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# INTERNATIONAL ASSET PRICING WITH TIME-VARYING RISK PREMIA

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Recent empirical evidence that forward exchange rates are biased predictors of future spot rates can also be interpreted as evidence against the hypothesis of a constant risk premium. Consequently, reconciliation of this evidence with efficient international capital markets requires the existence of time-varying risk premia. This paper modifies Kouri's (1977) asset pricing model to allow time-varying exected returns on assets, eliminates the assumption of a risk-free real asset and derives the characteristics of the risk premia in the forward market as well as the equilibrium yield relationships among the equities and riskless nominal bonds of all countries.

## 1. Introduction

The advent of flexible exchange rates in the 1970s was the impetus for considerable research into the efficiency of the market for foreign exchange. As a theoretical proposition, the existence of risk premia which would prevent forward exchange rates from being unbiased predictors of future spot rates was derived in numerous ways.[1] Nevertheless, the empirical literature of as late as 1978 provided evidence only that forward rates were relatively poor predictors of future spot rates, not that they were biased.[2] More recent empirical research by Geweke and Feige (1979) and Hansen and Hodrick (1980a) among others, has provided comparatively strong evidence that forward rates are biased predictors of future spot rates. The nature of the bias is that information which is readily available to the market when the forward rate is set significantly improves the forecast of the future spot rate. There is also considerable evidence in these papers against the hypothesis that a constant risk premium is the source of the bias. Consequently, if the

[1]Early theoretical discussions of the relationship between forward rates and expected future spot rates include Feldstein (1968) and Siegel (1972). More recent work includes Solnik (1974), Grauer, Litzenberger and Stehle (1976), Kouri (1977), Stockman (1978), Fama and Farber (1979), Frankel (1979), Roll and Solnik (1979), and Stulz (1980a, b).

[2]See Levich (1979) for a survey of the empirical literature addressing the relationship of forward rates to future spot rates and other issues in foreign exchange market efficiency.

empirical evidence is to be consistent with market efficiency, time-varying risk premia must separate forward rates from expected future spot rates.

The purpose of this paper is to investigate the modifications to an existing theoretical model that one must consider if time-varying risk premia are to be taken seriously. This is necessary because, on the one hand, most theoretical models are two-period maximization problems which are consequently poorly equipped to discuss the nature of time-varying risk premia while, on the other hand, Kouri's (1977) model investigates international asset pricing in a stationary environment with infinitely-lived agents, constant expected rates of return on all assets, and constant risk premia. The following analysis modifies the Kouri model to allow for significant intertemporal variation in the expected rates of return on assets. It also generalizes the model to allow for uncertain rates of inflation in all countries eliminating the unrealistic assumption of the existence of a riskless real asset. The resulting asset pricing relationships indicate the possible nature of the time-varying risk premia separating the forward premium from the expected rate of depreciation of the exchange rate as well as the nature of the relationships between expected rates of return on a variety of other assets.

The building blocks of the model are discussed in section 2. Investors in many countries choose their consumption and their optimal holdings of money, riskless nominal bonds of each country, and equities whose real returns are uncertain. Optimal portfolio proportions are derived in section 3. These are used to derive the equilibrium yield relationships among assets in section 4 and to determine the nature of the time-varying risk premia in the forward market. Section 5 provides some concluding remarks and a discussion of the implications of the paper for empirical analysis of risk premia.

## 2. The model

In this section the basic model of Kouri (1977) is modified to allow for significant intertemporal variation in the expected rates of return on assets and to eliminate the assumption of the existence of a riskless real asset. The model is essentially an extension of the analyses of Merton (1973), Breeden (1979), and Richard (1979) to an international environment.[3]

The model is partial equilibrium in the sense that unspecified stochastic state variables are postulated to drive the dynamic movement of the world economy. This simplification allows for a tractable analysis of the problem while presenting difficulties in empirical implementation of the theory. These issues are discussed in the conclusion of the paper.

[3]Stulz (1980a) has extended the Breeden model to an international environment with traded and nontraded goods.

Let there be $K$ state variables, $S(t)$, which provide the fundamental dynamic relations of the model. The dynamic movement of the state variables is characterized by the stochastic differential equations

$$\mathrm{d}S(t) = R_s S(t)\mathrm{d}t + \Sigma_s \mathrm{d}y, \tag{1}$$

where $R_s$ is a $K \times K$ matrix of constants with negative eigenvalues, and $\Sigma_s$ is a $K \times K$ diagonal matrix with the constant standard deviations $\sigma_s^i$ on the diagonals.[4] The term $\mathrm{d}y$ is a $K$-dimensional vector of Weiner processes where $\rho_{ij}\mathrm{d}t$ is the constant instantaneous correlation coefficient between $\mathrm{d}y_i$ and $\mathrm{d}y_j$.

Assume that there are $N$ countries in the world. Let there be one common real output of these countries, and let the $N$-dimensional vector of domestic currency prices of the output evolve through time according to

$$\frac{\mathrm{d}P}{P} = R_p S(t)\mathrm{d}t + \Sigma_p \mathrm{d}z, \tag{2}$$

where $R_p$ is an $N \times K$ matrix of constants, and $\Sigma_s$ is an $N \times N$ diagonal matrix with the constant standard deviations $\sigma_p^i$ on the diagonal. The term $\mathrm{d}z$ is an $N$-dimensional vector of Weiner processes where $\eta_{ij}\mathrm{d}t$ is the instantaneous constant correlation coefficient between $\mathrm{d}z_i$ and $\mathrm{d}z_j$, and $\xi_{ij}\mathrm{d}t$ is the instantaneous constant correlation between $\mathrm{d}z_i$ and $\mathrm{d}y_j$.

If we assume that there are no impediments to trade, the single good in the model must sell at the same price in any currency. As a result of traded goods arbitrage, the exchange rate of currency $i$ for currency $j$ must be given by

$$E_{ij} = P_i/P_j \tag{3}$$

and the dynamics of the exchange rate can be determined from the stochastic processes in (2) and Itô's Lemma.

Assume that the residents of any country can borrow or lend in any currency at a riskless instantaneous nominal interest rate. Let $B_j$ be the price of the discount bill so that the rates of return on the $N$ bills are given by

$$\mathrm{d}B/B = R_b S(t)\mathrm{d}t, \tag{4}$$

where $R_b$ is an $N \times K$ matrix of constants.

[4]Throughout the paper the covariance matrices are assumed to be constant and therefore not dependent on the state variables. This simplifies the presentation and results in asset pricing relationships that are potentially empirically tractable.

Agents in the model can also hold the equities of the various countries. Assume that there are $N_q \gg N$ equities in the world. Then, with only capital gains as a source of return, the vector of real rates of return is given by the percentage change in the real price of the equities:

$$\frac{\mathrm{d}Q}{Q} = R_q S(t)\mathrm{d}t + \Sigma_q \mathrm{d}v, \tag{5}$$

where $R_q$ is an $N_q \times K$ matrix of constants, and $\Sigma_q$ is an $N_q \times N_q$ diagonal matrix of constants with the standard deviations $\sigma_q^i$ on the diagonal. The term $\mathrm{d}v$ is an $N_q$-dimensional vector of Weiner processes where $\nu_{ij}\mathrm{d}t$ is the constant instantaneous correlation coefficient between $\mathrm{d}v_i$ and $\mathrm{d}v_j$, $\psi_{ij}$ is the constant instantaneous correlation coefficient between $\mathrm{d}y_i$ and $\mathrm{d}v_j$, and $\zeta_{ij}$ is the constant instantaneous correlation coefficient between $\mathrm{d}z_i$ and $\mathrm{d}v_j$.

The final asset which agents in the model hold is fiat money. There are assumed to be $N$ fiat monies in existence, and each money is assumed to have a zero nominal rate of return. The stochastic properties of the real rate of return on money will therefore be identical to the stochastic properties of the real rate of return on the nominal bills. Since in this case money would not be held unless it possessed some nonpecuniary convenience yield, it is assume that real balances of country $j$ money, $M_j/P_j$, yield utility to agents of country $j$.[5] In this case residents of country $j$ will hold the money of country $j$ and no other monies.

Real wealth for the representative resident of country $j$ consists of the real values of his money holdings, his bill holdings, and his equity holdings. Let $G_i^j$ be the number of units of the discount bill of country $i$ held by the residents of country $j$, and, similarly, let $K_i^j$ be defined as the holdings of the $i$th equity by the residents of $j$. Real wealth is therefore

$$W^j = \frac{M_j}{P_j} + \sum_{i=1}^{N} \frac{B_i G_i^j}{E_{ij} P_j} + \sum_{i=1}^{N_q} Q_i K_i^j. \tag{6}$$

If $b_i^j$ is the share of $j$'s real wealth invested in the $i$th bill for $i \neq j$ and the share invested in the $j$th bill and the share in real balances, $m^j$, when $i=j$, and $q_i^j$ is the share of $j$'s real wealth invested in the $i$th equity, then

$$1 = \sum_{i=1}^{N} b_i^j + \sum_{i=1}^{N_q} q_i^j. \tag{7}$$

[5]Ideally, one would like to examine the issues discussed in this paper in a model which captures the important microeconomic features of a world in which countries use different fiat monies and economic agents are motivated to buy and sell the currencies in an organized forward market. Kareken and Wallace (1981) have investigated multiple fiat money models in an overlapping-generations framework. That approach allows monies to be perfect substitutes in some cases while the approach taken in this paper ignores problems of currency substitution entirely.

A similar wealth constraint exists for the other $N-1$ countries.

The accumulation of real wealth occurs when the real capital gains on assets which are assumed to be the only sources of income for the investor exceed his instantaneous real consumption, $C^j$:

$$\mathrm{d}W^j=\left\{m^j\left(\frac{-\mathrm{d}B_j}{B_j}\right)+\sum_{i=1}^{N} b_i^j\left(\frac{\mathrm{d}(B_i/P_i)}{(B_i/P_i)}\right)+\sum_{i=1}^{N_q} q_i^j\left(\frac{\mathrm{d}Q_i}{Q_i}\right)\right\}W^j-C^j\mathrm{d}t. \quad (8)$$

The real rates of return on bills in (8) can be obtained from (2) and (4) with the use of Itô's Lemma:

$$\frac{\mathrm{d}(B_i/P_i)}{(B_i/P_i)}=[(R_b^i-R_p^i)S(t)+(\sigma_p^i)^2]\mathrm{d}t-\sigma_p^i\mathrm{d}z_i, \qquad i=1,\ldots,N. \quad (9)$$

In order to simplify notation, let $R_j=R_b^jS(t)$, $r_i=(R_b^i-R_p^i)S(t)+(\sigma_p^i)^2$, and $\alpha_i=R_q^iS(t)$. This allows (8) to be rewritten in vector notation as

$$\mathrm{d}W^j=\left\{-m^jR_j\mathrm{d}t+b^{j\prime}r\mathrm{d}t+q^{j\prime}\alpha\mathrm{d}t-b^{j\prime}\Sigma_p\mathrm{d}z+q^{j\prime}\Sigma_q\mathrm{d}v\right\}W^j-C^j\mathrm{d}t, \quad (10)$$

where $b^j$, $q^j$, $r$, and $\alpha$ are the column vectors with elements $b_i^j$, $q_i^j$, $r_i$ and $\alpha_i$, respectively.

The representative individual of country $j$ is assumed to maximize the expectation of the infinite discounted value of an instantaneous utility function which depends on consumption and his holdings of real balances. Let $\delta$ be the discount rate of individuals which is assumed to be constant and equal across all countries. The expectation is taken conditional on the values of the state variables and the stock of real wealth. The consumer's problem is therefore

$$\max_{\{C^j,m^j,b^j,q^j\}} E_0\left[\int_0^\infty \mathrm{e}^{-\delta t}U^j(C^j(t),m^jW^j)\mathrm{d}t\right] \quad (11)$$

subject to (7), (10), and the constraints that consumption and holdings of real balances are non-negative for all time.[6]

Let $V^j(W^j(t),S(t))$ represent the value function, i.e. the solution to (11) for given levels of real wealth and the state variables. Merton demonstrates that the optimal choices of consumption and the portfolio shares must satisfy the

[6]I assume that the instantaneous utility function is sufficiently concave to guarantee an internal solution.

following Bellman equation:

$$0=\max_{\{C^j,m^j,b^j,q^j\}}\Big\{U^j(C^j,m^jW^j)-\delta V^j$$
$$+V^j_w[W^j(-m^jR_j+b^{j\prime}r+q^{j\prime}\alpha)-C^j]$$
$$+\tfrac{1}{2}V^j_{ww}(W^j)^2[b^{j\prime}\Omega b^j-2b^{j\prime}\Gamma q^j+q^{j\prime}\Theta q^j]$$
$$+W^j(b^{j\prime}\Phi_b+q^{j\prime}\Phi_q)V^j_{ws}+\lambda^j(1-(b^{j\prime}\vdots q^{j\prime})\cdot \mathbf{1})\Big\}. \tag{12}$$

The new terms introduced here are $V^j_w$ and $V^j_{ww}$, the first and second partial derivatives of the value function with respect to real wealth; $V^j_{ws}$, a vector of second partial derivatives of the value function with respect to real wealth and the $K$ state variables; $\Omega$, the covariance matrix of the $N$ inflation rates; $\Gamma$, the covariance matrix of the $N$ inflation rates with the $N_q$ equity returns; $\Theta$, the covariance matrix of the $N_q$ equity returns; and $\Phi_b$ and $\Phi_q$, the covariance matrices of inflation rates and equity returns with the $K$ state variables. The symbol $\mathbf{1}$ is an $(N+N_q)$-dimensional vector of ones.

The first-order conditions are:

$$U^j_c-V^j_w=0, \tag{13}$$

$$U^j_m-V^j_wR_j=0, \tag{14}$$

$$V^j_wr+V^j_{ww}W^j[\Omega b^j-\Gamma q^j]+\Phi_bV^j_{ws}-\lambda^j/W^j=0, \tag{15}$$

$$V^j_w\alpha+V^j_{ww}W^j[-\Gamma b^j+\Theta q^j]+\Phi_qV^j_{ws}-\lambda^j/W^j=0, \tag{16}$$

$$1-(b^{j\prime}\vdots q^{j\prime})\mathbf{1}=0, \tag{17}$$

where $U^j_c$ and $U^j_m$ represent the partial derivatives of the instantaneous utility function with respect to consumption and real balances. These conditions (13)–(17) are also sufficient for an optimum because of the assumed strict concavity of the utility function.

## 3. The optimal portfolio

This section uses the results in (13)–(17) to derive the consumer-investor's optimal consumption and portfolio shares. From (13) and (14) we obtain the

standard result that

$$U^j_m/U^j_c = R_j, \tag{18}$$

which indicates that the marginal rate of substitution of real balances for consumption equals the nominal interest rate. Here, contrary to Kouri, the nominal interest rate can fluctuate through time and the marginal utility of consumption will be affected by variables other than the wealth of the individual. As the state variables fluctuate, investment opportunities change, and the desirability of consumption holding real wealth constant fluctuates. The finding that the demand for real balances depends only on desired consumption and the nominal interest rate remains an implication of this model. This is true, as in Kouri's model, because the rate of return on real balances has the same stochastic properties as the rate of return on the real bill holdings for the same currency. Hence, agents are concerned only with the aggregate of the two. As Kouri notes, residents of country $j$ always perfectly hedge their holdings of money by borrowing an equal amount in local currency. What is different about this model is that the nominal interest rate and the anticipated rates of return on all assets can fluctuate through time and the proportions of wealth invested in these other assets depend on the covariances of the rates of return among themselves as in Kouri but also on the covariances of asset returns with the state variables as in Merton, Richard, and Breeden.

The optimal portfolio shares are found by solving (15) and (16) simultaneously:

$$\begin{bmatrix} b^j \\ q^j \end{bmatrix} = A^j H^{-1} \begin{bmatrix} r \\ \alpha \end{bmatrix} + H^{-1} \begin{bmatrix} \Phi_b \\ \Phi_q \end{bmatrix} T^j + F^j H^{-1} \cdot \mathbf{1}, \tag{19}$$

where

$A^j \equiv \dfrac{-V^j_w}{V^j_{ww} W^j}$, the Arrow–Pratt measure of relative risk aversion,

$T^j \equiv \dfrac{-V^j_{ws}}{V^j_{ww} W^j}$ a $K$-dimensional vector,

$F^j \equiv \dfrac{\lambda^j}{V^j_{ww} (W^j)^2}$, and

$H \equiv \begin{bmatrix} \Omega & -\Gamma \\ -\Gamma & \Theta \end{bmatrix}$, the covariance matrix of all asset returns.

If the rates of return on the assets were all uncorrelated with the state variables and a risk-free real bill was available, (19) would reduce to Kouri's (17.1) and (17.2). When these terms are nonzero, the portfolio demands differ and the resulting asset pricing relationships are no longer as simple.

## 4. Equilibrium yield relationships among assets

In this section an intertemporal, international asset pricing model is presented. The details of the derivation are provided in the appendix since it follows straightforwardly but with some modifications from Richard (1979).

Since all assets in the model are risky in real terms, there is no risk-free asset which can be used in the asset pricing relationships. Instead, extending Black (1972) to an intertemporal environment, the expected rates of return on assets are priced relative to the expected rate of return on the zero beta portfolio which in this case is that minimum variance portfolio which is uncorrelated with the rate of return on the world market porfolio and the $K$ state variables. Let the expected rate of return on this portfolio be $r_0$.

It is clear from (19) that agents will care about the covariance of the rates of return on assets with changes in the $K$ state variables of the model. These influences are captured in the asset pricing relationship by finding portfolios with shares $y^k$ that replicate the covariance structure of all assets with the $k$th state variable. Let the expected real rate of return on this portfolio be $\alpha_{yk}$. With $\alpha_m$ defined to be the expected real rate of return on the world market portfolio, the intertemporal, international asset pricing model for an arbitrary asset is

$$\alpha_i - r_0 = \beta_i^m(\alpha_m - r_0) + \sum_{k=1}^{K} \beta_i^k(\alpha_{yk} - r_0), \tag{20}$$

where $\beta_i^m$ and the $\beta_i^k$'s are regression coefficients that could be recovered from a regression of $(\alpha_i - r_0)$ on the $(K+1)$ expected real rates of return on the right-hand-side of (20).

Eq. (20) states that the expected real rate of return on an asset is equal to the expected real rate of return on the portfolio of assets that has zero correlation with the world market portfolio and with the state variables plus terms that compensate for the systematic risk induced by correlation of the rate of return of the asset with the market portfolio and the state variable portfolios.

Although the expected real rate of return on any asset is equal across countries, the expected real rate of return on a bill denominated in currency $i$ generally will be different from the expected real rate of return on a bill denominated in currency $j$. From (20), the difference in the expected real

rates of return on two bills denominated in different currencies is

$$r_i - r_j = (\beta_i^m - \beta_j^m)(\alpha_m - r_0) + \sum_{k=1}^{K} (\beta_i^k - \beta_j^k)(\alpha_{yk} - r_0). \tag{21}$$

Since the rates of inflation in the two countries generally have different covariances with the rate of return on the market portfolio and with the $K$ state variable portfolios, the expected real rates of return will differ.

From (21) we can derive the relationships between any two nominal interest rates or between the forward premium and the expected rate of depreciation of an exchange rate for two currencies.

Using the definitions of the expected real interest rates, the differential expected real rate of return on the bills of two countries is

$$r_i - r_j = (R_b^i - R_p^i)S(t) + (\sigma_p^i)^2 - (R_b^j - R_p^j)S(t) - (\sigma_p^j)^2. \tag{22}$$

Since the exchange rate of country $i$ money for country $j$ money is $E_{ij} - P_i/P_j$, the rate of change of the exchange rate can be found from Itô's Lemma:

$$\frac{\mathrm{d}E_{ij}}{E_{ij}} = \frac{\mathrm{d}P_i}{P_i} - \frac{\mathrm{d}P_j}{P_j} - \frac{\mathrm{d}P_i}{P_i}\frac{\mathrm{d}P_j}{P_j} + \left(\frac{\mathrm{d}P_j}{P_j}\right)^2. \tag{23}$$

Using (23) and the processes defined in (2), the expected rate of change of the exchange rate is

$$e_{ij} = (R_p^i - R_p^j)S(t) - \eta_{ij}\sigma_p^i\sigma_p^j + (\sigma_p^j)^2.^7 \tag{24}$$

Substituting (24) and (22) into (21) gives

$$R_i - R_j = e_{ij} - (\sigma_p^i)^2 + \eta_{ij}\sigma_p^i\sigma_p^j + (\beta_i^m - \beta_j^m)(\alpha_m - r_0) + \sum_{k=1}^{K} (\beta_i^k - \beta_j^k)(\alpha_{yk} - r_0). \tag{25}$$

[7]In (24) notice that the expected rate of depreciation of the exchange rate is not simply the expected inflation differential. Similarly, in (22) the expected real return on a nominal bond is not the nominal interest rate minus the expected rate of inflation. The differences arise because in an explicitly stochastic model the expected real rate of return on a nominal asset is its nominal rate of return minus the expected rate of change in the purchasing power of the money which is $1/P$. From Itô's Lemma,

$$\frac{\mathrm{d}(1/P_j)}{(1/P_j)} = -\left(\frac{\mathrm{d}P_j}{P_j}\right) + (\sigma_p^j)^2.$$

Consequently, holding the expected rate of inflation and the covariances of all processes constant, an increase in the variance of the price level increases the expected real rate of return on all nominal assets denominated in that currency and decreases the expected rate of depreciation of the currency.

The left-hand side of (25) is the nominal interest differential which can vary through time contrary to Kouri's (29) while $e_{ij}$ represents the time-varying expected rate of depreciation of currency $i$ relative to currency $j$.

In the absence of transactions costs, the nominal interest rate differential equals the forward premium.[8] Thus, from (25), the forward premium equals the expected rate of depreciation of the currency plus a time-varying risk premium. This risk premium arises because of the systematic risk associated with investing in the nominal assets of different countries. These systematic risks will be significant if the inflation rates of the countries are correlated with either the real rate of return on the world market portfolio or the stochastic movement in the fundamental state variables of the system. In Kouri's model the deviation of the forward premium from the expected rate of depreciation of the exchange rate was a constant since expected real rates of return were constant. In this model expected real rates of return can vary, but an additonal term arises in the asset pricing relationship for each state variable that is added to the model.

## 5. Conclusions

In this paper an international capital asset pricing model with time-varying expected rates of return and time-varying risk premia has been developed. The essential insight of the model is that the systematic risk of an asset is related to the covariance of its rate of return with the rate of return on the world market portfolio and its covariance with the state variables of the system that drive the stochastic rates of return of all assets. It is clear that extensions of this model to full general equilibrium are necessary if we are to understand the true nature of the relationship since only in such a model will the state variables be revealed. At an aggregate level it is possible to speculate that shocks to aggregate demand from the government sector and to aggregate supply from weather and technological change in each country are candidates for state variables. Another possibility might be the probability of war which certainly fluctuates and affects portfolio choice across countries and assets. The number of state variables may therefore be quite large.

One possible solution to this dimensionality problem was proposed by Breeden (1979) who demonstrated how to collapse Merton's (1973) model into a single beta framework. At a point in time the expected rate of return on an asset in the Breeden model is equal to the risk-free rate plus a 'consumption beta' times the deviation of the risk-free rate of return from the expected rate of return on an asset that is perfectly correlated with

[8]No forward market is explicitly introduced since borrowing domestic money, purchasing foreign currency, and lending the foreign currency is equivalent to buying the foreign currency in the forward market ignoring transactions costs.

the percentage change in aggregate consumption. The consumption beta is the covariance of the rate of return on the asset with the percentage change in aggregate consumption divided by the variance of the percentage change in consumption. Stulz (1980a) has analyzed the Breeden model in an international environment. One problem with the consumption beta approach is that the relationship is not stable through time whereas the betas of the model in (20) other than the market beta are stable because of the assumed structure of the model.[9] Specification of a more general model with state dependent covariances would lead to an instantaneous relationship like (20) but with time-varying betas. In either case empirical testing of the proposition that time-varying risk premia are present in international asset pricing is a difficult task.

Since explicit testing of relationships derived in the intertemporal, international asset pricing literature is difficult, it is unlikely that we will soon have a definitive demonstration that time-varying risk premia are the source of the empirical findings of Geweke and Feige (1979) and of Hansen and Hodrick (1980a). In Hansen and Hodrick (1980b) an investigation is conducted of the cross-currency restrictions on a vector autoregression of the rates of return to speculation of a number of major currencies that arise if a single time-varying factor is the source of the above mentioned empirical findings. While the results of that study lend support to the idea that risk premia are the source of the deviation of forward rates from expectations of future spot rates, the results of this paper demonstrate that a great deal more work must be done in this area before we can conclusively argue that international asset markets are efficient.

## Appendix

This appendix provides some supplemental steps in moving from (19) to the international asset pricing relationship in (20). As such, it draws heavily on Richard (1979). For convenience (19) is rewritten here:

$$\begin{bmatrix} b^j \\ q^j \end{bmatrix} = A^j H^{-1} \begin{bmatrix} r \\ \alpha \end{bmatrix} + H^{-1} \Phi T^j + F^j H^{-1} \cdot \mathbf{1} \tag{A1}$$

and

$$\Phi = \begin{bmatrix} \Phi_b \\ \Phi_q \end{bmatrix}.$$

[9]The value of various assets in the market portfolio will fluctuate, which implies that the covariance of a particular asset with the market portfolio will not be constant.

Multiplying (A1) by $W^j$ and summing across all countries gives the demands for all assets:

$$d=\sum_{j=1}^{N} W^j\begin{bmatrix} b^j \\ q^j \end{bmatrix}=AH^{-1}\begin{bmatrix} r \\ \alpha \end{bmatrix}+H^{-1}\Phi T+FH^{-1}\cdot \mathbf{1}, \tag{A2}$$

where

$$A=\sum_{j=1}^{N} W^j A^j; \qquad T=\sum_{j=1}^{N} W^j T^j \quad \text{and} \quad F=\sum_{j=1}^{N} W^j F^j.$$

Rewrite (A2) as

$$\begin{bmatrix} r \\ \alpha \end{bmatrix}=\frac{1}{A}Hd-\Phi\frac{T}{A}-\frac{F}{A}\cdot \mathbf{1}, \tag{A3}$$

which is $N+N_q$ equations in the $K+2$ parameters $(1/A, T'/A, F/A)$. Since these parameters are unobservable, it is desirable to rework the equations to eliminate the unobservables and describe the asset prices in terms of variables which are potentially estimable.

In order to accomplish this task we must reduce the dimensionality of the system of equations to $K+2$. Define the vector of shares of the assets in the world market portfolio as $w\equiv d/(d\cdot \mathbf{1})$. Then the expected rate of return on the world market portfolio is

$$\alpha_m=w'\begin{bmatrix} r \\ \alpha \end{bmatrix}=-\frac{F}{A}+\frac{M}{A}\sigma_{mm}-w'\Phi\frac{T}{A}, \tag{A4}$$

where $\sigma_{mm}=w'Hw$, the variance of the world market portfolio and $M=d'\cdot \mathbf{1}$, the value of the portfolio of aggregate world wealth. This is the first of the $K+2$ equations.

Another $K$ equations will involve the portfolios of assets which replicate the covariance matrix $\Phi$. Let $\bar{y}^k$ be the vector of portfolio weights that solves

$$H\bar{y}^k=\Phi^k, \tag{A5}$$

where $\Phi^k$ is the $k$th column of $\Phi$. Since $H$ is nonsingular, (A5) has a unique solution, but $\bar{y}^k$ will not in general be portfolio shares. These may be defined by setting $y^k=\bar{y}^k/(\bar{y}^{k\prime}\cdot \mathbf{1})$. The vector of portfolio shares $y^k$ has the property that the covariance of each asset with portfolio $k$ is equal to the covariance of the asset with state variable $k$. Define $Y$ to be the matrix of portfolio shares with columns $y^k$ which replicates $\Phi$, i.e. $HY=\Phi$.

Now, construct a minimum variance portfolio with shares $w_0$ such that the covariance of the portfolio with the market portfolio is zero and the covariances of the portfolio with the state variables are zero. From (A3) and the properties of $w_0$, the rate of return on this zero beta portfolio is

$$w_0'\begin{bmatrix} r \\ \alpha \end{bmatrix} \equiv r_0 = \frac{M}{A} w_0' H w - w_0' \Phi \frac{T}{A} - \frac{F}{A} = -\frac{F}{A}. \tag{A6}$$

Recognize that

$$w'\Phi = w'HY = (\sigma_{m1}, \ldots, \sigma_{mK}), \tag{A7}$$

where $\sigma_{mk}$ is the covariance of the market portfolio with the $k$th state variable. Eq. (A4) can therefore be rewritten as

$$\alpha_m - r_0 = \frac{M}{A}\sigma_{mm} - (\sigma_{m1}, \ldots, \sigma_{mK})\frac{T}{A}. \tag{A8}$$

By defining $\sigma_{ik}$ to be the covariance of the $i$th portfolio of shares $y^i$ with the $k$th state variable, we can write the expected rate of return on this portfolio as

$$\alpha_{yi} - r_0 = \frac{M}{A}\sigma_{m1} - (\sigma_{i1}, \ldots, \sigma_{iK})\frac{T}{A}, \qquad i = 1, \ldots, K. \tag{A9}$$

The $(K+1)$ equations (A8) and (A9) can now be solved simultaneously to determine $(M/A, T'/A)$. Write these equations as a system

$$\begin{bmatrix} \alpha_m - r_0 \\ \alpha_{y1} - r_0 \\ \vdots \\ \alpha_{yK} - r_0 \end{bmatrix} = \begin{bmatrix} \sigma_{mm} & \sigma_{m1} & \ldots & \sigma_{mK} \\ \sigma_{m1} & \sigma_{11} & \ldots & \sigma_{1K} \\ \vdots & \vdots & & \\ \sigma_{mK} & \sigma_{K1} & \ldots & \sigma_{KK} \end{bmatrix} \begin{bmatrix} \dfrac{M}{A} \\ -\dfrac{T}{A} \end{bmatrix}, \tag{A10}$$

and define $\bar{\Sigma}$ to be the square matrix on the right-hand side of (A10).

Consequently,

$$\begin{bmatrix} M/A \\ -T/A \end{bmatrix} = \bar{\Sigma}^{-1} \begin{bmatrix} \alpha_m - r_0 \\ \alpha_{yl} - r_0 \\ \vdots \\ \alpha_{yK} - r_0 \end{bmatrix}. \tag{A11}$$

Define $\gamma_{im}$ to be the covariance of the $i$th particular asset with the market portfolio, and in a manner similar to (A7) let $\gamma_{ik}$ be the covariance of the $i$th asset with the $k$th state variable. Using these definitions, (A3) and (A6), we can write the expected return on the $i$th asset as

$$\alpha_i - r_0 = (\gamma_{im}\ \gamma_{i1} \cdots \gamma_{iK}) \begin{bmatrix} M/A \\ -T/A \end{bmatrix}. \tag{A12}$$

Substituting from (A11) into (A12) gives

$$\alpha_i - r_0 = (\gamma_{im}\ \gamma_{i1} \cdots \gamma_{ik}) \bar{\Sigma}^{-1} \begin{bmatrix} \alpha_m - r_0 \\ \alpha_{yi} - r_0 \\ \vdots \\ \alpha_{yK} - r_0 \end{bmatrix}, \tag{A13}$$

a result that holds for all assets. As Richard (1979) notes, (A13) has the form of a regression equation in that it can be rewritten as

$$\alpha_i - r_0 = \beta_i^m (\alpha_m - r_0) + \sum_{k=1}^{K} \beta_i^k (\alpha_{yk} - r_0), \tag{A14}$$

where the $\beta_i^k$'s are coefficients that would be recovered in the estimation of $(\alpha_i - r_0)$ on $(\alpha_m - r_0)$ and the $K$ returns $(\alpha_{yk} - r_0)$.

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