A Model of Dynamic Sovereign Borrowing
Effects of Credit History and Sanctions

Ganlin Chang
gc61@columbia.edu

and
Suresh M. Sundaresan
ms122@columbia.edu

Graduate School of Business
Columbia University
New York City
New York 10027

First draft: August 1999
Revised: January 2001
Abstract

A dynamic model of sovereign borrowing is used to examine optimal consumption and default strategies of sovereign borrower. Potential sanctions and seizable collateral constitute the basic economic ingredients that drive sovereign lending. Furthermore opportunity to borrow again in the future and arrears from past indebtedness also play an important role in determining sovereign yield spreads and the borrowing country’s optimal policy: in this sense a country’s credit history matters in sovereign lending. In our framework, more severe the expected sanctions are, greater is the initial loan amount (for the same level of promised coupons) and lower are the sovereign spreads. Increased collateralization not only lowers the spreads and increases the initial loan amount but also reduces the risk shifting incentives of the sovereign borrower. Higher level of arrears, cetaris paribus, leads to higher spreads and less conservative consumption and default policies for the sovereign borrower.

1 Introduction

The last decade has witnessed an enormous growth of net capital inflow to emerging markets. Among them, sovereign debt (including both loans and bonds) is the major part of the inflow and constitutes a major source of capital financing for less developed countries. The following table illustrates the net capital inflow to emerging market in 1990s’. It clearly shows the
important role that sovereign debt plays in the international capital markets.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Total</td>
<td>75.4</td>
<td>82.4</td>
<td>129.5</td>
<td>144.0</td>
<td>165.3</td>
<td>228.5</td>
<td>297.2</td>
<td>157.4</td>
<td>178.5</td>
</tr>
<tr>
<td>Equities</td>
<td>5.6</td>
<td>7.2</td>
<td>11.9</td>
<td>18.0</td>
<td>11.2</td>
<td>16.4</td>
<td>26.2</td>
<td>9.4</td>
<td>23.2</td>
</tr>
<tr>
<td>Loans</td>
<td>55.9</td>
<td>50.7</td>
<td>54.9</td>
<td>69.4</td>
<td>106.5</td>
<td>110.2</td>
<td>137.8</td>
<td>67.7</td>
<td>68.4</td>
</tr>
<tr>
<td>Bonds</td>
<td>13.9</td>
<td>24.3</td>
<td>62.7</td>
<td>56.5</td>
<td>57.6</td>
<td>101.9</td>
<td>133.2</td>
<td>80.2</td>
<td>87.0</td>
</tr>
</tbody>
</table>

Table 1. Gross Private Market Financing to Emerging Market
(Gross, US $ billions)

Source: IMF (2000)

Unlike other types of debtors, a sovereign debtor is unique in the sense that historically sovereign has the right which is absolute in nature. It has the power to do many things in a state: - to make laws, to impose and collect taxes, to form treaty with foreign nations, and the like. To better understand the uniqueness of a sovereign loan (or bond), we begin our analysis by briefly comparing the sovereign lending with domestic corporate lending.

1.1 Sovereign Lending vs. Corporate Lending

Sovereign lending is distinguished from domestic corporate lending in fundamental respects. The major difference between them is the legal system which regulates the relationship between the borrower and the lender in the event of default. The domestic corporate debt is an enforceable contract within the rubric of a bankruptcy code. Whenever the borrower fails to deliver in a timely fashion full contractual amount, the lender has the right to initiate action against the borrower under a specified bankruptcy code. The crux of this bankruptcy code is to provide a formal access to the assets of the borrower. The sovereign debt on the other hand may not necessarily be a legally enforceable contract. In the event of default, the ability of an
international court to enforce settlement is severely limited due to the nature of sovereignty. Furthermore, there is little collateral that the sovereign lender can control to secure the value of the debt. Only those assets inside the creditor’s border (for example, the liquid assets in the creditor country’s bank, the export shipment at creditor country’s harbor) may be seized by the lender in the event of default. Such assets are typically worth only a small fraction of the total outstanding sovereign debt. One frequently mentioned possibility is to apply liens to export. However enforcement remains a problem. If the borrowing country successfully exports goods to a third country, the creditor may still be unable to obtain possession of exports on the third country’s soil. For example, the creditors of the Polish “Copper Loan” of 1977 which was supposed to be secured on export earnings, could not obtain favorable treatment in the Polish debt restructuring. According to UNCTAD, 83% of the total external debt of all less developed countries in 1997 are essentially non-collateralized.

Another difference between sovereign lending and domestic corporate lending is the lengthy and costly restructuring process in sovereign debt crisis. The process for resolving a corporate default is subject to far less uncertainties because the domestic bankruptcy code (for example, the Receivership code in UK and the Chapter 11 in US) makes the renegotiation a well defined processes. Essentially, the bankruptcy code provides incentives for the debtors and its creditor to renegotiate contracts and reach a voluntary agreement on the terms of restructuring. Provisions such as the “Automatic Stay” and “Super-Majority Voting” in Chapter 11 are designed to encourage an efficient renegotiation between the debtor and creditor. In contrast, there is no such judicial authority to supervise the whole process in sovereign setting. Sovereign debt recontacting is still largely based on achieving consensus without any credible outside options such as liquida-
tion or bankruptcy as in corporate debt.

In the sovereign loan markets, there are two informal mechanisms that are used for rescheduling debt. The London Club is an informal network of commercial banks which reschedule the debt claims of sovereign borrowers. The Paris Club is an analogous institution consisting of members who are lender-countries which reschedule their loans to sovereign borrowers. The difficulty with the process of obtaining consensus is that there can be “holdouts”. Schwarcz (2000) cites an example of a Brazilian debt restructuring in which the fourth largest creditor held out leading to a litigation that lasted over a decade. The Paris Club also attempts to maintain the paradigm of “comparable treatment” by committing its members to seek debt relief from private creditors that is as generous as the official relief granted by the Paris Club. Often, in sovereign debt market IMF has tended to aid the process of restructuring. The presence of IMF lending can lead to moral hazard problems as the borrowers anticipating an IMF-led bail out may engage in excessive risk-taking and excessive leverage.

The rapid growth of the sovereign bond markets in relation to the sovereign loan markets and the conversion of loans to Brady bonds has further exacerbated the collective action problem of reaching an agreement amongst creditors. Haldane (1999) notes that Collective Action Clauses (CAC) are used in several sovereign bond contracts. CACs provide an ex-post coordinating mechanism for bondholders. The purpose of CACs is to bind the creditors together so that “holdout” problems are mitigated. Haldane (1999) notes that just under 50% of the stock of emerging market Eurobonds issued under the UK law include CACs. This is in sharp contrast to the bonds issued under the US law which do not have CACs. Unlike a corporate bankruptcy code there are no “automatic stays” or “standstill” agreements in the sovereign debt markets. Miller and Zhang (1999) have argued for the
need to enforce payments standstill.

1.2 Literature Review on Sovereign Debt

The absence of a well established bankruptcy code, the limited access to borrower’s collateral and a lengthy, costly restructuring process drive a wedge between sovereign debt and corporate debt. This wedge implies that there should be some incentives aside from moral obligation to make the sovereign borrower repay his contractual obligations. Much of the existing sovereign debt literature is devoted to identifying such incentives so as to explain why sovereign debt markets exist at all.

One stream of research points to the need for the borrower to access the credit markets repeatedly as a motivation for sovereign borrower to repay their debts. In these models, the sovereign borrower will be excluded from the world credit market once they repudiate their debt obligations. Hence to maintain access to future loans, they have to develop a “good reputation” by repaying the existing debt. This stream of research was pioneered by Eaton and Gersovitz (1981). They model a utility maximizing country which uses international credit market to share the risk of stochastic domestic output. Default on existing debt in their model will cause the country be excluded from future borrowing and the country must use costlier alternative to offset the fluctuation in domestic output (such as stockpiling). They show that a positive lending equilibrium can be sustained by such a reputation incentive under certain conditions. Similar to Eaton and Gersovitz (1981), Grossman and Van Huyck (1988) also model the risk sharing through sovereign debt using the argument of reputation. In their model, the lender can distinguish the “excusable” default, which is associated with implicitly understood contingency, from the “non-excusable” default. Although reduced payments (partial default) do occur along the equilibrium path, they are not consid-
erred as violation of the initial agreement and the lenders do not exclude the sovereign country from future borrowing. The total (excusable) default happens only at the very bad states. Atkeson (1991) extended the work of Grossman and Van Huyck (1988) by modeling the asymmetric information between the borrower and the lender, under which the latter can not monitor the former’s activity. Along the equilibrium path, a bad state is always associated with a punishment phase in order to preserve the borrower’s incentive not to cheat in the future. There are other papers which examine the reputation as the repayment incentive [see for example Cole, Dow and English (1995) and Manuelli (1986)].

Papers based on reputation explicitly and implicitly assume that the sovereign country is unable to enter into another financial agreement after it defaults on the initial debt service. Relaxing such an assumption might destroy the self-enforcing equilibrium based on reputation. Bulow and Rogoff (1989b) argue that if a sovereign country is able to enter a particular cash-in-advance contract, which is irrespective of its behavior regarding its debt contract, no positive equilibrium can be sustained if the sole threat by the lender is cutting off future loans. In another seminal paper, Bulow and Rogoff (1989a) argue that instead of the reputation for repayment, it is the threat of economic and political sanction held by the lender that enforces a positive lending equilibrium. Relying on a bargaining-theoretic approach, they show that positive debt is sustainable and lenders will impose a debt ceiling on the sovereign country. The intuition conveyed by their theory may be summarized as follows: the borrower cannot commit full repayment \textit{ex-ante} and the lender also cannot commit to sanctions \textit{ex-post}. What the lender will do whenever borrower defaults is to salvage more value from the borrower by renegotiation. In a full information world, lenders will limit the amount of credit and will enforce a debt ceiling on the borrower.
Unfortunately, there is limited empirical evidence regarding whether the enforceability of sovereign debt contracts is by sovereign borrower’s desire for good reputation or through the potential sanction threat held by the lender. From the empirical perspective, the answer to the basic question: “Why do sovereign debtors ever repay their debts?” is still far from conclusive. On one hand, Conklin (1998) finds extensive evidence consistent with sanction-based theory such as Bulow and Rogoff (1989b) in his analysis of the lending by a Genoese-led cartel to Philip II of Spain (1556-1598). Using literature on sovereign debt to interpret history events, Conklin (1998) shows that the king tried to renege on his debt and the Genoese applied an additional penalty to enforce their claims, and the king ultimately repaid. According to the author’s estimate, the King’s observed debt ceiling, cost of enduring the penalty and its ability to repay are all consistent with predictions of Bulow and Rogoff (1989b). On the other hand, English (1996) reached the opposite conclusion based on his analysis of American States default during the years 1840’s. He shows that even though eight States defaulted between 1841 and 1843, most of them chose not to do so despite the fact that creditors could not enforce repayment by imposing military or trade sanctions. He concludes that states repaid their debts in order to maintain their reputation in debt markets.

Although it still remains an open question for the borrowing country’s repayment incentive, many empirical researches do agree that a country’s credit history has a significant impact on its borrowing cost. By studying the U.S. capital market and foreign lending in 1920 - 1955, Eichengreen and Portes (1987) noticed that compared to countries which maintained debt service throughout, countries which defaulted in the 1930s were no less able to borrow in the 1940s and 1950s. However those countries which had a history of default had to pay a higher interest rate in order to access the
capital market. Eichengreen and Mody (1998) reached a similar conclusion in a study of 1,000 developing-country bonds issued in the years of 1991-1996. They noticed that the history of rescheduling has a weak positive effect on the probability of an issue while significantly increases the spread that successful issuers are forced to pay. Canton and Packer (1996) also found that of the large number of criteria used by Moody’s and Standard and Poor’s in their assignment of sovereign rating, a sovereign country’s credit history, along with other five factors (per capita income, GDP growth, inflation, external debt and economic development) plays a significant role in determining a country’s rating. All these papers suggest that although the past repayment record may have little to do with a country’s ability to borrow, investors do demand a higher premium for the debt issued by the counties with bad credit history.

As we have noticed, the sovereign debt literature has primarily focused on the repayment incentive of the borrower and the existence of a positive lending equilibrium. Unlike the literature in corporate debt\(^2\), there are surprisingly few papers which focus on the pricing of sovereign debt\(^3\). Most papers in sovereign debt literature assume an exogenous interest rate and do not characterize the sovereign yield spread over risk free rate. The influential work by Eaton and Gersovitz (1981) and Bulow and Rogoff (1989a) also make the same assumption.

The purpose of this paper is two fold. First, we want to endogenously determine the sovereign yield spreads along with the borrowing country’s optimal behavior. In particular, we want to see how the institutional nature of sovereign debt will affect the sovereign yield spreads and the borrowing country’s optimal behavior. Second we also want to present a model of dynamic borrowing and show how a country’s credit history affects its borrowing cost and optimal policy. By doing so, we are able to explicitly
examine the role of “credit history” in sovereign lending. We focus on is a small country which can be treated as a risk averse representative agent who maximizes its’s life time expected utility from consumption. The country is endowed with a risky production technology and wants to enter the world credit market to expand its' investment scale and increase it’s life time expected utility. We adopt the insights of Bulow and Rogoff (1989a): the essential repayment incentive for the sovereign borrower is the potential political or economical sanctions held by the lender. We explore the effect of “credit history” by allowing the sovereign country to borrow multiple rounds in the world credit markets. In particular we characterize a special situation where the borrower can optimally borrow one more round after signing the initial debt contract. One should note that such a setup is very different from those traditional reputation based models such as Eaton and Gersovitz (1981). In our setup, the country with bad credit history will not be totally excluded from the capital market, instead they are still allowed to borrow again. The exception is that the arrears from debt service in the past will have an effect on their cost of borrowing. The wealth of the country is not a sufficient state variable. The wealth together with the arrears constitute the state of the economy for the borrower and the lender. We hope that such a treatment of “reputation” will be more in line with the stylized fact which we described earlier.

We simultaneously determine the optimal consumption (investment) strategy and optimal default strategy for the sovereign borrower by solving his intertemporal optimal consumption problem. We characterize the value of the risky sovereign loan and derive the optimal debt level for the borrowing country. The optimal consumption behavior exhibits an interesting pattern depending on how close the sovereign country is to default and how much collateral the lender is able to seize. When the sovereign country experiences
a sequence of negative shocks and moves closer to the default boundary, it normally cuts its consumption rate in an effort to reduce the probability of costly default. When the country is very close to default however, it may actually gear up the consumption rate. This behavior is akin to the “over investment” problem in the corporate finance literature. It may cause a higher \textit{ex-ante} yield spread and thus make the risky debt a less efficient contract for the borrower. We are able to show that by improving collateralization, it may be possible to mitigate the “over investment” problem and increase the efficiency of the \textit{ex-ante} loan contract. Our model also sheds some light on what “credit history” means in a sovereign debt contract. The borrower’s credit history and the option to borrow again in the future will affect the borrower’s optimal consumption rate and the optimal default boundary, and in turn affect the valuation of the loan. We find that borrowers who have good credit history tend to be more conservative in their consumption and are more cautious about default. We examine how sanctions, levels of accessible collateral, arrears and risk free rates affect the valuation of sovereign debt contract, the borrower’s optimal behavior and the efficiency of the \textit{ex-ante} contract. Our model has several implications on \textit{ex-ante} sovereign contract design and \textit{ex-post} efficient bankruptcy procedure.

The paper is organized as follows: Section 2 describes the setup of our model. In section 3 we characterize the major results. Section 4 wraps up the paper and discusses some possible future extensions.

2 The Model

We focus on a small country (sovereign borrower) who cannot affect the world risk-free interest rate. The country is assumed to maximize life-time
discounted expected utility of consumption:

\[ E_0 \left[ \int_0^\infty e^{-\rho t} u(c_t) dt \right] \quad (1) \]

where \( \rho \) is the rate of time preference and the utility function \( u(.) \) is strictly increasing and concave. In this paper, we will focus on a special class of utility functions as specified below:

\[ u(c) = \frac{1}{1 - \gamma} e^{\gamma - 1}, \quad \gamma > 0, \quad \gamma \neq 1 \quad (2) \]

We can think of a “representative agent” in the economy who represents the aggregated utility of its citizens\(^5\). There is a single commodity in the country which is the numeraire. The country has access to a risky production technology. When an amount \( W_t \) of the good is invested in the technology at time \( t \), it will evolve as a diffusion process:

\[ \frac{dW_t}{W_t} = \mu_t dt + \sigma_t dz_t \quad (3) \]

The process \( z_t \) is a standard Brownian Motion on the underlying probability space \((\Omega, \mathcal{F}, \mathcal{P})\) which models all the uncertainty in the economic environment of the country. The risky production technology summarizes all the investment opportunities for the country, including both domestic productions and international trade gains with other countries. We do not explicitly model the exchange rate in our model. All the terms in our model are real and measured in the unit of the consumption good. The fluctuation of exchange rate can be interpreted as implicitly embedded in the exogenous shock \( dz_t \). For example a decline in the price of oil would be expected to cause a significant depreciation of an oil-exporting country’s currency, which will be regarded as an exogenous negative shock \( dz_t \) to the country’s production technology. For simplicity, the instantaneous expected rate of return \( \mu \) and the diffusion coefficient \( \sigma \) are assumed to be exogenous positive constants. Hence the country is endowed with a constant opportunity set which will not be affected by its’ debt level, exchange rate etc.\(^6\)
If there is no borrowing and lending, the country will be in autarchy and the whole problem will be reduced to Merton’s (1971) intertemporal optimal consumption problem. At each instant, the representative agent decides the optimal consumption rate \( c^* \) and invest the rest into risky technology. It is well understood that for the case of power utility and constant opportunity set, the optimal consumption \( c^* \) and the value function \( J_0(.) \) has the form:

\[
J_0(W) = \frac{\lambda^{-\gamma}}{1-\gamma} W^{1-\gamma}, \\
c^*(W) = \lambda W. 
\] (4)

where the constant \( \lambda = \frac{1}{\gamma} [\rho - (1-\gamma)(\mu - \frac{\sigma^2}{2})] \).

In our model the sovereign country has access to the world credit market. Under very mild conditions (for example, a high growth investment opportunity: i.e. a relative large \( \mu \)), the country has an incentive to borrow from world credit markets and expand the scale of its investment. In our paper we assume that the country can borrow from abroad by signing a specific debt contract of the form \( \{C^{(1)}, I|_{C^{(1)}}\} \) at time zero: where \( C^{(1)} \) is the perpetual flow rate coupon the debtor country required to pay and \( I|_{C^{(1)}} \) is the loan amount associated with this coupon rate. The loan amount \( I|_{C^{(1)}} \) will be endogenously determined\(^8\). Suppose the country’s initial wealth level is \( x_0 \), then after signing the loan contract it will augment its’ wealth level to \( W_0 = x_0 + I|_{C^{(1)}} \). In our paper we assume that there exists a competitive credit market and a constant risk-free world interest rate \( r \). There are many competitive rational investors in the capital market who are willing to provide the loan \( \{C^{(1)}, I|_{C^{(1)}}\} \) at a “fair rate”. If contracts are perfectly enforceable and there is no uncertainty about the future repayment, such a loan is simply a riskless loan and its’ “fair rate” will be the world interest rate \( r \). However given the nature of sovereign debt: complete information, costless contract enforcement and negligible negotiating costs are unrealistic
assumptions. Hence the key question our model will answer is how to determine the “fair rate” for the sovereign loan when some of these assumptions are relaxed.

To ensure a non-trivial borrowing equilibrium, we follow the arguments of Bulow and Rogoff (1989a). The primary motivation for repayment is the threat of direct sanctions that the lender can impose on the sovereign entity. In our paper we assume that the creditor possess the ability to penalize the sovereign debtor if the country unilaterally repudiates its debt: the lender can impose political or economic sanctions on the sovereign borrower. Such sanctions will adversely affect the sovereign country’s ability to trade in the world market and will make the sovereign borrower less likely maintain its growth rate. Specifically we assume once the creditor impose the sanctions, the growth rate of the debtor country will drop from $\mu$ to $\hat{\mu}$. In general, the effect of the sanctions $\Delta \mu = \mu - \hat{\mu}$ will depend on a lot of factors: the sovereign country’s need to import technology, the export exposure to the creditor, etc. For example a country whose economy heavily relies on a single commodity export may be expected to experience a more severe damage from sanctions than a country who has a more diversified industrial structure; a creditor country who has a close economic relationship with a borrower (like United States and Mexico) will have a more credible threat of sanctions on the borrower.

In addition to the sanctions which the lender can impose, we further assume that the lender can seize a fixed amount of asset $K$ from the sovereign country in the event of default. Such an asset can be interpreted as the accessible collateral to the lender. As we have discussed in the introduction, accessible collateral is rare in sovereign lending. Hence one would expect that $K$ will be a relatively small number compared to the value of the loan. The modeling of sanctions and accessible collateral follows the spirit of Gibson
and Sundaresan (2000).

A rational lender may not impose penalties when default occurs, instead he will use the threat to extract part of the repayment from the sovereign borrower. Obviously the amount that the lender can extract from the borrower will depend on the bargaining process between these two parties and their outside options. Let us denote $S$ as the strategic debt service, which is the outcome of the renegotiation process. One can expect that $S$ depends on the credibility of the threat and the bargaining strengths of each party. We will characterize such a bargaining process in detail later. In our full information model, the sovereign borrower fully understands the actions that will be taken by the lender once he defaults. By taking these potential future actions into account, he will optimally choose the timing of default.

One should note that during this recontracting process, the strategic debt service amount $S$ is strictly dominated by the original contractual coupon payment $C^{(1)}$ since it is the sovereign borrower who optimally determines whether he will default or not. Otherwise the sovereign borrower would rather pay the contractual amount $C^{(1)}$ instead of bargaining with the lender. As a consequence, the sovereign borrower is accruing arrears in the renegotiation process. Let us denote the total accrued arrears at time $t$ be $A_t$. Such a state variable can be viewed as a proxy for a sovereign country’s credit history: a country which has never been defaulted will have $A = 0$ and a country who have bad credit history will carry positive arrears $A$. Combine with the country’s wealth level $W$, the pair of $(W, A)$ will fully describe the state of a sovereign borrower.

Besides the optimal default boundary, the borrower has another important decision to make in our model: when should he re-borrow again. During the initial contractual region, the borrower may experience a sequence of positive shocks to the economy and his wealth increase. At a certain point, the
sovereign country may have the incentive to borrow more and expand it’s investment opportunity. In our model we assume that the sovereign country will re-enter the credit market and sign a second loan contract \((C^{(2)}, I|_{C^{(2)}})\) with another creditor when the pair of state variables \((W, A)\) reaches a certain threshold. Our model is cast in a dynamic borrowing framework\(^{10}\).

Often sovereign loan contracts have a “negative pledge” clause which precludes the issuance of additional debt at the same or higher level of seniority without the consent of existing debt holders. Hence we assume that the new debt is *subordinated* to the old debt. Furthermore we make an important assumption regarding the re-borrowing process: *in order to re-sign such a junior loan contract, the sovereign country must fully (or at least partially) clear the arrears \(A\) to the senior lender.* A very important implication of such a dynamic borrowing framework is that it enables us to explore the “reputation effect” in sovereign lending, where models with one shot static borrowing opportunity are unable to address. As noted earlier, there is ample empirical evidence which suggests that the country’s default history matters in determining its borrowing costs. Our setup implies that the re-borrowing opportunities are different for sovereign countries who carry different arrears. A higher arrears \(A\) implies a more costly re-borrowing opportunity for the sovereign borrower since he must fully (or partially) clear the higher accrued arrears. We will show that such a different re-borrowing opportunity will not only affect the sovereign country’s re-borrowing policy, but also affect its default policy, consumption policy and its loan spreads. We explore in detail this dynamic problem later.

Now let us focus on the borrower’s problem at the initial contractual region, where he pays a coupon rate of \(C^{(1)}\). In this region, the accrued arrears remains a constant and the dynamics of the state variables follows
the following process:

\[
\begin{cases}
    dA_t = 0 \\
    dW_t = \left[ \mu W_t - c_t - C^{(1)} \right] dt + \sigma W_t dz_t
\end{cases}
\]  

(5)

The sovereign country will optimally choose the consumption policy \( c^*_t(W, A) \) plus the default boundary \( W^B(A) \) and the re-borrowing boundary \( W^U(A) \) to maximize its lifetime expected utility. From Chang and Sundaresan (2000), it can be shown that the sovereign country’s valuation function \( J(W, A) \) is the unique \( C^2 \) class solution which satisfy the following HJB equation:

\[
pJ = \frac{1}{2} \sigma^2 W^2 J_{WW} + (\mu W - C^{(1)}) J_W + \max_c \left( u(c) - c J_W \right)
\forall \ W^U(A) > W > W^B(A)
\]  

(6)

with lower boundary conditions:

\[
J(W^B, A) = J_B(W^B, A), \quad \lim_{W \to W^B+} J_W(W, A) = \frac{\partial J_B}{\partial W^B}
\]  

(7)

and upper boundary conditions:

\[
J(W^U, A) = J_U(W^U, A), \quad \lim_{W \to W^U-} J_W(W, A) = \frac{\partial J_U}{\partial W^U}
\]  

(8)

where \( J_U \) [or \( J_B \)] is the corresponding value function after re-borrowing [or default]. The first part of the upper [lower] boundary condition is the “value matching” condition, while the second part is the so-called “smooth pasting” condition which ensures the optimality of the upper [lower] boundary \( W^U \) [\( W^B \)]. Typically for a moving boundary problem with two state variables, the “smooth pasting” conditions should involve derivatives of \( J \) up to the second order. However, since the state variable \( A \) remains a constant within the domain, the problem can be simplified as if we only have single state variable \( W \) and treat the arrears \( A \) as an exogenous parameter.
Furthermore the first order condition of the HJB equation yields that the optimal consumption $c^*$ can be written as a feed back form:

$$c^*(W, A) = (J_W)^{-\frac{1}{\tau}}$$ (9)

The problem will be solved through backward induction. First we solve the two sub-problems: the subproblem after re-borrowing and the subproblem after default. The corresponding value functions $J_U$ and $J_B$ are then used as boundary conditions for the HJB equation (6) at the initial stage. Hence we will first present the two sub-problems and then go back to the initial stage to present the main problem.

2.1 The Sub-Problem after Default:

Once the borrower default (i.e. he is unable/unwilling to meet the contractual obligation), the borrower and the lender will continuously bargain over the original debt service. Such a renegotiation process will continue until the sovereign borrower recovers from the economic distress and is willing to provide the contractual debt service again. In this renegotiation period, the sovereign borrower will pay a strategic flow rate $S(W, A)$, which depends on the outcome of the bargaining process. In our model we assume that the lender holds all the bargaining power. Such an assumption is made solely for analytical tractability. Hence the amount of strategic debt service $S(W, A)$ will make the sovereign borrower indifferent between accepting the penalty or continuing to serve the debt in a strategic manner. It can be shown that if the lender DOES impose the penalty at $(W, A)$, the sovereign borrower’s value function will be $\hat{J}(W - K)$ where $\hat{J} (\cdot)$ is the solution of the following HJB equation:

$$\rho \hat{J} = \frac{1}{2} \sigma^2 W \hat{J}_{WW} + \mu W \hat{J}_W + \max(u(c) - c\hat{J}_W)$$ (10)
Hence the sovereign country’s value function $J_B$ by serving the strategic debt service will equal to $\hat{J}(W - K)$:

$$J_B(W, A) = \hat{J}(W - K) = \frac{\hat{\lambda}^{-\gamma}}{1 - \gamma} (W - K)^{1 - \gamma} \quad (11)$$

where the constant $\hat{\lambda} = \frac{1}{\gamma}[\rho - (1 - \gamma)(\bar{\mu} - \frac{\sigma^2}{2})]$.

Meanwhile in this recontracting region, the dynamics of the state variables $(W, A)$ is given by:

$$\begin{align*}
\frac{dA_t}{dt} &= [C(1) - S(W, A)]dt \\
\frac{dW_t}{dt} &= [\mu W_t - c_t - S(W, A)]dt + \sigma W_t dz_t \quad (12)
\end{align*}$$

From dynamic programming theory, we know that the value function $J_B$ is also the solution of the following Bellman equation:

$$\rho J_B = \frac{1}{2}\sigma^2 W^2 J_{WW} + (\mu W - S(W, A))J_{BW} + \max(u(c) - cJ_{BW}) \quad (13)$$

Substitute (12) into (13), we get a closed form solution for the strategic debt service $S$:

$$S(W, A) = (\mu W - \frac{\gamma \sigma^2}{2} \frac{W^2}{W - K}) - (\bar{\mu} - \frac{\gamma \sigma^2}{2})(W - K) \quad (14)$$

and the optimal consumption $c^*(W, A)$ in the renegotiation period:

$$c^*(W, A) = \hat{\lambda}(W - K) \quad (15)$$

From expression (11) through (15), we can see that the borrower’s value function $J_B$, strategic debt service $S$ and optimal consumption rate $c^*$ in the renegotiation region are all independent of the state variable $A$. Such a result is not surprising once we understand that all the results are basically derived through the bargaining process which has nothing to do with the accrued arrears $A$ in our model. One should note that the bargaining process in sovereign debt setting tends to be lengthy and costly compared
to the renegotiation process in domestic corporate setting. To capture such a costly *ex-post* sovereign debt restructuring process, we assume that there is a constant flow of bargaining cost: $bc$. After taking account into such a deadweight cost, the net inflow of strategic debt service $\tilde{S}(W, A)$ for the lending party will be:

$$\tilde{S}(W, A) = S(W, A) - bc$$  \hspace{1cm} (16)

There is no guarantee that the net strategic debt service $\tilde{S}(W, A)$ will be always positive. For example when the wealth level $W$ drops close to $K$, net debt service payment $\tilde{S}(W, A)$ can become negative, which means that the lender has to inject fresh cash at very bad states to keep the solvency for the borrowing country. At this point, a rational lender will have no incentive to continue such a costly bargaining process. To rule out such “negative cash flow” for the lender, we assume that the lender will pull the penalty trigger for sanction at $W^*$ whenever the net proceeds $\tilde{S}(W, A)$ falls below zero\textsuperscript{12}. Hence by solving $\tilde{S}(W^*, A) = 0$ we get:

$$W^* = K + \frac{bc - \hat{r}K + \sqrt{(bc - \hat{r}K)^2 + 2K^2\gamma\sigma^2(\mu - \hat{\mu})}}{2(\mu - \hat{\mu})}$$  \hspace{1cm} (17)

where $\hat{r} = \mu - \gamma\sigma^2$.

Clearly, the trigger point $W^*$ depends on the bargaining cost $bc$, sanction effect $\mu - \hat{\mu}$, accessible collateral held by the lender $K$ and the volatility of the sovereign country’s production technology $\sigma$. The following table illustrates the comparative statics of $W^*$ with respect to such variables:

<table>
<thead>
<tr>
<th></th>
<th>Bargaining Cost: $bc$</th>
<th>Sanction Effect: $\mu - \hat{\mu}$</th>
<th>Accessible Collateral: $K$</th>
<th>Volatility $\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W^*$</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>

Table 2
The higher the bargaining cost, the less severe the expected sanction, the more collateral held by the lender and the more volatile the technology, the earlier will the lender enforce sanctions.

Next we show a result which illustrates an important special case where the lender does not hold any collateral:

**Sovereign Lending with no Collateral:** When the lender does not hold any collateral, the strategic debt service can be simplified as:

\[
\frac{S(W, A)}{W} = \mu - \hat{\mu}
\]

which is the difference between the expected return of the sovereign country’s technology with sanction and without sanction, i.e. the effect of the sanction. Furthermore the trigger point \(W^*\) when the lender imposes the sanction can be simplified as:

\[
W^* = \frac{bc}{\mu - \hat{\mu}}
\]

which is proportional to the bargaining cost and inversely related to the sanction effect. A costless bargaining process \((bc = 0)\) will keep the lender always engaged in negotiations \((W^* = 0)\).

2.2 The Sub-Problem after Re-Borrowing:

Suppose the sovereign country with arrears \(A\) wants to re-enter the credit market at wealth level \(W^U\). At this point the sovereign borrower must repay the senior lender an amount of \(\theta A\) \((0 < \theta \leq 1)\) in order to fully (or partially) clear its arrears. Such an amount of \(\theta A\) can be regarded as an equivalent re-borrowing cost for the sovereign country. The higher the arrears \(A\), the higher is the equivalent re-borrowing cost for the sovereign borrower. After the borrowing country clears the arrears, it will be able to sign a junior loan contract \((C^{(2)}, I_{[C^{(2)}})\) with the creditor and augment its’ wealth level from
\( W^U \) to \( W^U + I(\theta A) \). From then on the total outstanding contractual coupon rate will be \( C = C^{(1)} + C^{(2)} \).

In this sub-problem after re-borrowing, we assume that a cross-default provision is in effect whenever the sovereign falls into financial distress. Like “negative pledge” clause, “cross default” clause is also standard in sovereign loan contracts. Data on Euromoney Bondware database shows that during 1997 - 1998, 70% emerging bond issues has “cross default” clause while 77% has “negative pledge” clause. Similar to the problem at initial stage, the sovereign borrower will then optimally choose the boundary \( W^{II}_B \) at which he is unwilling to service the contractual coupon \( C^{(1)} \) and \( C^{(2)} \). In those bad realizations, we assume that all the creditors will form a creditor committee to negotiate with the debtor. This is a metaphor for the Paris club or the CACs that we described earlier in the paper. The sanction threat held by the joint committee is denoted by \( \Delta \mu_{II} = \mu - \hat{\mu}_{II} \) and the total collateral held by the creditor committee is \( K_{II} = K_1 + K_2 \), where \( K_1 \) and \( K_2 \) are the collateral accessible to each debt holder. The total strategic debt service \( S_{II}(W) \) and the penalty trigger point \( W^{II}_* \) have expressions similar to (14) and (17) respectively. The sharing rule between the senior creditor and the junior creditor is assumed to be \( \eta \); i.e. the senior creditor can receive \( \eta \tilde{S}_{II}(W) \) of the strategic debt service while the rest \( (1 - \eta)\tilde{S}_{II}(W) \) goes to the junior creditor. The seniority of the senior loan contract requires \( \frac{1}{2} \leq \eta < 1 \).

Such a sub-problem actually is equivalent to an one-shot static borrowing problem which does not allow dynamic re-borrowing as in Chang and Sundaresan (2000) and the borrower’s value function \( J_{II}(W) \) satisfies the following HJB equation:

\[
\rho J_{II} = \frac{1}{2} \sigma^2 W^2 J_{IIWW} + (\mu W - C) J_{IIW} + \max_c (u(c) - cJ_{IIW})
\]
\[ J_{II}(W_{II}^B) = J_{IIB}(W_{II}^B), \quad \lim_{W \to W_{II}^B} J_{IIW}(W) = \frac{\partial J_{IIB}(W_B)}{\partial W_{II}^B} \] (20)

where the value function after default \( J_{IIB}(\cdot) \) will be determined by the bargaining process between two parties according equation (14). Using a finite difference scheme, equation (20) can be numerically solved and the optimal consumption for the sovereign borrower will be \( c^*_II(W) = (J_{IIW})^{-\frac{1}{\lambda}} \)

However, in order to obtain the boundary conditions for the problem at initial contractual region, what we need is not only the value function as a function of the total wealth \( W \), but also as a function of the re-borrowing boundary \( W^U \). Since \( J_{U}(W^U, A) = J_{II}(W^U + I_{C(2)}) \), what we need at this point is the market value of the junior loan \( I_{C(2)} \). From contingent-claim pricing theory, the “fair value” of the senior or junior loan \( I^{(i)}(i = 1, 2) \) as a function of wealth \( W \) satisfies the following ODE:

\[-r^{(i)} + (rW - c^*_II(W) - d(W)) (\frac{dW}{dW})^i + \frac{1}{2} \sigma^2 W^2 I^{(i)}_{WW} + \bar{d}^{(i)}(W) = 0\]

\[ I^{(i)}(W_{II}^{**}) = K_i \] (21)

where \( d(W) = \begin{cases} C^{(1)} + C^{(2)}, & W_{II}^B < W \\ S_{II}(W), & W_{II}^* < W \leq W_{II}^B \end{cases} \) is gross payout from the debtor and \( \bar{d}^{(i)}(W) = \begin{cases} C^{(i)}, & W_{II}^B < W \\ \eta^{(i)}[S_{II}(W) - bc], & W_{II}^* < W \leq W_{II}^B \end{cases} \) is the net in-flow creditor \( i \) can receive.

Although the debt valuation ODE (21) looks similar to those found in corporate debt pricing literature [for example Merton (1974) or Leland (1994)], there are major differences between our valuation scheme and theirs. First, the consumption rate in our model is derived from the borrower’s optimal intertemporal optimization problem. As a consequence, the payout rate \( c^*(W) + d(W) \) in our model is optimally and endogenously determined. However, much of the corporate bond pricing literature assume an exogenous payout ratio. For example, a popular treatment in corporate debt pricing lit-
erature is a linear exogenous payout ratio. Such an exogenous treatment will cause a negative consumption (dividend) when the borrower’s wealth falls and default becomes imminent. In corporate debt setting, it is interpreted as new equity issuance. Whereas in our utility maximization setting as long as we assume an appropriate utility function, negative consumption never occurs.

Since the sovereign borrower augments the wealth level from $W^U$ to $W^U + I^*|_{C(2)} - \theta A$, the market valuation of the junior loan $I^*|_{C(2)}$ satisfies the following fixed-point requirement:

$$I^{(2)}(W^U + I^*|_{C(2)} - \theta A) = I^*|_{C(2)}$$

(22)

The solution to this fixed-point equation will provide the upper boundary conditions in the initial contractual region: $J^U(W^U, A) = J_{II}(W^U + I^*|_{C(2)})$.

It is clear that the market valuation of the junior loan $I^*|_{C(2)}$ depends on the country’s arrears: the higher the arrears $A$, the lower the value of the loan. As a consequence, the upper boundary condition $J^U(W^U, A)$ will also be affected by the sovereign borrower’s arrears through the fixed point equation (22).

2.3 The Main Problem at Initial Contractual Region:

Given the two boundary conditions $J_B$ and $J_U$, the “double moving boundary” problem at the initial contractual region is well defined and the HJB equation (6) can be numerically solved for different level of arrears $A$. Because the upper boundary $W^U(A)$, the lower boundary $W^B(A)$ and the optimal consumption policy $c^*(W, A)$ are determined simultaneously through the HJB equation (6), we expect that all the variables will be influence by the borrower’s arrears $A$. In this sense, a country’s credit history and reputation do matter in our model.
For a country in the initial contractual region with arrears $A_0$, it will enter the re-contracting region as long as its wealth level fall below $W^B(A_0)$. The country will accrue new arrears in the re-contracting region and will return to the initial contractual region $C(1)$ at $(W, A)$ as long as the state variables $(W, A)$ satisfy $W > W^B(A)$. Unlike entering the re-contracting region, from contractual region $C(1)$ to contractual region $C(1) + C(2)$ is an one way traffic. Once the sovereign country decides to re-borrow at $W^U(A_0)$, he will never return to the initial contractual region $C(1)$.

The “fair value” of the senior loan at $(W^0, A^0)$ is then the function $I$, which is the solution to the following partial differential equation, valued at $(W^0, A^0)$:

$$-rI + fI_A + [rW - c^* - d]I_W + \frac{1}{2}\sigma^2W^2I_{WW} + \tilde{d} = 0$$  \hspace{1cm} (23)

where $f(W, A) = \begin{cases} 0, & W^B(A) < W \\ C(1) - S(W), & W^* < W \leq W^B(A) \end{cases}$,

d$(W, A) = \begin{cases} C(1), & W^B(A) < W < W^U(A) \\ S(W), & W^* < W \leq W^B \end{cases}$ is gross payout from the borrower and $\tilde{d}(W) = \begin{cases} C(1), & W^B < W < W^U \\ S(W) - bc, & W^* < W \leq W^B \end{cases}$ is the net inflow the creditor can receive. Clearly, the market valuation of the loan $I$ depends not only on the country’s wealth level but also on its’ accrued arrears $A$, i.e. the country’s credit history. We will discuss such an impact later by solving a numerical example.

At last, for a country with initial endowment $x_0$, the market valuation of the senior loan at time zero should satisfy the following fixed point requirement:

$$I(x_0 + I|_{C(1)}, 0) = I|_{C(1)}$$  \hspace{1cm} (24)

Before moving to the section on numerical illustration, we summarize the setting of our model in figure 1a and figure 1b, where figure 1a illus-
trates borrowers with good credit history \((A = 0)\) while figure 1b illustrates borrowers with bad credit history \((A > 0)\).

Figure 1a: Borrowers with Good Credit History \((A = 0)\)
3 Numerical Results

3.1 Benchmark Setting

In this section, we will numerically solve the borrower’s optimization problem under a set of benchmark parameters. The parameters are meant to illustrate how institutional features influence the valuation of sovereign debt. We will look at a high growth developing country whose production technology has an expected growth rate $0.2$ and a diffusion coefficient $0.05$. The relative risk aversion of the sovereign country is assumed to be $2.0$ and the time discount factor is $0.05$. We further assume that the risk-free world interest rate is $0.1$ and the initial wealth level of the sovereign country is $1.0$. For such a sovereign country with a high growth technology, one can expect that it has strong incentive to enter the world credit market to expand its investment scale.

For the costs of sanctions, we will assume that the sanction effect $\mu - \hat{\mu}$ by the senior lender is $0.05$ and the accessible collateral held by the senior lender is $0.02^{14}$. Once the sovereign borrower signed the junior debt contract, we assume that the joint sanction effect by both lenders is $0.06$ and each creditor has access to a collateral of $0.02$. The sharing rule between the senior lender and the junior lender is assumed to be $.75$. The following table summarizes all the benchmark parameters used in this section.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>The Risk Aversion Coefficient ( \gamma )</td>
<td>2.0</td>
</tr>
<tr>
<td>The Time Discount Factor ( \rho )</td>
<td>0.05</td>
</tr>
<tr>
<td>Expected Growth Rate ( \mu )</td>
<td>0.2</td>
</tr>
<tr>
<td>The Diffusion Coefficient ( \sigma^2 )</td>
<td>0.05</td>
</tr>
<tr>
<td>World Interest Rate ( r )</td>
<td>0.1</td>
</tr>
<tr>
<td>Sanction Effect by Senior Lender ( \mu - \hat{\mu} )</td>
<td>0.05</td>
</tr>
<tr>
<td>Sanction Effect by Joint Lenders ( \mu - \hat{\mu}_{II} )</td>
<td>0.06</td>
</tr>
<tr>
<td>Accessible Collateral by Senior Lender ( K_1 )</td>
<td>0.02</td>
</tr>
<tr>
<td>Accessible Collateral by Junior Lender ( K_2 )</td>
<td>0.02</td>
</tr>
<tr>
<td>Sharing Rule between Two Lenders ( \eta )</td>
<td>0.75</td>
</tr>
<tr>
<td>Bargaining Cost ( bc )</td>
<td>0.005</td>
</tr>
<tr>
<td>Re-borrowing Requirement ( \theta )</td>
<td>0.5</td>
</tr>
</tbody>
</table>

**Table 3**

We first study the general property of the sovereign loan (optimal consumption policy, optimal default and the re-borrowing policy) by examining a special contract with coupon rate \( C^{(1)} = C^{(2)} = 0.1 \). After that we characterize the *optimal contract* for the sovereign borrower, which maximize its life-time expected utility from consumption. In the second part of this section, we also report the comparative statics for sanction effect \( \Delta \mu \), accessible collateral \( K \), world interest rate \( r \), etc.

**Optimal Policy for the Sovereign Borrower:**

We first numerically solve the HJB equation (6) and characterize the sovereign borrower’s optimal consumption, default and re-borrowing policy under the benchmark setting. The following figure illustrates the borrower’s
optimal default [re-borrowing] boundary for different arrears $A$:

In our benchmark setting, the upper and lower boundaries for a borrower with good credit history ($A = 0$) are: $W^U(0) = 2.583$ and $W^B(0) = 1.209$. As $A$ increase, both re-borrowing boundary $W^U$ and default boundary $W^B$ will increase correspondingly. Such a result is intuitive: higher arrears $A$ implies an unfriendly re-borrowing opportunity hence the borrower will wait longer and build up a higher wealth level before re-entering the credit market. As a consequence, the optimal default boundary will also be pushed up. When $A$ goes to infinity [which is equivalent to the static setting with one shot borrowing opportunity] we have: $\lim_{A \to \infty} W^U(A) \to \infty$ and $\lim_{A \to \infty} W^B(A) = 1.287$.

Because optimal consumption and default boundary (re-borrowing boundary): $W^B( W^U)$ are determined simultaneously through HJB equation (6), we expect the consumption policy of the sovereign country also to be affected by its arrears. The following figure illustrates the normalized optimal
consumption $\frac{c^*(W)}{W}$ of the sovereign borrower for different arrears $A$:

![Graph showing consumption behavior](image)

Figure 3

The country’s consumption pattern generally can be divided into two regions. Typically in the relatively high wealth region, the borrower tends to cut its consumption rate as wealth decreases. Due to costly default (sanction effect and collateral seizure), this kind of behavior reflects the borrower’s desire to reduce the possibility of costly default in poor states of the world. However, for a country closer to default boundary, it’s consumption behavior is exactly the reverse: instead of cutting the consumption rate, the country *gears up* the consumption rate when the wealth drops. The size of such a region depends on the collateral accessible to the lender. Typically, the region will diminish as the collateral accessible to lender increases. We will discuss this result in detail in a later section.

The figure also shows that the arrears $A$ affect the borrower’s consumption behavior in a significant manner. As we can see from the figure, the more valuable for the future re-borrowing opportunity, the more conservative is the consumption policy. For a borrower with a “good” credit reputation,
the re-borrowing opportunity is more valuable and hence his consumption rate is lower and default boundary is lower.

The different optimal default [re-borrowing] boundary and the different consumption pattern between the borrowers with different arrears imply that the probability of default [re-borrowing] and the conditional expected time to default [re-borrow] will also be different for them even if they have the same level of promised coupon and wealth. One would expect that the borrower with good credit history (a lower arrears \( A \)) will have a lower probability of default [or high probability of re-borrowing] and a longer conditional expected time to default [or shorter conditional expected time to re-borrow]. The reason is that the borrower with good credit history has a more conservative consumption pattern and a lower default [re-borrowing] boundary. Using a Monte Carlo simulation approach\(^\text{15}\), we illustrate this point in Table 4 for our benchmark setting. The initial wealth for both borrowers has been set as \( W_0 = 1.8 \).

| A         | \( P_{\text{default}} \) % | \( P_{\text{re-borrow}} \) % | \( E(T|\text{default}) \) (in years) | \( E(T|\text{re-borrow}) \) (in years) |
|-----------|-----------------------------|-----------------------------|---------------------------------|---------------------------------|
| good      | 29.3%                       | 71.7%                       | 2.62                            | 2.45                            |
| credit     |                             |                             |                                 |                                 |
| history    |                             |                             |                                 |                                 |
| \( A = 0.00 \) |                             |                             |                                 |                                 |
| bad       | 35.6%                       | 64.4%                       | 2.26                            | 3.75                            |
| credit     |                             |                             |                                 |                                 |
| history    |                             |                             |                                 |                                 |
| \( A = 0.20 \) |                             |                             |                                 |                                 |

Table 4

As we can see from Table 4, the borrower with good credit history is more likely to hit the re-borrowing boundary before touching the lower default boundary. There is a difference of 6% in probability between the borrower
with good credit history and the borrower with bad credit history. Furthermore conditional on default, the expected time to default is 2.62 years for the borrower with good credit history while it is 2.26 years for the borrower with bad credit history; conditional on re-borrowing, the expected time to re-borrow is 2.45 years for the borrower with good credit history, which it is 3.75 years for the borrower with bad credit history.

**Debt Valuation:**

The valuation of the debt is derived through the PDE (23), which can be numerically solved by an implicit finite different scheme. The following figure shows the sovereign spreads as a function of wealth for three different levels of $A$:

![Graph showing sovereign spreads as a function of wealth](image)

Figure 4

As we have discussed in the earlier section, the optimal default and consumption policies are different for countries with different arrears: countries with higher arrears tend to have higher default boundaries and consumption rates. Hence it is not surprising that the valuation of the sovereign
loan will also depend on its arrears A. For our benchmark setting, higher arrears implies a lower loan valuation and higher yield spreads. Our result is consistent with empirical evidence such as Canton and Packer (1996) and Eichengreen and Portes (1987) etc: a country’s default history does matter in determining its borrowing cost.

For the market valuation of the senior loan at time zero: \( I^*|_{C(x_o = 1)} \), we substitute the solution \( I(W, 0) \) into the fixed-point requirement (24). For our benchmark setting, the market value of the sovereign loan is priced as \( I^*|_{C(x_o = 1)} = 0.796 \), which corresponds a yield spread of 256 basis points and an initial wealth level of \( W_0 = 1.796 \) for the sovereign borrower. As we have discussed, the driving force for such a sovereign spread is the presence of potential sanction and the accessibility of collateral held by the lender. Compared to a corporate debt where there exists a well established bankruptcy code to provide the lender with a formal access to the assets of the borrower under default, such a sovereign yield spread is expected to be higher.

**Optimal Debt Contract:**

In our model, the primary motivation for a sovereign borrower to seek international funding is to expand the scale of its investment. By signing a contract with coupon rate \( C^{(1)} \), the sovereign augments its initial investment from \( x_0 \) to \( x_0 + I^*|_{C^{(1)}} \). Correspondingly, the life time expected utility will change from \( J_0(x_0) \) to \( J(x_0 + I^*|_{C^{(1)}}, 0) \). To examine how much utility the country can gain by borrowing, we define the relative “certainty equivalence” as:

\[
CE(x_0) = \frac{J_0^{-1}[J(x_0 + I^*, 0)]}{x_0}
\]  

(25)

which measures the normalized utility change for a country with initial wealth level \( x_0 \). Relying on the concavity of the value function \( J \) and \( J_0 \), we
can show that the relative “Certainty Equivalence” $CE(\cdot)$ is a well defined continuous function.

The objective of the sovereign country is to choose the optimal coupon rate $C^{(1)}$ which will maximize life-time expected utility from consumption $J(x_0 + I^*|_{C^{(1)}}, 0)$ (or equivalently the “certainty equivalence” $CE(x_0)$). For simplicity, we assume that the coupon rate $C^{(2)}$ on junior debt is set equal to the senior coupon rate $C^{(1)}$. Figure 5 shows the “certainty equivalence” $CE(x_0)$ as a function of coupon rate $C^{(1)}$ for a sovereign country with unit initial endowment: [i.e. $x_0 = 1$].

![Figure 5](image_url)

For the benchmark setting, the relative “certainty equivalence” starts from unity when coupon rate $C^{(1)} = 0$. It first decreases then bounces back and reaches the peak when coupon rate equals 0.118. The gain in the relative “certainty equivalence” is around 6.7% at the optimal coupon rate. Additional calculation yields that for such a contract $C^* = 0.118$, the debt is valued as $I^*|_{C^*} = 0.866$ at the competitive world capital market, which
corresponds to a yield spread of 360 basis points. Instead of making the sovereign country better off, a heavy debt burden (i.e. a too large coupon rate $C^{(1)}$) will cause utility loss for the country because of high probability of default and its associated deadweight losses.

We also plot the “certainty equivalence” as a function of coupon rate for the static borrowing setting where no re-borrowing is allowed. The optimal coupon rate in the static setting $C^*_{\text{static}} = 0.122$ is larger than the corresponding optimal coupon rate for our dynamic benchmark setting while the maximum “certainty equivalence” $CE_{\text{static}} = 1.0596$ is smaller than the corresponding $CE_{\text{benchmark}} = 1.0673$. The debt in static borrowing setting is valued as $I^*_{\text{static}}|C^* = 0.876$ with a yield spread of 393 basis points, which corresponds a higher spreads compared to the benchmark dynamic setting. Such a result is intuitive: since the sovereign country can optimally choose the timing of junior debt offering in our dynamic model, they can add more debt in the future. This flexibility of repeated borrowing permits the sovereign borrower to reduce the initial load of debt and hence the associated probability of financial distress. In a static model, the sovereign country can only borrow once at time zero because of the inflexible debt structure. The optimal coupon rate is consequently higher and hence the utility gain is lower in the static model.

### 3.2 Comparative Statics

In this subsection, we study how predictions of our model can be affected by the severity of sanctions $\mu - \hat{\mu}$, accessible collateral $K$ and world interest rate $r$. To simplify our analysis, all the comparative statics are based on a sovereign borrower with a “good” credit history: i.e. $A = 0$. 

35
Sanction Effect $\mu - \hat{\mu}$:

Sanction effect $\Delta \mu = \mu - \hat{\mu}$ measures the severity of the penalty held by the lender in the event of default. We would expect that a different $\mu - \hat{\mu}$ implies a different *ex-post* borrower’s behavior and a different *ex-ante* optimal contract. For the benchmark loan contract, Table 5a and Figure 6 present the borrower’s optimal policy and the debt valuation for different sanction effect $\Delta \mu$ under our benchmark setting when the senior coupon rate $C^{(1)} = 0.1$.

<table>
<thead>
<tr>
<th>$\Delta \mu$</th>
<th>$W^B$</th>
<th>$W^U$</th>
<th>$I^*$</th>
<th>Sprd(bps)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04</td>
<td>1.443</td>
<td>2.953</td>
<td>0.586</td>
<td>707</td>
</tr>
<tr>
<td>0.05</td>
<td>1.209</td>
<td>2.583</td>
<td>0.796</td>
<td>256</td>
</tr>
<tr>
<td>0.06</td>
<td>1.039</td>
<td>2.162</td>
<td>0.878</td>
<td>139</td>
</tr>
</tbody>
</table>

Table 5a

<table>
<thead>
<tr>
<th>$\Delta \mu$</th>
<th>Optimal Coupon: $C^*$</th>
<th>Certainty Equivalence: $CE^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04</td>
<td>0.078</td>
<td>1.0294</td>
</tr>
<tr>
<td>0.05</td>
<td>0.118</td>
<td>1.0673</td>
</tr>
<tr>
<td>0.06</td>
<td>0.164</td>
<td>1.1297</td>
</tr>
</tbody>
</table>

Table 5b
Figure 6a

Figure 6b
Typically, a high $\mu - \hat{\mu}$ implies a more severe potential penalty hence a more costly default. To avoid such a costly repudiation, the sovereign borrower facing a higher $\Delta \mu$ tends to consume less in a bad state (Figure 6b) and tries not to repudiate too early (a lower default boundary $W^B$). By the same token, a lender with high $\Delta \mu$ can extract more concession from the borrower in the “renegotiation region”, which means a higher strategic debt service $S(W)$ (Figure 6a). These two effects together result in a more valuable sovereign debt in the world capital market (a higher $I^*$). In our numerical example, the yield spread increases to 707 bps with a 0.01 increase in $\Delta \mu$ and decreases to 139 bps with a 0.01 increase in $\Delta \mu$.

Our model also predicts an interesting implication of sanction effect $\Delta \mu$ on optimal contract. Given a sovereign contract in place, a higher $\Delta \mu$ (a more costly default) is unfavorable to the borrower. But the fully informed lender is willing to provide a higher upfront loan for the same coupon rate $C$ if he holds a more severe potential penalty. Hence the borrower can augment more initial capital for a higher $\Delta \mu$. For a country with high growth rate, such a compensation may exceed the utility loss from a more costly default. Table 5b illustrates this point. The optimal coupon rate increases from 0.118 to 0.164 and the relative utility gain increased from 1.0673 to 1.1297 when the $\Delta \mu$ increased from 0.05 to 0.06. For such a case, an effective sanction threat (a higher $\Delta \mu$) can be welfare improving for both the lender and the borrower.

**Accessible Collateral $K$:**

The collateral held by the lender influences the loan amount as well as the spreads. The effect of $K$ on the optimal contract is similar to the sanction effect $\Delta \mu$. Table 6 and Figure 7 show the result for our benchmark setting when $K^{(1)} = K^{(2)} = 0.02$, 0.05 and 0.10. For a sovereign contract with higher accessible collateral $K$, the loan is more valuable and the spreads are
lower. There are two reasons for this: first the lender can secure more asset in the event of default and will have a stronger bargaining position in the renegotiation process; secondly, the sovereign borrower will consume more conservatively for a higher $K$. Moreover the welfare of the borrower increase as well with increase in $K$. These results are similar to the ones obtained by Gibson and Sundaresan (2000) in the context of static borrowing.

<table>
<thead>
<tr>
<th>$K$</th>
<th>$W_B$</th>
<th>$W_U$</th>
<th>$I^*$</th>
<th>$Sprd(bps)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>1.209</td>
<td>2.583</td>
<td>0.796</td>
<td>256</td>
</tr>
<tr>
<td>0.05</td>
<td>1.197</td>
<td>2.527</td>
<td>0.811</td>
<td>233</td>
</tr>
<tr>
<td>0.10</td>
<td>1.178</td>
<td>2.428</td>
<td>0.832</td>
<td>202</td>
</tr>
</tbody>
</table>

Table 6a

<table>
<thead>
<tr>
<th>$K$</th>
<th>Optimal Coupon: $C^*$</th>
<th>Certainty Equivalence: $CE^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>0.118</td>
<td>1.0673</td>
</tr>
<tr>
<td>0.05</td>
<td>0.124</td>
<td>1.0685</td>
</tr>
<tr>
<td>0.10</td>
<td>0.130</td>
<td>1.0698</td>
</tr>
</tbody>
</table>

Table 6b

Accessible collateral plays a very important role in determining the sovereign’s optimal consumption. In the earlier section it has been shown that the sovereign country will effectively dissipate the collateral as the wealth gets too close to the default boundary. Such a consumption pattern has an adverse impact on the valuation of the sovereign loan. Here the borrower is substituting tomorrow’s good investment opportunity for today’s consumption. As a result, the default probability increases and the sovereign country behaves in an opportunistic fashion. The accessible collateral $K$ determines how large this region will be. Typically, the larger the accessible collateral, the smaller the collateral dissipation region will be. When the accessible
collateral reaches a certain level, such a region can even disappear. Figure 7 shows that such a collateral dissipation region does disappear for our benchmark setting when $K = 0.4$.

![Figure 7](image)

Such an observation has an very important implication for ex-ante contract design in the sovereign borrowing. If the borrowing country has no or very little collateral that is accessible to the lender then the loan will carry a high premium. Hence, improved collateralization will make the sovereign borrower more easily enter the global credit markets since the lender knows ex-ante that the collateral held by him will encourage the sovereign to undertake more efficient investment-consumption to preserve the value of the loan. Further it has the additional benefits of reducing the incentives of borrower to indulge in “overinvestment”.

**World Interest Rate $r$:**

The change of world interest rate $r$ will directly affect the value of the
sovereign loan. For a more attractive interest rate (smaller \( r \)), the value of the sovereign loan will be higher and the country can gain more utility by entering the credit market. For our benchmark setting, Table 7a shows that the relative utility gain increases from 1.0098 to 1.1275 when world interest rate decreases from 0.12 to 0.08.

<table>
<thead>
<tr>
<th>( r )</th>
<th>( W^B )</th>
<th>( W^U )</th>
<th>( I^* )</th>
<th>( Sprd(bps) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.08</td>
<td>1.192</td>
<td>2.375</td>
<td>0.885</td>
<td>330</td>
</tr>
<tr>
<td>0.10</td>
<td>1.209</td>
<td>2.583</td>
<td>0.796</td>
<td>256</td>
</tr>
<tr>
<td>0.12</td>
<td>1.255</td>
<td>2.825</td>
<td>0.710</td>
<td>208</td>
</tr>
</tbody>
</table>

Table 7a

<table>
<thead>
<tr>
<th>( r )</th>
<th>Optimal Coupon: ( C^* )</th>
<th>Certainty Equivalence: ( CE^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.08</td>
<td>0.110</td>
<td>1.1275</td>
</tr>
<tr>
<td>0.10</td>
<td>0.118</td>
<td>1.0673</td>
</tr>
<tr>
<td>0.12</td>
<td>0.124</td>
<td>1.0098</td>
</tr>
</tbody>
</table>

Table 7b

The effect of interest rate \( r \) on the borrower’s optimal behavior is more subtle since the interest rate \( r \) does not directly enter the HJB equation (6). However, the change of interest rate \( r \) will implicitly affect the borrower’s optimal policy through the upper boundary conditions (8) because the valuation of the junior loan \( I_{C(2)} \) will be affected. A lower \( r \) will yield a more valuable re-borrowing option in the future and will make the borrower put more effort to preserve such an option. Hence we see a lower default boundary \( W^B \) and a more conservative consumption pattern for a smaller world interest rate \( r \). In the static setting, the borrower’s optimal consumption policy \( c^* \) and optimal default boundary \( W^B \) will not be affected by the interest rate \( r \).
4 Conclusion

In a model of repeated borrowing, we characterized the borrower’s optimal behavior (optimal consumption rate and optimal default strategy), the value for the sovereign loan and the optimal debt contract for the sovereign borrower. In our model, the primary repayment incentive for the borrower is the potential sanction and seizable collateral held by the lender. However a country’s credit history also plays an important role in determining a country’s optimal policy and credit spread because of his need to re-enter the credit market in future. We discussed the implications of sanctions, accessible collateral, level of world interest rate on the debt valuation, borrower’s optimal behavior and \textit{ex-ante} optimal contract design.

Our work may be extended in several possible directions. In our model the lender is assumed to hold all the bargaining power. One would expect
to see a more active borrower in the bargaining process which allows a Rubinstein (1985) type bargaining game. Second, our model is an one factor model where all the uncertainty is embedded in the production technology of the borrowing country. Allowing multiple factors which capture stochastic interest rate and exchange rate will make the problem richer. Finally, our dynamic model allows the sovereign borrower to re-enter the world credit market once in the future. The challenge is to build a dynamic model which allows for multiple rounds of borrowing opportunities for the sovereign borrower. We leave these possible extensions for future research.
References


Notes

1 We thank the participants in seminars presented at Georgia Institute of Technology and Federal Reserve Board for their comments and suggestions. We also thank Charles Calomiris, Prajit Dutta and Zhenyu Wang for their constructive comments and criticisms. We remain responsible for the opinions expressed.

2 For example, the structure corporate pricing model by Merton (1974), Leland (1994), Anderson and Sundaresan (1996) and the reduced form pricing model by Duffie and Singleton (1999). A detailed survey of corporate debt pricing literature can be found in Sundaresan (2000).

3 Recently there have been several papers focusing on the sovereign yield spreads. Among them, the work of Gibson and Sundaresan (1999) is closest to ours. They construct a structural model in which they compute the optimal level of sovereign debt and the spreads when the lender can seize exports and enforce sanctions. They do not consider dynamic borrowing. Nor do they explore optimal consumption policies of the sovereign borrower. Duffie, Pedersen and Singleton (2000) also model the sovereign yield spreads in a reduced form setup.

4 Kulatilaka and Marcus (1987) also investigated the sovereign borrower’s optimal default strategy in a continuous time framework. However, the yield spread in their model is exogenously given and they do not solve for the optimal consumption pattern of the borrower. Moreover, their model is a static one which makes them unable to study the effect of “reputation” in sovereign debt markets.

5 See Bhagwati (1982) and Krueger (1974) for a discussion of DUP
and rebt seeking.

6Our model can be easily extended to a stochastic opportunity set. Furthermore, the opportunity technology may also be set to be relevant to the sovereign’s indebtedness (for example, the growth rate may be a function of the country’s debt level). We leave this extension to future research.

7The value function $J$ is defined as the supremum of the expected utility over the set of admissible controls: $J(W_0) = \sup_{A(W_0)} E_0 \left[ \int_0^\infty e^{-\rho t} u(c_t) dt \right]

8Alternatively given a loan amount $I_{C(1)}$ we can solve for the level of coupon $C(1)$ endogenously.

9Our setup for debt obligation in arrears is similar to Emanuel (1983), in which he modeled the dividend obligation in arrears for a firm with preferred stock.

10Traditionally, it is a non-trivial task to model dynamic capital structure in a continuous-time setting. Leland (1998) and Goldstein, Ju and Leland (1999) have addressed this problem by assuming a “scaling property” to deal with the upward and downward debt restructuring. This scaling property assumes that the leverage ratio has a homogeneity property. However they are unable to justify the optimality of such a scaling property. Furthermore, the lack of homogeneity in our formulation makes the “scaling property” unsuitable for our model. In light of this, we simplify our model by allowing the sovereign country to borrow one more round after signing the initial contract $(C(1), I_{C(1)})$. Of course the restriction that the sovereign country can only re-borrow one more time is not necessary in our setting. Using the backward methodology
in this paper, we can extend the re-borrowing opportunity to a finite number of times. It significantly increases the computational complexity without qualitatively changing our conclusions.

Ideally, one can extend our model to a Rubinstein (1985) type bargaining game which allows the parties to make offers and counteroffers. However for reasons of tractability, we assume that one party holds all the bargaining power. Eaton and Fernandez (1995) have argued: “While bargaining is one way in which to deal with the problem of renegotiation, it is well known that bargaining models are quite sensitive to that exact specification of the bargaining game.” Fernandez and Rosenthal (1990) construct a model that examines the bilateral monopoly question without introducing bargaining in the form of an offer-counteroffer model. Their result lends support to models that have assumed that all the bargaining power lies with the banks since they obtain exactly the same result in an explicitly strategic model. Hence in this paper we assume an unilateral bargaining power on lender’s side and leave the two way bargaining for future research.

An alternative way to model the penalty trigger point \( W^* \) is to assume that the lender will abandon renegotiation once the market value of debt \( I \) falls below zero. Both methods yield similar results.

In particular, the strictly increasing and strictly concave utility function \( u(.) \) satisfy the Inada condition: \( \lim_{c \to 0} u'(c) = +\infty \).

Although estimates for the costs of sanction and the accessible collateral by the lender in economic literature are few and usually imprecise, we have tried to keep the parameters in line with the limited existing empirical evidence. For example, Bulow and Rogoff (1989a)
reported that during 1980’s, the World Bank estimates that trading disruption costing around 3% of some of the Latin American countries’ (Argentina, Brazil and Mexico) GDP would be more costly than making payments of 5 percent of their total external debt. Hence sanctions of half this magnitude (1.5 percent of GDP per annum) would support the market value of these loans by their model. In another extreme example, Hufbauer and Schott (1985) estimated the cost to Rhodesia of the trade sanctions imposed against it to be just over 15 percent of GNP per annum.

15 We simulate the wealth process in equation (5) starting from a fixed wealth level \( W_0 \) by a discretized Euler Scheme. For each path \( i \), the wealth process before default [or re-borrowing] was approximated by:

\[
W^{i}_{(j+1)h} = W^{i}_{jh} + (\mu W^{i}_{jh} - c^*(W^{i}_{jh}) - C^{(1)})h + \sigma W^{i}_{jh} \sqrt{h}X Z^{i}_{j+1}
\]

for \( W^{U}(A) > W^{i}_{jh} > W^{B}(A) \)

on a discrete time grid \( \{0, h, 2h, ...\} \), where \( Z^{i}_{j} \) are i.i.d. standard normal random variables.