0.16%	-2.05%	-9.15%	-32.08%	-121.32%
0.6	0.01%	0.09%	0.35%	0.59%	0.19%	-1.81%	-7.67%	-24.30%	-78.78%
0.7	0.02%	0.13%	0.42%	0.62%	0.21%	-1.58%	-6.48%	-19.06%	-54.94%
0.8	0.03%	0.17%	0.46%	0.63%	0.22%	-1.38%	-5.52%	-15.33%	-40.34%
0.9	0.04%	0.20%	0.49%	0.63%	0.24%	-1.21%	-4.72%	-12.56%	-30.78%
1.0	0.05%	0.23%	0.51%	0.62%	0.24%	-1.05%	-4.07%	-10.44%	-24.19%
1.1	0.07%	0.25%	0.51%	0.60%	0.25%	-0.91%	-3.52%	-8.78%	-19.46%
1.2	0.08%	0.27%	0.51%	0.59%	0.25%	-0.79%	-3.06%	-7.46%	-15.95%
1.3	0.09%	0.28%	0.50%	0.56%	0.25%	-0.69%	-2.67%	-6.40%	-13.27%
1.4	0.10%	0.28%	0.49%	0.54%	0.25%	-0.60%	-2.34%	-5.52%	-11.18%
1.5	0.10%	0.29%	0.48%	0.52%	0.25%	-0.52%	-2.06%	-4.80%	-9.53%
1.6	0.11%	0.29%	0.47%	0.50%	0.25%	-0.45%	-1.81%	-4.20%	-8.19%
1.7	0.12%	0.29%	0.45%	0.48%	0.24%	-0.39%	-1.61%	-3.69%	-7.10%
1.8	0.12%	0.28%	0.44%	0.46%	0.24%	-0.33%	-1.43%	-3.26%	-6.20%
1.9	0.12%	0.28%	0.42%	0.44%	0.23%	-0.29%	-1.27%	-2.89%	-5.44%
2.0	0.13%	0.28%	0.41%	0.42%	0.23%	-0.25%	-1.13%	-2.58%	-4.81%

Table 7: Heston Model: Relative errors of the App-LR method

	60	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**
0.1	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	-0.042%	-4.966%
0.2	0.000%	0.000%	0.000%	0.000%	-0.001%	-0.001%	0.000%	-0.015%	-1.091%
0.3	0.000%	0.000%	0.000%	-0.001%	-0.004%	-0.005%	-0.003%	-0.005%	-0.280%
0.4	0.000%	0.000%	0.000%	-0.003%	-0.008%	-0.011%	-0.008%	-0.004%	-0.087%
0.5	0.000%	0.000%	-0.001%	-0.005%	-0.013%	-0.018%	-0.015%	-0.008%	-0.033%
0.6	0.000%	0.000%	-0.001%	-0.007%	-0.017%	-0.025%	-0.023%	-0.015%	-0.018%
0.7	0.000%	0.000%	-0.002%	-0.009%	-0.022%	-0.031%	-0.032%	-0.025%	-0.019%
0.8	0.000%	0.000%	-0.003%	-0.011%	-0.025%	-0.037%	-0.040%	-0.035%	-0.027%
0.9	0.000%	-0.001%	-0.004%	-0.013%	-0.028%	-0.042%	-0.048%	-0.046%	-0.038%
1.0	0.000%	-0.001%	-0.004%	-0.014%	-0.030%	-0.045%	-0.054%	-0.055%	-0.049%
1.1	0.000%	-0.001%	-0.005%	-0.016%	-0.031%	-0.048%	-0.059%	-0.063%	-0.060%
1.2	0.000%	-0.001%	-0.006%	-0.016%	-0.032%	-0.049%	-0.063%	-0.069%	-0.069%
1.3	0.000%	-0.002%	-0.006%	-0.017%	-0.033%	-0.050%	-0.065%	-0.074%	-0.077%
1.4	0.000%	-0.002%	-0.007%	-0.017%	-0.033%	-0.050%	-0.066%	-0.077%	-0.083%
1.5	0.000%	-0.002%	-0.007%	-0.018%	-0.033%	-0.050%	-0.066%	-0.079%	-0.088%
1.6	0.000%	-0.002%	-0.007%	-0.018%	-0.032%	-0.049%	-0.066%	-0.080%	-0.090%
1.7	0.000%	-0.002%	-0.008%	-0.017%	-0.031%	-0.048%	-0.065%	-0.080%	-0.092%
1.8	-0.001%	-0.003%	-0.008%	-0.017%	-0.031%	-0.047%	-0.063%	-0.079%	-0.092%
1.9	-0.001%	-0.003%	-0.008%	-0.017%	-0.030%	-0.045%	-0.062%	-0.077%	-0.091%
2.0	-0.001%	-0.003%	-0.008%	-0.017%	-0.029%	-0.044%	-0.059%	-0.075%	-0.090%

Table 8: Heston Model: Cubic spline interpolation to approximate $\mathcal{K}'(z)$. $z \in [-15, 15]$ and step size 1

	60	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**
0.1	0.000%	0.000%	0.000%	0.002%	-0.007%	-0.276%	-8.979%	-68.560%	-528.702%
0.2	0.000%	0.000%	-0.002%	0.000%	-0.023%	-0.001%	-0.520%	-7.643%	-33.650%
0.3	0.000%	-0.001%	-0.003%	-0.001%	-0.043%	-0.005%	-0.003%	-0.174%	-2.398%
0.4	0.000%	-0.002%	0.000%	-0.003%	-0.063%	-0.011%	-0.008%	-0.001%	0.013%
0.5	-0.001%	-0.001%	-0.001%	-0.005%	-0.081%	-0.017%	-0.015%	-0.006%	0.003%
0.6	0.000%	0.000%	-0.001%	-0.007%	-0.097%	-0.025%	-0.023%	-0.014%	-0.003%
0.7	0.000%	0.000%	-0.002%	-0.009%	-0.110%	-0.027%	-0.032%	-0.025%	-0.012%
0.8	0.000%	0.000%	-0.003%	-0.011%	-0.120%	-0.036%	-0.039%	-0.035%	-0.024%
0.9	0.000%	-0.001%	-0.004%	-0.013%	-0.127%	-0.037%	-0.047%	-0.044%	-0.036%
1.0	0.000%	-0.001%	-0.004%	-0.014%	-0.132%	-0.025%	-0.051%	-0.055%	-0.048%
1.1	0.000%	-0.001%	-0.005%	-0.015%	-0.136%	-0.019%	-0.059%	-0.061%	-0.058%
1.2	0.000%	-0.001%	-0.006%	-0.016%	-0.137%	-0.028%	-0.060%	-0.069%	-0.069%
1.3	0.000%	-0.002%	-0.006%	-0.017%	-0.138%	-0.047%	-0.057%	-0.072%	-0.076%
1.4	0.000%	-0.002%	-0.007%	-0.017%	-0.137%	-0.042%	-0.059%	-0.073%	-0.081%
1.5	0.000%	-0.002%	-0.007%	-0.018%	-0.135%	0.030%	-0.064%	-0.077%	-0.087%
1.6	0.000%	-0.002%	-0.007%	-0.018%	-0.133%	0.249%	-0.066%	-0.080%	-0.089%
1.7	0.000%	-0.002%	-0.008%	-0.017%	-0.131%	0.973%	-0.058%	-0.077%	-0.088%
1.8	-0.001%	-0.003%	-0.008%	-0.017%	-0.128%	5.383%	-0.046%	-0.071%	-0.087%
1.9	-0.001%	-0.003%	-0.008%	-0.017%	-0.125%	2034.383%	-0.034%	-0.068%	-0.089%
2.0	-0.001%	-0.003%	-0.008%	-0.016%	-0.122%	5.170%	-0.026%	-0.067%	-0.090%

Table 9: SVJ Model: (1) Analytic values (2) Relative errors of the LR method. Parameters: $S_0 = 100$, $v_0 = 4\%$, $\kappa = 2$, $\theta = 4\%$, $\sigma = 0.2$, $\rho = -20\%$, $r = 3\%$, $\mu_J = -3\%$, $\sigma_J = 2\%$, $\lambda = 100\%$, maturities (column), strikes (row)

(1)	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**
0.1	40.180	30.210	20.241	10.406	2.703	0.207	0.004	3.3E-05	1.5E-07
0.2	40.359	30.419	20.506	11.041	3.901	0.766	0.084	0.006	3.3E-04
0.3	40.538	30.632	20.816	11.693	4.851	1.379	0.274	0.042	0.005
0.4	40.716	30.853	21.159	12.322	5.672	1.985	0.539	0.121	0.024
0.5	40.896	31.083	21.521	12.923	6.412	2.573	0.851	0.243	0.063
0.6	41.077	31.321	21.893	13.497	7.094	3.142	1.192	0.402	0.125
0.7	41.260	31.567	22.268	14.047	7.733	3.692	1.551	0.591	0.210
0.8	41.444	31.819	22.644	14.576	8.337	4.225	1.922	0.804	0.317
0.9	41.631	32.075	23.019	15.086	8.913	4.744	2.301	1.037	0.445
1.0	41.820	32.335	23.392	15.581	9.465	5.248	2.684	1.287	0.590
1.1	42.010	32.596	23.761	16.060	9.997	5.740	3.069	1.550	0.752
1.2	42.202	32.859	24.126	16.527	10.511	6.220	3.455	1.824	0.928
1.3	42.395	33.123	24.488	16.982	11.009	6.690	3.841	2.107	1.117
1.4	42.589	33.388	24.845	17.425	11.493	7.150	4.226	2.397	1.318
1.5	42.785	33.653	25.198	17.859	11.965	7.601	4.610	2.693	1.529
1.6	42.981	33.917	25.547	18.283	12.425	8.044	4.992	2.993	1.748
1.7	43.177	34.181	25.893	18.699	12.875	8.479	5.372	3.298	1.976
1.8	43.374	34.445	26.234	19.107	13.315	8.906	5.749	3.605	2.211
1.9	43.572	34.708	26.571	19.508	13.746	9.326	6.124	3.915	2.452
2.0	43.769	34.969	26.904	19.901	14.168	9.740	6.496	4.227	2.698

(2)	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**
0.1	0.000%	0.000%	0.000%	-0.004%	-0.043%	-0.136%	-0.241%	-0.323%	-0.376%
0.2	0.000%	0.000%	-0.001%	-0.008%	-0.043%	-0.116%	-0.209%	-0.296%	-0.362%
0.3	0.000%	0.000%	-0.002%	-0.012%	-0.046%	-0.113%	-0.197%	-0.275%	-0.338%
0.4	0.000%	0.000%	-0.003%	-0.015%	-0.050%	-0.115%	-0.194%	-0.268%	-0.325%
0.5	0.000%	0.000%	-0.004%	-0.017%	-0.054%	-0.119%	-0.196%	-0.267%	-0.322%
0.6	0.000%	-0.001%	-0.005%	-0.020%	-0.058%	-0.122%	-0.199%	-0.269%	-0.324%
0.7	0.000%	-0.001%	-0.006%	-0.022%	-0.061%	-0.124%	-0.200%	-0.271%	-0.327%
0.8	0.000%	-0.001%	-0.007%	-0.025%	-0.064%	-0.125%	-0.200%	-0.272%	-0.330%
0.9	0.000%	-0.002%	-0.008%	-0.027%	-0.066%	-0.126%	-0.198%	-0.270%	-0.331%
1.0	0.000%	-0.002%	-0.010%	-0.029%	-0.067%	-0.125%	-0.195%	-0.266%	-0.329%
1.1	0.000%	-0.003%	-0.011%	-0.031%	-0.068%	-0.123%	-0.191%	-0.260%	-0.325%
1.2	-0.001%	-0.003%	-0.012%	-0.032%	-0.069%	-0.121%	-0.185%	-0.253%	-0.318%
1.3	-0.001%	-0.004%	-0.013%	-0.033%	-0.069%	-0.119%	-0.180%	-0.245%	-0.310%
1.4	-0.001%	-0.004%	-0.014%	-0.034%	-0.069%	-0.116%	-0.174%	-0.237%	-0.300%
1.5	-0.001%	-0.005%	-0.015%	-0.035%	-0.068%	-0.113%	-0.168%	-0.228%	-0.289%
1.6	-0.001%	-0.005%	-0.016%	-0.036%	-0.067%	-0.110%	-0.162%	-0.219%	-0.278%
1.7	-0.002%	-0.006%	-0.016%	-0.036%	-0.067%	-0.107%	-0.156%	-0.211%	-0.267%
1.8	-0.002%	-0.006%	-0.017%	-0.036%	-0.066%	-0.104%	-0.151%	-0.202%	-0.256%
1.9	-0.002%	-0.007%	-0.018%	-0.037%	-0.065%	-0.101%	-0.145%	-0.194%	-0.246%
2.0	-0.002%	-0.007%	-0.018%	-0.037%	-0.063%	-0.098%	-0.140%	-0.186%	-0.236%

Table 10: SVJ Model: Relative errors of the App-LR method

	60	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**
0.1	N/A	0.000%	0.000%	-0.004%	-0.043%	-0.136%	-0.259%	-1.084%	-0.456%
0.2	0.252%	0.004%	0.000%	-0.008%	-0.043%	-0.116%	-0.212%	-0.559%	-1.282%
0.3	0.012%	0.007%	-0.001%	-0.012%	-0.046%	-0.113%	-0.197%	-0.346%	-0.940%
0.4	0.040%	0.007%	-0.002%	-0.015%	-0.050%	-0.115%	-0.194%	-0.288%	-0.626%
0.5	0.103%	0.005%	-0.003%	-0.017%	-0.054%	-0.119%	-0.196%	-0.273%	-0.458%
0.6	0.133%	0.004%	-0.005%	-0.020%	-0.058%	-0.122%	-0.199%	-0.271%	-0.381%
0.7	0.106%	0.002%	-0.006%	-0.022%	-0.061%	-0.124%	-0.200%	-0.272%	-0.351%
0.8	0.067%	0.001%	-0.007%	-0.025%	-0.064%	-0.125%	-0.200%	-0.272%	-0.339%
0.9	0.039%	0.000%	-0.008%	-0.027%	-0.066%	-0.126%	-0.198%	-0.270%	-0.335%
1.0	0.022%	-0.001%	-0.009%	-0.029%	-0.067%	-0.125%	-0.195%	-0.266%	-0.331%
1.1	0.013%	-0.002%	-0.011%	-0.031%	-0.068%	-0.123%	-0.191%	-0.260%	-0.325%
1.2	0.007%	-0.003%	-0.012%	-0.032%	-0.069%	-0.121%	-0.185%	-0.253%	-0.318%
1.3	0.004%	-0.004%	-0.013%	-0.033%	-0.069%	-0.119%	-0.180%	-0.245%	-0.310%
1.4	0.002%	-0.004%	-0.014%	-0.034%	-0.069%	-0.116%	-0.174%	-0.237%	-0.300%
1.5	0.001%	-0.005%	-0.015%	-0.035%	-0.068%	-0.113%	-0.168%	-0.228%	-0.289%
1.6	0.000%	-0.005%	-0.016%	-0.036%	-0.067%	-0.110%	-0.162%	-0.219%	-0.278%
1.7	-0.001%	-0.006%	-0.016%	-0.036%	-0.067%	-0.107%	-0.156%	-0.211%	-0.267%
1.8	-0.001%	-0.006%	-0.017%	-0.036%	-0.066%	-0.104%	-0.151%	-0.202%	-0.256%
1.9	-0.002%	-0.007%	-0.018%	-0.037%	-0.065%	-0.111%	-0.145%	-0.194%	-0.246%
2.0	-0.002%	-0.007%	-0.018%	-0.037%	-0.063%	-0.098%	-0.140%	-0.186%	-0.236%

Table 11: SVJ Model: the LR method and gamma-based approximation with $\nu = 4$ for strikes 90, 100

	LR(90)	**Gamma(90)**	**LR(100)**	**Gamma(100)**
0.1	-0.004%	0.006%	-0.043%	-0.074%
0.2	-0.008%	0.006%	-0.043%	-0.065%
0.3	-0.012%	0.004%	-0.046%	-0.060%
0.4	-0.015%	0.003%	-0.050%	-0.059%
0.5	-0.017%	0.001%	-0.054%	-0.059%
0.6	-0.020%	-0.002%	-0.058%	-0.060%
0.7	-0.022%	-0.004%	-0.061%	-0.061%
0.8	-0.025%	-0.006%	-0.064%	-0.062%
0.9	-0.027%	-0.008%	-0.066%	-0.064%
1.0	-0.029%	-0.010%	-0.067%	-0.065%
1.1	-0.031%	-0.012%	-0.068%	-0.065%
1.2	-0.032%	-0.014%	-0.069%	-0.066%
1.3	-0.033%	-0.016%	-0.069%	-0.066%
1.4	-0.034%	-0.017%	-0.069%	-0.066%
1.5	-0.035%	-0.018%	-0.068%	-0.066%
1.6	-0.036%	-0.019%	-0.067%	-0.065%
1.7	-0.036%	-0.020%	-0.067%	-0.065%
1.8	-0.036%	-0.021%	-0.066%	-0.064%
1.9	-0.037%	-0.021%	-0.065%	-0.063%
2.0	-0.037%	-0.022%	-0.063%	-0.063%

Table 12: Scott Model: (1) Analytic values (2) Relative errors of the LR method. Parameters: $S_0 = 100$, $y_0^1 = \theta_1 = 3\%$, $y_0^2 = \theta_2 = 2\%$, $\kappa_1 = 5$, $\kappa_2 = 0.4$, $\sigma = 1$, $\sigma_1 = 0.23$, $\sigma_2 = 0.1$, $\rho = -26\%$, $\mu_J = -4\%$, $\sigma_J = 1\%$, $\lambda = 100\%$, maturities (column), strikes (row)

(1)	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**
0.1	40.299	30.349	20.399	10.519	2.484	0.111	0.001	2.0E-06	2.8E-07
0.2	40.597	30.697	20.806	11.186	3.654	0.536	0.034	0.001	3.1E-05
0.3	40.893	31.043	21.230	11.872	4.611	1.072	0.148	0.014	0.001
0.4	41.188	31.390	21.669	12.542	5.459	1.640	0.339	0.052	0.007
0.5	41.481	31.738	22.117	13.190	6.237	2.216	0.592	0.125	0.022
0.6	41.773	32.088	22.570	13.818	6.965	2.790	0.890	0.234	0.053
0.7	42.064	32.439	23.023	14.426	7.655	3.359	1.221	0.378	0.103
0.8	42.354	32.790	23.476	15.018	8.316	3.919	1.578	0.553	0.174
0.9	42.644	33.143	23.928	15.594	8.951	4.472	1.952	0.757	0.266
1.0	42.932	33.495	24.377	16.156	9.565	5.017	2.341	0.984	0.380
1.1	43.219	33.847	24.822	16.705	10.161	5.554	2.740	1.233	0.514
1.2	43.506	34.198	25.264	17.243	10.741	6.084	3.146	1.500	0.668
1.3	43.792	34.549	25.702	17.769	11.307	6.606	3.558	1.782	0.839
1.4	44.077	34.899	26.137	18.286	11.861	7.121	3.974	2.078	1.027
1.5	44.360	35.248	26.567	18.794	12.403	7.630	4.393	2.385	1.231
1.6	44.643	35.595	26.993	19.293	12.934	8.133	4.815	2.701	1.447
1.7	44.925	35.941	27.415	19.785	13.457	8.629	5.238	3.027	1.677
1.8	45.206	36.285	27.833	20.268	13.970	9.120	5.661	3.359	1.918
1.9	45.486	36.627	28.247	20.745	14.475	9.606	6.085	3.698	2.169
2.0	45.765	36.968	28.657	21.215	14.972	10.087	6.509	4.042	2.429

(2)	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**
0.1	0.000%	0.000%	0.000%	0.000%	0.002%	-0.025%	-4.501%	-12.167%	-99.217%
0.2	0.000%	0.000%	0.000%	0.000%	-0.002%	-0.017%	-0.045%	-0.063%	-0.459%
0.3	0.000%	0.000%	0.000%	-0.001%	-0.008%	-0.029%	-0.058%	-0.076%	-0.109%
0.4	0.000%	0.000%	0.000%	-0.002%	-0.011%	-0.034%	-0.066%	-0.091%	-0.108%
0.5	0.000%	0.000%	0.000%	-0.003%	-0.013%	-0.034%	-0.064%	-0.095%	-0.118%
0.6	0.000%	0.000%	-0.001%	-0.004%	-0.013%	-0.032%	-0.059%	-0.089%	-0.117%
0.7	0.000%	0.000%	-0.001%	-0.004%	-0.013%	-0.029%	-0.052%	-0.079%	-0.108%
0.8	0.000%	0.000%	-0.001%	-0.004%	-0.012%	-0.025%	-0.044%	-0.068%	-0.094%
0.9	0.000%	0.000%	-0.001%	-0.004%	-0.011%	-0.022%	-0.038%	-0.058%	-0.080%
1.0	0.000%	0.000%	-0.001%	-0.004%	-0.010%	-0.019%	-0.032%	-0.049%	-0.068%
1.1	0.000%	0.000%	-0.001%	-0.004%	-0.009%	-0.017%	-0.028%	-0.041%	-0.057%
1.2	0.000%	0.000%	-0.001%	-0.004%	-0.008%	-0.015%	-0.024%	-0.035%	-0.048%
1.3	0.000%	0.000%	-0.001%	-0.003%	-0.007%	-0.013%	-0.021%	-0.030%	-0.041%
1.4	0.000%	0.000%	-0.001%	-0.003%	-0.007%	-0.012%	-0.018%	-0.026%	-0.035%
1.5	0.000%	0.000%	-0.001%	-0.003%	-0.006%	-0.010%	-0.016%	-0.022%	-0.030%
1.6	0.000%	0.000%	-0.001%	-0.003%	-0.005%	-0.009%	-0.014%	-0.020%	-0.026%
1.7	0.000%	0.000%	-0.001%	-0.003%	-0.005%	-0.008%	-0.012%	-0.017%	-0.022%
1.8	0.000%	0.000%	-0.001%	-0.002%	-0.004%	-0.007%	-0.011%	-0.015%	-0.020%
1.9	0.000%	0.000%	-0.001%	-0.002%	-0.004%	-0.006%	-0.010%	-0.013%	-0.017%
2.0	0.000%	0.000%	-0.001%	-0.002%	-0.004%	-0.006%	-0.009%	-0.012%	-0.015%

Table 13: Scott Model: Relative errors of the App-LR method using (1) separate approximations for $\hat{z}$ and $\tilde{z}$, and (2) $\hat{z}$ and the identity $\tilde{z} = \hat{z} - 1$.

(1)	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**
0.1	-246.923%	N/A	-0.001%	0.000%	0.002%	-0.025%	-9.641%	246.595%	-2.956%
0.2	N/A	0.011%	-0.001%	0.000%	-0.002%	-0.017%	-0.204%	-9.696%	363.989%
0.3	N/A	0.033%	0.000%	-0.001%	-0.008%	-0.029%	-0.063%	-1.502%	-4.076%
0.4	0.015%	0.010%	0.000%	-0.002%	-0.011%	-0.034%	-0.066%	-0.219%	-3.503%
0.5	0.020%	0.003%	0.000%	-0.003%	-0.013%	-0.034%	-0.064%	-0.105%	-0.698%
0.6	0.018%	0.001%	0.000%	-0.004%	-0.013%	-0.032%	-0.059%	-0.090%	-0.198%
0.7	0.007%	0.001%	-0.001%	-0.004%	-0.013%	-0.029%	-0.052%	-0.079%	-0.119%
0.8	0.002%	0.000%	-0.001%	-0.004%	-0.012%	-0.025%	-0.044%	-0.068%	-0.096%
0.9	0.001%	0.000%	-0.001%	-0.004%	-0.011%	-0.022%	-0.038%	-0.058%	-0.081%
1.0	0.000%	0.000%	-0.001%	-0.004%	-0.010%	-0.019%	-0.032%	-0.049%	-0.068%
1.1	0.000%	0.000%	-0.001%	-0.004%	-0.009%	-0.017%	-0.028%	-0.041%	-0.057%
1.2	0.000%	0.000%	-0.001%	-0.004%	-0.008%	-0.015%	-0.024%	-0.035%	-0.048%
1.3	0.000%	0.000%	-0.001%	-0.003%	-0.007%	-0.013%	-0.021%	-0.030%	-0.041%
1.4	0.000%	0.000%	-0.001%	-0.003%	-0.007%	-0.012%	-0.018%	-0.026%	-0.035%
1.5	0.000%	0.000%	-0.001%	-0.003%	-0.006%	-0.010%	-0.016%	-0.022%	-0.030%
1.6	0.000%	0.000%	-0.001%	-0.003%	-0.005%	-0.009%	-0.014%	-0.020%	-0.026%
1.7	0.000%	0.000%	-0.001%	-0.003%	-0.005%	-0.008%	-0.012%	-0.017%	-0.022%
1.8	0.000%	0.000%	-0.001%	-0.002%	-0.004%	-0.007%	-0.011%	-0.015%	-0.020%
1.9	0.000%	0.000%	-0.001%	-0.002%	-0.004%	-0.006%	-0.010%	-0.013%	-0.017%
2.0	0.000%	0.000%	-0.001%	-0.002%	-0.004%	-0.006%	-0.009%	-0.012%	-0.015%

(2)	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**
0.1	0.121%	N/A	0.000%	0.000%	0.002%	-0.025%	-4.499%	-12.211%	-99.224%
0.2	N/A	0.000%	0.000%	0.000%	-0.002%	-0.017%	-0.045%	-0.056%	-0.589%
0.3	N/A	0.000%	0.000%	-0.001%	-0.008%	-0.029%	-0.058%	-0.075%	-0.105%
0.4	0.001%	0.000%	0.000%	-0.002%	-0.011%	-0.034%	-0.066%	-0.090%	-0.103%
0.5	0.000%	0.000%	0.000%	-0.003%	-0.013%	-0.034%	-0.064%	-0.095%	-0.117%
0.6	0.000%	0.000%	-0.001%	-0.004%	-0.013%	-0.032%	-0.059%	-0.089%	-0.117%
0.7	0.000%	0.000%	-0.001%	-0.004%	-0.013%	-0.029%	-0.052%	-0.079%	-0.108%
0.8	0.000%	0.000%	-0.001%	-0.004%	-0.012%	-0.025%	-0.044%	-0.068%	-0.094%
0.9	0.000%	0.000%	-0.001%	-0.004%	-0.011%	-0.022%	-0.038%	-0.058%	-0.080%
1.0	0.000%	0.000%	-0.001%	-0.004%	-0.010%	-0.019%	-0.032%	-0.049%	-0.068%
1.1	0.000%	0.000%	-0.001%	-0.004%	-0.009%	-0.017%	-0.028%	-0.041%	-0.057%
1.2	0.000%	0.000%	-0.001%	-0.004%	-0.008%	-0.015%	-0.024%	-0.035%	-0.048%
1.3	0.000%	0.000%	-0.001%	-0.003%	-0.007%	-0.013%	-0.021%	-0.030%	-0.041%
1.4	0.000%	0.000%	-0.001%	-0.003%	-0.007%	-0.012%	-0.018%	-0.026%	-0.035%
1.5	0.000%	0.000%	-0.001%	-0.003%	-0.006%	-0.010%	-0.016%	-0.022%	-0.030%
1.6	0.000%	0.000%	-0.001%	-0.003%	-0.005%	-0.009%	-0.014%	-0.020%	-0.026%
1.7	0.000%	0.000%	-0.001%	-0.003%	-0.005%	-0.008%	-0.012%	-0.017%	-0.022%
1.8	0.000%	0.000%	-0.001%	-0.002%	-0.004%	-0.007%	-0.011%	-0.015%	-0.020%
1.9	0.000%	0.000%	-0.001%	-0.002%	-0.004%	-0.006%	-0.010%	-0.013%	-0.017%
2.0	0.000%	0.000%	-0.001%	-0.002%	-0.004%	-0.006%	-0.009%	-0.012%	-0.015%

Figure 1: Graphs of $\mathcal{K}(z)$ and $\mathcal{K}'(z)$ with $T = 1$ in the Heston model

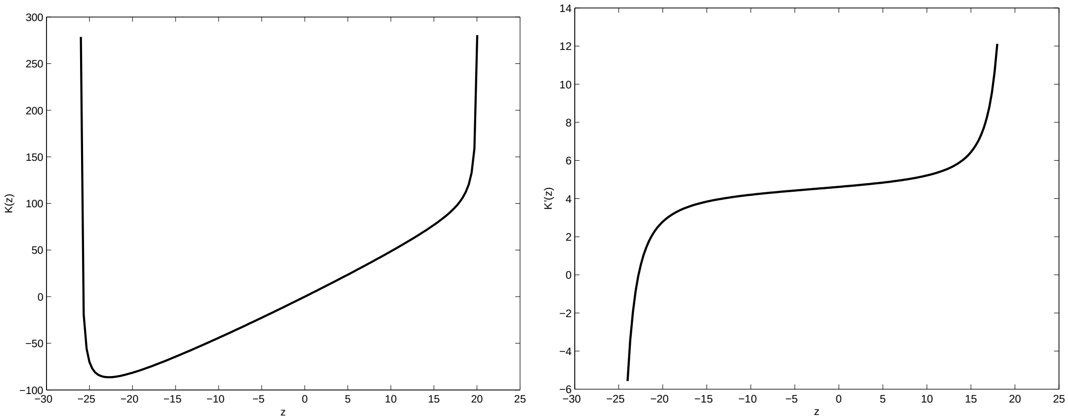

Figure 2: Graphs of $yz - \mathcal{K}(z)$, $y - \mathcal{K}'(z)$ where $y = \ln c$, $T = 0.1$, $c = 60$ in the SVJ model

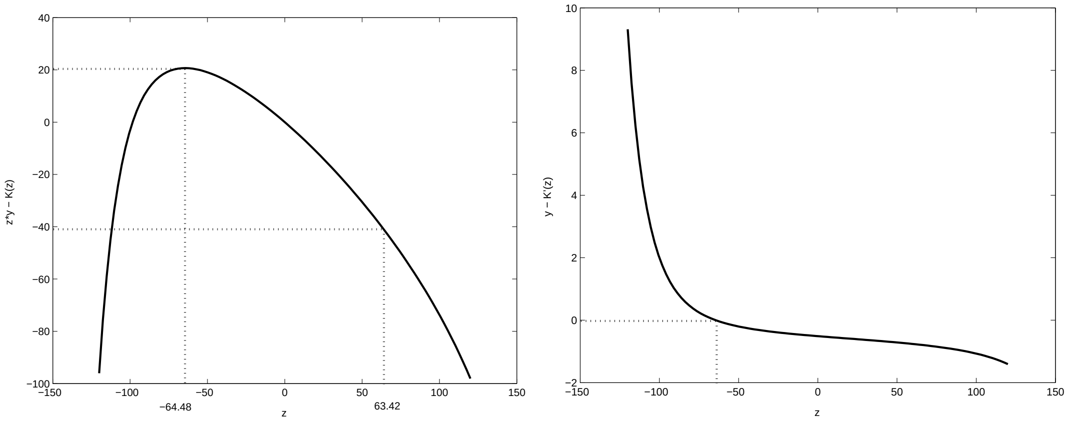

Figure 3: Effect of the jump arrival rate in the SVJ Model

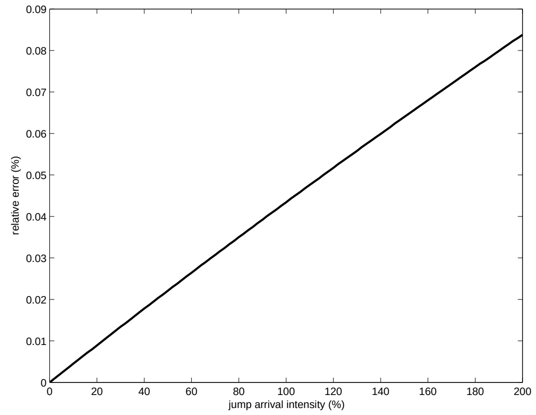

Figure 4: Effect of the mean of the jump size in the SVJ Model

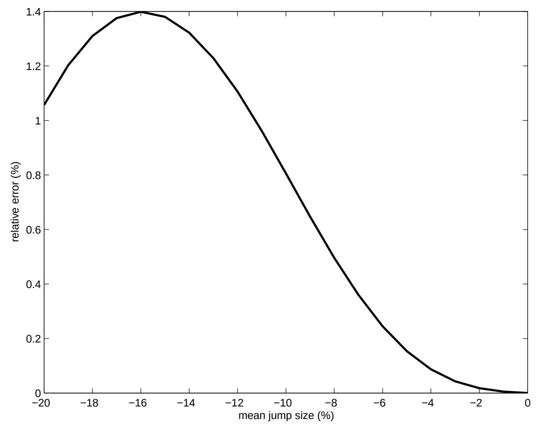

Figure 5: Effect of the volatility of the jump size in the SVJ Model