Cap and swaption approximations in Libor market models with jumps

Paul Glasserman
403 Uris Hall, Graduate School of Business, Columbia University, New York, NY 10027

Nicolas Merener
Department of Applied Physics and Applied Mathematics, Columbia University, New York, NY 10027

This paper develops formulas for pricing caps and swaptions in Libor market models with jumps. The arbitrage-free dynamics of this class of models were characterized in Glasserman and Kou (2003) in a framework allowing for very general jump processes. For computational purposes, it is convenient to model jump times as Poisson processes; however, the Poisson property is not preserved under the changes of measure commonly used to derive prices in the Libor market model framework. In particular, jumps cannot be Poisson under both a forward measure and the spot measure, and this complicates pricing. To develop pricing formulas, we approximate the dynamics of a forward rate or swap rate using a scalar jump-diffusion process with time-varying parameters. We develop an exact formula for the price of an option on this jump-diffusion through explicit inversion of a Fourier transform. We then use this formula to price caps and swaptions by choosing the parameters of the scalar diffusion to approximate the arbitrage-free dynamics of the underlying forward or swap rate. We apply this method to two classes of models: one in which the jumps in all forward rates are Poisson under the spot measure, and one in which the jumps in each forward rate are Poisson under its associated forward measure. Numerical examples demonstrate the accuracy of the approximations.

1 Introduction

This paper develops formulas for pricing caps and swaptions in jump-diffusion Libor market models. The success of the original Libor market models and their extensions (including Brace, Gatarek, and Musiela, 1997; Miltersen, Sandmann, and Sondermann, 1997; Jamshidian, 1997; Andersen and Andreasen, 2000; Joshi and Rebonato, 2001; and Zühlsdorff, 2000) is due in part to the availability of formulas and fast numerical methods for pricing caps and swaptions within this framework. These are necessary for model calibration. The Libor market model framework was extended to include jumps in Glasserman and Kou (2003) and, using a different approach, in Jamshidian (1999). Glasserman and Kou (2003) focus on characterizing the arbitrage-free dynamics of forward rates in the

The authors thank Professor Steve Kou for helpful discussions in developing the approximations in this paper. This work is supported by NSF grant DMS0074637.

presence jumps. They identify a tractable subclass of models in which caplet prices are given through a blending of the formulas of Black (1976) and Merton (1976), but they leave open the question of simultaneous pricing of swaptions and also the question of cap and swaption pricing for other specifications of the jump process. We take up these questions here.

Including jumps in a model of the Libor term structure introduces an issue not present in purely diffusive models: the most convenient model of jumps is a Poisson process, but the Poisson property is not preserved by the changes of measure (eg, between the spot measure, the forward measure, and the swap measure) that lie at the heart of the market model framework. In formulating a model, one must choose the measure under which jumps will be Poisson. Glasserman and Kou (2003) obtain a caplet formula by requiring that each Libor rate have Poisson jumps under its associated forward measure; we refer to this choice here as a *forward Poisson* (FP) specification. For other purposes, it may be desirable to have jumps in all forward Libor rates follow Poisson processes simultaneously under a single measure, so here we also consider cap and swaption prices in a *spot Poisson* (SP) specification that accomplishes this under the spot measure. The arbitrage-free dynamics of forward Libor under both specifications follow immediately from more general results in Glasserman and Kou (2003).

Because we consider cap and swaption prices in both the SP and FP specifications, we deal with four basic pricing problems. Our strategy for all four cases may be summarized as follows:

- Express the option (caplet or swaption) as an expectation involving only a single underlying rate (forward rate or swap rate) through appropriate choice of measure (forward measure or swap measure).
- Approximate the dynamics of the underlying rate under the chosen measure using a scalar jump-diffusion with time-varying parameters, Poisson jump times, and lognormal jump-size factors.
- Apply an extension of Merton's (1976) formula to value an option on the approximating jump-diffusion process.

We comment briefly on each of these steps. The first uses ideas now standard in the market model literature; we review these in Section 2. In the last step, the extension is needed to handle time-varying parameters. The approximating jump-diffusion process has the property that its logarithm has independent increments, and this lends itself to option pricing through Fourier transform inversion. We obtain an option pricing result by inverting the transform explicitly. This formula is the key to our cap and swaption approximations. The formula could also be used to incorporate jumps in the pricing of equity or currency options.

In the three-step strategy outlined above, only the second step involves approximations – the first and third are exact. To approximate a more general process (forward rate or swap rate) using a simpler jump-diffusion, we need, in each case, to choose the approximating drift, diffusion coefficient, Poisson arrival rate, and jump-size parameters. We do this by analyzing the dynamics of forward rates and swap rates under the appropriate measures.

The presence of jumps suggests that we are working in an incomplete market and thus raises the question of the sense in which a formula can legitimately be interpreted as a price. We are, in effect, assuming that the market is made "nearly" complete by derivative prices. Because caps and swaptions are actively traded, the purpose of pricing formulas lies in calibrating model parameters (such as jump rates) to market prices – the formulas are most useful in reverse. In a complete market, the prices of derivatives would completely determine the parameters; in practice, we expect that market prices would substantially constrain though not completely determine parameter values. The remaining indeterminacy would be removed by imposing additional restrictions on the parameterization. This is standard practice in, for example, calibrating a diffusion model to a volatility surface.

The rest of the paper is organized as follows. Section 2 reviews Libor market models and discusses derivatives pricing and changes of numeraire. Section 3 introduces an exact formula for the price of a European option on a particular type of jump-diffusion with time-varying coefficients. Section 4 discusses general arbitrage restrictions on market models with jumps, and the dynamics under relevant pricing measures. In Section 5, we develop caplet and swaption formulas in the SP specification and support the method through numerical experiments. We treat the FP specification in Section 6. By construction, caplets are priced exactly in this setting, but we go beyond Glasserman and Kou (2003) in allowing for time-varying parameters. We derive a swaption approximation and support it through numerical results. Section 7 summarizes the models and formulas presented. All proofs are collected in the Appendix.

2 Libor market models and derivatives

We consider Libor market models of the term structure, of the type developed by Brace, Gatarek, and Musiela (1997); Miltersen, Sandmann, and Sondermann (1997); and Jamshidian (1997). We take a discrete tenor structure – a finite set of dates $0 = T_0 < T_1 < \ldots < T_{M+1}$, with $T_{i+1} - T_i \equiv \delta$. The fixed accrual period δ is expressed as a fraction of a year; for instance, $\delta = 1/2$ represents six months. Each tenor date T_k is the maturity of a zero-coupon bond; $B_k(t)$ denotes the price of that bond at time $t \in [0, T_k]$ and $B_k(T_k) \equiv 1$. Forward Libor rates $L_1, \ldots, L_M$ may be defined from the bond prices by setting

$$L_k(t) = \frac{1}{\delta}\left(\frac{B_k(t)}{B_{k+1}(t)} - 1\right), \quad t \in [0, T_k], \quad k = 1, \ldots, M \tag{1}$$

Similarly, $L_0(0)$ is the rate for $[0, T_1]$. Let $\eta(t) = \inf\{j \geq 0\colon T_j \geq t\}$ so that $\eta(t)$ is the index of the next maturity as of time t. Bond prices can then be written in terms of Libor rates and the next bond to mature $B_{\eta(t)}$ as

$$B_k(t) = B_{\eta(t)}(t) \prod_{j=\eta(t)}^{k-1} \frac{1}{1+\delta L_j(t)} \tag{2}$$

Absence of arbitrage by trading in bonds is essentially equivalent to the existence of a numeraire $M(t)$ and an associated measure under which discounted asset prices $B_j(t)/M(t)$ are martingales. For a general class of payoffs, determined by the state of the set of forward rates at time T, the price $C(0)$ of a contingent claim with payoff $C(T)$ becomes

$$C(0) = M(0)\mathsf{E}\left[\frac{C(T)}{M(T)}\right]$$

the expectation taken with respect to the measure associated with numeraire $M(t)$. There are several numeraires that have been shown to be convenient in the pricing of derivatives. The spot martingale measure P, introduced by Jamshidian (1997), uses a discretely compounded money market account as numeraire,

$$B(t) = B_{\eta(t)}(t) \prod_{j=0}^{\eta(t)-1} \left(1+\delta L_j(t)\right) \tag{3}$$

Under this measure, the price of a caplet with strike K, for the period $[T_n, T_{n+1}]$, which pays $\delta(L_n(T_n) - K)^+$ at time T_{n+1} is

$$C_n(0) = \delta \mathsf{E}^P\left[\prod_{j=0}^{n} \frac{1}{1+\delta L_j(T_j)} \left(L_n(T_n) - K\right)^+\right]$$

For pricing derivatives tied to just one forward rate (such as a caplet), it is often convenient to choose as numeraire a zero-coupon bond maturing at the end of the accrual period associated with the forward rate. Thus, to price a claim contingent on L_n, we take as numeraire the bond B_{n+1}. Observe from (1) that

$$\delta L_n(t) = \frac{B_n(t) - B_{n+1}(t)}{B_{n+1}(t)}$$

is the ratio of a portfolio of assets to $B_{n+1}(t)$, so that under the measure associated with B_{n+1} as numeraire, $L_n(t)$ is a martingale. This in fact is why this particular choice of numeraire is convenient. The measure associated with this numeraire is usually called the *forward measure* or *terminal measure* for maturity T_{n+1}; see Musiela and Rutkowski (1997) for background. Writing $\mathsf{E}^{P_{n+1}}$ for expectation under this measure, we have

$$C_n(0) = \delta B_{n+1}(0)\mathsf{E}^{P^{n+1}}\left[\left(L_n(T_n) - K\right)^+\right] \tag{4}$$

The nth caplet price is thus determined by the dynamics of L_n under its associated forward measure.

Swaptions are also widely traded derivatives. A payer's swaption maturing at T_n is an option to enter a fixed-for-floating swap between dates T_n and T_{M+1}. If the option is exercised, the holder makes fixed payments δK and receives floating payments $\delta L_i(T_i)$ at T_{i+1}, $i = n, \ldots, M$. The swaption value at expiration is

$$C_{n,M}(T_n) = \delta \sum_{j=n}^{M} B_{j+1}(T_n)\left(S_{n,M}(T_n) - K\right)^+$$

where the swap rate $S_{n,M}$ is

$$S_{n,M}(t) \equiv \sum_{j=n}^{M} b_j(t) L_j(t) \quad \text{with} \quad b_j(t) \equiv \frac{B_{j+1}(t)}{\sum_{i=n}^{M} B_{j+1}(t)} \tag{5}$$

The time-0 swaption price under the spot martingale measure is

$$C_{n,M}(0) = \delta \mathsf{E}^P\left[\sum_{j=n}^{M} B_{j+1}(T_n)\left(S_{n,M}(T_n) - K\right)^+\right] \tag{6}$$

but it is often convenient to change the pricing measure and take, as Jamshidian (1997) did,

$$M(t) = \delta \sum_{j=n}^{M} B_{j+1}(t) \tag{7}$$

as numeraire, associated to the measure $P^{n,M}$. This leads to an alternative representation for the swaption price, as the deterministically discounted expected value of a European payoff,

$$C_{n,M}(0) = \delta \sum_{j=n}^{M} B_{j+1}(0) \mathsf{E}^{P^{n,M}}\left[\left(S_{n,M}(T_n) - K\right)^+\right] \tag{8}$$

Furthermore, from the representation,

$$S_{n,M}(t) = \frac{B_n(t) - B_{M+1}(t)}{\delta \sum_{j=n}^{M} B_{j+1}(t)}$$

and the form of the numeraire in (7), it is evident that $S_{n,M}$ is a martingale under $P^{n,M}$.

We have seen that, under the appropriate changes of measure, both caplets and swaptions become European options with trivial discounting, on forward and swap rates respectively. This naturally suggests a strategy to develop pricing formulas. The idea is to approximate the dynamics of each of the relevant rates, under the corresponding pricing measure, by dynamics simple enough to lead to formulas for option prices. To carry this out, in the section we develop an option pricing formula for relatively simple scalar jump-diffusion. This formula will then be the tool we use to develop cap and swaption pricing formulas.

3 A jump-diffusion option formula

In this section, we digress from our discussion of term structure models to develop our key pricing tool. We consider the pricing of a European option on a scalar jump-diffusion with Poisson jump times and lognormal jump-size factors. This is a process used by Merton (1976) to model a stock price, except that we allow the parameters of the model to vary with time. This extension is essential for our intended application. We restrict the time dependence of the coefficients by allowing them to change only at tenor dates while remaining constant during each accrual period. This restriction is computationally convenient and consistent with the way interest rate volatilities are calibrated in practice. One could presumably extend the pricing formula below to continuously varying parameters by appropriately replacing the sums with integrals.

Consider, then, the process

$$\frac{\mathrm{d}G(t)}{G(t-)} = a(t)\mathrm{d}t + \gamma(t)\,\mathrm{d}W(t) + \mathrm{d}\left(\sum_{j=1}^{N(t)}(Y_j - 1)\right), \quad 0 \le t \le T_M \tag{9}$$

with $W(t)$ a standard Brownian motion, $\gamma(t)$ deterministic and bounded, N a Poisson process with rate $\lambda(t)$, and $Y_j \in (0, \infty)$ independent with lognormal density $f(\cdot, t)$ having mean $1 + m(t)$, and with W, N, and $\{Y_1, Y_2, \ldots,\}$ mutually independent. We take the G to be right-continuous and denote by $G(t-)$ the left limit at t. As the marks Y_j are positive, the form of the jump term ensures that $G(t)$ also remains positive, given positive $G(0)$. We parameterize each lognormal density $f(\cdot, t)$ through the mean $\mu(t)$ and standard deviation $\sigma(t)$ of the associated normal density. In particular, $m = \mathrm{e}^{\mu+\sigma^2/2} - 1$. By piecewise constant coefficients we mean that $a(t) = a_k$, $\gamma(t) = \gamma_k$, $\lambda(t) = \lambda_k$ and $f(y, t) = f_k(y)$ for $T_{k-1} < t \le T_k$, $k = 1, \ldots, M$. At this point, G has no specific financial meaning; later, we use it to model or approximate a forward rate or a swap rate.

Imposing a piecewise constant parameterization on dynamics (9) leads to closed form European option prices through the Fourier transform approach, successfully used to price contingent claims by Heston (1993); Duffie, Pan, and Singleton (2000); Carr and Madan (1999); and Scott (1997) among others. The Fourier inversion approach is based on the transform of the logarithm of the asset (G in our case) defined as

$$\psi(z) \equiv \mathsf{E}\left[\mathrm{e}^{z\log(G(t))}\right], \quad z \in \mathbb{C}$$

In particular, we need the existence, in analytic form, of $\psi(z)$ at $z = 1 + iu$ and $z = iu$, with $u \in \mathbb{R}$ and $i = \sqrt{-1}$. The choice of a lognormal jump density f is standard in financial modeling and is also computationally convenient. The same method could be used with other choices of f; Kou (1999) uses a double-exponential distribution for log jump sizes. The key point is that the logarithm of G has independent increments and piecewise constant coefficients, so its distribution is given by a convolution and is thus convenient for transform analysis. The pricing formula for a European option maturing at T_n, $n \le M$, is as follows:

PROPOSITION 3.1 With G defined as in (9), the expected value of a European payoff is

$$\mathsf{E}\left[(G(T_n)) - K)^+\right] = G(0)\Pi_1 - K\Pi_2 \tag{10}$$

with

$$\Pi_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{e^{B_1(u)} \sin\left(B_2(u) - u\log\left(\frac{K}{G(0)}\right)\right)}{u} \, du \tag{11}$$

$$\Pi_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{e^{B_3(u)} \sin\left(B_4(u) - u\log\left(\frac{K}{G(0)}\right)\right)}{u} \, du \tag{12}$$

where $w_i = \mu_i + \sigma_i^2$, $\alpha_i = a_i - \gamma_i^2/2$ and

$$B_1(u) = \delta\sum_{i=1}^{n}\lambda_i\left[e^{\mu_i + \sigma_i^2(1-u^2)/2}\cos(w_i u) - 1\right] - \lambda_i m_i - \gamma_i^2 u^2/2$$

$$B_2(u) = \delta\sum_{i=1}^{n}\lambda_i e^{\mu_i + \sigma_i^2(1-u^2)/2}\sin(w_i u) + \alpha_i u + \gamma_i^2 u$$

$$B_3(u) = \delta\sum_{i=1}^{n}\lambda_i\left[e^{-\sigma_i^2 u^2/2}\cos(\mu_i u) - 1\right] - \gamma_i^2 u^2/2$$

$$B_4(u) = \delta\sum_{i=1}^{n}\lambda_i e^{-\sigma_i^2 u^2/2}\sin(\mu_i u) + \alpha_i u$$

This result is proved in the Appendix. The integrals in (11) and (12) can be computed to very high accuracy in very short time, so the result is essentially a closed-form expression.

In allowing time-varying coefficients, the formula above extends the Merton formula (1976) for the price of a European option on a lognormal jump-diffusion. The Merton formula was used by Glasserman and Kou (2003) (in a manner analogous to the use of the Black, 1976, formula in Brace, Gatarek and Musiela, 1997) to build a jump-diffusion model that prices caplets (or swaptions) exactly by forcing the forward (or swap) rate to be, under its own forward (or swap) measure, a scalar lognormal jump-diffusion with Poisson distributed jump times and constant coefficients over the life of the rate. In the spirit of Glasserman and Kou we obtain Proposition 3.1 as an exact caplet pricing formula for those model specifications in which the dynamics of a forward rate under its forward measure are exactly of the form (9). Proposition 3.1 is also potentially useful for equity and currency options, where jumps can be used to reproduce a skew in implied volatilities.

Although we do not pursue this line in this paper, we could also build models, with piecewise constant coefficients, that price swaptions exactly through Proposition 3.1. However, as noted by Glasserman and Kou (2003), and as will become clear later from the dynamics of the rates under different measures, exact prices cannot be achieved for caplets and swaptions within the same model specification, because the law of the jumps cannot be Poisson under both forward and swap measures. A similar problem arises in the pure diffusion case, where it is well-known that there is no model in which the forward and swap rates are simultaneously lognormal.

As explained in the Introduction, we develop caplet formulas through the dynamics of each forward rate under its associated forward measure. Depending on the particular model specification the dynamics are, either exactly or approximately, of the form (9). Caplets are then priced using the formula in Proposition 3.1. For swaption formulas, we work along a line well developed in the pure diffusion setting by Andersen and Andreasen (2000); Brace, Dun, and Barton (2001); Hull and White (2000); Jäckel and Rebonato (2000); and Kawai (2000) among others, who characterized the volatility of the swap rate in terms of the volatilities of the forward rates. We use the fact that a swap rate is approximately a linear combination of simple forward rates and develop approximations for the dynamics of the swap rate under the swap measure, with the form (9) and its coefficients written in terms of those in the dynamics of the forward rates. We then approximate swaption prices through Proposition 3.1.

To approximate a forward rate or swap rate by a process of the form (9), we need to choose the parameters of the approximating process. A substantial part of the paper consists in choosing these parameters. In particular, we will take the density of the jump sizes f in (9) to approximately match the first two moments of the conditional expected jump size of the rate of interest. For later use, we record here that the expected jump size in (9), conditional on being at a jump time τ, is

$$\mathsf{E}[G(\tau) - G(\tau-) \mid \tau, G(\tau-)] = G(\tau-)\int_0^\infty (y-1) f(y)\,\mathrm{d}y = G(\tau-)m \tag{13}$$

and the conditional second moment is

$$\begin{aligned}\mathsf{E}\left[(G(\tau) - G(\tau-))^2 \middle| \tau, G(\tau-)\right] &= G^2(\tau-)\int_0^\infty (y-1)^2 f(y)\,\mathrm{d}y \\ &= G^2(\tau-)\left[\mathrm{e}^{\sigma^2(\tau-)}(1+m(\tau-))^2 - 2m(\tau-) - 1\right]\end{aligned} \tag{14}$$

Later, we match these expressions to corresponding moments for jumps in forward rates or swap rates in order to select parameters.

4 Jump-diffusion Libor models

4.1 Modeling jumps

Glasserman and Kou (2003) characterized the arbitrage-free dynamics of Libor market models with jumps. Before entering into the details of their formulation and possible model specifications, we briefly review some tools used to model jumps, most importantly the notions of *marked point process* (MPP) and associated *intensity*. For general mathematical background on MPPs see, eg, Brémaud (1981), and for their use in term structure modeling see Björk, Kabanov, and Runggaldier (1997).

We describe an MPP through a sequence of pairs of times and marks $\{(\tau_j, X_j), j = 1, 2, \ldots\}$, with the interpretation that the mark X_j arrives at τ_j. The τ_j take values in $(0, \infty)$ and are strictly increasing in j. The marks X_j take values in a subset of $\mathbb{R}^D$. Let $N(t)$ be the number of points in (0, t]: $N(t) = \sup\{j \geq 0\colon \tau_j \leq t\}$. From an MPP we construct jump processes by choosing a function H: $\mathbb{R}^D \times (0, \infty) \to \mathbb{R}$ and defining

$$J(t) = \sum_{j=1}^{N(t)} H\left(X_j, \tau_j\right)$$

The function H transforms the mark X_j into a jump magnitude, and so different jump processes can be generated from one MPP by different choices of H. The MPP $\{(\tau_j, X_j)\}$ is assumed to admit an intensity process $\nu(\mathrm{d}x, t)$ interpreted as the arrival rate of marks in dx, conditional on the history of the MPPs and the Brownian motion $W(t)$ up to $t-$. More precisely, the intensities have the property that, for all bounded H,

$$\sum_{j=1}^{N(t)} H\left(X_j, \tau_j\right) - \int_0^t \int_{\mathbb{R}^D} H(x,s)\nu(\mathrm{d}x,s)\,\mathrm{d}s$$

is a martingale in t.

Marked point processes are the source of the jumps in the forward rate models we will consider. The evolution of the rate maturing at T_k takes the general form

$$\frac{\mathrm{d}L_k(t)}{L_k(t-)} = \alpha_k(t, L(t-))\mathrm{d}t + \gamma_k(t, L(t-))\mathrm{d}W(t) + \mathrm{d}J_k(t)$$

for W a d-dimensional Brownian motion, and deterministic functions $\alpha_k\colon [0, \infty) \times \mathbb{R}^M \to \mathbb{R}$ and $\gamma_k\colon [0, \infty) \times \mathbb{R}^M \to \mathbb{R}^d$. The processes L_k are right-continuous and L denotes the vector $(L_1, \ldots, L_M)$. The jump term is driven by r MPPs, with

$$J_k(t) = \sum_{i=1}^{r} \sum_{j=1}^{N^{(i)}(t)} H_{ki}\left(X_j^{(i)}, \tau_j^{(i)}\right)$$

and deterministic functions $H_{ki}\colon \mathbb{R}^D \times (0, T_M) \to \mathbb{R}$, $k = 1, \ldots, M$, $i = 1, \ldots, r$.

Most relevant for practical applications is the case of models driven by MPPs with a "Markovian" property – namely, that $\nu(\mathrm{d}x, t) = \lambda(\mathrm{d}x, L(t-), t)$ for some *deterministic* λ. In other words, the intensity depends on the history of the process only through the current state L. Models considered in this work will be Markovian, and we will further assume the existence of a mark density conditional on the current state. In this case, the intensity can be written as $\lambda(\mathrm{d}x, L(t-), t) = \lambda(x, L(t-), t)\,\mathrm{d}x$.

Observe that integrating over all possible marks we get the total jump arrival rate at t conditional on $L(t-)$, $\int_{\mathbb{R}^D} \lambda(x, L(t-), t)\,\mathrm{d}x$. Also, the conditional probability density of the mark, given a jump time τ and $L(\tau-)$ is

$$\frac{\lambda(x, L(\tau-), \tau)}{\int_{\mathbb{R}^D} \lambda(x, L(\tau-), \tau)\,\mathrm{d}x}$$

A subclass within the class of Markovian MPPs with mark density are processes for which the intensity is independent of $L(t-)$. The intensity can then be written as $\nu(\mathrm{d}x, t) = \lambda_0(t), f(x, t)\,\mathrm{d}x$ with $f(\cdot, t)$ a probability density on $\mathbb{R}^D$. Thus, the arrival times follow a Poisson process with deterministic intensity $\lambda_0(t)$, and the marks are independent and distributed with density $f(\cdot, t)$ at time t. Process (9) belongs to this subclass.

4.2 Dynamics

With the tools of the previous section, we present now a simplified version of Theorem 3.1 in Glasserman and Kou (2003). This is a characterization of the arbitrage-free dynamics of simple forward rates under the spot martingale measure P. (This is a minor modification of the usual risk-neutral measure, but based on a discretely compounded rather than continuously compounded numeraire.)

The building blocks are a d-dimensional Brownian motion $W_{P(t)}$ and r marked point processes $\{(\tau_j^{(i)}, X_j^{(i)}), j = 1, 2, \ldots\}$, $X_j^{(i)} \in \mathbb{R}^D$, $i = 1, \ldots, r$ with intensities $\nu_P^{(i)}$. For each $n = 1, \ldots, M$ let $\gamma_n(\cdot)$ be a bounded, adapted, $\mathbb{R}^d$-valued process and let H_{ni}, $i = 1, \ldots, r$, be functions from $\mathbb{R}^D \times [0, T_M]$ to $[-1, \infty)$. The model

$$\frac{\mathrm{d}L_n(t)}{L_n(t-)} = \alpha_n(t)\mathrm{d}t + \gamma_n(t)\mathrm{d}W_P(t) + \mathrm{d}J_n(t), \quad 0 \le t \le T_n, \quad n = 1, \ldots, M \tag{15}$$

with

$$J_n(t) = \sum_{i=1}^{r} \sum_{j=1}^{N^{(i)}(t)} H_{ni}\left(X_j^{(i)}, \tau_j^{(i)}\right) \tag{16}$$

is arbitrage-free if $\alpha_n(t)$ is given by

$$\gamma_n(t) = \sum_{k=\eta(t)}^{n} \frac{\delta \gamma_k(t)^{\mathsf{T}} L_k(t-)}{1 + \delta L_k(t-)}$$

$$-\int_{\mathbb{R}^D} \sum_{i=1}^{r} H_{ni}(x,t) \prod_{k=\eta(t)}^{n} \frac{1+\delta L_k(t-)}{1+\delta L_k(t-)(1+H_{ki}(x,t))} \nu_P^{(i)}(\mathrm{d}x,t) \tag{17}$$

We have implicitly taken $W_P(t)$ to be a column vector, $\gamma_k(t)$ a row vector and $\gamma_k(t)^\mathsf{T}$ its transpose. Observe that the first term in (17) is the familiar drift restriction for purely diffusive Libor market models; the second term is the additional restriction imposed by jumps.

As mentioned in Section 2, the pricing of the nth caplet is simplified if we take B_{n+1} as numeraire. We will use this fact to develop caplet approximations, so we discuss now the dynamics of L_n under P^{n+1}. Under this measure, L_n becomes a martingale and therefore the drift is determined by the characteristics of the jump process. A consequence of the Girsanov theorem generalized to processes with jumps (as in Björk, Kabanov, and Runggaldier, 1997) changing from the spot measure P to the forward measure P^{n+1} transforms the intensity of an MPP. For the class of models in Glasserman and Kou (2003), the intensities $\nu_{Pn+1}^{(i)}$ and $\nu_P^{(i)}$ for the ith MPP under the forward and spot measures are related as in the following result (proved in the Appendix):

LEMMA 4.1 *Under P^{n+1}, the intensity of the ith marked point process is given by*

$$\nu_{P^{n+1}}^{(i)}(\mathrm{d}x,t) = \prod_{j=\eta(t)}^{n} \frac{1+\delta L_j(t-)}{1+\delta L_j(t-)(1+H_{ji}(x,t-))} \nu_P^{(i)}(\mathrm{d}x,t)$$

As with caplets, there is also a privileged measure for the pricing of swaptions, which we will use to develop swaption approximations. Under the swap measure $P^{n,M}$ the swap rate becomes a martingale, so identification of the jump part of the dynamics determines the drift. The change of intensity associated to the change from the spot measure P to the swap measure $P^{n,M}$ is given by

LEMMA 4.2 *Under $P^{n,M}$, the intensity of the ith marked point process is given by*

$$\nu_{P^{n,M}}^{(i)}(\mathrm{d}x,t) = \sum_{j=n}^{M} b_j(t-) \prod_{k=\eta(t)}^{j} \frac{1+\delta L_k(t-)}{1+\delta L_k(t-)(1+H_{ki}(x,t-))} \nu_P^{(i)}(\mathrm{d}x,t)$$

with $b_j(t)$ as in (5).

From these two lemmas we see that the changes of measure considered do not preserve the Poisson property because the new intensity is derived from the original intensity through multiplication by a stochastic factor. We noted at the end of Section 4.1 that the Poisson case is characterized by a deterministic intensity. The changes of measure do, however, preserve the Markovian feature because the stochastic factor by which the intensity is multiplied is a function only of the current levels of the L_ks. If, for example, jumps were Poisson under the spot measure P, then Lemma 4.1 tells us that under the forward measure P^{n+1} both the arrival rate of jumps and the sizes of jumps would depend on the evolution of the forward rates.

4.3 Two model specifications

The class of models defined by (15)–(17) is very wide. Practical implementation of a pricing model requires the specification of the diffusion volatility and characteristics of the jump process. In particular, we have seen that while it is desirable for pricing purposes to have Poisson jumps, this feature is restricted by the state dependent intensities introduced when changing measures. Therefore, a key point in the specification of a model is determining what rate will have Poisson jumps under which measure. We focus our investigation on two different specifications of the model (15)–(17), each motivated by its own computational and modeling advantages. The first specification we consider postulates a Poisson process for the jump times of the rates under the spot martingale measure and independent marks to generate the actual jump magnitudes. We refer to this as the *spot Poisson* (SP) specification. In this case, Monte Carlo methods are straightforward to implement because Poisson jumps are easy to simulate, and this is useful for the pricing of exotic options. However, as pointed out in Glasserman and Kou (2003), and as is apparent in Lemma 4.1, the jumps in each rate under its own forward measure cease to be Poisson, as the distribution of the jump times and marks becomes state dependent; this complicates caplet pricing. A similar complication arises under the swap measure, and the jumps in the swap rate cease to be Poisson. We develop caplet and swaption formulas for this model specification approximating the forward and swap rate, under the forward and swap measures respectively, with processes of the form (9). Proposition 3.1 then gives the prices.

Our second specification was developed in Glasserman and Kou (2003) and further investigated in Glasserman and Merener (2003). The idea is to choose the diffusion volatilities and jump law in (15)–(17) in a way that leads to exact closed formulas for caplet prices. This is achieved by forcing each rate to evolve *exactly* as in (9) under its own forward measure. We refer to this as the it *forward Poisson* (FP) specification. In this case, there is no common measure under which all rates follow a Poisson process because the changes of measure introduce state dependent jump arrival rates. For instance, it is clear from Lemma 4.1 that the jumps of the rates under the spot martingale measure are not Poisson as both the arrival rate of the jumps, and the density from which the jump magnitudes are sampled, are state dependent. This is consequential because pricing of exotic deals must be done in general by simulation methods, typically involving the evolution of the whole term structure under a common measure. This task is complicated by the non-Poisson nature of the jumps. The issue is discussed in detail in Glasserman and Merener (2003), where thinning algorithms for path discretization methods are proposed and tested in realistic settings and efficient simulation of the model is explored.

By construction, the FP specification prices caplets exactly. We address the pricing of swaptions, for which we derive a formula through the introduction of an approximate swap rate process of the form (9) and subsequent application of Proposition 3.1.

5 Spot Poisson (SP) specification

5.1 Dynamics

We formulate in this section a lognormal jump-diffusion Libor model in which the law of the jump times, under the spot martingale measure, is Poisson. At a jump time, a vector mark $X \in \mathbb{R}_+^D$ is sampled. This mark has independent components, each generated from a standard lognormal distribution (meaning that the logarithm of each component of X has a standard normal distribution). The map that translates $x \in \mathbb{R}_+^D$ into the actual jump magnitude is

$$H_k(x, t) = \prod_{j=1}^{D} \beta_{jk}(t) x_j^{\sigma_{jk}(t)} - 1 \tag{18}$$

with β and σ deterministic and nonnegative, a choice that we will discuss after introducing the dynamics of the rates. This specification is a special case of (15)–(17) with just one marked point process so $r = 1$ in (15). Furthermore, this point process is Poisson. To summarize, the dynamics of the rates under the spot measure are

$$\frac{\mathrm{d}L_n(t)}{L_n(t-)} = \alpha_n(t)\mathrm{d}t + \gamma_n(t)\mathrm{d}W_P(t) + \mathrm{d}J_n(t), \quad 0 \le t \le T_n, \quad n = 1, \ldots, M \tag{19}$$

with

$$J_n(t) = \sum_{j=1}^{N(t)} H_n(X_j, \tau_j) \tag{20}$$

and $\alpha_n(t)$ equal to

$$\begin{aligned} \gamma_n(t) = & \sum_{k=\eta(t)}^{n} \frac{\delta \gamma_k(t)^{\mathsf{T}} L_k(t-)}{1 + \delta L_k(t-)} \\ & - \int_{\mathbb{R}_+^D} H_n(x,t) \prod_{k=\eta(t)}^{n} \frac{1 + \delta L_k(t-)}{1 + \delta L_k(t-)(1 + H_k(x,t))} \nu_P(\mathrm{d}x, t) \end{aligned} \tag{21}$$

where γ_n is deterministic, H_n is as in (18) and

$$\nu_P(\mathrm{d}x, t) = \lambda(t) f(x, t) \mathrm{d}x$$

with $\lambda(t)$ a bounded deterministic arrival rate and $f(\cdot, t)$ multivariate lognormal on $\mathbb{R}^D$. The marks X_j, $j = 1, 2\ldots$ are independent and W_P, N, X_j are mutually independent.

For the purposes of later applying Proposition 3.1, we will take γ, λ, β and σ to be constant between tenor dates, though writing in what follows a general dependance on t to avoid additional indices. From a practical point of view and calibration considerations, a piecewise constant specification is sufficiently general.

The form of H_k in (18) is motivated by noticing that the jump in the logarithm of the nth rate at a jump time τ is

$$\log\left(\frac{L_n(\tau)}{L_n(\tau-)}\right) = \log(1 + H_n(X,\tau))$$
$$= \sum_{j=1}^{D}\left\{\log(\beta_{jn}(\tau)) + \sigma_{jn}(\tau)\log(X^{(j)})\right\}$$

and therefore, conditional on τ, distributed as a multivariate normal because $\log(X^{(1)}),\ldots,\log(X^{(D)})$ are independent, normally distributed ($X^{(j)}$ is the jth component of the vector X). Different choices of β and σ generate different correlations between the jump magnitudes of the rates. This is analogous to the role of γ in determining the correlations introduced through the diffusion term in a multifactor model.

5.2 Simulation

Contingent claim pricing by Monte Carlo simulation entails the generation of sample paths of the model presented above. As discussed in Glasserman and Merener (2003), in order to minimize biases, it is often convenient to take the logarithm of the rates as the variables to solve numerically. We have implemented a first order scheme for the logarithm of the rates on a discretization grid that includes all jump times of the Poisson process, which may be computed in advance (independently for each path) as their law is independent of the state of the system. Glasserman and Merener (2003) and Mikulevicius and Platen (1988) discuss path discretization methods for stochastic differential equations with jumps, and Kloeden and Platen (1992) give an extensive treatment of discretization methods for pure diffusions. Notice that the drift term in (19) includes an integral in the mark space. This integral is recomputed at every time step of the time discretization grid, based on the state of the discretized solution at the time. We have implemented Monte Carlo simulation for a model specification in which the mark is sampled from $\mathbb{R}_+$, that is, taking $D = 1$. We computed numerically the integral in the mark space with a grid uniformly spaced up to a sufficiently large cutoff. Recomputing this integral at every point of the time grid, which includes all jump times, is probably unnecessary when jumps happen often. Some of the methods in Kloeden and Platen (1992) may be useful in reducing this computational overhead.

5.3 A caplet formula

In this section we derive a simple approximation for the price of a caplet associated with the rate L_n. It is clear from (4) that it is convenient to start from the dynamics of L_n under the forward measure P^{n+1}. Changing numeraire leaves the diffusion volatility unchanged. Furthermore, the martingale feature of L_n under P^{n+1} dictates the drift term once the intensity of the jump process under the forward measure is identified. The dynamics of L_n under P^{n+1} are

$$\frac{\mathrm{d}L_n(t)}{L_n(t-)} = \gamma_n(t)\mathrm{d}W_{P^{n+1}}(t) - \int_{\mathbb{R}_+^D} H_n(x,t)\,\nu_{P^{n+1}}(\mathrm{d}x,t)\mathrm{d}t + \mathrm{d}J_n(t) \tag{22}$$

with

$$J_n(t) = \sum_{j=1}^{N(t)} H_n(X_j, \tau_j)$$

with $\gamma_n \in \mathbb{R}^d$ deterministic, H_n as in (18) and where the arrival rates of the jumps under P^{n+1} is given by Lemma 4.1,

$$\nu_{P^{n+1}}(\mathrm{d}x,t) = \prod_{k=\eta(t)}^{n} \frac{1+\delta L_k(t-)}{1+\delta L_k(t-)\prod_{j=1}^{D}\beta_{jk}(t)x_j^{\sigma_{jk}(t)}}\lambda(t)f(x,t)\mathrm{d}x \tag{23}$$

The dynamics (22) and (23) do not lead to exact closed caplet prices because the law of the jumps $\nu_{P^{n+1}}(\mathrm{d}x, t)$ depends on the path of L. We derive a pricing formula by introducing a martingale process $\hat{L}$ of the form (9) that approximates L_n under P^{n+1} in a distribution sense. We consider

$$\frac{\mathrm{d}\hat{L}(t)}{\hat{L}(t-)} = \hat{\gamma}(t)\mathrm{d}W(t) - \int_0^\infty (y-1)\hat{\lambda}(t)\hat{f}(y,t)\mathrm{d}y\mathrm{d}t + \mathrm{d}\left(\sum_{j=1}^{\hat{N}(t)}(Y_j-1)\right), \quad 0 \le t \le T_n \tag{24}$$

where $\hat{\gamma}$ is deterministic, W is a standard (scalar) Brownian motion, $\hat{N}$ is a Poisson process with arrival rate $\hat{\lambda}(t)$ and marks sampled from the lognormal density $\hat{f}(\cdot, t)$ with parameters $\hat{\sigma}(t)$, $\hat{\mu}(t)$. W, $\hat{N}$, and $\hat{Y}_j$, $j = 1, 2, \ldots$ are independent, with $\hat{\gamma}$, $\hat{\lambda}$, $\hat{\sigma}$, and $\hat{\mu}$ to be determined. The initial condition is $\hat{L}(0) = L_n(0)$.

We specify the coefficients of the approximate process in terms of those in the dynamics of L as follows. First, we take

$$\hat{\gamma}(t) = [\gamma_n(t)\gamma_n(t)^{\mathsf{T}}]^{1/2} \tag{25}$$

a natural choice, leading to exact prices in the absence of jumps.

Next, we look at the jump term. We will determine the jump law of $\hat{L}$ in two steps. First, we will define $\hat{\lambda}$, the total jump arrival rate of $\hat{L}$, as an approximation of the total arrival rate of L_n under P^{n+1}. Second, we will identify $\hat{f}$, the conditional jump probability density of $\hat{L}$, by approximately matching it with the conditional jump probability density of L_n.

For the total arrival rate, one would like to define $\hat{\lambda}$ as the total jump arrival rate of L_n under P^{n+1}, which at each time t is given by the integral over dx of (23). Unfortunately, this is not a deterministic process because it depends on L. We introduce then an approximation that will become standard throughout the rest of the paper, by fixing the rates at time zero. Under this approximation, and writing ν_{Pn+1} explicitly as in (23) we define

$$\hat{\lambda}(t) \equiv \int_{\mathbb{R}_+^D} \prod_{k=\eta(t)}^{n} \frac{1+\delta L_k(0)}{1+\delta L_k(0)\prod_{j=1}^{D}\beta_{jk}(t)x_j^{\sigma_{jk}(t)}} \lambda(t) f(x,t)\mathrm{d}x \tag{26}$$

Notice that the explicit time dependence of the coefficients β, σ, λ is preserved.

Last, we identify the parameters of the jump size probability density $\hat{f}$ by approximately matching its first two moments with those of the conditional probability density of the jumps of L_n. From our discussion about intensities in Section 4, and dynamics (22), we know that the conditional distribution of marks of L_n, at a jump time τ, is

$$\frac{\nu_{P^{n+1}}(\mathrm{d}x,\tau-)}{\int_{\mathbb{R}_+^D} \nu_{P^{n+1}}(\mathrm{d}x,\tau-)}$$

which can be approximated as $\nu_{P^{n+1}}(\mathrm{d}x, \tau-)/\hat{\lambda}(\tau-)$. Therefore, the conditional expectation of the jump size is

$$\begin{aligned}
\mathsf{E}^{P^{n+1}}\left[L_n(\tau)-L_n(\tau-)\middle|\tau, L(\tau-)\right] &= L_n(\tau-)\int_{\mathbb{R}_+^D} H_n(x,\tau-)\frac{\nu_{P^{n+1}}(\mathrm{d}x,\tau-)}{\int_{\mathbb{R}_+^D}\nu_{P^{n+1}}(\mathrm{d}x,\tau-)} \\
&\approx L_n(\tau-)\int_{\mathbb{R}_+^D} H_n(x,\tau-)\frac{\nu_{P^{n+1}}(\mathrm{d}x,\tau-)}{\hat{\lambda}(\tau-)}
\end{aligned}$$

This motivates fixing the rates at time zero in the last integral above and, writing H_n and ν_{Pn+1} explicitly, defining

$$\begin{aligned}
I_1(t) \equiv \int_{\mathbb{R}_+^D} &\left(\prod_{j=1}^{D}\beta_{jn}(t)x_j^{\sigma_{jn}(t)} - 1\right) \\
&\times \prod_{k=\eta(t)}^{n} \frac{1+\delta L_k(0)}{1+\delta L_k(0)\prod_{j=1}^{D}\beta_{jk}(t)x_j^{\sigma_{jk}(t)}}\frac{\lambda(t)}{\hat{\lambda}(t)} f(x,t)\mathrm{d}x
\end{aligned} \tag{27}$$

The conditional expectation of the jump size of $\hat{L}$ at jump time s (under the measure associated to $\hat{L}$) follows from (13) because $\hat{L}$ is of the form (9),

$$\mathsf{E}\left[\hat{L}(s)-\hat{L}(s-)\middle| s, \hat{L}(s-)\right] = \hat{L}(s-)\int_0^\infty (y-1)\hat{f}(y,s-)\mathrm{d}y = \hat{L}(s-)\hat{m}(s-)$$

This suggests matching (27) and $\hat{m}$, to define the latter in terms of the parameters of the original model

$$\hat{m}(t) \equiv I_1(t) \tag{28}$$

We proceed in identical way for the second moment of the conditional jump probability. We have

$$\mathsf{E}^{P^{n+1}}[(L_n(\tau) - L_n(\tau-))^2 | \tau, L(\tau-)] \approx L_n^2(\tau-) \int_{\mathbb{R}_+^D} H_n^2(x, \tau-) \frac{\nu_{P^{n+1}}(\mathrm{d}x, \tau-)}{\hat{\lambda}(\tau-)}$$

which motivates defining I_2 as

$$I_2(t) \equiv \int_{\mathbb{R}_+^D} \left(\prod_{j=1}^{D} \beta_{jn}(t) x_j^{\sigma_{jn}(t)} - 1 \right)^2 \times \prod_{k=\eta(t)}^{n} \frac{1 + \delta L_k(0)}{1 + \delta L_k(0) \prod_{j=1}^{D} \beta_{jk}(t) x_j^{\sigma_{jk}(t)}} \frac{\lambda(t)}{\hat{\lambda}(t)} f(x, t) \mathrm{d}x$$

and matching it to the integral in the right side of

$$\mathsf{E}\left[\left(\hat{L}(s) - \hat{L}(s-) \right)^2 \middle| s, L(s-) \right] = \hat{L}(s-)^2 \int_0^\infty (y-1)^2 \hat{f}(y, s-) \mathrm{d}y$$

Using (14) we get

$$\mathrm{e}^{\hat{\sigma}^2(t)} (1 + \hat{m}(t))^2 - 2\hat{m}(t) - 1 = I_2(t)$$

Further algebra involving (28) gives

$$\hat{\sigma} = \left[\log \left(\frac{I_2 + 1 + 2I_1}{(1 + I_1)^2} \right) \right]^{\frac{1}{2}} \qquad \hat{\mu} = \log(1 + I_1) - \frac{\hat{\sigma}^2}{2} \tag{29}$$

To summarize, the dynamics of $\hat{L}$ are given by (24), (25), (26), and (29), and the caplet is priced using Proposition 3.1 applied to $\hat{L}$. The integrals in (26) and (29) are computed numerically.

5.4 A swaption formula

The derivation of a swaption formula for model (18)–(21) follows along the lines of last section. We recall that, by using $\sum_{j=n}^{M} \delta B_j(t)$ as numeraire, the complex swaption payoff is transformed into a simple European payoff on the swap rate with a deterministic discount factor. Furthermore, the swap rate is a martingale under this measure. As in the caplet case, we will obtain a swaption formula through Proposition 3.1, by approximating the swap rate $S_{n,M}$ dynamics under the associated swap measure $P^{n,M}$ with a process $\hat{S}$ defined as

$$\frac{\mathrm{d}\hat{S}(t)}{\hat{S}(t)} = \hat{\gamma}(t)\mathrm{d}W - \int_0^{\infty}(y-1)\hat{\lambda}\hat{f}(y,t)\mathrm{d}y\,\mathrm{d}t + \mathrm{d}\left[\sum_{j=1}^{\hat{N}(t)}(Y_j-1)\right] \tag{30}$$

with W a (scalar) standard Brownian motion, $\hat{\gamma}(t)$ deterministic, $\hat{N}(t)$ a Poisson process with intensity $\hat{\lambda}(t)$ and Y_j distributed as $\hat{f}(\cdot,\ t)$, lognormal with parameters $\hat{\sigma}(t)$, $\hat{\mu}(t)$, and W, $\hat{N}$, and $\hat{Y}_j$, $j = 1, 2, \ldots$ independent. The initial condition is $\hat{S}(0) = S_{n,M}(0)$.

As the dynamics of the swap rate are determined by the forward rates (5), we begin by considering the dynamics of the forward rates under the swap measure $P^{n,M}$,

$$\frac{\mathrm{d}L_k(t)}{L_k(t-)} = \gamma_k(t)\mathrm{d}W_{P^{n,M}}(t) + \ldots\,\mathrm{d}t + \mathrm{d}J_k(t) \tag{31}$$

with

$$J_k(t) = \sum_{i=1}^{N(t)} H_k(X_i,\tau_i)$$

and where the arrival rates of the jumps under $P^{n,M}$ is given by Lemma 4.2

$$\nu_{P^{n,M}}(\mathrm{d}x,t) = \sum_{j=n}^{M} b_j(t-)\prod_{i=\eta(t)}^{j}\frac{1+\delta L_i(t-)}{1+\delta L_i(t-)(1+H_i(x,t-))}\lambda(t-)f(x,t-)\mathrm{d}x \tag{32}$$

We have omitted the drift term in (31) because we will force $\hat{S}$ to be a martingale by construction so the drift of the forward rates is irrelevant for our purposes. Notice, however, that the forward rates are not martingales under the swap measure. We will approximate the swap rate as a linear combination of simple forward rates by fixing at time zero the weights $b_j(t)$ in (5). We write b_j for $b_j(0)$ and we have

$$S_{n,M}(t) \approx \sum_{j=n}^{M} b_j L_j(t)$$

and apply Itô's formula on the right side of the last equation to investigate how to approximate the dynamics of the swap rate. The diffusion term is

$$\sum_{j=n}^{M} b_j L_j(t)\gamma_j(t)\mathrm{d}W_{P^{n,M}}$$

with $\gamma_j \in \mathbb{R}^d$, which has quadratic variation

$$\sum_{j=n}^{M}\sum_{k=n}^{M} b_j b_k L_j(t) L_k(t)\gamma_j(t)\gamma_k(t)^{\mathsf{T}}\mathrm{d}t$$

This diffusion term is to be approximated by the diffusion term in (30), $\hat{S}(t)\,\hat{\gamma}(t)\,\mathrm{d}W$, with $\hat{\gamma} \in \mathbb{R}$ and quadratic variation $\hat{\gamma}^2 \hat{S}(t)^2 \mathrm{d}t$. Fixing the rates at time zero and matching quadratic variations as in Jäckel and Rebonato (2000), we define

$$\hat{\gamma}(t) \equiv \frac{1}{S_{n,M}(0)} \left[\sum_{j=n}^{M} \sum_{k=n}^{M} b_j b_k L_j(0) L_k(0) \gamma_j(t) \gamma_k(t)^{\mathsf{T}} \right]^{\frac{1}{2}} \tag{33}$$

where we have used that $\hat{S}(0) = S_{n,M}(0)$. This would hold exactly (under the assumption of constant weights) if all forward rates had the same volatility.

Next we focus on the jump contribution. As we did in the caplet formula of Section 5.3, we will (approximately) match the total jump arrival rates and the conditional jump probability densities of $\hat{S}$ and the process $\sum_{j=n}^{M} b_j L_j(t)$. Notice that while the forward rates that contribute to a swap rate can potentially jump with different magnitudes, they all jump at the same times as they are driven by the same marked point process. Therefore, the total arrival rate of the jumps of $\sum_{j=n}^{M} b_j L_j(t)$ is the integral of $\nu_{P^{n,M}}(\mathrm{d}x, t)$ over $\mathbb{R}_+^D$. As in the caplet case, this is dependent on the path of L so we fix the rates L and weights b at time zero to define

$$\hat{\lambda}(t) \equiv \int_{\mathbb{R}_+^D} \sum_{j=n}^{M} b_j \prod_{k=\eta(t)}^{j} \frac{1+\delta L_k(0)}{1+\delta L_k(0)(1+H_k(x,t))} \lambda(t) f(x,t)\mathrm{d}x \tag{34}$$

with $H_k(x, t)$ as in (18).

Last, we identify $\hat{f}$ in (31) by approximately matching its first two moments with those of the jump size of the swap rate, conditional on being at a jump time. As in Section 5.3, the conditional jump probability density is

$$\frac{\nu_{P^{n,M}}(\mathrm{d}x,t)}{\int_{\mathbb{R}_+^D} \nu_{P^{n,M}}(\mathrm{d}x,t)} \approx \frac{\nu_{P^{n,M}}(\mathrm{d}x,t)}{\hat{\lambda}(t)}$$

Therefore, at a jump time τ of $S_{n,M}$ we have

$$\begin{aligned} &\mathsf{E}^{P^{n,M}}\left[S_{n,M}(\tau) - S_{n,M}(\tau-) \,\middle|\, \tau, L(\tau-)\right] \\ &\approx \int_{\mathbb{R}_+^D} \sum_{j=n}^{M} b_j(\tau-)\, L_j(\tau-)\, H_j(x,\tau-) \frac{\nu_{P^{n,M}}(\mathrm{d}x,\tau-)}{\hat{\lambda}(\tau-)} \end{aligned}$$

which motivates fixing the rates and weights at time zero on the right side above, writing $\nu_{P^{n,M}}$ explicitly and defining

$$I_1(t) \equiv \int_{\mathbb{R}_+^D} \sum_{j=n}^{M} b_j\, L_j(0)\, H_j(x,t) \prod_{k=\eta(t)}^{n} \frac{1+\delta L_k(0)}{1+\delta L_k(0)(1+H_k(x,t))} \frac{\lambda(t)}{\hat{\lambda}(t)} f(x,t)\mathrm{d}x \tag{35}$$

For $\hat{S}$, which is of the form (9), we use (13) and write

$$\mathsf{E}\left[\hat{S}(s)-\hat{S}(s-)\,\middle|\,s,\hat{S}(s-)\right]=\hat{S}(s-)\int_0^\infty (y-1)\hat{f}(y,s-)\,\mathrm{d}y=\hat{S}(s-)\hat{m}(s-) \quad (36)$$

which suggests matching (35) and the rightmost expression in (36) with $\hat{S}$ fixed at time 0 to define $\hat{m}$ as

$$\hat{m}(t)\equiv\frac{I_1(t)}{S_{n,M}(0)} \quad (37)$$

We proceed in the same way for the second moment. Define

$$I_2(t)=\int_{\mathbb{R}_+^D}\left(\sum_{j=n}^{M} b_j\,L_j(0)\,H_j(x,t)\right)^2\prod_{k=\eta(t)}^{n}\frac{1+\delta L_k(0)}{1+\delta L_k(0)(1+H_k(x,t))}\frac{\lambda(t)}{\hat{\lambda}}f(x,t)\,\mathrm{d}x$$

and match it to the right side of

$$\mathsf{E}\left[\left(\hat{S}(s)-\hat{S}(s-)\right)^2\,\middle|\,s,L(s-)\right]=\hat{S}(s-)^2\int_0^\infty (y-1)^2\hat{f}(y,s-)\,\mathrm{d}y$$

with $\hat{S}(s-)$ replaced by $\hat{S}(0)$. Using (14) again we get

$$\mathrm{e}^{\hat{\sigma}^2(t)}(1+\hat{m}(t))^2-2\hat{m}(t)-1=\frac{I_2(t)}{S_{n,M}^2(0)}$$

Further algebra gives,

$$\hat{\sigma}=\left[\log\left(\frac{I_2\big/S_{n,M}^2(0)+1+2\hat{m}}{(1+\hat{m})^2}\right)\right]^{\frac{1}{2}}\qquad\hat{\mu}=\log(1+\hat{m})-\frac{\hat{\sigma}^2}{2} \quad (38)$$

with $\hat{m}$ as in (37).

To summarize, the dynamics of $\hat{S}$ are given by (30), (33), (34), and (38) and the swaption is priced by applying Proposition 3.1 to $\hat{S}$.

5.5 Numerical results

We are now ready to test the caplet and swaption formulas developed in the previous section and compare the prices they give to prices obtained by Monte Carlo simulation. For simplicity, we take a one factor diffusion, and jump marks sampled from a standard lognormal so $D = 1$ in (18).

Recall that the parameters of the model are piecewise constant, ie, remain constant between tenor dates. We denote this explicitly, with $\gamma_k(T_j)$ being the diffusion volatility of the kth rate across the accrual period ending at T_j, and same convention for the rest of the parameters. Using expressions for the moments of a lognormal density and the fact that the jumps are Poisson, we compute the total instantaneous volatility of rate k in the jth accrual period under the spot martingale measure,

$$\left[\gamma_k(T_j)^2 + \lambda(T_j)\left(e^{\sigma_k(T_j)^2}(1+m_k(T_j))^2 - 2(1+m_k(T_j))+1\right)\right]^{\frac{1}{2}}$$

with $1 + m_k(T_j) = \beta_k(T_j)\,e^{(\sigma_k(T_j))^2/2}$. The instantaneous volatility is approximately $[\gamma_k(T_j)^2 + \lambda_k(T_j)\sigma_k(T_j)^2]^{1/2}$ for $\beta_k(T_j) = 1$ and $\sigma_k(T_j) \ll 1$. The first parameter set we consider is this:

Parameter set A: The initial term structure is flat, $L_k(0) = 0.06$, $\forall k$, $\delta = 0.5$ and the parameters are, for all j and k,

$$\gamma_k(T_j) = 0.1,$$
$$\lambda_k(T_j) = 5 \times 0.99^{j-1}, \quad \sigma_k(T_j) = 0.1 \times 1.01^{j-1}, \quad \beta_k(T_j) = 1.0$$

As times evolves (j increases), jumps happen less often but are larger in size. For some sense on the size of the typical forward rate jump we compute the standard deviation of $X^{\sigma_k(T_j)}$ (as we are taking $\beta = 1$) for X sampled from a lognormal density associated to a standard normal. This is approximately 0.1, so a typical forward rate jump size is 10% of the value of the rate. The total instantaneous volatility, for all rates during the first accrual period, is approximately 0.245. The squared diffusion volatility is 0.01, the jump squared volatility is 0.06 so that most of the rate volatility is due to jumps. Also, as the parameters of the jump magnitudes do not depend on the rate index k, all rates jump with the same percentage magnitude.

Parameter set B: The initial term structure is increasing, $L_k(0) = \log(1.051271 + 0.0011178 \times k)$, $\delta = 0.5$. This gives $L(0) = 0.05$ and $L(20) = 0.07$. The parameters are, for all j and k,

$$\gamma_k(T_j) = 0.1,$$
$$\lambda_k(T_j) = 5 \times 1.01^{j-1}, \qquad \sigma_k(T_j) = 0.2 \times 0.95^{k-j}, \quad \beta_k(T_j) = 1.0$$

In this specification, rates jump more often as time evolves (j increasing). Also, we have taken a stationary specification for the conditional distribution of marks at jump times, by making σ (and β, trivially) dependent on time to maturity $k - j$. Rates closer to maturity jump with wider amplitude than rates far from maturity. The total volatility of the rate that matures next, as seen from time zero, is 0.45. This would correspond in practice to a very high volatility period.

We compare our option price formulas against simulation results. The simulation uses a grid with time step 0.5, which coincides with the tenor dates; but since the grid also includes all jump times, the effective time step is actually much smaller. Monte Carlo simulation is subject to some error due to model

TABLE 1 Approximate vs simulated caplet prices, parameter set A.

Spot Poisson Caplet prices, Parameter set A			
Caplet maturity (years)	**Strike**	**Simulation (95% conf. interval)**	**Approximate price**
2	0.05	116.93 (0.11)	116.96
2	0.06	71.02 (0.09)	71.04
2	0.07	41.53 (0.07)	41.56
5	0.05	127.56 (0.14)	127.60
5	0.06	94.89 (0.13)	94.93
5	0.07	70.60 (0.11)	70.64
10	0.05	121.47 (0.13)	121.38
10	0.06	100.76 (0.12)	100.69
10	0.07	84.19 (0.12)	84.13

TABLE 2 Approximate vs simulated caplet prices, parameter set B.

Spot Poisson Caplet prices, Parameter set B			
Caplet maturity (years)	**Strike**	**Simulation (95% conf. interval)**	**Approximate price**
2	0.044	151.82 (0.51)	152.15
2	0.054	113.02 (0.47)	113.32
2	0.064	84.71 (0.42)	84.99
5	0.050	176.96 (0.63)	176.68
5	0.060	149.59 (0.60)	149.28
5	0.070	127.45 (0.57)	127.13
10	0.061	169.97 (0.56)	169.65
10	0.071	152.63 (0.54)	152.37
10	0.081	137.80 (0.52)	137.59

discretization. As a simple check, we have compared bond prices estimated from simulation with bonds calculated from the initial forward rates. The error in the simulated bond prices is sufficiently small to be ignored, which suggests that we may safely compare our approximations against option prices computed through simulation.

Results for caplets are presented in Tables 1 and 2, results for swaptions are presented in Tables 3 and 4. The differences between approximate caplet prices and simulated caplet prices are less than the half-widths of the rather tight 95% confidence intervals for the simulation results, except perhaps for the in-the-money, two-year caplet of parameter set B. For all other caplets, the relative pricing error, defined as the difference between approximate and exact price divided by the exact price, is smaller than the half-widths of the 95% confidence intervals divided by the exact price, therefore relative errors are less than 0.5%.

The absolute swaption errors in parameter set A are all smaller than the half-widths of the 95% confidence intervals, which implies relative errors smaller

TABLE 3 Approximate vs simulated swaption prices, parameter set A.

Spot Poisson swaption prices, Parameter set A			
Swaption length (years)	**Strike**	**Simulation (95% conf. interval)**	**Approximate price**
3 x 3	ITM 0.05	342.94 (0.94)	342.45
3 x 3	ATM 0.06	229.51 (0.82)	229.59
3 x 3	OTM 0.07	151.48 (0.89)	151.61
3 x 7	ITM 0.05	714.89 (2.07)	713.88
3 x 7	ATM 0.06	478.96 (1.80)	478.29
3 x 7	OTM 0.07	315.67 (1.99)	315.48
5 x 5	ITM 0.05	559.59 (1.03)	560.22
5 x 5	ATM 0.06	415.89 (0.93)	416.52
5 x 5	OTM 0.07	309.05 (1.01)	309.68

TABLE 4 Approximate vs simulated swaption prices, parameter set B.

Spot Poisson swaption prices, Parameter set B			
Swaption length (years)	**Strike**	**Simulation (95% conf. interval)**	**Approximate price**
3 x 3	ITM 0.049	439.77 (1.05)	440.94
3 x 3	ATM 0.059	340.15 (0.97)	341.20
3 x 3	OTM 0.069	264.56 (1.02)	265.47
3 x 7	ITM 0.053	849.83 (1.74)	861.15
3 x 7	ATM 0.063	632.13 (1.58)	641.87
3 x 7	OTM 0.073	471.13 (1.70)	478.31
5 x 5	ITM 0.055	702.05 (1.46)	708.36
5 x 5	ATM 0.065	571.72 (1.36)	577.42
5 x 5	OTM 0.075	468.37 (1.45)	473.21

than 0.7%. Therefore the swaption pricing problem, when all rates are driven by the same MPP and have the same percentage jump (as σ does not depend on k in this case), is solved satisfactorily. In parameter set B present, the accuracy of the pricing formulas continues to be excellent for 3×3 and 5×5 swaptions. While the relative pricing errors is larger for options out of the money, the relative error is not more than 1% in all cases. The approximate formula is a bit less accurate for 3×7 options. Comparison between 5×5 and 3×7 swaptions suggests that increasing swap length has more impact on the quality of the approximation than longer maturity. Relative pricing error in the 3×7 case is largest for the out-of-the-money swaption, but even the worst case error is less than 2% of the option price and thus less than typical bid-ask spreads.

Causes of the comparatively larger errors in set B with respect to set A could be the large volatility of parameter set B and also the fact that in specification A all rates jump with the same percentage jump so that $\sum_{k=n}^{M} b_k L_k(\tau-)$ jumps to $\sum_{k=n}^{M} b_k L_k(\tau-) X^{\sigma}$ with X lognormally distributed. As σ does not depend on k, the percentage jump of $\sum_{k=n}^{M} b_k L_k$ is also lognormal, consistent with the approximate process $\hat{S}$ that has lognormal jumps by construction. This is not the case in specification B, in which the percentage jump size depends on the forward rate and therefore $\sum_{k=n}^{M} b_k L_k(\tau-) X^{\sigma_k}$ is not lognormally distributed.

6 Forward Poisson (FP) specification

6.1 Dynamics

We turn now to a specification of the model (15)–(17) in which each forward rate has Poisson jumps under its associated forward measure. We will take the coefficients of the model to be constant between tenor dates, though we continue writing a general dependance on t to avoid additional indices.

We model the evolution of M rates using M marked point processes with marks in $(0, \infty)$ and spot intensities $\nu_{P(i)}$, $i = 1, \ldots, M$, and a d-dimensional Brownian motion $W(t)$. At time t, the nth forward rate L_n is affected by the ith MPP for $i = n + 1 - \eta(t)$, $n + 2 - \eta(t), \ldots, M$. The impact of the jumps of the marked point processes on the forward rates is stationary because of the dependence on time to maturity $n - \eta(t)$. With this choice, the rate that will mature next, $L_{\eta(t)}$, is sensitive to all M marked point processes, and if some rate L_k jumps then all rates maturing earlier than T_k also jump. The function H_{ni} transforms the abstract marks of the ith marked point process into jump magnitudes of the nth rate; with a view towards (9), we choose these to be

$$H_{ni}(x) = \begin{cases} x-1, & \text{for } i = n+1-\eta, n+2-\eta, \ldots, M; \\ 0, & \text{otherwise} \end{cases} \tag{39}$$

As shown in Proposition 3.1 of Glasserman and Kou (2003), (9) holds simultaneously for all forward rates under their respective forward measures if the intensities of the marked point processes under the spot martingale measure $\nu_P^{(i)}(\mathrm{d}y, t) = \nu_P^{(i)}(y, t)\mathrm{d}y$ satisfy

$$\sum_{i=n+1-\eta(t)}^{M} \nu_P^{(i)}(y,t) = \prod_{j=\eta(t)}^{n} \frac{1+\delta y L_j(t-)}{1+\delta L_j(t-)} \lambda_n(t) f_n(y,t) \tag{40}$$

for $\lambda_n(t)$ deterministic and bounded. We take $f_n(\cdot, t)$ to be lognormal with $\int_0^\infty y f_n(y, t)\mathrm{d}y = 1 + m_n(t)$. Finally, we choose a lognormal specification of the diffusion volatility. The dynamics of L_n under its own forward measure P^{n+1} are

$$\frac{\mathrm{d}L_n(t)}{L_n(t-)} = -\lambda_n(t) m_n(t)\mathrm{d}t + \gamma_n(t)\mathrm{d}W_{P^{n+1}}(t) + \mathrm{d}\left(\sum_{j=1}^{N_n(t)} \left(Y_j^{(n)} - 1 \right) \right) \tag{41}$$

with γ_n deterministic, $N_n(t)$ a Poisson process with arrival rate $\lambda_n(t)$, the marks $Y_j^{(n)}$, $j = 1, 2, \ldots$, with density $f_n(\cdot, t)$ and $W_{P^{n+1}}$, N, and $Y_j^{(n)}$, $j = 1, 2, \ldots$ independent.

The jump processes driving each the rates are defined by (39) and (40).

Imposing these on the general model (15)–(17) leads to the dynamics of this specification under the spot martingale measure, given explicitly in Glasserman and Merener (2003). As discussed in Glasserman and Merener (2003), it is convenient to visualize the process in the following way: under the spot martingale measure the rate closest to maturity $L_{\eta(t)}$ is driven by a jump process with total intensity

$$\sum_{i=1}^{M} \nu_P^{(i)}(y,t) = \frac{1 + y\delta L_{\eta(t)}(t)}{1 + \delta L_{\eta(t)}(t)} \lambda_{\eta(t)}(t) f_{\eta(t)}(y,t)$$

and if $L_{\eta(t)+j}$ jumps then $L_{\eta(t)+j+1}$ jumps with probability

$$\frac{\sum_{i=j+2}^{M} \nu_P^{(i)}(y,t)}{\sum_{i=j+1}^{M} \nu_P^{(i)}(y,t)} = \frac{1 + y\delta L_{\eta(t)+j+1}(t)}{1 + \delta L_{\eta(t)+j+1}(t)} \frac{\lambda_{\eta(t)+j+1}(t) f_{\eta(t)+j+1}(y,t)}{\lambda_{\eta(t)+j}(t) f_{\eta(t)+j}(y,t)} \tag{42}$$

Notice that, although different forward rates accept a given mark y with different state-dependent probabilities, all rates that jump, at a jump time, do it with the same mark y.

For the probabilities defined in (42) to be less than or equal than one, ie, for the intensities $\nu_P^{(i)}$ to be nonnegative, some restrictions apply on the parameters of the lognormal densities f and the functions $\lambda(t)$. In particular,

$$\lambda_j(t) f_j(y,t) < \lambda_k(t) f_k(y,t) \quad \text{for} \quad j > k \tag{43}$$

These restrictions are discussed in detail in Glasserman and Kou (2003) and Glasserman and Merener (2003). It suffices to state here that the numerical experiments in this paper are performed on parameter specifications that satisfy these restrictions.

6.2 Caplet pricing

The dynamics of each forward rate under its own forward measure is, by construction, of the form (9) with piecewise constant coefficients, as shown in Proposition 3.1 of Glasserman and Kou (2003). Therefore, in this specification, caplet prices are computed exactly through Proposition 3.1.

6.3 A swaption formula

In this section we develop a formula for the price of a swaption in the FP specification. We will take advantage again of the powerful algorithm introduced in Proposition 3.1. To apply this, we need to identify a martingale process $\hat{S}$ of the form (9) that approximates, in a distribution sense, the dynamics of $S_{n,M}$ under $P^{n,M}$. We will achieve this in two steps. First, we will introduce a process $\hat{L}$ that approximates the dynamics of the forward rates under $P^{n,M}$, as the exact dynamics of L under $P^{n,M}$ are very complicated. Second, we will derive the dynamics of $\hat{S}$, based on the dynamics of $\hat{L}$.

For $\hat{L}$ we will a postulate a process in which each $\hat{L}_k$ (only $k = n, \ldots, M$ are relevant to the swap) evolves exactly as L_k does under P^{k+1}. In other words, we will approximate the dynamics of each forward rate under $P^{n,M}$ with the dynamics under its own forward measure. This is clearly a crude approximation. A way to motivate this choice is as follows. We look at the change of intensity arising when changing from the forward measure P^{k+1} (the one under which L_k has Poisson jumps) to the swap measure $P^{n,M}$. The Radon–Nikodym derivative that relates the measures is proportional to the ratio

$$\frac{M(t)}{B_{k+1}(t)} = \frac{\delta \sum_{j=n}^{M} B_{j+1}(t)}{B_{k+1}(t)}, \quad k = n, \ldots, M$$

If this ratio remains constant in time, then the intensities are unaffected by the change of measure because of Girsanov's theorem (Theorem 3.12 in Björk, Kabanov, and Runggaldier, 1997). This is trivially true for $n = M$. But we also expect this ratio to remain approximately constant for $n \gg M - n + 1$, because bonds in the numerator and denominator mature close to each other, and therefore tend to fluctuate together. Intuitively, the dynamics of $\hat{L}$ should be a good approximation to the dynamics of L under $P^{n,M}$ when $M - n + 1$, the swap length, is much smaller than the maturity date n.

In addition to specifying the marginal law of each $\hat{L}_k$, we need to specify the dependence among their jumps. For this we devise a mechanism that approximates the FP construction in (42). We specify a Poisson process N_n of jumps in $\hat{L}_n$ (the first rate relevant to the swap); each $\hat{L}_j$, $j = n + 1, \ldots, M$, then accepts or rejects a jump conditional on a jump in $\hat{L}_{j-1}$. More explicitly, we define

$$\frac{d\hat{L}_j(t)}{\hat{L}_j(t-)} = -\lambda_j(t) m_j(t) dt + \gamma_j(t) dW(t) + d\left(\sum_{i=1}^{N_n(t)} \left(Y_i^{(j)} - 1 \right) \right) \tag{44}$$

with W a d-dimensional Brownian motion, N_n a Poisson process with rate $\lambda_n(t)$, and $\hat{L}_j(0) = L_j(0)$. The mark $Y_i^{(n)}$ is lognormally distributed as $f_n(\cdot, t)$ and, for $j = n + 1, \ldots, M$,

$$\text{if} \quad Y_i^{(j-1)} = y, \quad \text{then} \quad Y_i^{(j)} = \begin{cases} y, & \text{with probability } \dfrac{\lambda_j(t) f_j(y,t)}{\lambda_{j-1}(t) f_{j-1}(y,t)} \\ 1, & \text{otherwise} \end{cases}$$

$$\text{if} \quad Y_i^{(j-1)} = 1, \quad \text{then} \quad Y_i^{(j)} = 1 \tag{45}$$

This jump distribution is an approximation to the exact specification in (42). All rates evolve simultaneously under $P^{n,M}$ with joint Poisson dynamics. Notice from (45) that the probability that $\hat{L}_j$ jumps with mark y given that $\hat{L}_n$ jumps with mark y is

$$\frac{\lambda_j(t) f_j(y,t)}{\lambda_n(t) f_n(y,t)} \tag{46}$$

which is well-defined because of the restrictions on f and λ discussed at the beginning of this section. Since the jumps of $\hat{L}_n$ arrive with intensity $\lambda_n(t)f_n(y, t)$ it follows that the effective jump process driving $\hat{L}_j$ has intensity $\lambda_j(t)f_j(y, t)$. Therefore in this approximate process each rate evolves exactly as it would under its own forward measure. The dependence in the jumps of the $\hat{L}_j$s approximates the dependence in the jumps of the L_js.

Our next goal is to approximate the dynamics of the swap rate under the $P^{n,M}$ measure by a process $\hat{S}$ following

$$\frac{\mathrm{d}\hat{S}(t)}{\hat{S}(t)} = -\hat{\lambda}(t)\hat{m}(t)\mathrm{d}t + \hat{\gamma}(t)\mathrm{d}W + \mathrm{d}\left[\sum_{j=1}^{\hat{N}(t)}(Y_j - 1)\right] \tag{47}$$

with $\hat{\gamma}(t)$ deterministic, $\hat{N}(t)$ a Poisson process with intensity $\hat{\lambda}(t)$, Y_j distributed as $\hat{f}(\cdot,\, t)$, lognormal with parameters $\hat{\mu}(t)$, $\hat{\sigma}(t)$ and W, $\hat{N}$, and Y_j, $j = 1, 2, \ldots$ independent and $\hat{S}(0) = S_{n,M}(0)$. Fixing the weights $b_i(t)$ in (5) at their time zero values b_i we have

$$S_{n,M}(t) \approx \sum_{j=n}^{M} b_j L(t) \tag{48}$$

We apply Itô's rule on the right side of (48) to motivate the choice of parameters in the dynamics of $\hat{S}$. First, we look at the diffusion term

$$\sum_{j=n}^{M} b_j L_j(t)\gamma_j(t)\,\mathrm{d}W_{P^{n,M}}$$

Matching this with the diffusion term in (47) we fix the forward rates at their initial values $L_j(0)$ and using the initial condition of $\hat{S}$ define

$$\hat{\gamma}(t) \equiv \frac{1}{S_{n,M}(0)}\left[\sum_{j=n}^{M}\sum_{k=n}^{M} b_j b_k L_j(0) L_k(0)\gamma_j\gamma_k^{\mathsf{T}}\right]^{\frac{1}{2}} \tag{49}$$

where the deterministic time dependence of the parameters $\gamma_i(t)$ (through $\eta(t)$) is preserved.

Next, we focus on the jump term. The goal is to have a jump law in (47) that reproduces the total jump arrival rate and the conditional jump size probability of $\sum_{j=n}^{M} b_j \hat{L}_j(t)$, which is our proxy for the swap rate.

First, the total arrival rate. It follows from (45) that every jump time of $\hat{L}_n$ is also a jump time of $\sum_{j=n}^{M} b_j \hat{L}_j(t)$, and that there are no other jump times. Therefore, the arrival rate of jump events is

$$\hat{\lambda}(t) \equiv \lambda_n(t) \tag{50}$$

We continue with the jump magnitudes of $\hat{S}$. We identify $\hat{f}$ lognormal that generates jump sizes approximately distributed as those of $\sum_{j=n}^{M} b_j \hat{L}_j(t)$, by approximately matching moments of the jump magnitudes of $\hat{S}$ and $\sum_{j=n}^{M} b_j \hat{L}_j(t)$. For the latter, conditional on the state of the system at a jump time τ we have

$$\begin{aligned}
&\mathsf{E}\left[\sum_{j=n}^{M} b_j \hat{L}_j(\tau) - \sum_{j=n}^{M} b_j \hat{L}_j(\tau) \,\middle|\, \tau, \hat{L}(\tau-)\right] \\
&= \sum_{j=n}^{M} b_j \hat{L}_j(\tau-) \mathsf{E}\left[Y^{(j)} - 1 \,\middle|\, \tau, \hat{L}(\tau-)\right] \\
&= \sum_{j=n}^{M} b_j \hat{L}_j(\tau-) \frac{\lambda_j(\tau-)}{\lambda_n(\tau-)} m_j(\tau-)
\end{aligned} \tag{51}$$

where we have used the conditional jump probability of $\hat{L}_j$ (46) and the fact that $\int_0^{\infty}(y-1)f_j(y,t)\mathrm{d}y = m_j(t)$. The expected jump size in (47), conditional on $\hat{S}$ at the jump time s is

$$\begin{aligned}
\mathsf{E}\left[\hat{S}(s) - \hat{S}(s-) \mid s, S(s-)\right] &= \hat{S}(s-)\mathsf{E}\left[Y_j - 1 \,\middle|\, s, \hat{S}(s-)\right] \\
&= \hat{S}(s-)\hat{m}
\end{aligned}$$

which suggests, by comparison with (51), fixing the rates at time zero an defining $\hat{m}$ as

$$\hat{m}(t) \equiv \frac{\sum_{j=n}^{M} b_j \hat{L}_j(0) \frac{\lambda_j(t)}{\lambda_n(t)} m_j(t)}{\sum_{j=n}^{M} b_j \hat{L}_j(0)} \tag{52}$$

Next, we focus on matching the second moment of the jump magnitude conditional on being at a jump time. For $\sum_{j=n}^{M} b_j \hat{L}_j(t)$ we have

$$\begin{aligned}
&\mathsf{E}\left[\left(\sum_{j=n}^{M} b_j \hat{L}_j(\tau) - \sum_{j=n}^{M} b_j \hat{L}_j(\tau-)\right)^2 \,\middle|\, \tau, \hat{L}(\tau-)\right] \\
&= \mathsf{E}\left[\left(\sum_{j=n}^{M} b_j \left(\hat{L}_j(\tau-)(Y^{(j)} - 1)\right)\right)^2 \,\middle|\, \tau, \hat{L}(\tau-)\right] \\
&= \sum_{i=n}^{M}\sum_{j=n}^{M} b_i b_j \hat{L}_i(\tau-)\hat{L}_j(\tau-)\, \mathsf{E}\left[(Y^{(i)} - 1)(Y^{(j)} - 1) \,\middle|\, \tau, \hat{L}(\tau-)\right]
\end{aligned} \tag{53}$$

Notice that (45) implies that if $i < j$ then $Y^{(j)} \neq 1$ implies $Y^{(i)} = Y^{(j)}$; in words, at a jump time, all rates maturing earlier than a jumping rate are also jumping rates, with the same mark. Therefore, denoting $\xi \equiv \max\{i, j\}$

$$
\begin{aligned}
&\mathsf{E}\left[(Y^{(i)}-1)(Y^{(j)}-1)\,\middle|\,\tau,\hat{L}(\tau-)\right] \;=\; \mathsf{E}\left[(Y^{(\xi)}-1)^2\,\middle|\,\tau,\hat{L}(\tau-)\right]\\
&=\frac{\lambda_\xi(t)}{\lambda_n(t)}\int_0^\infty (y-1)^2 f_\xi(y,\tau-)\mathrm{d}y\\
&=\frac{\lambda_\xi(t)}{\lambda_n(t)}\left(\mathrm{e}^{(\sigma_\xi^2(\tau-))}\left(1+m_\xi(\tau-)\right)^2-2m_\xi(\tau-)-1\right)
\end{aligned}
\tag{54}
$$

Using (54), we write (53) as

$$
\sum_{i=n}^{M}\sum_{j=n}^{M} b_i b_j \hat{L}_i(\tau-)\hat{L}_j(\tau-)\frac{\lambda_\xi(t)}{\lambda_n(t)}\left(\mathrm{e}^{(\sigma_\xi^2(\tau-))}\left(1+m_\xi(\tau-)\right)^2-2m_\xi(\tau-)-1\right) \tag{55}
$$

We match this with the expected squared jump size of $\hat{S}$, at jump time s

$$
\begin{aligned}
&\hat{S}(s-)^2\,\mathsf{E}\left[(Y_j-1)^2\,\middle|\,s,\hat{S}(s-)\right]\\
&=\hat{S}(s-)^2\left(\mathrm{e}^{(\hat{\sigma}^2(s-))}\left(1+\hat{m}(s-)\right)^2-2\hat{m}(s-)-1\right)
\end{aligned}
$$

fix the rates at time zero, and use that $\hat{S}(0)=\sum_{j=n}^{M} b_j\hat{L}_j(0)$ to enforce

$$
\begin{aligned}
&\mathrm{e}^{(\hat{\sigma}^2(t))}\left(1+\hat{m}(t)\right)^2-2\hat{m}(t)-1\\
&=\frac{\sum_{i=n}^{M}\sum_{j=n}^{M} b_i b_j\hat{L}_i(0)\hat{L}_j(0)\frac{\lambda_\xi(t)}{\lambda_n(t)}\left(\mathrm{e}^{\left(\sigma_\xi^2(t)\right)}\left(1+m_\xi(t)\right)^2-2m_\xi(t)-1\right)}{\sum_{i=n}^{M}\sum_{j=n}^{M} b_i b_j\hat{L}_i(0)\hat{L}_j(0)}
\end{aligned}
$$

Finally, using that $\hat{m}(t)=\mathrm{e}^{(\hat{\mu}(t)+(\hat{\sigma}(t))^2)/2)}-1$ we get

$$
\begin{aligned}
\hat{\sigma}(t)^2 = \log\Bigg(&\frac{\sum_{i=n}^{M}\sum_{j=n}^{M} b_i b_j\hat{L}_i(0)\hat{L}_j(0)\frac{\lambda_\xi(t)}{\lambda_n(t)}\left(\mathrm{e}^{\left(\sigma_\xi^2(t)\right)}\left(1+m_\xi(t)\right)^2-2m_\xi(t)-1\right)}{\left(1+\hat{m}(t)\right)^2\sum_{i=n}^{M}\sum_{j=n}^{M} b_i b_j\hat{L}_i(0)\hat{L}_j(0)}\\
&+\frac{1+2\hat{m}}{\left(1+\hat{m}(t)\right)^2}\Bigg)
\end{aligned}
\tag{56}
$$

and

$$
\hat{\mu}(t)=\log\left(1+\hat{m}(t)\right)-\frac{\hat{\sigma}(t)^2}{2} \tag{57}
$$

where $\hat{m}$ is given by (52). The evolution of $\hat{S}$ is given then by (47) with (49), (50), (56) and (57). The swaption price is then computed using Proposition 3.1 on the rate $\hat{S}$.

6.4 Numerical results

We investigate now the accuracy of the swaption formula introduced in the previous section. Results are then compared to swaption prices obtained using (6), where the expected discounted payoff is computed via Monte Carlo simulation of the term structure under the spot martingale measure. We generate the sample paths of the term structure using a first order discretization scheme on the logarithm of the forward rates as implemented and discussed in Glasserman and Merener (2003). With the accrual period fixed at $\delta = 0.5$, taking the time step to be less than 0.1 ensures (empirically) that the bias in the simulated price is less that 0.1% of the exact price. Simulated swaption prices are then very good approximation to the exact price and provide a benchmark for testing the approximate method of Section 6.3.

For simplicity, we perform numerical tests driven by a one factor diffusion and M marked point processes, although the method can handle multidimensional diffusions. The parameters are held constant between tenor dates. Moreover, we make the model stationary in the sense that the parameters of the kth rate depend on the number of accrual periods to maturity, $k - \eta(t)$. As before, $\gamma_k(T_j)$ denotes the diffusive volatility of L_k during the accrual period ending at T_j. A consequence of this stationary specification is that all rates follow, under their respective forward measures and for a fixed distance to their own maturities, the same stochastic differential equation. The initial term structure is increasing, $L_k(0) = \log(1.051271 + 0.0011178 \times k)$ and the accrual period is $\delta = 0.5$. The choice of parameters we present next are close to those of model specification SP but not identical as we need to satisfy restriction (43). The parameters are as follows.

Parameter Set A: For all j, k,

$$\gamma_k(T_j) = 0.1,$$
$$\lambda_k(T_j) = 5 \times 0.9^{k-j}, \quad \sigma_k(T_j) = 0.1 \times 0.95^{k-j}, \quad \mu_k(T_j) = 0.0$$

The total instantaneous volatility of the rate next to mature ($k = j$) is approximately 0.245, the diffusion square volatility is 0.01 and the jump squared volatility is 0.06 so most of the volatility of the maturing rate is due to jumps. The jump volatility decreases as rates are more distant from maturity.

Parameter Set B: For all j, k,

$$\gamma_k(T_j) = 0.1,$$
$$\lambda_k(T_j) = 5 \times 0.9^{k-j}, \quad \sigma_k(T_j) = 0.2 \times 0.9^{k-j}, \quad \mu_k(T_j) = -0.1$$

The total instantaneous volatility of the rate next to mature is approximately 0.46, the diffusion square volatility is 0.01 and the jump square volatility is 0.2, therefore almost all the volatility of the rate next to mature is due to jumps. As in parameter set A, jump volatility decreases as rates are more distant from maturity.

Swaption prices for parameters A and B are presented in Tables 5 and 6. Pricing formulas are competitive for practical applications. For 3×3 and 5×5 swaptions, the differences between simulated and approximate prices are not more than 1% of the exact price. This is also the case for 3×7 options in the parameter set A, and in the 3×7 options in B the relative error does not exceed 1.2%.

TABLE 5 Approximate vs simulated swaption prices, FP specification, parameter set A.

Forward Poisson swaption prices, Parameter set A			
Swaption length (years)	**Strike**	**Simulation (95% conf. interval)**	**Approximate price**
3 x 3	ITM 0.049	284.65 (0.31)	285.11
3 x 3	ATM 0.059	152.00 (0.25)	152.30
3 x 3	OTM 0.069	74.55 (0.28)	74.46
3 x 7	ITM 0.053	557.49 (0.53)	560.50
3 x 7	ATM 0.063	268.08 (0.41)	270.69
3 x 7	OTM 0.073	111.59 (0.47)	112.29
5 x 5	ITM 0.055	422.52 (0.65)	424.68
5 x 5	ATM 0.065	245.90 (0.54)	247.46
5 x 5	OTM 0.075	134.91 (0.61)	135.44

TABLE 6 Approximate vs simulated swaption prices, FP specification, parameter set B.

Forward Poisson swaption prices, Parameter set B			
Swaption length (years)	**Strike**	**Simulation (95% conf. interval)**	**Approximate price**
3 x 3	ITM 0.049	362.50 (0.64)	363.79
3 x 3	ATM 0.059	244.95 (0.55)	245.73
3 x 3	OTM 0.069	161.15 (0.61)	161.26
3 x 7	ITM 0.053	653.02 (0.99)	660.36
3 x 7	ATM 0.063	390.36 (0.81)	394.85
3 x 7	OTM 0.073	219.69 (0.93)	220.00
5 x 5	ITM 0.055	521.61 (0.88)	526.11
5 x 5	ATM 0.065	361.82 (0.77)	364.86
5 x 5	OTM 0.075	246.37 (0.86)	247.38

7 Summary

We conclude by summarizing the characteristics of the model specifications and pricing formulas we have presented.

SP specification: All forward rates have Poisson jumps under a common measure, the spot martingale measure. Caplet prices are approximated using Proposition 3.1 and (24) with (25), (26), and (29). Swaption prices are approximated using Proposition 3.1 and (30) with (33), (34), and (38).

FP specification: Each forward rate has Poisson jumps under its own forward measure. Caplets are priced exactly using Proposition 3.1 and (41). Swaption prices are approximated using Proposition 3.1 and (47) with (49), (50), (56) and (57).

Appendix

Proof of proposition 3.1

The transform of $X(t) \equiv \log(G(t))$ at $t = T_n$ is

$$\psi(z) = \mathsf{E}[e^{zX(T_n)}], \quad z \in \mathbb{C} \tag{58}$$

with $G(t)$ evolving as in (9). Then, the expected payoff of a European option maturing at T_n with strike K, ie, $C(0) = \mathsf{E}[(G(T_n) - K)^+]$, can be written as in Heston (1993), and Carr and Madan (1999),

$$C(0) = [G(0)\Pi_1 - K\Pi_2] \tag{59}$$

with

$$\Pi_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Im}\frac{\psi(1+iu)e^{-iu(\log(K))}}{\psi(1)u}\,\mathrm{d}u \tag{60}$$

$$\Pi_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Im}\frac{\psi(iu)e^{-iu(\log(K))}}{u}\,\mathrm{d}u \tag{61}$$

All that remains is the calculation of ψ in (58). Application of Itô's rule and (9) (with piecewise constant coefficients) yields the dynamics of $X(t)$ (where the index in the coefficients denotes the accrual period)

$$\mathrm{d}X(t) = \alpha_i \mathrm{d}t + \gamma_i \mathrm{d}W(t) + \mathrm{d}\left(\sum_{j=N(T_{i-1})+1}^{N(t)} \hat{Y}_{j,i}\right), \quad t \in (T_{i-1}, T_i], \quad i = 1, \ldots, n \tag{62}$$

with $\alpha_i \equiv a_i - \gamma_i^2/2$, $N(t)$ a Poisson process with intensity λ_i. The marks $\hat{Y}_{j,i}$ are normally distributed with mean μ_i and standard deviation σ_i because jumps in X are jumps in the logarithm of G. The process $X(t)$ has independent increments so it is convenient to express the terminal state as

$$X(T_n) = X(0) + \sum_{i=1}^{n}\left(\Delta_i^c + \Delta_i^d\right)$$

where the increment in the ith accrual period due to the continuous part of the path is

$$\Delta_i^c = \delta\alpha_i + \delta^{\frac{1}{2}}\gamma_i Z_i$$

with $\{Z_1, Z_2, \ldots, Z_n\}$ independent and standard normally distributed. The increment in the ith accrual period due to the jumps is the sum of the magnitudes of the jumps occurring in $(T_{i-1}, T_i]$

$$\Delta_i^d = \sum_{j=N(T_{i-1})+1}^{N(T_i)} \hat{Y}_{j,i}$$

and the total number of jumps in $(T_{i-1}, T_i]$, denoted by N_i, is a Poisson random variable with parameter $\delta\lambda_i$. Since the $\{\Delta_i^d, \Delta_i^d\}$ are mutually independent we have from (58)

$$\psi(z) = \mathrm{e}^{zX(0)} \prod_{i=1}^{n} \mathsf{E}\left[\mathrm{e}^{z\Delta_i^c}\right] \mathsf{E}\left[\mathrm{e}^{z\Delta_i^d}\right]$$

The transform of a Poisson number of normally distributed jumps is easily calculated and is given explicitly in Scott (1997):

$$\mathsf{E}\left[\mathrm{e}^{z\Delta_i^d}\right] = \exp\left[\delta\lambda_i\left(\mathrm{e}^{(z\mu_i + z^2\sigma_i^2/2)} - 1\right)\right]$$

Because Δ_i^c is normally distributed we write

$$\mathsf{E}\left[\mathrm{e}^{z\Delta_i^c}\right] = \mathrm{e}^{[\delta(z\alpha_i + z^2\gamma_i^2/2)]}$$

Therefore,

$$\psi(z) = \exp\left[zX(0) + \sum_{i=1}^{n} \delta\lambda_i\left(\mathrm{e}^{(z\mu_i + z^2\sigma_i^2/2)} - 1\right) + z\delta\alpha_i + z^2\delta\gamma_i^2/2\right] \tag{63}$$

Next, we insert (63) with $z = 1 + iu$, and (63) with $z = 1$, in (60), and insert (63) with $z = iu$ in (61), and work tedious but straightforward algebra to obtain (11)–(12). The time-zero European option price is given by $[G(0)\Pi_1 - K\Pi_2]$. □

Proof of Lemma 4.1

Proof of Lemma 4.1 is a special case of the proof of Lemma 4.2 as the numeraire associated with P^{n+1} is proportional to the numeraire associated with $P^{n,n}$

Proof of Lemma 4.2

First, some notation. An MPP can be described through a random measure $\mu(\mathrm{d}x, \mathrm{d}t)$ on the product of the mark space and the time axis assigning unit mass to each point (τ_j, X_j). This representation makes it possible to write $\sum_{j=1}^{N(t)} H(X_j, \tau_j) = \int_0^t \int_{\mathbb{R}^D} H(x, s)\mu(\mathrm{d}x, \mathrm{d}s)$.

Now the proof. We want to identify the change in intensity associated with changing from the spot measure P to the swap measure $P^{n,M}$. Define

$$Z(t) = \frac{M(t)}{B(t)}$$

with $M(t)$ as in (7). Because M is a linear combination of asset prices, Z is a martingale under the spot measure. From (1) and (7) it follows that

$$Z(t) = \prod_{j=0}^{\eta(t)-1} \frac{1}{1+\delta L_j(T_j)} \sum_{k=n}^{M} \delta \prod_{j=\eta(t)}^{k} \frac{1}{1+\delta L_j(t)}$$

Itô's formula and the dynamics of L (15)–(17) give the dynamics of Z. At τ, a jump time of the ith MPP, with mark X, the percentage jump in Z is

$$\frac{Z(\tau)-Z(\tau-)}{Z(\tau-)} = \frac{\sum_{k=n}^{M} \delta \prod_{j=\eta(t)}^{k} \frac{1}{1+\delta L_j(\tau-)(1+H_{ji}(X,\tau))}}{\sum_{k=n}^{M} \delta \prod_{j=\eta(t)}^{k} \frac{1}{1+\delta L_j(\tau-)}} - 1$$

which can be written as

$$\frac{\sum_{k=n}^{M} \delta \prod_{j=\eta(t)}^{k} \frac{1}{1+\delta L_j(\tau-)} \prod_{j=\eta(t)}^{k} \frac{1+\delta L_j(\tau-)}{1+\delta L_j(\tau-)(1+H_{ji}(X,\tau))}}{\sum_{k=n}^{M} \delta \prod_{j=\eta(t)}^{k} \frac{1}{1+\delta L_j(\tau-)}} - 1$$

Multiplying numerator and denominator by $B_{\eta(t)}$ and recalling the definition of the weight b_k in (5) this expression simplifies to

$$\sum_{k=n}^{M} b_k \prod_{j=\eta(t)}^{k} \frac{1+\delta L_j(\tau-)}{1+\delta L_j(\tau-)(1+H_{ji}(X,\tau))} - 1$$

Now we aggregate the (non-simultaneous) jumps from the r MPPs driving L, and write the dynamics of Z in the random measure notation, where the drift follows from the martingale condition

$$\frac{\mathrm{d}Z(t)}{Z(t-)} = \ldots \, \mathrm{d}W_P + \sum_{i=1}^{r} \int_{\mathbb{R}^D} \left(\sum_{k=n}^{M} b_k \prod_{j=\eta(t)}^{k} \frac{1+\delta L_j(t-)}{1+\delta L_j(t-)(1+H_{ji}(x,t))} - 1 \right) \times \left(\mu_P^{(i)}(\mathrm{d}x, \mathrm{d}t) - \nu_P^{(i)}(\mathrm{d}x, t) \right) \mathrm{d}t \tag{64}$$

Next we define the swap measure $P^{n,M}$ through the Radon–Nikodym derivative

$$\left(\frac{\mathrm{d}P^{n,M}}{\mathrm{d}P} \right)_t = Z(t) \frac{B(0)}{M(0)}$$

and invoke Girsanov's theorem (in the form in Theorem 3.12 in Björk, Kabanov, and Runggaldier, 1997) to identify the change in intensity associated to the change of measure. Girsanov's theorem states that given a P-martingale process $A(t)$, (P not necessarily being the spot measure) with

$$\frac{\mathrm{d}A(t)}{A(t-)} = \Gamma \mathrm{d}W^P + \int_{\mathbb{R}^D} (\Phi(x,t)-1)\big[\mu_P(\mathrm{d}x,\mathrm{d}t) - \nu_P(\mathrm{d}x,\mathrm{d}t)\big]$$

and under technical conditions, there is an equivalent measure Q with Radon–Nikodym derivative

$$\left(\frac{\mathrm{d}Q}{\mathrm{d}P}\right)_t = A(t)$$

such that $\mathrm{d}W^P = \Gamma \mathrm{d}t + \mathrm{d}W^Q$ with W^Q a Brownian motion under Q, and the intensity of the driving jump process under Q is $\nu_Q(\mathrm{d}x, t) = \Phi(x, t)\nu_P(\mathrm{d}x, t)$. As we have r MPPs, we invoke Lemma A.1 in Appendix A of Glasserman and Kou (2003) which uses Girsanov's theorem to get the intensity of each MPP,

$$\nu^{(i)}_{P^{n,M}}(\mathrm{d}x,t) = \left(\sum_{k=n}^{M} b_k(t) \prod_{j=\eta(t)}^{k} \frac{1+\delta L_j(t)}{1+\delta L_j(t)(H_{ji}(x,t)+1)}\right) \nu^{(i)}_P(\mathrm{d}x,t)$$

with $b_k(t) = B_{k+1}(t)/M(t)$. □

REFERENCES

Andersen, L., and Andreasen, J. (2000). Volatility skews and extensions of the Libor market model. *Applied Mathematical Finance* **7**, 1–32.

Björk, T., Kabanov, Y., and Runggaldier, W. (1997). Bond market structure in the presence of marked point processes. *Mathematical Finance* **7**, 211–39.

Black, F. (1976). The pricing of commodity contracts. *Journal of Financial Economics* **3**, 167–79.

Brace, A., Dun, T., and Barton, G. (2001). Towards a central interest rate model. In *Handbooks in Mathematical Finance. Topics in Option Pricing, Interest Rates and Risk Management*. Cambridge University Press.

Brace, A., Gatarek, D., and Musiela, M. (1997). The market model of interest rate dynamics. *Mathematical Finance* **7**, 127–55.

Brémaud, P. (1981). *Point Processes and Queues: Martingale Dynamics*. Springer, New York.

Carr, P., and Madan, D. (1999). Option valuation using the fast Fourier transform. *Journal of Computational Finance* **2** (Summer), 61–74.

Duffie, D., Pan, J., Singleton, K. (2000). Transform analysis and asset pricing for affine jump-diffusions. *Econometrica* **68**, 1343–76.

Glasserman, P., and Kou, S. G. (2003). The term structure of simple forward rates with jump risk. *Mathematical Finance*, July, 383–410.

Glasserman, P., and Merener, N. (2003). Numerical solution of jump-diffusion LIBOR market models. Forthcoming in *Finance and Stochastics* **7**, 1–27.

Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The Review of Financial Studies* **6**, 327–43.

Hull, J., and White, A. (2000). Forward rate volatilities, swap rate volatilities and the implementation of the LIBOR market model. *Journal of Fixed Income* **10** (September), 46–62.

Jäckel, P., and Rebonato, R. (2000). Linking caplet and swaption volatilities in a BGM/J framework: Approximate solutions. Working paper, QUARC, Royal Bank of Scotland, London. Available at www.rebonato.com.

Jamshidian, F. (1997). LIBOR and swap market models and measures. *Finance and Stochastics* **1**, 293–330.

Jamshidian, F. (1999). LIBOR Market model with semimartingales. Working paper, NetAnalytic Ltd., London.

Joshi, M., and Rebonato, R. (2001). A stochastic-volatility, displaced-diffusion extension of the LIBOR market model. Working paper, QUARC, Royal Bank of Scotland, London. Available at www.rebonato.com.

Kawai, A. (2000). Swaption pricing in the LIBOR market model. Working paper, School of Mathematics, University of New South Wales.

Kloeden, P. E., and Platen, E. (1992). *Numerical Solution of Stochastic Differential Equations*. Springer-Verlag, Berlin.

Kou, S. G. (1999). A jump diffusion model for option pricing with three properties: leptokurtic feature, volatility smile, and analytical tractability. Working paper, Columbia University, Available at www.ieor.columbia.edu/~kou.

Merton, R. (1976). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics* **3**, 125–44.

Mikulevicius, R., and Platen, E. (1988). Time discrete Taylor approximations for Ito processes with jump component. *Mathematische Nachrichten* **138**, 93–104.

Miltersen, K. R., Sandmann, K, and Sondermann, D. (1997). Closed-form solutions for term structure derivatives with lognormal interest rates. *Journal of Finance* **52**, 409–30.

Musiela, M., and Rutkowski, M., Martingale (1997). *Methods in Financial Modeling*. Springer, New York.

Scott, L. O. (1997). Pricing stock options in a jump-diffusion model with stochastic volatility and interest rates: application of Fourier inversion methods. *Mathematical Finance* **7**, 413–26.

Zühlsdorff, C. (2000). Extended Libor market models with affine and quadratic volatility. Discussion paper, Department of Statistics, University of Bonn Available at www.finasto.unibonn.de.