Asset Prices and Default-Free Term Structure in an Equilibrium Model of Default

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Abstract

We present an equilibrium production economy in which default occurs in equilibrium. The borrower chooses optimal default and consumption policies taking into account the fact that default is costly and that the lender will gain access to the technology upon default. We derive asset prices, and default premia in this economy. The borrower’s relative risk aversion in wealth increases with decreases in wealth due to the increased possibility of default at low wealth levels. This produces a time-varying pricing kernel and a counter-cyclical equity premium. We thus provide an equilibrium rationale for the default premium to influence expected asset returns.

1 Introduction

We construct an equilibrium production model of default with two agents in this paper. We lay the groundwork for linking the asset pricing with default and offer a theoretical rationale for default premium to influence asset returns. This property was assumed by Jagannathan and Wang (1996) and other scholars in the asset pricing literature. The general equilibrium production model of Cox, Ingersoll and Ross (1985) provides our basic frame of reference. The borrower in the economy has exclusive access to the only risky production technology when the economy begins. The borrower is endowed with limited initial endowment of the only good which is not storable. The lender does not have any access to the risky technology on the initial date but is endowed with the good which he can decide either to lend to the borrower or simply consume. Since the good is not storable the only way for the lender to consume over time is to lend the good to the borrower in exchange for a stream of promised payments. To keep the focus sharply on default we assume that the borrower offers a debt contract to the lender which promises a constant flow rate of payments to the lender forever. The
borrower faces a cost associated with default: when the promised payments are not paid the borrower loses a fraction of his wealth plus a fixed amount to the lender. Moreover, upon default the lender is able to access the risky technology permitting him to become a fully utility maximizing participant in the economy. The lender chooses his optimal strategy at time 0 also by comparing the expected utility associated with lending with the utility of consuming the good today. When the lender optimally decides to lend at time 0, the borrower is able to augment his endowment and determine optimally his consumption and default policies. This is the equilibrium that we study in the paper. We characterize the feasible loans and an equilibrium in which there is lending with welfare improvements to both lender and borrower. The decision to accept the loan and to default later is endogenous in the model. The borrower chooses the optimal time of default to maximize his expected life-time utility.

Our approach has some merits and some drawbacks relative to other contributions in the literature. We contribute at a methodological level by computing allocations and prices in an economy where there is endogenous default. We use optimal stopping time methods to do this. We offer a framework in which asset prices depend on default premium. We derive an ICAPM to make this relationship explicit. Finally, we exploit the production technology to draw some predictions about how default-free term structure might be influenced by the probability of default. The drawbacks are the following: we exogenously impose a participation constraint on the lender and we are unable to permit some agents to default while at the same time allowing other agents to be solvent. We also impose a specific debt contract as a means to augment initial endowment, although we explore later [see Figure 1] under what circumstances borrower will want debt and under what circumstances he will use equity. We focus our attention on allocat-
tions and prices before default because under many realistic conditions, the economy operates with a positive probability of default but actually does not experience default for very long periods of time. We believe that this focus is reasonable: we often focus on asset prices of firms before default to examine default premium. This is one of the distinct contributions of our paper.

The subject that we study in this paper has been investigated by some scholars recently. Two recent theoretical papers by Zhang (1997) and Alvarez and Jermann (2000) have explored the importance of default risk on asset pricing using models of endogenous solvency constraints. These models draw on the insights of Kocherlakota (1996) and Kehoe and Levine (1993) by incorporating participation constraints and are able to shed some light on the risk sharing implications of default risk. But by construction these models of solvency constraints eliminate the possibility of default in equilibrium. In our framework there is default in equilibrium. In models of solvency constraints no default-risky loan is modelled and hence default premium is less direct to compute. Moreover the determination of default-free term structure is not addressed in models of solvency constraints. Our approach also differs in a major way in the nature of risk sharing. In the model of Alvarez and Jermann (2000) default leads to autarchy with no risk sharing possibilities. We accommodate this as a special case but in general permit the risk sharing to continue even after default. A second strand of literature exemplified by Geanakoplos and Zame (1998) explores default in an endowment economy with two periods wherein a durable good is used as collateral to borrow and the collateral is seized by lenders upon default. Our paper uses a production economy similar to Cox, Ingersoll and Ross (1985) and treats the case of perishable good. Kubler and Schmedders (2001) consider an economy with a perishable good. The productive asset plays the role of
collateral. They offer a computational framework to study the equilibrium properties. Zame (1993) offers a framework in which default actually helps to complete the market.\textsuperscript{3}

Our approach allows us to shed some light on the following issues and questions:

a) \textit{What are the properties of optimal default strategies in an equilibrium model?} A key result here is that there is default in equilibrium in our model. This is in sharp contrast to the results in the existing literature on equilibrium models with solvency constraints which we review later. In addition, the optimal default boundary depends on the costs associated with default and the lender's status after default. If the lender participates in the economy as a maximizing agent after default, we show that the risk-sharing possibilities effectively reduce the cost of default and hence leads to a higher optimal default boundary. If the lender and borrower have identical preferences, default effectively leads to autarky. For most part we focus on this latter case as it is closer to much of the equilibrium literature on defaults.

b) \textit{What is the effect of costs of default [transfer payments to the lender] on the risk aversion of borrower?} An important result in this context is that the borrower becomes much more risk averse as his wealth level drops and approaches the optimal default boundary. He reduces the optimal flow rate of consumption in order to reduce the likelihood of default. Eventually when the wealth drops further and is very close to the optimal default boundary the borrower becomes much less risk averse and starts to dissipate his wealth by increasing his optimal flow rate of consumption. The presence of these two regions is shown to be robust whether the borrower returns to autarky after default or is allowed to share his risk with the lender after default.

c) \textit{How does the presence of default risk affects the default-free term structure?} We show that the presence of default risk induces an extra value
to the risk-free asset in the economy. The borrower’s increasing risk aversion as the default boundary is approached leads to a lower shadow risk-free rate. This effect is mitigated by the possibility of risk-sharing with the lender after default. The term structure of default-free interest rates becomes steeper in the presence of default. In the second region the risk-free rate increases and the term structure becomes inverted. Many recent papers have documented that the Treasury rates reflect a flight to quality premium and have argued that perhaps collateralized rates such as the repo rates or swap rates are better proxies for risk-free rates at different maturity sectors. To our knowledge, this is the first paper to formally demonstrate this effect.

d) **What is the relationship between default premium and asset returns?**

We derive a CAPM with default risk and show that equity premium depends on two factors: the first factor is the covariance of consumption with wealth, which is the standard prediction. The second factor is the covariance of consumption to the household indebtedness. We characterize these factors and show two key results. First, the presence of default risk generally increases the equity premium in the economy. Second, there is a positive association between equity risk premium and default premium.

The paper is organized as follows: the next section develops the basic equilibrium model of default and motivates its construction. Section 3 characterizes the properties of optimal debt contract, borrower’s risk aversion in wealth, optimal consumption policies and the default-free term structure. We also characterize the feasible regions of lending. In section 4 we develop the CAPM with endogenous default risk. We simulate the model to present some implications for asset pricing. Here we show how default risk influences equity premium. We provide a theoretical rationale here for the assumption that the equity premium depends on default risk. This assumption is used, for example, by Jagannathan and Wang (1996). Section 5 concludes. In
section 6, we provide an appendix which collects all our technical results.

2 The Model and the Nature of Default

We consider a production economy setting with two agents: a borrower and a lender. In the next subsection, we describe the technology and the economic environment. The subsequent sections will describe the optimization problem and our notion of equilibrium in this economy.

2.1 Economic Environment

The economic environment consists of production technology, preferences, punishment mechanisms associated with default, and the manner in which equilibrium prevails in the economy. We proceed to describe each in turn.

2.1.1 Production Technology and its Access

There is a single good in the economy and it serves as the numeraire. The production sector has a risky technology. Once an amount \( q_t \) of the good is invested in the technology at time \( t \), the output will evolve as shown below:

\[
\frac{dq_t}{q_t} = \mu dt + \sigma dz_t
\]

where the instantaneous expected rate of return \( \mu \) and the diffusion coefficient \( \sigma \) are exogenous positive constants.\(^4\) The process \( z_t \) is a standard Brownian Motion on the underlying probability space \((\Omega, \mathcal{F}, \mathbb{P})\).

2.1.2 Preferences and Endowments

The risk-averse borrower (consumer) is endowed with an initial wealth of \( x_0 \) and has an exclusive access to the risky production technology. He maximizes his life-time discounted expected utility of consumption: \( E_0[\int_0^\infty e^{-\rho t} \]
\[ u(c)dt \] where \( u \) is his von Neumann-Morgenstern utility function and \( \rho \) is his time preference rate. In this paper we examine a special class of utility functions whose relative risk aversion in consumption is a positive constant:\(^5\)

\[ u(c) = \frac{1}{1-A}c^{1-A}, \quad A > 0, \quad A \neq 1 \quad (2) \]

This specification has been widely used in the theory of intertemporal consumption-portfolio selection problems, default-free term structure theory and in asset pricing.

The second agent in our model is the lender. He is restricted from participating in the production technology at time zero. He arrives at time zero with an initial endowment. The good is not storable. Hence the only way for the lender to consume over time is to lend to the borrower in exchange for a stream of promised future payments.

### 2.1.3 The Loan Contract and Punishment Mechanism

In our model, we assume a specific loan contract \( \{C; \alpha, K; I_0\} \) which is described below:

The borrower can borrow an amount \( I_0 \) from the lender at time zero.\(^6\) But this requires him to pay the lender a flow rate of \( C \) per unit time until default.\(^7\) If the borrower decides to default at time \( \tau \geq 0 \) then he loses a fraction \( (1 - \alpha) \) of his wealth plus a lump sum of \( K \) and the exclusive access to the risky technology.\(^8\) The lender, in exchange for lending \( I_0 \) at time zero, derives utility by consuming the contractual debt payments \( C \) per unit time until default. During this period the lender is outside the economy due to the participation constraint: he can neither invest in the risky technology, nor can he trade with the borrower. Upon default, the lender collects the amount of \( (1 - \alpha)W + K \) from the borrower and becomes a maximizing agent in the economy. He will have full access to the risky technology and
there will be active risk-free lending and borrowing after default. There are no deadweight losses in our economy. One should note that the parameters \{α, K\} reflects the sharing rule of wealth between lender and borrower upon default, which is governed by the relevant bankruptcy codes that are applicable in the economy and the relative bargaining positions of the lender and the borrower. We have not explicitly modelled the trade-offs between equity and debt contract in this model. We briefly address this issue later to show that under some circumstances debt may not be the optimal contract.

2.2 Optimization Problem and Equilibrium

We now describe the equilibrium in this economy. The equilibrium after default will be a standard two-person dynamic equilibrium in a production economy (similar to the one studied by Dumas [1989]). Although the lender remains outside of the economy until default, he can still significantly influence the equilibrium before default through his participation in the economy after default. By letting the lender and borrower to be identical, we can reduce the problem after default to autarky which is the standard assumption imposed by Alvarez and Jermann (2000).

Given the loan contract \{\overline{\nu}; α, K\}, the controls of the borrower will be the amount \(q_t\) invested in the risky technology, the consumption rate \(c_t\) and the optimal default level \(W^*\). Define the \(\{\mathcal{F}_t\}\)-stopping time: \(τ = \inf\{t \geq 0 | W_t ≤ W^*\}\). The wealth dynamics facing the borrower can be formally represented as:

\[
dW_t = \left[ r_t(W_t - q_t) - c_t - \overline{\nu} \right] dt + \mu q_t dt + \sigma q_t dz_t \quad \text{for } 0 ≤ t < τ \tag{3}
\]

where \(r_t\) is the default free interest rate. Let us denote the set of admissible controls by \(\mathcal{A}(W_0)\). The objective function facing the borrower is the expected life-time discounted utility maximization. Formally, the borrower
maximizes the value function $J$ which is defined as the supremum of the expected utility over the set of admissible controls:

$$J(W_0) = \sup_{A(W_0)} E_0 \left[ \int_0^\infty e^{-\rho t} u(c_t) dt \right]$$

(4)

Let us denote the optimal policy be $(c^*, q^*, W^*)$. The equilibrium in this economy is defined as following:

**Definition 1** An equilibrium $\{ (r, c^*, q^*), I_0^* \}$ is a set of stochastic processes $(r; c^*, q^*)$ and an initial borrowing amount $I_0^*$ which satisfy:

(i). Market clearing condition: $q_t^* = W_t$.

(ii). Borrower’s loan valuation condition:

$$I(x_0 + I_0^*) = I_0^*$$

(5)

where $I(.)$ is the borrower’s valuation function for the loan.

(iii). Feasibility for Borrower:

$$J(x_0 + I_0^*) > J_0(x_0)$$

(6)

where $J_0(.)$ is the borrower’s valuation function in autarchy.$^9$

(iv). Feasibility for Lender:

$$J^L > J_0^L$$

(7)

where $J^L$ is the lender’s expected utility at time zero by lending and $J_0^L$ is the corresponding expected utility when he simply consumes the endowment.

The market clearing condition (i) implies that there is no risk-free lending or borrowing at equilibrium, as in CIR (1985). The borrower’s loan valuation condition (ii) is a fixed-point requirement which says that the equilibrium level of borrowing must be such that the amount borrowed $I_0^*$ is equal to the borrower’s valuation of the loan contract at time zero. The borrower’s
feasibility condition (iii) states that the borrower is willing to borrow from
the lender only if the loan is sufficiently attractive to him at time zero,
i.e. his life time expected utility by borrowing is larger than the life time
expected utility in autarchy. For example when the coupon rate \( \overline{C} \) is very
unfavorable to the borrower [\( \overline{C} \) is very large relative to his initial endowment
\( x_0 \)] or the recovery rate to the lender is very favorable, then it is likely
that the borrower will choose not to borrow from the lender. The lender's
feasibility condition (iv) requires that the lender also has to be better off by
providing such a loan contract. Otherwise the lender will simply consume
his initial endowment at time zero. For a risk averse lender, condition (iv)
will be trivially satisfied because lending is the only way for him to smooth
consumption over time.

Critical to the characterization of our equilibrium with default is the opti-
mal default boundary \( W^* \). We show in the appendix that the value function
satisfies the following properties: (i) \( J(\cdot) \) is strictly increasing and strictly
concave. (ii) \( J(\cdot) \) is continuous on \( [W^*, \infty) \) with \( J(W^*) = J_B(\alpha W^* - K) \),
where \( J_B(\cdot) \) is the borrower's valuation function after default. (iii) (smooth
pasting condition) \( \lim_{W \to W^+} J'(W) = \frac{\partial J_B(\alpha W^* - K)}{\partial W} \). (iv) (dynamic pro-
gramming principle)

\[
J(W) = \sup_{\mathcal{A}(W_0)} \mathbb{E}_0 \left[ \int_0^\tau e^{-\rho t} u(c_t) dt + e^{-\rho \tau} J_B(\alpha W^* - K) \right]
\]

We also show in the appendix that for any \( t < \tau \), the value function \( J(\cdot) \) is the unique \( C^2(W^*, +\infty) \) solution of the Bellman equation:

\[
\rho J = \frac{1}{2} \sigma^2 W^2 J_{WW} + (\mu W - \overline{C}) J_W + \max_{c \geq 0} \left[ u(c) - c J_W \right] \quad (W > W^*)
\]

with boundary condition \( J(W^*) = J_B(\alpha W^* - K) \) and \( \lim_{W \to W^+} J'(W) = \frac{\partial J_B(\alpha W^* - K)}{\partial W} \). And the optimal policy \( c^*_t \) is given by:

\[
c^*(W) = (u')^{-1}(J_W(W))
\]
The approach to solve this problem is by backward induction. We first solve the two-person general equilibrium after default. An example of such a two-person general equilibrium is the one studied by Dumas (1989) in which the borrower and lender are both risk averse but different [As in Dumas (1989), one agent has a power utility function and the other has a log utility]. The wealth sharing rule is obtained by maximizing the welfare function [which is a weighted sum of the utilities of the two agents] as in Dumas (1989). Since we know at default the share of wealth of borrower and lender, we can precisely compute the constant weight $\lambda^*$ [used in the welfare function] for a given default boundary. Using this weight we determine the value function $J_B(\alpha W^* - K)$ of the borrower upon default. The value function of the borrower will reflect the risk-sharing possibilities after default. In particular, the risk aversion of lender will influence the value function of the borrower. We then use this value function of the borrower as the boundary condition to search for the optimal default boundary of the borrower as explained in the appendix.\textsuperscript{10} We have designed and implemented a finite difference scheme to numerically solve such a free boundary problem. The formal analysis that leads to the determination of optimal default boundary is presented in the appendix. There we also present and discuss the technical results which characterize the properties of the equilibrium. These results show that the economy that we study has a well defined equilibrium and that the value function and its derivatives converge to their counterparts in a general equilibrium model with identical consumers. We also present in detail in the appendix the numerical procedure that we use in the paper to compute the equilibrium.

We now directly proceed to illustrate our numerical results in the next few sections. In view of the computational complexity we will simply focus on a baseline case where the borrower and the lender are identical. For an
active risk-sharing lender, who is different from the borrower, the results are qualitatively similar to the baseline case.  

3 Optimal Default, Time-Varying Risk Aversion & Term Structure

We examine a baseline case where we set the borrower’s subjective discount factor $\rho = 0.05$ and the risk aversion parameter $A = 2.0$. We also assume for the baseline case that the lender is identical to the borrower. At time zero the borrower is endowed with one unit of consumption good: $x_0 = 1.0$. For the risky technology, we assume the instantaneous expected rate of return $\mu = 0.10$ and the diffusion coefficient $\sigma^2 = 0.02$. We further assume that the sharing rule between borrower and lender upon default is $\alpha = 0.25$ and $K = 0.05$. In the following context, we first describe the optimal coupon rate for the borrower under the baseline setting. Given the fact that such an optimal contract is sustainable by both borrower and lender, we next characterize the property of the equilibrium for our model under such a contract.

3.1 Lending and Risk Sharing

We first characterize the feasibility of a certain loan contract for the borrower. Note that the borrower is willing to take on the loan only if his life time expected utility by borrowing is larger than the life time expected utility in autarchy [condition (iii) of the equilibrium]. To examine how much utility he can gain by taking on the loan, we define the relative certainty equivalence as:

$$CE(x_0) = \frac{J_{0}^{-1}[J(x_0 + I^*)]}{x_0}$$

which measures the normalized utility change for the borrower with an
initial wealth level $x_0$. Relying on the concavity of the value function $J$ and $J_0$, the relative certainty equivalence $CE(\cdot)$ is a well defined continuous function. A borrower is willing to accept a loan contract if and only if his relative certainty equivalence $CE > 1$.

The following figure shows the borrower’s relative certainty equivalence for different coupon rate $\overline{C}$ and lump sum cost $K$. For our baseline setting $K = 0.05$, the relative certainty equivalence reaches the maximum level when coupon rate $\overline{C} = 0.027$. Typically the relative certainty equivalence is smaller for a borrower with a higher lump sum cost $K$. When the lump sum cost is very high (for example $K > 0.26$), the maximum level of certainty equivalence will be smaller than one. In this situation, the borrower prefers to remaining autarchy and does not borrow from the lender. Equity will be the preferred mechanism for any scale expansion under these circumstances.

In these situations the two agents can become equity holders in the expanded production opportunity set.

![Figure 1.](image-url)

The relationship between the proportional cost $1 - \alpha$ and the relative certainty equivalence $CE$ is similar to the relationship between the lump
sum cost $K$ and the relative certainty equivalence. For the sake of brevity this result is not presented in the paper.

3.2 Relative Risk Aversion in Wealth

In this section we characterize the behavior of the indirect value function. In particular, we plot in figure 2, the relative risk aversion in wealth ($RRA$) of the borrower as measured using his value function in our economy. Note that the relative risk aversion in wealth for the general equilibrium economy with no default under our hypothesized assumptions is simply a constant given by $A = 2.0$. In figure 2, we plot the wealth along the X-axis and the relative risk aversion in wealth along the Y-axis. We find that the relative risk aversion increases in a significant manner as the wealth level drops from $\infty$ to $\hat{W}_{RRA_{\text{max}}} = 0.782$, where the relative risk aversion in wealth reaches its peak $RRA_{\text{max}} = 3.8$. A further decrease of wealth leads to a reduction in $RRA$ until it reaches the default boundary $W^* = 0.381$. Thus there are two regions in this economy. In one region the relative risk aversion increases with decreases in wealth. As the wealth drops the probability of default increases and the borrower becomes more risk averse in this region. We call this region as “flight to quality”. This is a metaphor for the borrower’s implicit preference for less risky assets and his aversion for the more risky assets. We will show later that in the flight to quality region the borrower’s shadow risk-free rates falls with decreases in wealth. The second region where the borrower’s relative risk aversion decreases with decreases in wealth is a manifestation of the overinvestment distortions in our economy. In this region he has an implicit preference for risky asset. We will show later that in this region the borrower increases his rate of consumption and thus dissipates the collateral. Hence we call this region as the “collateral dissipation” region. Under this set of parameters, the initial

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augmented wealth for the borrower will be \( W_0 = 1.557 \). The corresponding relative risk aversion coefficient at time zero is \( RRA_0 = 3.14 \). The relative importance of these two regions depends on the magnitude of the lump sum costs \( K \). This will be discussed further in the paper later.

![Figure 2.](image-url)

In figure 2 we also plot the effect of the recovery rate parameter \( \alpha \) on relative risk aversion. Note that upon default the borrower keeps a fraction \( \alpha \) of his wealth. Naturally, as \( \alpha \) increases the recovery rate on the loan falls. We found that when the recovery rate increases, (namely, as \( \alpha \) decreases) there are two effects: first, the optimal default boundary decreases. The borrower is more careful about defaulting the loan which implies that the optimal default boundary \( W^* \) is decreasing as \( \alpha \) decreases. Simultaneously, the borrower becomes more risk averse and the relative risk aversion \( RRA \) increases.

Time varying risk aversion has played an important role in the asset pricing literature. Campbell and Cochrane (1999) show that models of habit formation which produce time varying risk aversion can help explain aggregate stock market behavior. We show that time varying risk aversion may also arise due to the presence of default. Our model implies that pronounced
increases in risk aversion may result in economies where risk sharing possibilities after default are limited and where the costs of default are high.

3.3 Equilibrium Default-Free Interest Rate

The instantaneous default-free instantaneous interest rate in our model is given by \( r(W) = \mu - \frac{WJW\sigma^2}{J_W} \). In figure 3, we plot the instantaneous risk-free rate as a function of wealth.

Default risk has two striking effects on the equilibrium risk-free rate. First, when the borrower is in the flight to quality region, the equilibrium risk-free rate is always below the one given by the default-free economy which equals to a constant: \( R = \mu - A\sigma^2 \). At wealth levels close to \( \tilde{W}_{r_{\min}} = \tilde{W}_{RR_{\max}} = 0.782 \), the equilibrium interest rate is well below the level implied in an economy with no default risk. In the illustration that we display in figure 3, the maximum difference is about 358 basis points at a wealth level \( \tilde{W}_{r_{\min}} = 0.782 \). Note that the presence of default risk has important pricing consequences for the default-free interest rates in this flight to quality region. These rates display a cyclical behavior: when the economy’s wealth decrease, the real risk-free rates go down and when the economy’s wealth increases, the real risk-free rates increase. A further decrease in wealth leads the economy to the region of collateral dissipation and the interest rate begins to rise as the wealth decrease. The region of collateral dissipation depends on \( K \). For very high wealth levels, i.e. as \( W \to \infty \), the risk-free rate approaches the level given by the model with no default risk.
In figure 3 we also plot the effect of the lump sum cost $K$ on equilibrium default-free interest rates. As $K$ increases, the interest rates fall and the region where collateral is dissipated becomes smaller. Thus we find that the overinvestment distortions are mitigated by lump sum costs of default as opposed to proportional costs of default. The intuition for this is the following: with proportional costs the borrower loses more when the wealth level at which he defaults is high. So he has an incentive to consume more when default is imminent. This way he leaves less collateral to the lender. With a lump sum cost of default this incentive is sharply curbed.

3.4 Default-Free Term Structure

We now characterize the default-free term structure in this economy.\textsuperscript{12} In standard equilibrium setting, the term structure will be flat: as the yield to mature for a zero coupon bond $R(t, T) = -(\ln P(t, T))/(T - t)$ is simply a constant equal to the instantaneous interest rate $\mu - \sigma^2$. However in our model, the shape of the term structure is wealth dependent and exhibits a rich pattern as shown in figure 4.
In the flight to quality region, the default free term structure becomes more expensive as the wealth goes down and the curve gets steeper. The impact of the default risk on default-free term structure is quite subtle: it arises from the implied relative risk-aversion in wealth and the optimal default boundary. We find that our model with a constant opportunity set is able to deliver fairly rich shapes of default-free term structure once default risk is admitted. The default risk (which is priced as a factor through the new equilibrium default-free instantaneous rate) permeates through the pricing of the yield curve.

Recently in the United States economy several practitioners have attributed the fall in the Treasury interest rates and the increase in the slope of the Treasury term structure to the increased risk aversion concerning the potential for costly defaults in the telecommunications sector of the economy. Our model’s implications are certainly consistent with this view.
3.5 Optimal Consumption

In the absence of default risk, the general equilibrium model implies that the optimal consumption is given by $kW$. Moreover, the elasticity of optimal consumption with respect to wealth is a constant. In figure 5a, we plot the normalized optimal consumption $\frac{C(W)}{kW}$ and in figure 5b the normalized elasticity of optimal consumption with respect to wealth. It is useful to note that the consumption elasticity in our model is the ratio of the RRA in wealth to the RRA in consumption. Since the RRA in wealth has already been characterized in figures 2, it is easy to interpret figures 5a and 5b.

In our model, the optimal consumption rate and the elasticity of consumption depends on how close the economy is to defaulting. In particular, they depend on which region the wealth level falls in. As wealth approaches infinity, our economy approaches that of an equilibrium model with no default. In this region, the consumption and elasticity is close to the baseline case. In the flight to quality region, as the wealth decreases, the rate of consumption falls and the wealth elasticity of consumption rises. The elasticity is generally higher than what is implied by the base case for moderate to high levels of wealth. Our prediction is that the wealth elasticity of consumption increases as the economy approaches the default boundary. This
is when the economy is behaving with greater caution to avoid default. Our results in this context are in conformity with the evidence reported by Olney (1999) concerning the consumption data in the United States in the great depression period. Olney (1999) reports that prior to the great depression the bankruptcy code favored the sellers of consumer durable on installment credit to the households. She argues that the households tried to avoid default by curtailing their consumption which in turn precipitated the depression. Our model predicts that the consumption elasticity is higher at lower wealth levels when the default probability is high. Note that when the lump sum costs are lower, the collateral dissipation region or the over-investment region increases.\textsuperscript{13}

4 Equity Premium

In this section we will discuss the equity premium in the economy. The possibility that the equity premium may be related to default risk has been recognized by many scholars in empirical asset pricing. Papers by Chen (1991), Fama and French (1989), Keim and Stambaugh (1986), and Ferson and Campbell (1991) have shown that the market risk premium is time-varying and it varies over the business cycle. Stock and Watson (1989) and Bernanke (1990) stress the superior ability of proxies of default premium to forecast business cycles. Jagannathan and Wang (1996) have assumed that the conditional equity risk premium is a linear function of the default premium in the economy. Their conditional CAPM (which also takes into account the returns from human capital) is able to explain the cross-sectional variations in equity returns more successfully. Empirical evidence also suggests that default risk proxies such as the junk bond spreads over default-free security yields are useful in explaining the returns on stocks and default-free bonds. Chen, Roll and Ross (1986) present evidence that the
spread on high-yield bonds explain the returns on stocks. All these papers suggest that the existence of default risk may affect the equity returns.

The value of equity in our economy is the wealth net of the market value of borrowing at any time. We denote this by \( E(W) = W - I(W) \), which can be viewed as a contingent claim with continuous payout \( c^* \). The risk premium in the underlying CIR (1985) setting is simply a constant: \( \mu - A\sigma^2 \). However it is wealth dependent in our model. The presence of default risk in our model has a strong effect on equity risk premium. As the probability of default begins to increase, the agent becomes more risk averse and consumes less to avoid the costliness of default. Such a behavior causes the equity risk premium to be systematically higher in our model when the wealth level in the region of flight to Quality.

Since \( I(.) \in C^2(W^*, \infty) \), we apply the Ito’s Lemma to \( E(W) \) and write the stochastic differential equation governing the movement of \( E(.) \) as:

\[
\frac{dE}{E} = (\mu_E - c^*)dt + V_E dz_t
\]  

(12)

Where the instantaneous rate of return on equity is given by \( \mu_E = \frac{(1-I_W)(\mu_W - \overline{z}) + I_W c^* - \frac{1}{2}I_W \sigma^2 W^2}{E} \). Following Theorem 2 of CIR (1985), we state the following Lemma without proof:

**Lemma 1** The instantaneous risk premium \( \mu_E(W) - r(W) \) satisfy a version of CAPM:

\[
\mu_E - r = \frac{1}{E} \left( \frac{-u_{cc}(c^*)}{u_c(c^*)} \right) (COV \ c^*, W) - \frac{1}{E} \left( \frac{-u_{cc}(c^*)}{u_c(c^*)} \right) (COV \ c^*, I) \]  

(13)

where \( (COV \ c^*, W) \) denotes the instantaneous covariance between optimal consumption and wealth.

The CAPM says that the equity premium depends on the covariance of consumption with wealth and the covariance of consumption with risky
debt value. If the latter covariance is negative, the equity premium will be higher, ceteris paribus.

We investigate the implications of our ICAPM in two ways: first, we explore how the two covariances influence the equity premium. This is reported in Figure 6.

Note that the second covariance term, which captures the covariance of consumption with the household debt is never positive [after incorporating the negative sign]. In the limit, when wealth increases to infinity, this covariance term vanishes. This covariance becomes more negative as the wealth goes down. On the other hand, since consumption is influenced by default, the first covariance term actually increases more than the decrease in the second covariance term as the wealth goes down, thereby causing a net increase in equity premium. This result is stable for a number of parameter configurations in the flight to quality region. In figure 7, we relate the default premium to the equity premium as both are simultaneously set in our economy.
Note in figure 7 that as wealth increases, both the default premium and the equity premium decline in our economy, although the default premium declines much more rapidly than the equity premium. As the wealth declines, the premia rise slowly at first and then much more sharply. We thus provide a framework that accounts for the co-movement of default and equity premia.\textsuperscript{14}

In their recent paper, Lettau and Ludvigson (2000) present evidence that allowing for time variation in risk premia may be essential to the success of conditional consumption CAPM. The source of such variations may come from such factors as habit formation, or labor earnings or as in this paper from default risk. All these approaches deliver a variation in risk aversion that is countercyclical: the risk aversion is high in recession and low in booms. Thus we have three competing alternative drivers to the time variation in risk premia. Future empirical work can test to what extent these drivers are useful in understanding the time variation in equity premia.
5 Conclusion

We have presented an equilibrium production model of default. This model extends the general equilibrium production model of Cox, Ingersoll and Ross (1985) to a case where there are two agents and presents an equilibrium in which default occurs with a positive probability. The model allows one to determine endogenously the optimal default boundary, optimal consumption, risk-free term structure and the default premium. A key implication of our model is that the risk aversion in wealth of the borrower displays time variation through endogenous wealth dependency. Our model predicts that there are two [endogenously determined] regions. In one region the risk aversion increases with decreases in wealth. In the other region the risk aversion increases with decreases in wealth. The model permits the borrower to be a lifetime expected utility maximizer. The lender is initially subjected to a participation constraint, which is removed upon default by the borrower when he becomes a utility maximizing and a risk-sharing player in the economy.

The model can be extended in many ways. We have chosen to model the lender through a participation constraint before default. Alternatively, the lender could be participating in the economy throughout the time period as a utility maximizer with access to either the risky or risk-free asset or both. This extension will take the model closer to a truly general equilibrium analysis with default. We have also focussed on the simple case of static borrowing. The case with dynamic borrowing opportunities is a natural extension to our framework.
6 Appendix

6.1 Characterization of the Equilibrium

In order to characterize this competitive economy, we first look at the planning problem with the same physical production opportunities but with no default-free borrowing and lending. In this situation the wealth process before default follows:

\[ dW_t = \left[ \mu W_t - c_t - \mathcal{C} \right] dt + \sigma W_t dz_t \quad \text{for} \quad 0 \leq t < \tau \tag{14} \]

and the central planner seeks to maximize the corresponding value function \( \bar{J} \):

\[ \bar{J}(W_0) = \sup_{\mathcal{A}(W_0)} E_0 \left[ \int_0^\infty e^{-\rho t} u(c_t) dt \right] \tag{15} \]

where \( \mathcal{A}(W_0) \) is the corresponding set of admissible controls.

It is evident that if \( J = \bar{J} \) and \( r = \rho - \frac{LW}{J_W} \), then the solution to the original competitive equilibrium will be exactly equivalent to this simple planning problem.\(^{15}\) So in the following context we will characterize the planner’s dynamic programming problem (15). For notational simplicity, we will not distinguish the variables in the planning economy and the competitive economy in the following context.

**Lemma 2** (i) \( J(\cdot) \) is strictly increasing and strictly concave.

(ii) \( J(\cdot) \) is continuous on \([W^*, \infty)\) with \( J(W^*) = J_B(\alpha W^* - K) \), where \( J_B(\cdot) \) is the borrower’s valuation function after default.

(iii) (smooth pasting condition) \( \lim_{W \to W^*} J'(W) = \frac{\partial J_B(\alpha W^* - K)}{\partial W^*} \).

(iv) (dynamic programming principle)

\[ J(W_0) = \sup_{\mathcal{A}(W_0)} E_0 \left[ \int_0^\tau e^{-\rho t} u(c_t) dt + e^{-\rho \tau} J_B(\alpha W^* - K) \right] \tag{16} \]
Proof. (i) and (ii) follows from Zariphopoulou (1994) Proposition 2.1. For (iii), see Dumas (1991) who provides an extensive discussion of “smooth pasting” or “super contact” conditions. The dynamic programming principle (iv) is presented with proof in Fleming and Soner (1993).

The next Lemma is a key result which we will be using to characterize the value function and the optimal default boundary.

**Lemma 3** For any \( t < \tau \), the value function \( J(\cdot) \) is the unique \( C^2(W^*, +\infty) \) solution of the Bellman equation:

\[
\rho J = \frac{1}{2} \sigma^2 W^2 J_{WW} + (\mu W - C) J_W + \max_{c \geq 0} \left[ u(c) - c J_W \right] \quad (W > W^*)
\]

(17)

with boundary condition \( J(W^*) = J_B(\alpha W^* - K) \) and \( \lim_{W \to W^+} J'(W) = \frac{\partial J_B(\alpha W^* - K)}{\partial W^*} \). And the optimal policy \( c^*_t \) is given by:

\[
c^*(W) = (u')^{-1}(J_W(W))
\]

(18)

Proof. Equation (17) is uniformly elliptic and hence has a unique smooth \( C^2(W^*, +\infty) \) solution (see Krylov (1987)). Applying the verification theorem (Fleming and Rishel (1975)) leads to our Lemma.

Unlike the standard CIR (1985) single agent economy, there is no closed form solution to the HJB equation (17). This is due to the presence of borrowing and lending, in particular the non-homogeneous term \( C J_W \) in equation (17). We have designed and implemented a finite difference scheme to numerically solve such a free boundary problem. Since the value function is \( C^2(\bar{W}, +\infty) \) smooth, the convergence of our numerical scheme directly follows the consistency and stability of the theory of finite-difference schemes (see Strikerda (1989)). A description of our procedure is outlined in the next section.
Although we cannot get an explicit solution to (17), intuition suggests that the economy with an active lending will converge to standard CIR single agent economy when \( W \) is very large. We formally state the following limiting results under a special case when the borrower is identical to the lender. Under this situation, the borrower’s valuation function after default \( J_B \) will simply be his valuation function in autarchy \( J_0 \).

**Proposition 4**

(i) \( \lim_{W \to \infty} J(W) = J_0(W) = \frac{k-A}{1-A} W^{1-A} \).

(ii) \( \lim_{W \to \infty} J_W(W) = (1-A) \frac{k-A}{1-A} W^{-A} \).

(iii) \( \lim_{W \to \infty} J_{WW}(W) = (1-A)(-A) \frac{k-A}{1-A} W^{-A-1} \).

**Proof.** (i) We first observe that \( J(W) \leq J_0(W) \) hence \( \lim_{W \to \infty} J(W) \leq J_0(W) \). So what we need to show is \( \lim_{W \to \infty} J(W) \geq J_0(W) \). Since \( W^* \) is optimal chosen by the borrower, we have \( J(W) \geq \tilde{J}(W) \) where \( \tilde{J}(W) \) is the solution to HJB equation (17) with \( W^* = 0 \). So it is sufficient to prove that \( \lim_{W \to \infty} \tilde{J}(W) = J_0(W) \). Let \( v^{(\varphi)}(W) = \varphi^{1-A} \tilde{J}(\frac{W}{\varphi}) \), \( \forall W \geq 0, \varphi > 0 \). Then we have \( \lim_{W \to 0} v^{(\varphi)}(W) = \frac{u(0)}{\rho} \) uniformly in \( \varphi \). Moreover we can see that \( v^{(\varphi)} \) is the unique \( C^2(0, +\infty) \) solution which satisfies:

\[
\rho v^{(\varphi)} = \frac{A}{1-A} (v^{(\varphi)}_W)^{1-A} + uW v^{(\varphi)}_W - \mathcal{C}_\varphi v^{(\varphi)}_W + \frac{1}{2} \sigma^2 W^2 v^{(\varphi)}_{WW}
\]

with \( \lim_{W \to 0} v^{(\varphi)}(W) = \frac{u(0)}{\rho} \) \( (19) \)

Note that \( v^{(\varphi)} \) can be interpreted as the value function for a borrower with coupon rate \( \mathcal{C}_\varphi \) and default level \( W^* = 0 \), it is obvious that \( v^{(\varphi)} \) perseveres all the properties of \( J \) and \( v^{(\varphi)} \leq J_0 \). Hence \( v^{(\varphi)} \) is locally uniformly bounded. Moreover \( v^{(\varphi)}_W \) is also locally uniformly bounded since \( v^{(\varphi)} \) is concave and locally Lipschitz. So there exists a subsequence \( v^{(\varphi_n)} \) which will converge to a function \( \mathcal{J} \) locally uniformly on \((0, \infty)\). In order to show that \( \mathcal{J} \) coincides with \( J_0 \), we need the stability properties of viscosity solutions. We record the following Lemma from Lions (1983):

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Lemma 5 Let $\varepsilon > 0$, $F^\varepsilon$ a continuous function from $R^+ \times R \times R \times R$ to $R$ and let $J^\varepsilon$ be viscosity solution of $F^\varepsilon(W, J^\varepsilon, J^{\varepsilon'}, J^{\varepsilon''}) = 0$ in $[0, \infty)$. We assume that $F^\varepsilon$ converges locally uniformly on $R^+ \times R \times R \times R$ to some function $F$ and that $J^\varepsilon$ converges locally uniformly on $[0, \infty)$ to some function $J$. Then $J$ is a viscosity solution of $F(W, J, J', J'') = 0$ in $[0, \infty)$.

So according to Lemma (5), we have $\mathbf{J}$ is the unique viscosity solution of (17). On the other hand, the value function $J_0$ is also a viscosity solution of (17). Therefore $\mathbf{J} = \lim_{\varphi_n \to 0} v(\varphi_n) = J_0$ which leads to $\lim_{W \to \infty} J(W) = J_0(W)$. Hence $J_0(W) \leq \lim_{W \to \infty} J(W) \leq J_0(W)$.

(ii) The result directly follows the fact that $J_W$ is also locally uniformly bounded.

(iii) Taking limit on both sides of the HJB equation (17), the converges of $J_{WW}$ is straightforward.

Q.E.D. $\blacksquare$

The convergence of $J, J_W, J_{WW}$ implies that not only the asymptotic behavior of value function $J$ converges to that of $J_0$, but also all the interesting variables which depend up to second derivative of $J$ will converge to those variables in the standard CIR economy. For example the shadow default-free (instantaneous) interest rate $r(W)$ and the optimal consumption policy will satisfy:

$$\lim_{W \to \infty} r(W) = \mu - A\sigma^2$$

$$\lim_{W \to \infty} \frac{c(W)}{W} = k$$

(20)

The result is important in the sense that it provides a formal proof that the economy that we are modelling approaches the classic general equilibrium production economy with no default as the wealth approaches infinity.
We also show that in the limit as the economy approaches the default boundary, the consumption policy and the default-free interest rates can be solved in closed form when the borrower is identical to the lender.

From the boundary condition \( J(W^*) = J_0(\alpha W^* - K) \) and the “smooth pasting” condition \( \lim_{W \to W^+} J'(W) = \frac{\partial J_0(\alpha W^* - K)}{\partial W} \), we can characterize the limiting behavior of \( r(W) \) and \( c(W) \) when wealth level is close to default:

\[
\begin{align*}
\lim_{W \to W^+} r(W) &= \frac{2\rho}{1 - A} \frac{W - K/\alpha}{W} - \mu + \frac{2C}{1 - A} \alpha^{1 - \frac{1}{A}} k \frac{W - K/\alpha}{W} \\
\lim_{W \to W^+} c(W) &= \alpha^{1 - \frac{1}{A}} k \left( \frac{W - K}{\alpha} \right)
\end{align*}
\]

(21)

With the value function and the optimal consumption rule determined, we can specify the borrower’s valuation of the loan next. When \( W \leq W^* \) the borrower will default and the value of the loan will simply be the leftover upon default: \((1 - \alpha) W - K\). When \( W > W^* \), the future payments for the loan can be summarized as: \( \mathcal{C} \), \( s < \tau \); and \((1 - \alpha) W^* - K, s = \tau \). Given the smoothness of the value function \( J \), the borrower’s value for such a loan at time \( t \) can be expressed as the expectation of the product of its future payoff, a time-discount factor \( e^{-\rho(s-t)} \) and a risk adjustment factor \( \frac{J(W^*, s)}{J(W, t)} \).

The existence and uniqueness of the valuation is guaranteed by the dynamic completeness of the market. In particular, when \( W > W^* \) the borrower will value the loan at time zero by:

\[
I(W) = \mathbb{E}_t \left[ \int_t^{\tau} e^{-\rho s} \frac{J(W(s,s), s)}{J(W(t), t)} \mathcal{C} ds + e^{-\rho \tau} \frac{J(W^*, \tau)}{J(W, t)} \{ (1 - \alpha) W^* - K \} \right] \tag{22}
\]

It can be shown that \( I(\cdot) \) also satisfies the following ordinary differential equation for \( W \geq W^* \):

\[
-r(W) I + (r(W) W - c^*(W) - \mathcal{C}) I_W + \frac{1}{2} \sigma^2 W^2 I_{WW} + \mathcal{C} = 0 \tag{23}
\]

From standard differential equation theory, for example Krylov (1987), we know that the uniformly elliptic ODE in (23) has a unique \( C^2(W^*, \infty) \)
class solution $I(\cdot)$. Combining this with fixed-point equation (7), we are able to determine the equilibrium borrowing amount at time zero $I^*_0(x_0|\overline{C}, \alpha, K)$ for a specific choice of $(\overline{C}, \alpha, K)$. The following theorem provides a formal proof for the existence of such a fixed point $I^*_0$.

**Theorem 6** For any level of initial endowment $x_0$, there always exists a $I^*_0$ associated with $(\overline{C}, \alpha, K)$, where $\overline{C} \geq 0$, $0 < \alpha \leq 1$ and $0 \leq \frac{K}{\alpha} < x_0$ such that $I^*_0$ satisfies equation (7).

**Proof.** First of all notice that $\forall W \geq W^*$ the function $I(\cdot)$ satisfy ODE (23) which is uniformly elliptic, hence $I(\cdot)$ is continuous on $[W^*, \infty)$. For $\frac{K}{\alpha} < W < W^*$ we have $I(\cdot) = (1 - \alpha)(\cdot - \frac{K}{\alpha})$ which is continuous. So $I(\cdot)$ is continuous on $(\frac{K}{\alpha}, \infty)$ which implies that $I(\cdot + x_0)$ is continuous on $(0, \infty)$ for $x_0 > \frac{K}{\alpha}$. Secondly, $I(\cdot)$ also satisfy the expectation form (22), hence $I(\cdot) > 0$. Also $I(\cdot)$ satisfies the boundary condition $\lim_{W \to \infty} I(W) = \frac{\overline{C}}{\overline{\pi}}$.

Now define function $g(x) = I(x_0 + x) - x$. Noting $g(\cdot)$ is also continuous and we have \[ \begin{cases} g(0) = I(x_0) > 0 \\ g(\infty) < 0 \end{cases} \], the existence of such a fixed point $I^*_0$ immediately follows. \[ \square \]

### 6.2 Numerical Solution to HJB Equation (17)

In this section we will describe the numerical procedure to solve the HJB equation (17). The approach will be backward induction: we first solve the two-person general equilibrium after default to get the borrower’s value function upon default; and then we input it as the boundary condition to solve the HJB equation (17).

For our baseline setting where the borrower and the lender are identical, the solution to the two-person general equilibrium problem after default is trivial: the borrower’s valuation function after default $J_B$ will simply be his valuation function in autarchy $J_0$. For the case when the lender has a
logarithmic utility, there will be no closed form solution for $J_B$. Following Dumas (1989), we first solve the central planner's problem who maximizes the welfare function which is a weighted average (with constant weight $\lambda$) of each individual’s utility function. The welfare optima will specify the wealth-sharing rule between the two agents. Because we also know the share of wealth of the borrower and the lender upon default, we are then able to determine the constant weight $\lambda^*$ for a given default boundary $W^*$ through the fixed-point requirement [As equation (18) in Dumas (1989)]. Using this particular weight $\lambda^*$, we then determine the borrower’s value function upon default $J_B(\alpha W^* - K)$ associated with such a default boundary $W^*$.

Once we have determined the borrower’s valuation function upon default, we will then use a finite difference scheme analogous to policy iteration to solve the HJB equation (17). First of all, for a fixed critical default boundary $W^*$ we introduce a discrete grid $\{W_0, W_1, W_2, \ldots, W_N\}$. The low boundary $W_0$ is set to $W^*$ and the upper boundary $W_N$ is an artificially chosen large number, hence the grid size $h$ will be $\frac{W_N - W^*}{N}$. A finite-difference approximation for $J_W$ and $J_{WW}$ is:

\[
J_i W = \frac{J_{i+1} - J_{i-1}}{2h}, \\
J_{iWW} = \frac{J_{i+1} - 2J_i + J_{i-1}}{h^2}, \quad i = 1, \ldots, N - 1
\]

We impose two Dirichlet boundary conditions: $J(W_0) = J_B(\alpha W^* - K)$ and $J_N = \frac{k-A}{1-A}W_N^{1-A}$. The second one comes from the asymptotic property of $J$. An alternative Neumann boundary condition $J_W(W_N) = 0$ has also been applied to check the robustness of our result. We conclude that these two boundary conditions leads to exact identical result except for very large wealth level close to upper boundary $W_N$.

We adopt the following “policy iteration” algorithm to solve the non-linear equation (17):
Step(0). First we guess an initial $J_i^{(0)}$. For example we can take the standard CIR value function $J_0(W)$ as the initial form of $J(W)$, i.e. $J_i^{(0)} = \frac{k-A}{1-A}W_i^{1-A}, \ i = 1, \ldots, N - 1$. Hence the initial policy $C_i^{(0)}$ is given by:

$$(u')^{-1}(J_i^{(0)}) = \left(\frac{J_i^{(0)} - J_{i+1}^{(0)}}{2h}\right)^{-\frac{1}{A}}, \ i = 1, \ldots, N - 1.$$

Step(k). Let $J^{(k-1)}$ denote the solution of k-th step of the iterative procedure and $C^{(k-1)}$ is the corresponding optimal policy where $C^{(k-1)} = (u')^{-1}(J_i^{(k-1)})$. Then $J^{(k)}$ is computed as a solution of the tridiagonal system:

$$
\rho J_i^{(k)} + \left(\frac{J_{i+1}^{(k)} - J_{i-1}^{(k)}}{2h}\right)(C_i^{(k-1)} - \mu W_i) - \frac{1}{2}\sigma^2 W_i^2 \left(\frac{J_{i+1}^{(k)} - 2J_i^{(k)} + J_{i-1}^{(k)}}{h^2}\right)
= u(C_i^{(k-1)}), \quad i = 1, \ldots, N - 1
$$

$J_0^{(k)} = J_B(\alpha W^* - K), \ J_N^{(k)} = kW_N^{1-A}$

The iteration procedure is repeated until $\max_i |J_i^{(k)} - J_i^{(k-1)}| < \varepsilon$, where $\varepsilon$ is the desired tolerance level.

After finishing the “policy iteration”, compute the error for “smooth pasting condition”:

$$error(W^*) = \left| \frac{J_1 - J_0}{h} - \frac{\partial J_B(\alpha W^* - K)}{\partial W^*} \right| \quad (25)$$

The optimal default level $W^*$ is determined by a line-search for the minimum of $error(W^*)$. The grid size $h$ is chosen small enough such that the finite-difference scheme is no longer sensitive to $h$. 
7 Reference


Notes

1 We thank the participants in seminars presented at Boston College, Carnegie-Mellon University, Columbia University, University of Chicago, International Monetary Fund, The Federal Reserve Bank of New York, Joint Columbia-New York University Workshop, London Business School, Princeton University, University of Maryland and University of Texas at Austin for their comments and suggestions. We thank Raghu Sundaram and the referee for many insightful comments and for alerting us to related contributions.

2 We discuss later the conditions under which equity will be used instead.

3 We thank the referee for bringing this paper to our attention.

4 We restrict attention to a constant opportunity set to get tractable results. The generalization to a stochastic opportunity set introduces significant computational complexity.

5 We would like to point out that all of our major results still hold for a general von Neumann-Morgenstern utility function $u$: which is a strictly increasing, strictly concave $C^3(0, +\infty)$ function with $\lim_{c\to 0} u'(c) = +\infty$ and $\lim_{c\to +\infty} u'(c) = 0$ and satisfies the condition $|u(c)| \leq M(1 + c)^\gamma$ for some positive constants $M$ and $\gamma$.

6 We do not consider dynamic borrowing opportunities. This implies that the borrower has no “reputational costs” associated with default. We will show that the consumer reduces the rate of consumption in poor states of the economy to stave off default. This may be interpreted as a “dynamic borrowing” action. We thank Patrick Bolton for pointing out this interpretation.

7 Given a $\overline{C}$ we will determine $I_0$ endogenously. We can equivalently take
exogenously and find the coupon level \( \bar{C} \) endogenously.

\[ I_0 \text{ in this sense, the risky production technology effectively serves as collateral to borrow money from the lender. This type of modeling has been used in the context of credit cycles by Kiyotaki and Moore (1997) and Krishnamurthy (1998).} \]

\[ \text{For the setup we have chosen (i.e. a constant opportunity set and a power utility function), Merton (1971) has shown the following results for autarky: if} \quad k = \frac{1}{T} \left[ \rho - (1 - A) \left( \mu - \frac{\sigma^2}{2} \right) \right] > 0, \text{ then the value function } J_0(W) \text{ will be given by: } J_0(W) = \frac{k-A}{1-A} W^{1-A} \]

\[ \text{It should be emphasized that the computational burden associated with solving this problem by backward induction is nontrivial: we have to solve the two-person general equilibrium model after default for every wealth level in order to determine the optimal default boundary.} \]

\[ \text{When the lender’s utility is log and thus is different from the borrower, the risk aversion results will be somewhat muted. In general for a lender who is different from the borrower, there will be active trading between the borrower and the lender after default. This leads to welfare gains to both the lender and the borrower. As a consequence, the borrower’s effective cost of default will be reduced and his optimal policy before default will be different. One would expect that for a different lender, the borrower’s relative risk aversion before default will become lower and the default boundary } W^* \text{ will become higher.} \]

\[ \text{Let us denote } P(t, T) \text{ as the price at time } t \text{ for a zero coupon bond which pays one unit consumption good at time } T. \quad P(t, T) \text{ will satisfy the following partial differential equation (PDE) with boundary conditions } P(W^*, \tau) = \]

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\[ e^{-R(T-r)}, P(W,T) = 1: \]
\[ -r(W)P + (r(W)W - c^*(W) - \overline{C})P_W + \frac{1}{2}\sigma^2 W^2 P_{WW} + P_t = 0 \quad (26) \]

13 We note that when the lender has a logarithmic utility and thus is distinct from the borrower, the borrower’s consumption policy is less conservative due to the possibility of more risk-sharing after default.

14 In the collateral dissipation region, the effects will be different for reasons that were discussed earlier.

15 Here we have assumed the existence of an interior equilibrium. The statement follows Theorem 1 of CIR (1985).

16 A one-dimensional differential equation is said to be uniformly elliptic if the coefficient of the second-order derivative \(a_{22}\) satisfies \(0 < \alpha_1([a,b]) \leq a_{22} \leq \alpha_2([a,b])\) for any interval \([a,b]\) where \(\alpha_1\) and \(\alpha_2\) are two constants depending only on \([a,b]\).