Exercises
in
Recursive Macroeconomic Theory

preliminary and incomplete

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Introduction

This is a first version of the solutions to the exercises in Recursive Macroeconomic Therory, First Edition, 2000, MIT press, by Lars Ljungqvist and Thomas J. Sargent. This solution manuscript is currently only available on the web. We invite the reader to bring typos and other corrections to our attention. Please email `sargent@stanford.edu`, `poweill@stanford.edu` or `svnieuwe@stanford.edu`. We will regularly update this manuscript during the following months. Some questions ask for computations in matlab. The program files can be downloaded from the ftp site `zia.stanford.edu/pub/sargent/rmtex`.

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CHAPTER 1

Time series

Exercise 1.1.

Consider the Markov Chain $(P, \pi_0) = \left(\begin{bmatrix} .9 & .1 \\ .3 & .7 \end{bmatrix}, \begin{bmatrix} .5 \\ .5 \end{bmatrix} \right)$, where the state space is $\overline{x} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$. Compute the likelihood of the following three histories for $t = 0, 1, 2, 3, 4$:

a. 1,5,1,5,1.

b. 1,1,1,1,1.

c. 5,5,5,5,5.

Solution

The probability of observing a given history up to $t = 4$, say $(x_{i_5}, x_{i_4}, x_{i_3}, x_{i_2}, x_{i_1}, x_{i_0})$, is given by

$$P(x_{i_4}, x_{i_3}, x_{i_2}, x_{i_1}, x_{i_0}) = P_{i_4,i_3} P_{i_3,i_2} P_{i_2,i_1} P_{i_1,i_0} \pi_{0i_0}$$

where $P_{ij} = \text{Prob}\left(x_{t+1} = \overline{x}_j | x_t = \overline{x}_i\right)$ and $\pi_{0i} = \text{Prob}\left(x_0 = \overline{x}_i\right)$.

By applying this formula one obtains the following results:

a. $P(1, 5, 1, 5, 1) = P_{21} P_{12} P_{21} P_{21} \pi_{01} = (.3)\,(.1)\,(.3)\,(.1)\,(.5) = .00045.$
b. $P(1, 1, 1, 1, 1) = P_{11} P_{11} P_{11} P_{11} \pi_{01} = (.9)^4\,(.5) = .3281.$
c. $P(5, 5, 5, 5, 5) = P_{22} P_{22} P_{22} P_{22} \pi_{02} = (.7)^4\,(.5) = .12.$

Exercise 1.2.

A Markov chain has state space $\overline{x} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$. It is known that $E\left(x_{t+1} | x_t = \overline{x}\right) = \begin{bmatrix} 1.8 \\ 3.4 \end{bmatrix}$ and that $E\left(x_{t+1}^2 | x_t = \overline{x}\right) = \begin{bmatrix} 5.8 \\ 15.4 \end{bmatrix}$. Find a transition matrix consistent with these conditional expectations. Is this transition matrix unique (i.e., can you find another one that is consistent with these conditional expectations)?

Solution

From the formulas for forecasting functions of a Markov chain, we know that

$$E\left(h(x_{t+1}) | x_t = \overline{x}\right) = Ph,$$

where $h(\overline{x})$ is a function of the state represented by an $n \times 1$ vector h. Applying this formula yields:

$$E\left(x_{t+1} | x_t = \overline{x}\right) = P\overline{x} \text{ and } E\left(x_{t+1}^2 | x_t = \overline{x}\right) = P\overline{x}^2.$$

This yields a set of 4 linear equations:

$$\begin{bmatrix} 1.8 \\ 3.4 \end{bmatrix} = P \begin{bmatrix} 1 \\ 5 \end{bmatrix} \text{ and } \begin{bmatrix} 5.8 \\ 15.4 \end{bmatrix} = P \begin{bmatrix} 1 \\ 25 \end{bmatrix},$$

which can be solved for the 4 unknowns. Alternatively, using matrix notation, we can rewrite this as $e = Ph$, where $e = [e_1, e_2]$, $e_1 = E\left(x_{t+1}|x_t = \overline{x}\right)$, $e_2 = E\left(x_{t+1}^2|x_t = \overline{x}\right)$ and $h = [h_1, h_2]$, where $h_1 = \overline{x}$ and $h_2 = \overline{x}^2$:

$$\begin{bmatrix} 1.8 & 5.8 \\ 3.4 & 15.4 \end{bmatrix} = P \begin{bmatrix} 1 & 1 \\ 5 & 25 \end{bmatrix}.$$

Then P is uniquely determined as $P = eh^{-1}$. Uniqueness follows from the fact that h_1 and h_2 are linearly independent. After some algebra we obtain a well-defined stochastic matrix:

$$P = \begin{bmatrix} .8 & .2 \\ .4 & .6 \end{bmatrix}.$$

Exercise 1.3.

Consumption is governed by an n state Markov chain P, π_0 where P is a stochastic matrix and π_0 is an initial probability distribution. Consumption takes one of the values in the $n \times 1$ vector $\bar{c}$. A consumer ranks stochastic processes of consumption $t = 0, 1 \ldots$ according to

$$E \sum_{t=0}^{\infty} \beta^t u(c_t),$$

where E is the mathematical expectation and $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ for some parameter $\gamma \geq 1$. Let $u_i = u(\bar{c}_i)$. Let $v_i = E[\sum_{t=0}^{\infty} \beta^t u(c_t)|c_0 = \bar{c}_i]$ and $V = Ev$, where $\beta \in (0, 1)$ is a discount factor.

a. Let u and v be the $n \times 1$ vectors whose ith components are u_i and v_i, respectively. Verify the following formulas for v and V: $v = (I - \beta P)^{-1} u$, and $V = \sum_i \pi_{0,i} v_i$.

b. Consider the following two Markov processes:

Process 1: $\pi_0 = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Process 2: $\pi_0 = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$, $P = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$.

For both Markov processes, $\bar{c} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$. Assume that $\gamma = 2.5, \beta = .95$. Compute unconditional discounted expected utility V for each of these processes. Which of the two processes does the consumer prefer? Redo the calculations for $\gamma = 4$. Now which process does the consumer prefer?

c. An econometrician observes a sample of 10 observations of consumption rates for our consumer. He knows that one of the two preceding Markov processes generates the data, but not which one. He assigns equal "prior probability" to the two chains. Suppose that the 10 successive observations on consumption are as follows: $1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1$. Compute the likehood of this sample under process 1 and under process 2. Denote the likelihood function $\text{Prob}(\text{data}|\text{Model}_i), i = 1, 2$.

d. Suppose that the econometrician uses Bayes' law to revise his initial probability estimates for the two models, where in this context Bayes' law states:

$$\text{Prob}(\text{Model}_i|\text{data}) = \frac{\text{Prob}(\text{data}|\text{Model}_i) \cdot \text{Prob}(\text{Model}_i)}{\sum_j \text{Prob}(\text{data}|\text{Model}_j) \cdot \text{Prob}(\text{Model}_j)}.$$

The denominator of this expression is the unconditional probability of the data. After observing the data sample, what probabilities does the econometrician place on the two possible models?

e. Repeat the calculation in part d, but now assume that the data sample is $1, 5, 5, 1, 5, 5, 1, 5, 1, 5$.

Solution

a. Given that $v_i = E\left[\sum_{t=0}^{\infty} \beta^t u(c_t)|c_0 = \overline{c}_i\right]$, we can apply the usual vector notation (by stacking):

$$v = E\left[\sum_{t=0}^{\infty} \beta^t u(c_t)|c_0 = \overline{c}\right].$$

To apply the forecasting function formula in the notes:

$$E\sum_{k=0}^{\infty} \beta^k \left(h(x_{t+k})|x_t = \overline{x}\right) = \left(I - \beta P\right)^{-1} h.$$

Let $h(\overline{x}) = u(\overline{c})$. Then it follows immediately that:

$$v = E\left[\sum_{t=0}^{\infty} \beta^t u(c_t)|c_0 = \overline{c}\right] = \left(I - \beta P\right)^{-1} u.$$

Second, to compute $V = Ev$, simply note that in general the unconditional expectation at time 0 of a foreasting function h is given by: $E(h(x_0)) = \sum_{i=1}^{n} h_i \pi_{0,i} = \pi_0^{'} h$, or, in particular:

$$V = \sum_{i=1}^{n} v_i \pi_{0,i}.$$

Also, you should be able to verify that $V = E\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right]$ by applying the law of iterated expectations.

b. the matlab program `exer0103.m` computes the solutions.

Process1 and Process 2: $V = -7.2630$ for $\gamma = 2.5$

Process1 and Process 2: $V = -3.36$ for $\gamma = 4$

Note that the consumer is indifferent between both of the consumption processes regardless of γ.
c. Applying the same logic as in exercise in, construct the likelihood function as the probability of having observed this partical history of consumption rates, conditional on the model.

$$\text{Prob}(\text{data}|\text{Model}_1) = (P_{1,1})^9\,(.5) = .5,$$

$$\text{Prob}(\text{data}|\text{Model}_2) = (P_{1,1})^9\,(.5) = .5^{10} = .0009765.$$

d. Applying Bayes' law:

$$\begin{aligned}\text{Prob}\left(\text{Model}_1|\text{data}\right) &= \frac{\text{Prob}(\text{data}|\text{Model}_1)\text{Prob}(\text{Model}_1)}{\sum_i \text{Prob}(\text{data}|\text{Model}_i)\text{Prob}(\text{Model}_i)}\\ &= \frac{.5\text{Prob}(\text{Model}_1)}{.5\text{Prob}(\text{Model}_1) + .000976\text{Prob}(\text{Model}_2)},\end{aligned}$$

and by the same logic:

$$\text{Prob}\left(\text{Model}_2|\text{data}\right) = \frac{.000976\text{Prob}(\text{Model}_2)}{.5\text{Prob}(\text{Model}_1) + .000976\text{Prob}(\text{Model}_2)}.$$

e. Consider the sample $(1, 5, 5, 1, 5, 5, 1, 5, 1, 5)$

$$\begin{aligned}\text{Prob}(\text{data}|\text{Model}_1) &= P_{21}P_{22}P_{12}P_{21}P_{22}P_{12}P_{21}P_{12}P_{21}(.5) =\\ &= 0,\end{aligned}$$

$$\begin{aligned}\text{Prob}(\text{data}|\text{Model}_2) &= P_{21}P_{22}P_{12}P_{21}P_{22}P_{12}P_{21}P_{12}P_{21}(.5)\\ &= .5^{10} = .0009765.\end{aligned}$$

Applying Bayes' law:

$$\begin{aligned}\text{Prob}\left(\text{Model}_1|\text{data}\right) &= \frac{\text{Prob}(\text{data}|\text{Model}_1)\text{Prob}(\text{Model}_1)}{\sum_i \text{Prob}(\text{data}|\text{Model}_i)\text{Prob}(\text{Model}_i)}\\ &= 0,\end{aligned}$$

which implies:

$$\text{Prob}\left(\text{Model}_2|\text{data}\right) = 1.$$

Exercise 1.4.

Consider the univariate stochastic process

$$y_{t+1} = \alpha + \sum_{j=1}^{4} \rho_j y_{t+1-j} + c w_{t+1}, \tag{1}$$

where w_{t+1} is a scalar martingale difference sequence adapted to $J_t = [w_t, \ldots, w_1, y_0, y_{-1}, y_{-2}, y_{-3}]$, $\alpha = \mu(1 - \sum_j \rho_j)$ and the ρ_j's are such that the matrix

$$A = \begin{bmatrix} \rho_1 & \rho_2 & \rho_3 & \rho_4 & \alpha \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

has all of its eigenvalues in modulus bounded below unity.

a. Show how to map this process into a first-order linear stochastic difference equation.

b. For each of the following examples, if possible, assume that the initial conditions are such that y_t is covariance stationary. For each case, state the appropriate initial conditions. Then compute the covariance stationary mean and variance of y_t assuming the following parameter sets of parameter values: *i.* $\rho = \begin{bmatrix} 1.2 & -.3 & 0 & 0 \end{bmatrix}$, $\mu = 10, c = 1$. *ii.* $\rho = \begin{bmatrix} 1.2 & -.3 & 0 & 0 \end{bmatrix}$, $\mu = 10, c = 2$. *iii.* $\rho = \begin{bmatrix} .9 & 0 & 0 & 0 \end{bmatrix}$, $\mu = 5, c = 1$. *iv.* $\rho = \begin{bmatrix} .2 & 0 & 0 & .5 \end{bmatrix}$, $\mu = 5, c = 1$.

v. $\rho = \begin{bmatrix} .8 & .3 & 0 & 0 \end{bmatrix}$, $\mu = 5, c = 1$. *Hint 1:* The Matlab program `doublej.m`

, in particular, the command `X=doublej(A,C*C')` computes the solution of the matrix equation $A'XA + C'C = X$. This program can be downloaded from `ftp://zia.stanford.edu/pub/sargent/webdocs/matlab`.

Hint 2: The mean vector is the eigenvector of A associated with a unit eigenvalue, scaled so that the mean of unity in the state vector is unity.

c. For each case in part b, compute the h_j's in $E_t y_{t+5} = \gamma_0 + \sum_{j=0}^{3} h_j y_{t-j}$.

d. For each case in part b, compute the $\tilde{h}_j$'s in $E_t \sum_{k=0}^{\infty} .95^k y_{t+k} = \sum_{j=0}^{3} \tilde{h}_j y_{t-j}$.

e. For each case in part b, compute the autocovariance $E(y_t - \mu_y)(y_{t-k} - \mu_y)$ for the three values $k = 1, 5, 10$.

Solution

a. To compute the solutions this problem , you can use the program `ex0104.m` .

Mapping the univariate stochastic process into a first-order linear stochastic difference equation:

The first-order linear difference equation corresponding to (1) is :

$$
(2) \qquad \begin{bmatrix} y_{t+1} \\ y_t \\ y_{t-1} \\ y_{t-2} \\ 1 \end{bmatrix} = \begin{bmatrix} \rho_1 & \rho_2 & \rho_3 & \rho_4 & \alpha \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ y_{t-3} \\ 1 \end{bmatrix} + \begin{bmatrix} c \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} w_{t+1},
$$

or, equivalently:

$$x_{t+1} = Ax_t + Cw_{t+1},$$

for $t = 0, 1, 2 \ldots$,where $x_t' = \begin{bmatrix} y_{t+1} & y_t & y_{t-1} & y_{t-2} & 1 \end{bmatrix}$, x_0 is a given initial condition, A is a 5×5 matrix and C is an 5×1 matrix.

b. Assume that the initial conditions are such that y_t is covariance stationary. Consider the initial vector x_0 as being drawn from a distibution with mean μ_0 and covariance matrix Σ_0.
Given stationarity, we can derive the unconditional mean of the process by taking unconditional expectations of eq.(1) :

$$\mu = \alpha + \mu \sum_{j=1}^{4} \rho_j,$$

or, equivalently:

$$\mu = \alpha \left(1 - \sum_{j=1}^{4} \rho_j \right).$$

This implies that we can write:

$$y_{t+1} - \mu = \sum_{j=1}^{4} \rho_j \left(y_{t+1-j} - \mu \right) + cw_{t+1},$$

or

$$\widetilde{x}_{t+1} = A\widetilde{x}_t + Cw_{t+1},$$

where $\widetilde{x}_{t+1} = x_{t+1} - \mu$ where $\mu' = \begin{bmatrix} \mu & \mu & \mu & \mu & 1 \end{bmatrix}$.
As you know, the second moments can be derived by calculating $C_x(0) = E\widetilde{x}_{t+1}\widetilde{x}_{t+1}'$, which produces a discrete Lyapunov equation:

$$C_x(0) = AC_x(0)A' + CC'.$$

Stationarity requires two conditions:

- All of the eigenvalues of A are less than unity in modulus, except possibly for the one associated with the constant term
- the initial condition x_0 needs to be drawn from the stationary distribution, described by its first two moments μ and $C_x(0)$

i. $\rho = \begin{bmatrix} 1.2 & -3 & 0 & 0 \end{bmatrix}$, $\mu = 10$,$c = 1$
This implies:

$$
\begin{bmatrix} y_{t+1} \\ y_t \\ y_{t-1} \\ y_{t-2} \\ 1 \end{bmatrix} = \begin{bmatrix} 1.2 & -.3 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ y_{t-3} \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} w_{1,t+1}. \tag{3}
$$

The eigenvalues are given by $\lambda = \begin{bmatrix} 0 & 0 & .35 & .84 & 1 \end{bmatrix}$. The relevant eigenvalues are smaller than unity. The first condition for stationarity is satisfied. Now, we can solve the discrete Lyupanov equation for $C_x(0)$.
Recall from the previous handout that:

$$
E_t\left(x_{t+j} - E(x_{t+j}|J_t)\right)\left(x_{t+j} - E(x_{t+j}|J_t)\right)' = \sum_{l=0}^{j-1} A^l CC' A^{l'}.
$$

The matlab program doublej.m calculates the $\lim_{j\to\infty}$ of the above expression (type help doublej.m to verify). As one would expect, if the system is stationary, this limit converges to the unconditional second moment:

$$
C_x(0) = \lim_{j\to\infty} E_t\left(x_{t+j} - E(x_{t+j}|J_t)\right)\left(x_{t+j} - E(x_{t+j}|J_t)\right)' = \sum_{l=0}^{\infty} A^l CC' A^{l'}.
$$

Note that CC' is a matrix of zeros in this case except for the (1,1)st element which is 1.
To calculate $C_x(0)$, simply type $V = \text{doublej}(A, CC')$:

$$
C_x(0) = \begin{bmatrix} 7.42 & 6.85 & 6.00 & 5.14 & 0 \\ 6.85 & 7.42 & 6.85 & 6.00 & 0 \\ 6.00 & 6.85 & 7.42 & 6.85 & 0 \\ 5.14 & 6.00 & 6.85 & 7.42 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
$$

Obviously, the diagonal elements (except for the zero element associated with the constant) contain the variance of y_t.
ii. $\rho = \begin{bmatrix} 1.2 & -3 & 0 & 0 \end{bmatrix}$, $\mu = 10$,$c = 2$
The 1st part of the answer to i. is still valid, as A has not changed. Its eigenvalues are bounded below untiy in modulus. There is a covariance stationary distribution associated with this system.

$$
C_x(0) = \begin{bmatrix} 29.71 & 27.42 & 24.00 & 20.57 & 0 \\ 27.42 & 29.71 & 27.42 & 24.00 & 0 \\ 24.00 & 27.42 & 29.71 & 27.42 & 0 \\ 20.57 & 24.00 & 27.42 & 29.71 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
$$

iii.$\rho = \begin{bmatrix} .9 & 0 & 0 & 0 \end{bmatrix}$, $\mu = 5$,$c = 1$
Consider the associated first-order difference equation:

$$\begin{bmatrix} y_{t+1} \\ y_t \\ y_{t-1} \\ y_{t-2} \\ 1 \end{bmatrix} = \begin{bmatrix} .9 & 0 & 0 & 0 & .5 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ y_{t-3} \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} w_{t+1}. \tag{4}$$

The eigenvalues are given by $\lambda' = \begin{bmatrix} 0 & 0 & 0 & 0.9 & 1 \end{bmatrix}$. Note that all the eigenvalues are bounded below unity except for the one associated with the constant term.

$$C_x(0) = \begin{bmatrix} 5.26 & 4.73 & 4.26 & 3.83 & 0 \\ 4.73 & 5.26 & 4.73 & 4.26 & 0 \\ 4.26 & 4.73 & 5.26 & 4.73 & 0 \\ 3.83 & 4.26 & 4.73 & 5.26 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and $\mu' = \begin{bmatrix} 5 & 5 & 5 & 5 & 1 \end{bmatrix}$.
In order for the sequence $\{x_t\}$ to satisfy stationarity, the intitial value x_0 needs to be drawn from the stationary distribution with μ and $C_x(0)$ as the unconditional moments.
iv. $\rho = \begin{bmatrix} .2 & 0 & 0 & .5 \end{bmatrix}$, $\mu = 5, c = 1$,

$$A = \begin{bmatrix} .2 & 0 & 0 & .5 & 1.5 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

As before, calculate the eigenvalues: $\lambda' = \begin{bmatrix} .8957 & .0496 + .8365i & .0496 - .8365i & -.7950 & 1 \end{bmatrix}$. Note that there are 2 complex eigenavlues. These invariably appear as complex conjugates:

$$\lambda_1 = a + bi; \lambda_2 = a - bi.$$

Rewrite it in polar coordinate form:

$$\lambda_1 = R\left[\cos\theta + i\sin\theta\right],$$

where R and θ are defined as:

$$R = \sqrt{a^2 + b^2}; \cos\theta = \frac{a}{R}; \sin\theta = \frac{b}{R},$$

R is the modulus of a complex numer. All of the relevant eigenvalues are bounded below unity in modulus ($R = \sqrt{a^2 + b^2} = .83$). Next, compute $C_x(0)$:

$$C_x(0) = \begin{bmatrix} 1.47 & .41 & .16 & .24 & 0 \\ .41 & 1.47 & .41 & .16 & 0 \\ .16 & .41 & 1.47 & .41 & 0 \\ .24 & .16 & .41 & 1.47 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and $\mu' = \begin{bmatrix} 5 & 5 & 5 & 5 & 1 \end{bmatrix}$.
v. $\rho = \begin{bmatrix} .8 & .3 & 0 & 0 \end{bmatrix}$, $\mu = 5, c = 1$,

$$A = \begin{bmatrix} .8 & .3 & 0 & 0 & 1.5 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and compute the eigenvalues: $\lambda' = \begin{bmatrix} 0 & .0 & -.27 & 1.07 & 1 \end{bmatrix}$. The 1st condition for stationarity is violated.
c. Note that in a linear model the conditonal expectation and the best linear predictor coincide. Recall the set of K orthogonality conditions defining the best linear predictor, i.e. the linear projection of $Y = y_{t+5}$ on $X = \begin{bmatrix} y_t & y_{t-1} & y_{t-2} & y_{t-3} & 1 \end{bmatrix}$:

$$E\left(X\left(Y - X'\beta\right)\right) = 0,$$

where $K = 5$ (# of parameters). Solving for β yields the following expression:

$$\beta = \left(E(XX')\right)^{-1} E\left(XY\right).$$

Importantly, no stationarity assumptions have been imposed. Two observations are worth mentioning here. First, note that $X = x_t$, as defined in part b. Keep in mind that $E(x_t - \mu)(x_t - \mu)' = C_x(0) = Ex_tx_t' - \mu\mu'$.

$$C_{x,t}(0) = E(XX') - E(X)E(X'),$$

which implies that:

$$E(XX') = C_{x,t}(0) + \mu_t\mu_t'.$$

Second, note that:

$$\begin{aligned} E\left(Xy_{t+5}\right) &= E\left(x_tx_{t+5}'G'\right) \\ &= \left(C_{x,t}(-5) + \mu_t\mu_{t+5}'\right)G', \end{aligned}$$

where $G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $C_{x,t}(-5) = C_{x,t}(5)' = C_{x,t}(0)'A^{5\prime}$.
Assuming stationarity, we obtain the following formula:

$$\begin{aligned} \beta &= \left(C_x(0) + \mu\mu'\right)^{-1}\left(C_x(-5) + \mu\mu'\right)G' \\ &= \left(C_x(0) + \mu\mu'\right)^{-1}\left(C_x(0)A^{5\prime} + \mu\mu'\right)G'. \end{aligned}$$

i. $\beta' = \begin{bmatrix} .7387 & -.26 & 0.0 & 0.0 & 5.21 . \end{bmatrix}$
ii. $\beta' = \begin{bmatrix} .7387 & -.26 & 0.0 & 0.0 & 5.21 \end{bmatrix}$.
iii. $\beta' = \begin{bmatrix} .5905 & 0.0 & 0.0 & 0.0 & 2.0476 \end{bmatrix}$.
iv. $\beta' = \begin{bmatrix} .2003 & .02 & .004 & .25 & 2.6244 \end{bmatrix}$.
d. Assume the eigenvalues of $.95A$ are bounded below untiy in modulus:

$$\begin{aligned} E_t \sum_{k=0}^{\infty} .95^k y_{t+k} &= E_t \sum_{k=0}^{\infty} .95^k G x_{t+k} \\ &= G \sum_{k=0}^{\infty} .95^k A^k x_t \\ &= G \left(I - .95A\right)^{-1} x_t. \end{aligned}$$

By the same reasoning as before, let $Y = G\left(I - .95A\right)^{-1} x_t$ and let $X' = \begin{bmatrix} y_t & y_{t-1} & y_{t-2} & y_{t-3} & 1 \end{bmatrix}$.. Solving for β yields the following expression:

$$\begin{aligned} \beta &= \left(E(XX')\right)^{-1} E\left(XY\right) \\ &= \left(C_x(0) + \mu\mu'\right)^{-1} E(x_t x_t') \left(I - .95A\right)^{-1\prime} G' \\ &= \left(C_x(0) + \mu\mu'\right)^{-1} \left(C_x(0) + \mu\mu'\right) \left(I - .95A\right)^{-1\prime} G' \\ &= \left(I - .95A\right)^{-1\prime} G', \end{aligned}$$

where $G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$.
i. $\beta' = \begin{bmatrix} 7.64 & -2.17 & 0.0 & 0 & 145.3155 \end{bmatrix}$.
ii. $\beta' = \begin{bmatrix} 7.64 & -2.17 & 0.0 & 0 & 145.3155 \end{bmatrix}$.
iii. $\beta' = \begin{bmatrix} 6.89 & 0.0 & 0.0 & 0 & 65.51 \end{bmatrix}$.
iv. $\beta' = \begin{bmatrix} 2.4829 & 1.0644 & 1.1204 & 1.1794 & 70.76 \end{bmatrix}$.
v. $.95A$ has eigenvalues: $\lambda' = \begin{bmatrix} 0 & 0 & -..26 & 1.02 & .95 \end{bmatrix}$; $E_t \sum_{k=0}^{\infty} .95^k y_{t+k}$ explodes.
e.To compute the autocovariances, recall that $C_x(j) = A^j C_x(0)$

i. $C_x(1)(1,1) = 6.85$, $C_x(5)(1,1) = 3.70$, $C_x(10)(1,1) = 1.59$.

ii. $C_x(1)(1,1) = 27.41$, $C_x(5)(1,1) = 14.81$, $C_x(10)(1,1) = 6.39$.

iii. $C_x(1)(1,1) = 4.73$, $C_x(5)(1,1) = 3.10$, $C_x(10)(1,1) = 1.83$.

iv. $C_x(1)(1,1) = .41$, $C_x(5)(1,1) = .36$, $C_x(10)(1,1) = .13$.

Exercise 1.5.

A consumer's rate of consumption follows the stochastic process

$$c_{t+1} = \alpha_c + \sum_{j=1}^{2} \rho_j c_{t-j+1} + w_{t+1} + \sum_{j=1}^{2} \delta_j z_{t+1-j} + \psi_1 w_{1,t+1}, \tag{5}$$

$$z_{t+j} = \sum_{j=1}^{2} \gamma_j c_{t-j+1} + \sum_{j=1}^{2} \phi_j z_{t-j+1}, + \psi_2 w_{2,t+1} \tag{6}$$

where w_{t+1} is a 2×1 martingale difference sequence, adapted to $J_t = \begin{bmatrix} w_t & \dots w_1 & c_0 & c_{-1} & z_0 & z_{-1} \end{bmatrix}$, with contemporaneous covariance matrix $E w_{t+1} w_{t+1}' | J_t = I$, and the coefficients $\rho_j, \delta_j, \gamma_j, \phi_j$ are such that the matrix

$$A = \begin{bmatrix} \rho_1 & \rho_2 & \delta_1 & \delta_2 & \alpha_c \\ 1 & 0 & 0 & 0 & 0 \\ \gamma_1 & \gamma_2 & \phi_1 & \phi_2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

has eigenvalues bounded strictly below unity in modulus.
The consumer evaluates consumption streams according to

$$V_0 = E_0 \sum_{t=0}^{\infty} .95^t u(c_t), \tag{2}$$

where the one-period utility function is

$$u(c_t) = -.5(c_t - 60)^2. \tag{3}$$

a. Find a formula for V_0 in terms of the parameters of the one-period utility function (3) and the stochastic process for consumption.

b. Compute V for the following two sets of parameter values: *i.* $\rho = \begin{bmatrix} .8 & -.3 \end{bmatrix}, \alpha_c = 1, \delta = \begin{bmatrix} .2 & 0 \end{bmatrix} \gamma = \begin{bmatrix} 0 & 0 \end{bmatrix}, \phi = \begin{bmatrix} .7 & -.2 \end{bmatrix}$, $\psi_1 = \psi_2 = 1$.

ii. Same as for part i except now $\psi_1 = 2, \psi_2 = 1$.

Hint: Remember `doublej.m` .

Solution

a. Consider the first-order linear difference equation:

$$\begin{bmatrix} c_{t+1} \\ c_t \\ z_{t+1} \\ z_t \\ 1 \end{bmatrix} = \begin{bmatrix} \rho_1 & \rho_2 & \delta_1 & \delta_2 & \alpha_c \\ 1 & 0 & 0 & 0 & 0 \\ \gamma_1 & \gamma_2 & \phi_1 & \phi_2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_t \\ c_{t-1} \\ z_t \\ z_{t-2} \\ 1 \end{bmatrix} + \begin{bmatrix} \psi_1 & 0 \\ 0 & 0 \\ 0 & \psi_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_{1,t+1} \\ w_{2,t+1} \end{bmatrix}. \tag{7}$$

Guess that V_0 is quadratic in x, the state vector:

$$V_0 = x' B x + d,$$

where d is an arbitrary state vector.

Then we know, from the definition of V_0, that:

$$V_0 = (x_0' G x_0) + \beta E_0 V_1, \tag{8}$$

where

$$G = \begin{bmatrix} -.5 & 0 & 0 & 0 & 30 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 30 & 0 & 0 & 0 & -1800 \end{bmatrix}.$$

Note that:

$$\begin{aligned} E_0 V_1 &= d + E_0 \left(A x_0 + C w_1\right)' B \left(A x_0 + C w_1\right) \\ &= x_0 A' B A' x_0 + E_0 (w_1' C' B C w_1) \\ &= x_0 A' B A' x_0 + \operatorname{tr}\left(B C E_0 w_1 w_1' C'\right) \\ &= d + x_0 A' B A' x_0 + \operatorname{tr}\left(B C C'\right). \end{aligned}$$

Substitute this result back into eq.(21):

$$\begin{aligned} x_0' B x_0 + d &= x_0' G x_0 + \beta \left[x_0' A' B A' x_0 + \operatorname{tr}\left(B C C'\right)\right] + \beta d \\ &= x_0' G x_0 + \beta \left[x_0' A' B A' x_0 + \operatorname{tr}\left(B C C'\right) + d\right]. \end{aligned}$$

Collecting terms, this yields two equations:

$$B = G + \beta \left[A' B A\right], \tag{9}$$

and

$$d(1 - \beta) = \beta \operatorname{tr}\left(C' B C\right). \tag{10}$$

i.Use `ex0105.m` to compare your solutions. Make sure not to forget the discount factor $\beta = .95$. The command to compute B is doublej $(\sqrt{.95}A, G)$, which produces:

$$\begin{aligned} B &= 10^4 * \begin{bmatrix} -1.3284 & -1.2803 & 0 & 0 & -0.6690 \\ -1.2803 & -1.2620 & 0 & 0 & -0.6356 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -0.6690 & -0.6356 & 0 & 0 & -0.3600 \end{bmatrix} \\ d &= -2.5240e + 006. \end{aligned}$$

ii. Note that only d changes (the risk premium):

$$d = -1.0096e + 007.$$

Exercise 1.6.

Consider the stochastic process $\{c_t, z_t\}$ defined by equations (1) in exercise 1.5. Assume the parameter values described in part b, item i. If possible, assume the initial conditions are such that $\{c_t, z_t\}$ is covariance stationary.

a. Compute the initial mean and covariance matrix that make the process covariance stationary.

b. For the initial conditions in part a, compute numerical values of the following population linear regression:

$$c_{t+2} = \alpha_0 + \alpha_1 z_t + \alpha_2 z_{t-4} + \epsilon_t$$

where $E\epsilon_t \begin{bmatrix} 1 & z_t & z_{t-4} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$.

Solution

a. Use `ex0105.m` to compare your solutions

$$C_x(0) = \begin{bmatrix} 1.97 & 1.24 & 0.24 & 0.48 & 0 \\ 1.24 & 1.97 & .07 & .24 & 0 \\ 0.24 & 0.07 & 1.57 & .92 & 0 \\ 0.48 & 0.24 & .92 & 1.57 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and $\mu' = \begin{bmatrix} .666 & .666 & 0 & 0 & .3333 \end{bmatrix} \times 3 = \begin{bmatrix} 2 & 2 & 0 & 0 & 1 \end{bmatrix}$.

b. Following the same line of reasoning as before, derive the orthogonality conditions:

$$EX'(Y - XB) = 0,$$

where $X' = \begin{bmatrix} 1 & z_t & z_{t-4} \end{bmatrix}$ and $Y = c_{t+2}$.Solving for β :

$$\beta = (E(XX'))^{-1} E(XY),$$

where

$$E(XX') = \begin{bmatrix} 1 & 1 & 1 \\ Ez_t^2 & \text{cov}(z_t, z_{t-4}) & 1 \\ \text{cov}(z_t, z_{t-4}) & Ez_{t-4}^2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1.57 & -.0336 & 1 \\ -.0336 & 1.57 & 1 \end{bmatrix}.$$

Note that $\text{cov}(z_t, z_{t-4})$ is the (3,3) element of $C_x(4) = A^4 C_x(0)$. Similarly,

$$E(XY) = \begin{bmatrix} E(c_{t+2}) \\ cov(c_{t+2}, z_t) \\ cov(c_{t+2}, z_{t-4}) \end{bmatrix} = \begin{bmatrix} 2 \\ 0.1155 \\ -.0497 \end{bmatrix}.$$

This implies

$$\beta' = \begin{bmatrix} 4.29 & 4.19 & -6.48 \end{bmatrix}.$$

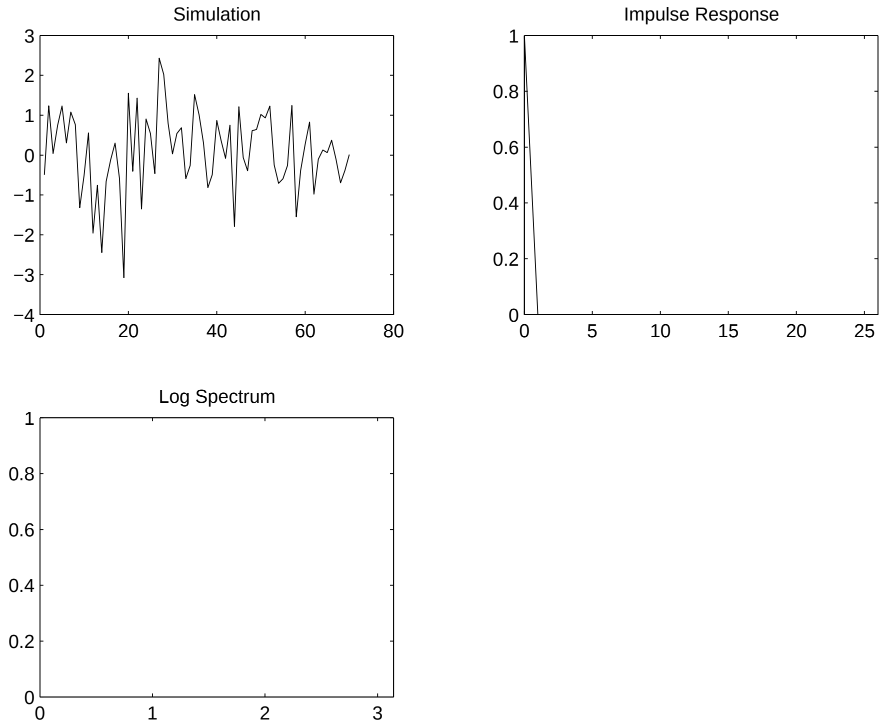


FIGURE 1. Exercise 1.7 a

Exercise 1.7.

Get the Matlab programs `bigshow.m` and `freq.m` .

Use `bigshow` to compute and display a simulation of length 80, an impulse response function, and a spectrum for each of the following scalar stochastic processes y_t. In each of the following, w_t is a scalar martingale difference sequence adapted to its own history and the initial values of lagged y's. a. $y_t = w_t$. b. $y_t = (1 + .5L)w_t$. c. $y_t = (1 + .5L + .4L^2)w_t$. d. $(1 - .999L)y_t = (1 - .4L)w_t$. e. $(1 - .8L)y_t = (1 + .5L + .4L^2)w_t$. f. $(1 + .8L)y_t = w_t$. g. $y_t = (1 - .6L)w_t$.

Study the output and look for patterns. When you are done, you will be well on your way to knowing how to read spectral densities.

Solution

a. $y_t = w_t$ see Figure 1.
b. $y_t = (1 + 0.5L)w_t$ see Figure 2.
c. $y_t = (1 + 0.5L + 0.4L^2)w_t$ see Figure 3.
d. $(1 - 0.999L)y_t = (1 - 0.4L)w_t$ see Figure 4.
e. $(1 - 0.8L)\ y_t = (1 + 0.5L + 0.4L^2)w_t$ see Figure 5.
f. $(1 + 0.8L)y_t = w_t$ see Figure 6.

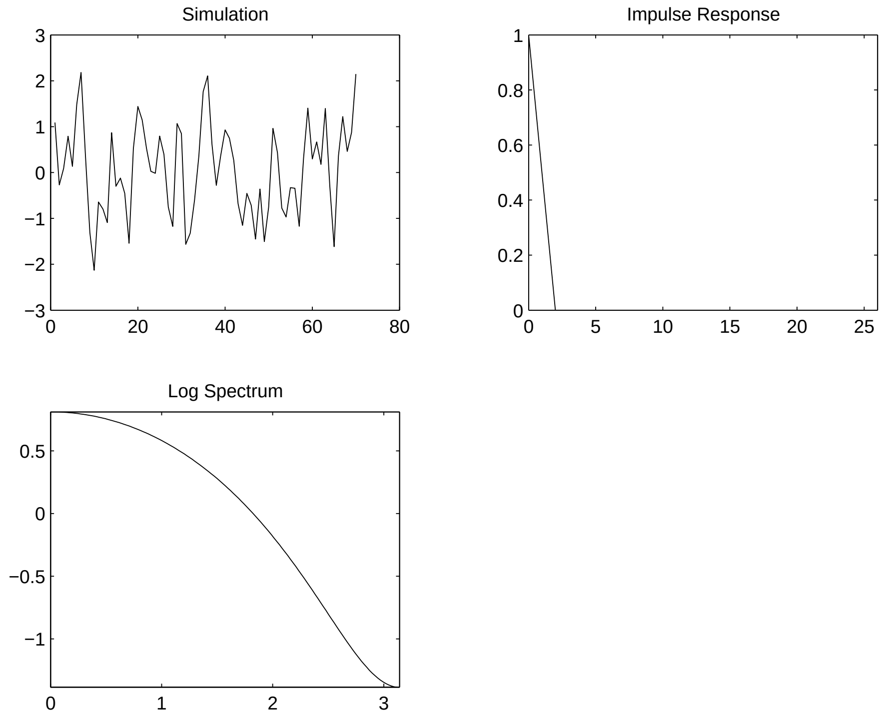


FIGURE 2. Exercise 1.7 b

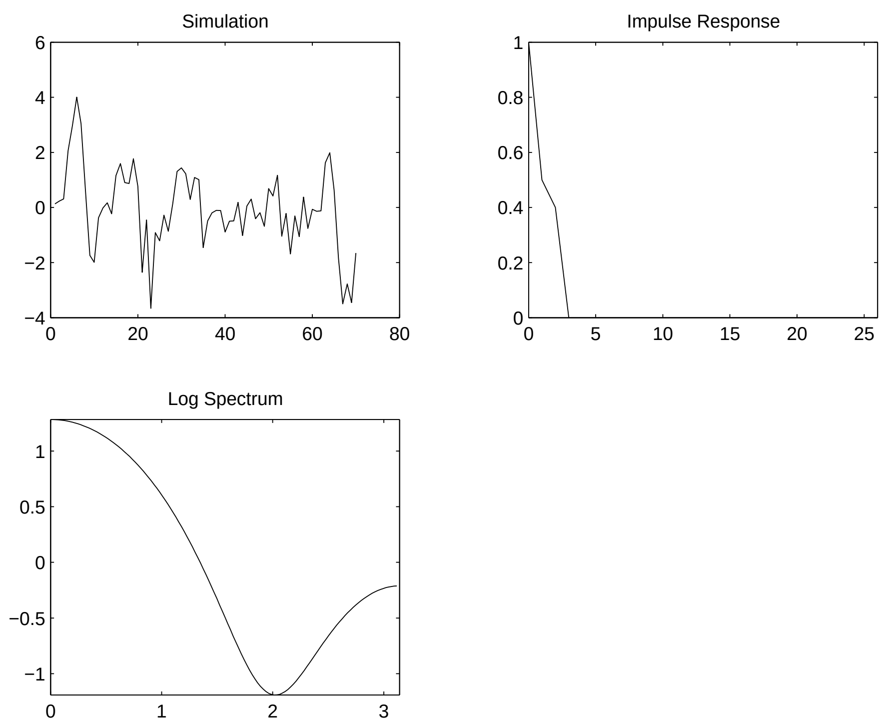


FIGURE 3. Exercise 1.7 c

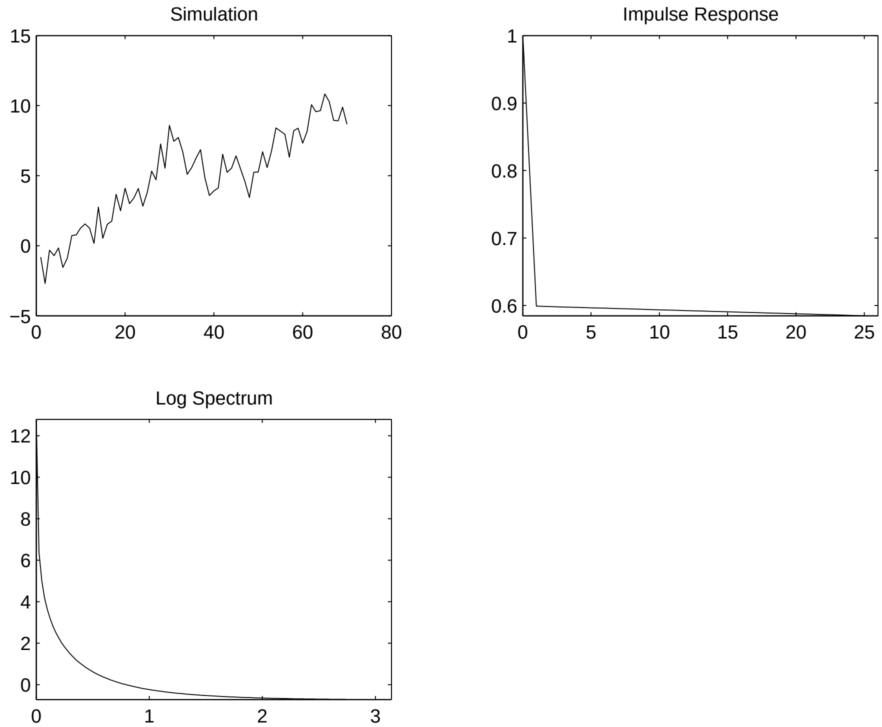


FIGURE 4. Exercise 1.7 d

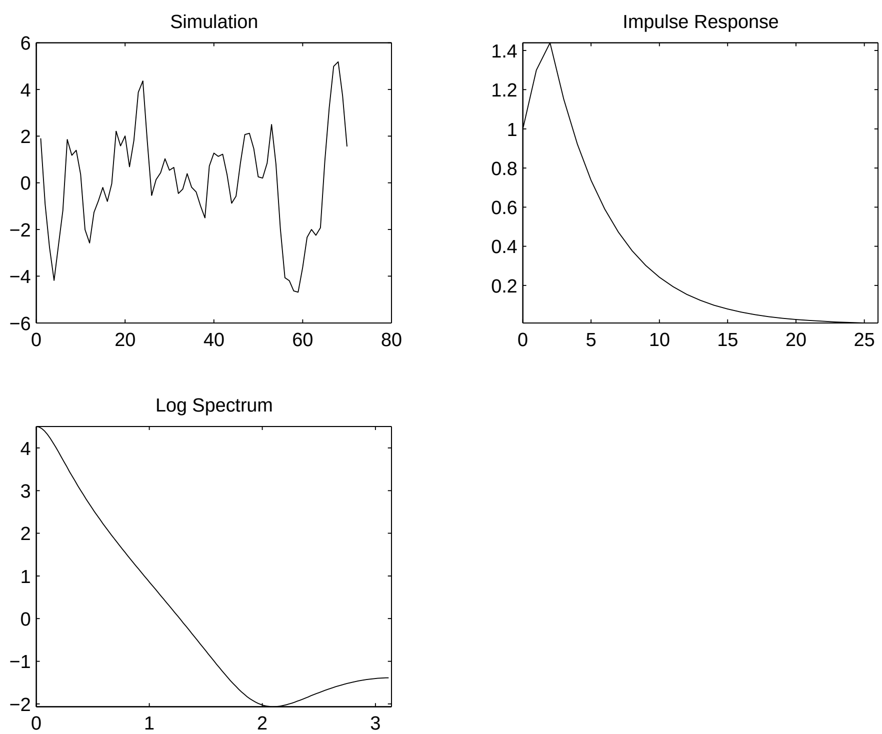


FIGURE 5. Exercise 1.7 e

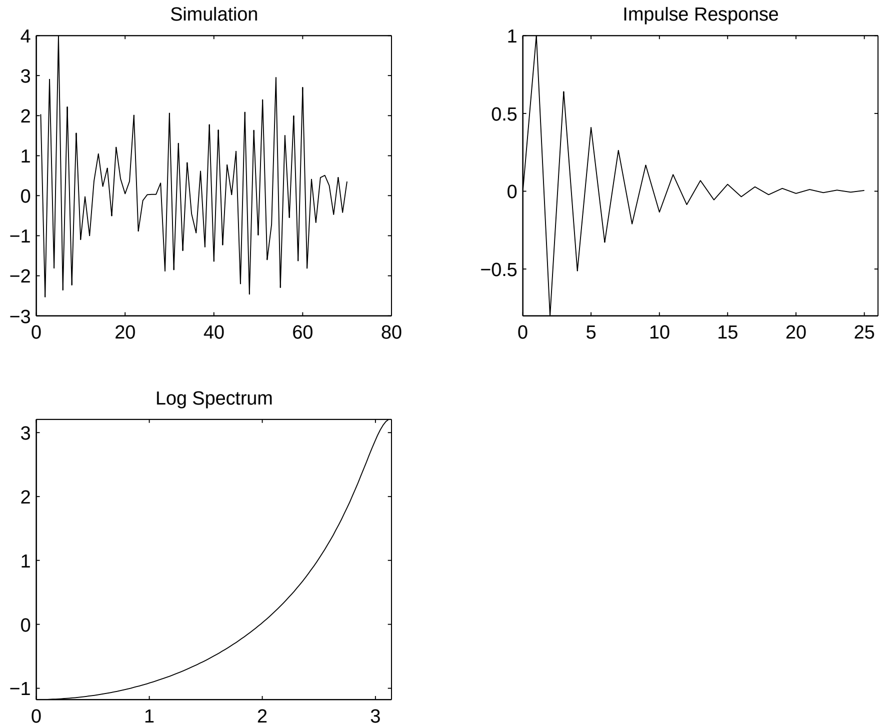


FIGURE 6. Exercise 1.7 f

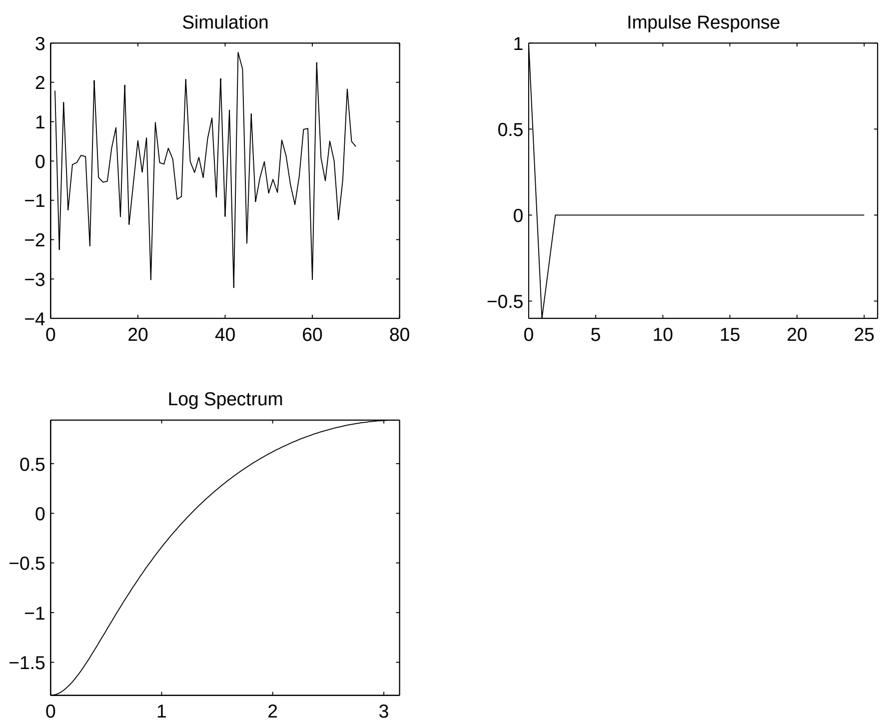


FIGURE 7. Exercise 1.7 g

g. $y_t = (1 - 0.4L)w_t$ see Figure 7.

Exercise 1.8.

This exercise deals with Cagan's money demand under rational expectations. A version of Cagan's (1956) demand function for money is

$$m_t - p_t = -\alpha(p_{t+1} - p_t), \alpha > 0,\ t \geq 0, \tag{1}$$

where m_t is the log of the nominal money supply and p_t is the price level at t. Equation (1) states that the demand for real balances varies inversely with the expected rate of inflation, $(p_{t+1} - p_t)$. There is no uncertainty, so the expected inflation rate equals the actual one. The money supply obeys the difference equation

$$(1 - L)(1 - \rho L)m_t^s = 0 \tag{2}$$

subject to initial condition for m_{-1}^s, m_{-2}^s. In equilibrium,

$$m_t \equiv m_t^s \quad \forall t \geq 0 \tag{3}$$

(i.e., the demand for money equals the supply). For now assume that

$$|\rho\alpha/(1 + \alpha)| < 1. \tag{4}$$

An *equilibrium* is a $\{p_t\}_{t=0}^{\infty}$ that satisfies equations (1), (2), and (3) for all t.

a. Find an expression an equilibrium p_t of the form

$$p_t = \sum_{j=0}^{n} w_j m_{t-j} + f_t. \tag{5}$$

Please tell how to get formulas for the w_j for all j and the f_t for all t.

b. How many equilibria are there?

c. Is there an equilibrium with $f_t = 0$ for all t? **d.** Briefly tell where, if anywhere, condition (4) plays a role in your answer to part a.

e. For the parameter values $\alpha = 1, \rho = 1$, compute and display all the equilibria.

Solution

a. First, consider the money demand equation and rewrite the demand for money as a function of the future time path of prices:

$$\begin{aligned} m_t &= \left((1 + \alpha) - \alpha L^{-1}\right) p_t \\ &= (1 + \alpha)\left(1 - \frac{\alpha}{1 + \alpha} L^{-1}\right) p_t. \end{aligned} \tag{11}$$

We know that in equilibrium: $m_t^s = m_t$ for all $t \geq 0$. This last observation together with equation (11) implies that the current price can be expressed as a function of the entire sequence of future money supplies:

$$(12) \qquad p_t = \frac{1}{1+\alpha}\sum_{j=0}^{\infty}\left(\frac{\alpha}{1+\alpha}\right)^j L^{-j}m_t^s + \left(\frac{1+\alpha}{\alpha}\right)^t c$$
$$= \frac{1}{1+\alpha}\sum_{j=0}^{\infty}\left(\frac{\alpha}{1+\alpha}\right)^j m_{t+j}^s + \left(\frac{1+\alpha}{\alpha}\right)^t c.$$

where c is some arbitrary nonnegative constant (naturally, we want to keep the price level positive for all t).
Next, let us turn to the money supply equation. Note that the money supply difference equation has a unit root which means we cannot simply apply the usual approach. In stead, working forward, starting at time 0, we get:

$$m_0^s - m_{-1}^s = \rho(m_{-1}^s - m_{-2}^s),$$

and, similarly, we find that at time 1 :

$$m_1^s - m_0^s = \rho(m_0^s - m_{-1}^s).$$

By substituting backwards repeatedly, we find that the money supply, in levels, is given by:

$$(13) \qquad m_t^s = \rho\left(1+\rho+\rho^2+\ldots+\rho^t\right)\left(m_{-1}^s - m_{-2}^s\right),$$

which, for $|\rho| < 1$, becomes:

$$m_t^s = \rho\frac{1-\rho^{t+1}}{1-\rho}\left(m_{-1}^s - m_{-2}^s\right).$$

The money supply at t can be written in terms of its two inital values. This money supply equation can be plugged back into the price level equation in (64), which produces:

$$(14) \quad p_t = \frac{1}{1+\alpha}\sum_{j=0}^{\infty}\left(\frac{\alpha}{1+\alpha}\right)^j \rho\frac{1-\rho^{t+j+1}}{1-\rho}\left(m_{-1}^s - m_{-2}^s\right) + \left(\frac{1+\alpha}{\alpha}\right)^t c$$
$$(15) \qquad = \left[\frac{\rho}{1-\rho} - \left(\frac{\rho^{t+2}}{1-\rho}\right)\frac{1}{1+\alpha(1-\rho)}\right]\left(m_{-1}^s - m_{-2}^s\right) + \left(\frac{1+\alpha}{\alpha}\right)^t c$$
$$(16) \qquad = \frac{\rho}{1-\rho}\left[1-\rho^{t+1}\frac{1}{1+\alpha(1-\rho)}\right]\left(m_{-1}^s - m_{-2}^s\right) + \left(\frac{1+\alpha}{\alpha}\right)^t c.$$

Hence

$$p_t = \sum_{j=0}^{n} w_j m_{t-j} + f_t,$$

we know that, for $|\rho| < 1$

$$
\begin{aligned}
w_j &= \frac{\rho}{1-\rho}\left[1-\rho^j\frac{1}{1+\alpha(1-\rho)}\right]\ for\ j=t+1 \\
&= -\frac{\rho}{1-\rho}\left[1-\rho^{j-1}\frac{1}{1+\alpha(1-\rho)}\right]\ for\ j=t+2 \\
&= 0 \text{ for } j\notin\{t+1,t+2\},
\end{aligned}
$$

and as for the second part:

$$
f_t=\left(\frac{1+\alpha}{\alpha}\right)^t c,
$$

where c is an arbitrary non-negative constant
b. Since we can pick any constant $c\geq 0$ in f_t, we can construct infinitely many sequences $\{p_t\}_{t=0}^{\infty}$ that satisfy the equilibrium condition at all $t\geq 0$.
c. There is an equilibrium with $f_t=0$ for all t, which is obtained by setting $c=0$. This immediately fixes the initial price level p_0 in terms of the initial money supplies:

$$
\begin{aligned}
p_0 &= \frac{\rho}{1-\rho}\left[1-\frac{\rho}{1+\alpha(1-\rho)}\right]\left(m_{-1}^s-m_{-2}^s\right) \\
&= \frac{\rho}{1-\rho}\left[\frac{(1+\alpha)(1-\rho)}{1+\alpha(1-\rho)}\right]\left(m_{-1}^s-m_{-2}^s\right) \\
&= \left[\frac{\rho(1+\alpha)}{1+\alpha(1-\rho)}\right]\left(m_{-1}^s-m_{-2}^s\right).
\end{aligned}
$$

d. This condition guarantees that

$$
\sum_{j=0}^{\infty}\left(\frac{\alpha}{1+\alpha}\right)^j\frac{\rho^{t+j+2}}{1-\rho},
$$

in (14) is bounded.
e. Set $\rho=1$ in equation (13) and you obtain:

$$
m_t^s=(t+1)\left(m_{-1}^s-m_{-2}^s\right).
$$

Now, recall that, for $\alpha=1$,

$$\begin{aligned}
p_t &= \frac{1}{1+\alpha}\sum_{j=0}^{\infty}\left(\frac{\alpha}{1+\alpha}\right)^j m^s_{t+j} + (2)^t c. \\
&= \frac{1}{1+\alpha}\sum_{j=0}^{\infty}\left(\frac{\alpha}{1+\alpha}\right)^j (t+1+j)\left(m^s_{-1} - m^s_{-2}\right) + (2)^t c. \\
&= (t+1)\left(m^s_{-1} - m^s_{-2}\right) + \frac{1}{1+\alpha}\sum_{j=0}^{\infty}\left(\frac{\alpha}{1+\alpha}\right)^j j\left(m^s_{-1} - m^s_{-2}\right) + (2)^t c \\
&= (t+1)\left(m^s_{-1} - m^s_{-2}\right) + \alpha\left(m^s_{-1} - m^s_{-2}\right) + (2)^t c \\
&= (t+2)\left(m^s_{-1} - m^s_{-2}\right) + (2)^t c,
\end{aligned}$$

for $c \geq 0$, where we have used :

$$\sum jx^j = x\frac{d}{dx}\sum x^j = x\frac{d}{dx}\frac{1}{1-x} = \frac{x}{(1-x)^2}.$$

Hence, we have constructed an infinite number of equilibria, each of which corresponds to a different $c \geq 0$.

Exercise 1.9.

The $n \times 1$ state vector of an economy is governed by the linear stochastic difference equation

(1) $$x_{t+1} = Ax_t + C_t w_{t+1}$$

where C_t is a possibly time varying matrix (known at t) and w_{t+1} is an $m \times 1$ martingale difference sequence adapted to its own history with $Ew_{t+1}w'_{t+1}|J_t = I$, where $J_t = \begin{bmatrix} w_t & \ldots & w_1 & x_0 \end{bmatrix}$. A scalar one-period payoff p_{t+1} is given by

(2) $$p_{t+1} = Px_{t+1}$$

The stochastic discount factor for this economy is a scalar m_{t+1} that obeys

(3) $$m_{t+1} = \frac{Mx_{t+1}}{Mx_t}.$$

Finally, the price at time t of the one-period payoff is given by $q_t = f_t(x_t)$, where f_t is some possibly time-varying function of the state. That m_{t+1} is a stochastic discount factor means that

(4) $$E(m_{t+1}p_{t+1}|J_t) = q_t.$$

a. Compute $f_t(x_t)$, describing in detail how it depends on A and C_t.

b. Suppose that an econometrician has a time series data set $X_t = \begin{bmatrix} z_t & m_{t+1} & p_{t+1} & q_t \end{bmatrix}$, for $t = 1, \ldots, T$, where z_t is a strict subset of the variables in the state x_t. Assume that investors in the economy see x_t even though the econometrician only sees a subset z_t of x_t. Briefly describe a way to use these data to test implication (4). (Possibly but perhaps not useful hint: recall the law of iterated expectations.)

Solution

a.

$$\begin{aligned}
f_t(x_t) &= q_t = E_t\left[m_{t+1}p_{t+1}\right] \\
&= E_t\left[\frac{Mx_{t+1}Px_{t+1}}{Mx_t}\right] \\
&= \frac{1}{Mx_t}E_t\left[M(Ax_t + C_t w_{t+1})P(Ax_t + C_t w_{t+1})\right] \\
&= \frac{1}{Mx_t}E_t\left[(MAx_t + MC_t w_{t+1})(PAx_t + PC_t w_{t+1})\right] \\
&= \frac{1}{Mx_t}E_t\left[MAx_t PAx_t + MC_t w_{t+1}PC_t w_{t+1}\right] \\
&= \frac{1}{Mx_t}E_t\left[MAx_t x_t' A' P' + MC_t w_{t+1}w_{t+1}' C_t' P'\right] \\
&= \frac{1}{Mx_t}\left\{MAx_t x_t' A' P' + MC_t E_t\left[w_{t+1}w_{t+1}'\right] C_t' P'\right\} \\
&= \frac{1}{Mx_t}\left\{MAx_t x_t' A' P' + MC_t C_t' P'\right\}.
\end{aligned}$$

b. Because $X_t \subset J_t$, and by the law of iterated expectations, rewrite the Euler equation $q_t = E_t\left[m_{t+1}p_{t+1}\right]$ as follows:

$$E\left[m_{t+1}p_{t+1} - f_t(x_t)|X_t\right] = 0.$$

This condition states that $m_{t+1}p_{t+1} - q_t$ is orthogonal to the information set X_t and hence to every subset of X_t such as z_t. Therefore:

$$E\left[\left(m_{t+1}p_{t+1} - f_t(x_t)\right)z_t\right] = 0.$$

We can test the Euler equation $q_t = E_t\left[m_{t+1}p_{t+1}\right]$ by testing the condition $E\left[\left(m_{t+1}p_{t+1} - q_t\right)z_t\right] = 0$. This can be tested by the econometrician by regressing $m_{t+1}p_{t+1} - q_t$ on z_t and checking whether the hypothesis that the coefficient on z_t, $\beta_z = 0$, cannot be rejected. *Exercise 1.10* Let P be a transition matrix for a Markov chain that has two distinct eigenvectors π_1, π_2 corresponding to unit eigenvalues of P. Prove for any $\alpha \in [0,1]$ that $\alpha\pi_1 + \alpha\pi_2$ is an invariant distribution of P.

Exercise 1.10.

Consider a Markov chain with transition matix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ .2 & .5 & .3 \\ 0 & 0 & 1 \end{bmatrix},$$

with initial distribution $\pi_0 = \begin{bmatrix}\pi_{1,0} & \pi_{2,0} & \pi_{3,0}\end{bmatrix}'$. Let $\pi_t = \begin{bmatrix}\pi_{1t} & \pi_{2t} & \pi_{3t}\end{bmatrix}'$ be the distribution over states at time t. Prove that for $t > 0$

$$\begin{aligned}\pi_{1t} &= \pi_{1,0} + .2\left(\tfrac{1-.5^t}{1-.5}\right)\pi_{2,0}\\ \pi_{2t} &= .5^t\pi_{2,0}\\ \pi_{3t} &= \pi_{3,0} + .3\left(\tfrac{1-.5^t}{1-.5}\right)\pi_{2,0}.\end{aligned}$$

Solution

The transition can be written as

$$\begin{bmatrix}\pi_{1,t+1}\\ \pi_{2,t+1}\\ \pi_{3,t+1}\end{bmatrix} = \begin{bmatrix}1 & 0.2 & 0\\ 0 & 0.5 & 0\\ 0 & 0.3 & 1\end{bmatrix}\begin{bmatrix}\pi_{1,t}\\ \pi_{2.t}\\ \pi_{3,t}\end{bmatrix}.$$

Looking at subsequent transitions, the first and third colums are left unchanged. We find that the second column changes as follows: the second row is simply 0.5^t because the other two elements on the second row are zero. The first row, second column element is given by: $p_{21}(1+p_{22}+p_{22}^2+p_{22}^3+...p_{22}^t) = p_{21}\left(\frac{1-p_{22}^t}{1-p_{22}}\right) = 0.2\left(\frac{1-0.5^t}{1-0.5}\right)$. The same logic holds true for the third row second column element: $0.3\left(\frac{1-0.5^t}{1-0.5}\right)$.
Therefore, the stationary distribution becomes

$$\begin{bmatrix}\pi_{1,t}\\ \pi_{2,t}\\ \pi_{3,t}\end{bmatrix} = \begin{bmatrix}1 & 0.2\left(\frac{1-0.5^t}{1-0.5}\right) & 0\\ 0 & 0.5^t & 0\\ 0 & 0.3\left(\frac{1-0.5^t}{1-0.5}\right) & 1\end{bmatrix}\begin{bmatrix}\pi_{1,0}\\ \pi_{2.t}\\ \pi_{3,t}\end{bmatrix}.$$

Exercise 1.11.

Let P be a transition matrix for a Markov chain. For $t = 1, 2, \ldots$, prove that the jth column of P^t is the distribution across states at t when the initial distribution is $\pi_{j,0} = 1, \pi_{i,0} = 0 \forall i \neq j$.

Solution

Without loss of generality we assume a 3-state Markov chain. The inital distribution is degenerate in that $\pi_{2,0} = 1$. We have

$$\begin{bmatrix}\pi_{1,1}\\ \pi_{2,1}\\ \pi_{3,1}\end{bmatrix} = \begin{bmatrix}p_{11} & p_{21} & p_{31}\\ p_{12} & p_{22} & p_{32}\\ p_{13} & p_{23} & p_{33}\end{bmatrix}\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}.$$

It is clear that The distribution over states in period 1 is given by the second column of the transition matrix. This is true for every following transition. The initial probability distribution selects off the second column of the transition matrix, which is P^j after j transitions.

CHAPTER 2

Dynamic programming

Exercise 2.1. *Howard's policy iteration algorithm*

Consider the Brock-Mirman problem: to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln c_t,$$

subject to $c_t + k_{t+1} \leq A k_t^{\alpha} \theta_t$, k_0 given, $A > 0$, $1 > \alpha > 0$, where $\{\theta_t\}$ is an i.i.d. sequence with $\ln \theta_t$ distributed according to a normal distribution with mean zero and variance σ^2.
Consider the following algorithm. Guess at a policy of the form $k_{t+1} = h_0(A k_t^{\alpha} \theta_t)$ for any constant $h_0 \in (0, 1)$. Then form

$$J_0(k_0, \theta_0) = E_0 \sum_{t=0}^{\infty} \beta^t \ln(A k_t^{\alpha} \theta_t - h_0 A k_t^{\alpha} \theta_t).$$

Next choose a new policy h_1 by maximizing

$$\ln(A k^{\alpha} \theta - k') + \beta E J_0(k', \theta'),$$

where $k' = h_1 A k^{\alpha} \theta$. Then form

$$J_1(k_0, \theta_0) = E_0 \sum_{t=0}^{\infty} \beta^t \ln(A k_t^{\alpha} \theta_t - h_1 A k_t^{\alpha} \theta_t).$$

Continue iterating on this scheme until successive h_j have converged.
Show that, for the present example, this algorithm converges to the optimal policy function in one step.

Solution

Under the policy $k_{t+1} = h_0 A k_t^{\alpha} \theta_t$, we get:

$$k_1 = h_0 A k_0^{\alpha} \theta_0 \text{ and } \ln k_1 = \ln A h_0 + \ln \theta_0 + \alpha \ln k_0.$$

Similarly, derive $\ln k_2, \ln k_3$...which yields the following recursive equation for $\ln k_t$:

$$\ln k_t = \ln (A h_0) \frac{1 - \alpha^t}{1 - \alpha} + \ln \theta_t + \alpha \ln \theta_{t-1} + \cdots + \alpha^{t-1} \ln \theta_0 + \alpha^t \ln k_0.$$

Plug this recursive formula for $\ln k_t$ into the objective function $E \sum_{t=0}^{\infty} \beta^t \ln (A k_t^{\alpha} \theta_t - h_0 A k_t^{\alpha} \theta_t)$ to derive $J_0(k_0, \theta_0)$:

$$\begin{aligned} J_0(k_0, \theta_0) &= \ln(1 - h_0)A + \ln \theta_0 + \alpha \ln k_0 + \beta \left[\ln(1 - h_0)A + E \ln \theta_1 + \alpha \ln k_1\right] \\ &\quad + \ldots \beta^t \left[\ln(1 - h_0)A + E \ln \theta_t + \alpha \ln k_t\right] + \ldots \\ &= H_0 + H_1 \ln \theta_0 + \frac{\alpha}{1 - \alpha\beta} \ln k_0, \end{aligned}$$

where H_0 and $H_1$are constants. Next, choose a policy h_1 to maximize

$$\begin{aligned} & \ln(Ak^{\alpha}\theta - k') + \beta E J_0(k_1, \theta_1) \\ = \ & \ln(Ak^{\alpha}\theta - k') + \beta E \left[H_0 + H_1 \ln \theta' + \frac{\alpha}{1-\alpha\beta} \ln k' \right]. \end{aligned}$$

The first-order condition for this problem is:

$$-\frac{1}{Ak^{\alpha}\theta - k'} + \frac{\alpha\beta}{1-\alpha\beta}\frac{1}{k'} = 0,$$

which yields: $h_1 = \alpha\beta$. Now, plug the new policy function $k' = h_1 Ak^{\alpha}\theta$ into $E\sum_{t=0}^{\infty}\beta^t \ln\left(Ak_t^{\alpha}\theta_t - h_1 Ak_t^{\alpha}\theta_t\right)$ to derive $J_1(k_0, \theta_0)$. Firts, note that:

$$\ln k_t = \ln\left(Ah_1\right)\frac{1-\alpha^t}{1-\alpha} + \ln\theta_t + \alpha\ln\theta_{t-1} + \cdots + \alpha^{t-1}\ln\theta_0 + \alpha^t \ln k_0 \text{ for t} \geq 1.$$

Using this recursive formula, calculate $J_1(k_0, \theta_0)$:

$$J_1(k_0, \theta_0) = K_0 + K_1 \ln\theta_0 + \frac{\alpha}{1-\alpha\beta}\ln k_0,$$

where K_0 and $K_1$are constants. Next, choose a policy h_2 to maximize

$$\begin{aligned} & \ln(Ak^{\alpha}\theta - k') + \beta E J_1(k_1, \theta_1) \\ = \ & \ln(Ak^{\alpha}\theta - k') + \beta E \left[K_0 + K_1 \ln \theta' + \frac{\alpha}{1-\alpha\beta} \ln k' \right]. \end{aligned}$$

The first-order condition for this problem is:

$$-\frac{1}{Ak^{\alpha}\theta - k'} + \frac{\alpha\beta}{1-\alpha\beta}\frac{1}{k'} = 0,$$

which yields: $h_2 = \alpha\beta$. That's exactly what he had obtained for h_1! We have verified that our improvement algoritm has in fact converged after just one iteration.

CHAPTER 3

Practical dynamic programming

Exercise 3.1. *Value Function Iteration and Policy Improvement Algorithm*

The goal of this exercise is to study, in the context of a specific problem, two methods for solving dynamic programs : value function iteration and Howard's policy improvement. Consider McCall's model of intertemporal job search. An unemployed worker draws one offer from a c.d.f. F, with $F(0) = 0$ and $F(B) = 1$, $B < \infty$. If the worker rejects the offer, she receives unemployment compensation c and can draw a new wage offer next period. If she accepts the offer, she works forever at wage w. The objective of the worker is to maximize the expected discounted value of her earnings. Her discount factor is $0 < \beta < 1$.

a. Write the Bellman equation. Show that the optimal policy is of the reservation wage form. Write an equation for the reservation wage w^*.

b. Consider the value function iteration method. Show that at each iteration, the optimal policy is of the reservation wage form. Let w_n be the reservation wage at iteration n. Derive a recursion for w_n. Show that w_n converges to w^* at rate β.

c. Consider Howard's policy improvement algorithm. Show that at each iteration, the optimal policy is of the reservation wage form. Let w_n be the reservation wage at iteration n. Derive a recursion for w_n. Show that the rate of convergence of w_n towards w^* is (locally) quadratic. Specifically use a Taylor expansion to show that, for w_n close enough to w^*, there is a constant K such that $w_{n+1} - w^* \cong K(w_n - w^*)^2$.

Solution

a. Let $V(w)$ be the value of an unemployed worker with offer w in hand and who behaves optimally. The Bellman equation is:

$$V(w) = \max_{\text{accept,reject}} \left\{ \frac{w}{1-\beta}, c + \beta \int V(w')dF(w') \right\}.$$

The right hand side takes the max of an increasing function and of a constant. Thus, the optimal policy is of the reservation wage form. There is a reservation wage w^* such that, for $w \leq w^*$, the increasing function is less than the constant and the worker rejects the offer. For $w \geq w^*$, the increasing function is greater than the constant and the worker accepts the offer. The reservation wage w^* solves :

$$\begin{aligned}
\tfrac{w^*}{1-\beta} &= c + \beta \textstyle\int V(w')dF(w') \\
&= c + \beta \textstyle\int_0^{w^*} \tfrac{w^*}{1-\beta} dF(w') + \beta \int_{w^*}^{B} \tfrac{w'}{1-\beta} dF(w') \\
&= c + \tfrac{\beta}{1-\beta} w^* F(w^*) + \tfrac{\beta}{1-\beta} w^*(1 - F(w^*)) + \tfrac{\beta}{1-\beta} \textstyle\int_{w^*}^{B} (1 - F(w'))dw' \\
&= c + \tfrac{\beta}{1-\beta} w^* + \tfrac{\beta}{1-\beta} \textstyle\int_{w^*}^{B} (1 - F(w'))dw',
\end{aligned}$$

where the last two equalities are obtained by doing an integration by part on $\int_{w^*}^B w' dF(w')$. Thus, the reservation wage is a solution (actually, the unique one) of the equation:

$$w^* = c(1-\beta) + \beta w^* + \beta \int_{w^*}^B (1 - F(w'))dw'. \tag{17}$$

b. The value function iteration algorithm iterates on the Bellman equation:

$$V^{n+1} = \max_{\text{accept,reject}} \left\{ \frac{w}{1-\beta}, c + \beta \int V^n(w')dF(w') \right\}.$$

As in the previous question, it is apparent that the optimal policy at order $n+1$ is of the reservation wage form. The reservation wage at order $n+1$ solves

$$\frac{w_{n+1}}{1-\beta} = c + \beta \int V^n(w')dF(w').$$

Manipulating this equation exactly as in question **a**, one shows that the sequence of reservation wage satisfies the recursion:

$$w_{n+1} = c(1-\beta) + \beta w_n + \beta \int_{w_n}^B (1 - F(w'))dw'. \tag{18}$$

To show convergence, we substract the equation (17) to equation (18). We obtain :

$$w_{n+1} - w^* = \beta(w_n - w^*) + \beta \int_{w_n}^{w^*} (1 - F(w'))dw'.$$

Observe that $w_n - w^* = -\int_{w_n}^{w^*} dw'$ to get:

$$w_{n+1} - w^* = -\beta \int_{w_n}^{w^*} F(w')dw'.$$

Since $0 \leq F(w') \leq 1$, this last equality implies :

$$|w_{n+1} - w^*| \leq \beta |w_n - w^*|.$$

This shows that the sequence w_n converges to w^* at linear rate β. Note that this linear rate is the one predicted by the contraction mapping theorem.

c. Assume that the optimal policy at iteration n of the policy improvement algorithm is of the reservation wage form. Let w_n be this reservation wage. Let V^n be the value of a worker who uses *forever* the reservation wage policy w_n. For $w \geq w_n$, the worker accepts the offer and $V^n(w) = \frac{w}{1-\beta}$. For $w \leq w^n$, the worker rejects the offer and $V^n(w) = \text{constant} \equiv Q_n$. The constant Q_n solves:

$$\begin{aligned} Q_n &= c + \beta \int_0^{w_n} Q_n dF(w') + \beta \int_{w_n}^B \tfrac{w'}{1-\beta} dF(w') \\ Q_n &= (1 - \beta F(w_n))^{-1} \left(c + \tfrac{\beta}{1-\beta} \int_{w_n}^B w' dF(w') \right). \end{aligned}$$

Observe that the value function at iteration n is not continuous. There is a "jump" at $w = w_n$. The jump expresses that the reservation wage policy w_n is suboptimal. Namely, at $w = w_n$, the worker is not indifferent between accepting or rejecting the offer. Let's do iteration $n + 1$. We need to solve:

$$\begin{aligned} \tilde{V}(w) &= \max_{\text{accept,reject}} \left\{ \tfrac{w}{1-\beta}, c + \beta \int V^n(w') dF(w') \right\} \\ &= \max_{\text{accept,reject}} \left\{ \tfrac{w}{1-\beta}, Q_n \right\}. \end{aligned}$$

It is apparent that the optimal policy is of the reservation wage form. The reservation wage at iteration $n + 1$ solves:

$$w_{n+1} = (1 - \beta F(w_n))^{-1} \left(c(1 - \beta) + \beta \int_{w_n}^B w' dF(w') \right) \equiv G(w_n). \tag{19}$$

Easy algebra shows that the optimal reservation wage w^* is a fixed point of this recursion. We won't show convergence here. To obtain the desired result on the speed of convergence we use a Taylor expansion. For w_n close enough to w^*, we have:

$$w_{n+1} - w^* \cong G'(w^*)(w_n - w^*) + 1/2 G''(w^*)(w_n - w^*)^2.$$

Using the fact that $w^* = G(w^*)$ to evaluate $G'(w^*)$ shows that $G'(w^*) = 0$. Thus, for w_n close enough to w^*, we have :

$$w_{n+1} - w^* \cong 1/2 G''(w^*)(w_n - w^*)^2.$$

The convergence rate is locally quadratic. This illustrates the "higher speed" of the policy improvement algorithm. The quadratic rate is characteristic of Newton's method. The speed of convegence of both methods is illustrated in figure 15.

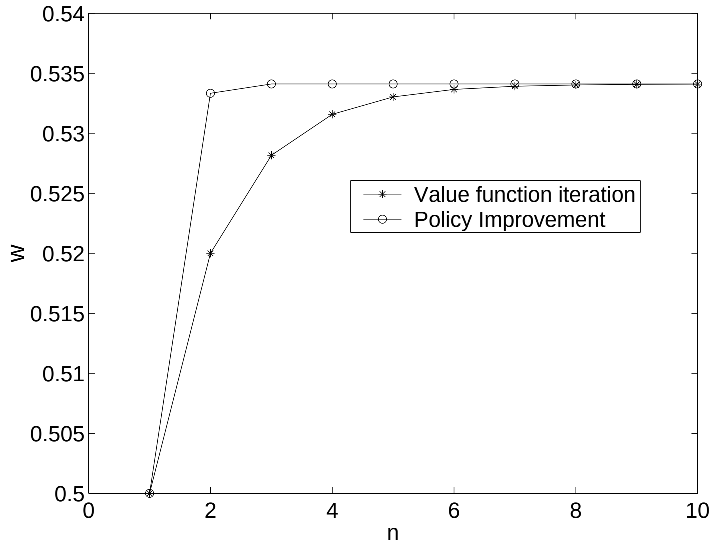


FIGURE 1. Exercise 3.1 : Value Function Iteration VS Policy Improvement

CHAPTER 4

Linear quadratic dynamic programming

Exercise 4.1.

Consider the modified version of the optimal linear regulator problem where the objective is to maximize

$$\sum_{t=0}^{\infty} \beta^t \left\{ x_t' R x_t + u_t' Q u_t + 2 u_t' H x_t \right\}$$

subject to the law of motion:

$$x_{t+1} = A x_t + B u_t.$$

Here x_t is an $n \times 1$ state vector, u_t is a $k \times 1$ vector of controls, and x_0 is a given initial condition. The matrices R, Q are negative definite and symmetric. The maximization is with respect to sequences $\{u_t, x_t\}_{t=0}^{\infty}$.

a. Show that the optimal policy has the form

$$u_t = -(Q + \beta B' P B)^{-1} (\beta B' P A + H) x_t,$$

where P solves the algebraic matrix Riccati equation

$$P = R + \beta A' P A - (\beta A' P B + H')(Q + \beta B' P B)^{-1} (\beta B' P A + H).$$

b. Write a Matlab program to solve equation 4 by iterating on P starting from P being a matrix of zeros.

Solution

a. Let x_t denote the n-dimensional state vector and let u_t denote the $k-$dimensional control vector. The stochastic discounted linear regulator problem is to choose a sequence $\{u_t\}_{t=0}^{\infty}$ to maximize:

$$Max_{\{u_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left\{ x_t' R x_t + u_t' Q u_t + 2 u_t' H x_t \right\},$$

subject to x_0 being given and the law of motion:

$$x_{t+1} = A x_t + B u_t.$$

R is an $n \times n$ negative semidefinite symmetric matrix, Q is an $k \times k$ negative semidefinite symmetrix matrix. A is $n \times n$ while B is $n \times k$.
Conjecture that the value function is quadratic in the state vector:

$$V(x) = x' P x,$$

where P is an $n \times n$ matrix[1].
Taking our guess seriously, we plug it in the Bellman equation and mechanically write out the terms, to obtain:

[1]Note that we can drop the scalar d in the non-stochastic case.

$$V(x) = \max_u x'Rx + u'Qu + 2u'Hx + \beta\left[(Ax+Bu)'P(Ax+Bu)\right].$$

Some basic algebra manipulations show the following :

$$\begin{aligned} V(x) &= \max_u x'Rx + u'Qu + 2u'Hx + \\ &\quad \beta\left[x'A'PAx + x'A'PBu + u'B'PAx + u'B'PBu\right] \\ &= \max_u x'Rx + u'Qu + 2u'Hx + \beta\left[\begin{array}{c} x'A'PAx + 2x'A'PBu \\ +u'B'PBu \end{array}\right], \end{aligned}$$

where the second equality follows from the fact that $u'B'PAx$ is a scalar and can be transposed (also, recall that Q, P and R are symmetric). We obtain the following version of the Bellman equation:

$$V(x) = \max_u x'Rx + u'Qu + 2u'Hx + \beta\left[x'A'PAx + 2x'A'PBu + u'B'PBu\right]. \tag{20}$$

Note that the max operator still appears on the r.h.s. Next, we derive the first order necessary conditions:

$$2Qu + 2Hx + 2\beta\left[B'PBu + B'PAx\right] = 0,$$

where I have used $\frac{\partial(x'Ax)}{\partial x} = (A + A')x$; $\frac{\partial(y'Bz)}{\partial y} = Bz$ (where y and z are colunmn vectors) and $\frac{\partial(y'Bz)}{\partial z} = B'y$.
Now we have a feedback rule for u :

$$u = -\left(Q + \beta B'PB\right)^{-1}\left(\beta B'PA + H\right)x,$$

or $u = -Fx$. The next part involves some tedious but straightforward algebra. First, substitute this back into eq.(20):

$$\begin{aligned} V(x) &= x'Rx + x'\left(\begin{array}{c} (\beta A'PB + H')(Q + \beta B'PB)^{-1}Q \\ (Q + \beta B'PB)^{-1}(\beta B'PA + H) \end{array}\right)x \\ &\quad -2x'(\beta A'PB + H')(Q + \beta B'PB)^{-1}Hx + \beta x'A'PAx \\ &\quad +\beta x'\left(\begin{array}{c} (\beta A'PB + H')(Q + \beta B'PB)^{-1} \\ (B'PB)(Q + \beta B'PB)^{-1} \end{array}\right)x \\ &\quad -2\beta x'(A'PB)(Q + \beta B'PB)^{-1}(\beta B'PA + H)x. \end{aligned} \tag{21}$$

Second, collect the second and fifth term in the above equation:

$$\begin{aligned} x' &\quad \left(\begin{array}{c} (\beta A'PB + H')(Q + \beta B'PB)^{-1}(Q + \beta B'PB) \\ (Q + \beta B'PB)^{-1}(\beta B'PA + H) \end{array}\right)x \\ &= x'(\beta A'PB + H')(Q + \beta B'PB)^{-1}(\beta B'PA + H)x. \end{aligned}$$

Plug this back into eq.(21) and you obtain:

$$
\begin{aligned}
V(x) \;=\;& x'Rx + x'\left(\beta A'PB + H'\right)\left(Q + \beta B'PB\right)^{-1}\left(B'PA + H\right)x \\
& -2x'\left(\beta A'PB + H'\right)\left(Q + \beta B'PB\right)^{-1}Hx \\
& +\beta x'A'PAx \\
& -2\beta x'\left(A'PB\right)\left(Q + \beta B'PB\right)^{-1}\left(\beta B'PA + H\right)x. \qquad (22)
\end{aligned}
$$

Next, take the transpose of the third term on the r.h.s. of eq.(22) and collect the third and fifth term :

$$
\begin{aligned}
& -2\beta x'\left(A'PB\right)\left(Q + \beta B'PB\right)^{-1}\left(\beta B'PA + H\right)x \\
& -2x'H'\left(Q + \beta B'PB\right)^{-1}\left(\beta B'PA + H\right)x,
\end{aligned}
$$

which equals:

$$-2x'\left(\beta A'PB + H'\right)\left(Q + \beta B'PB\right)^{-1}\left(B'PA + H\right)x.$$

Substitute this back into (22) and rearrange, which produces:

$$V(x) = x'Rx - x'\left(\beta A'PB + H'\right)\left(Q + \beta B'PB\right)^{-1}\left(\beta B'PA + H\right)x + \beta x'A'PAx.$$

Collecting terms in x and the constants, we obtain the Ricatti equation:

$$P = R - \left(\beta A'PB + H'\right)\left(Q + \beta B'PB\right)^{-1}\left(\beta B'PA + H\right) + \beta A'PA.$$

b. See matlab program `olrp.m`

Exercise 4.2.

Verify that equations (4.10) and (4.11) implement the policy improvement algorithm for the discounted linear regulator problem.

Solution

Step 1. Start with a given policy $u_0 = -F_0 x_0$ and assume this policy is used forever. That is start from a matrix F_0, check whether the eigenvalues of $(A - BF_0)$ are less than $\beta^{-0.5}$ in modulus. Denote the value of working forever with this policy by $x'P_0x$. The matrix P_0 is implicitly determined by

$$x'P_0x = x'Rx + x'F_0QF_0x + \beta x'(A + BF_0)'P_0(A + BF_0)x,$$

or simplified

$$P_0 = R + F_0QF_0 + \beta(A + BF_0)'P_0(A + BF_0).$$

Step2. Using the value function from he previous step, $x'P_0x$, perform a one-step Bellman iteration to find a new policy funtion F_1. The first order conditions of this maximization is

$$\left(Q + B'P_0B\right)u = -B'P_0Ax,$$

or $u = -F_1x$, where F_1 is given by

$$F_1 = \left(Q + B'P_0B\right)^{-1}B'P_0.$$

Iterating on step 1 and step 2 until convergence implements the policy improvement algorithm. it is equivalent to iterating on

$$\begin{aligned} P_j &= R + F_j Q F_j + \beta(A + BF_j)' P_j (A + BF_j) \\ F_{j+1} &= \left(Q + B' P_j B\right)^{-1} B' P_j. \end{aligned}$$

Exercise 4.3.

A household seeks to maximize

$$-\sum_{t=1}^{\infty} \beta^t \left\{ (c_t - b)^2 + \gamma i_t^2 \right\}$$

subject to

$$\begin{aligned} llc_t + i_t &= ra_t + y_t \\ a_{t+1} &= a_t + i_t \\ y_{t+1} &= \rho_1 y_t + \rho_2 y_{t-1}. \end{aligned}$$

Here c_t, i_t, a_t, y_t are the household's consumption, investment, asset holdings, and exogenous labor income at t; while $b > 0, \gamma > 0, r > 0, \beta \in (0,1)$, and ρ_1, ρ_2 are parameters, and y_0, y_{-1} are initial conditions. Assume that ρ_1, ρ_2 are such that $(1 - \rho_1 z - \rho_2 z^2) = 0$ implies $|z| > 1$.

a. Map this problem into an optimal linear regulator problem.

b. For parameter values $[\beta, (1+r), b, \gamma, \rho_1, \rho_2] = (.95, .95^{-1}, 30, 1, 1.2, -.3)$, compute the household's optimal policy function using your Matlab program from exercise 4.1.

Solution

Note that if the roots z that satisfy $(1 - \rho_1 z - \rho_2 z^2) = 0$ are outside the unit circle (i.e. $|z| > 1$), this implies the $\lambda_i' s$ that satisfy:

$$1 - \rho_1 L - \rho_2 L^2 = (1 - \lambda_1 L)(1 - \lambda_2 L),$$

are inside the unit circle and the system does not explode.

a. Map this into an optimal linear regulator problem:

There is not a unique way to set up the state-space representation. Here is the easiest way to proceed.
Let $u_t = i_t = a_{t+1} - a_t$ be our control. Then the 3-dimensional state vector x_t evolves according to:

$$x_{t+1} = Ax_t + Bu_t,$$

or equivalently:

$$\begin{bmatrix} a_{t+1} \\ y_{t+1} \\ y_t \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \rho_1 & \rho_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_t \\ y_t \\ y_{t-1} \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} i_t,$$

where A and B are defined accordingly. Our choice for the state vector and the control yields the following matrices in the quadratic one-period return function:

$$R = -\begin{bmatrix} r^2 & r & 0 & -br \\ r & 1 & 0 & -b \\ 0 & 0 & 0 & 0 \\ -br & -b & 0 & 1 \end{bmatrix}; Q = -1-\gamma; H = -\begin{bmatrix} -2r & -2 & 0 & 2b \end{bmatrix}'.$$

Verify that, given these choices for R, Q and H, the one period return function $-(c_t-b)^2-\gamma i_t^2$ can be written as:

$$-(c_t-b)^2-\gamma i_t^2 = x_t' R x_t + u_t' Q u_t + 2u_t' H x_t,$$

where I have used the fact that $c_t = ra_t + y_t - i_t$.
To compute the solution, type $[f,p] = olrp(\beta, A, B, R, Q, H)$.

b. For the parameter values $(\beta, (1+r), b, \gamma, \rho_1, \rho_2) = (.95, .95^{-1}, 30, 1, 1.2., -.3)$, we get a policy function $u = -Fx$, where F is given by:

$$F = \begin{bmatrix} -.0095 & -.8127 & -.0488 & 5.4423 \end{bmatrix},$$

and a value function $v(x) = x'Px$ with

$$P = 1.0e+4 \begin{bmatrix} 0 & 0 & 0 & 0.0051 \\ 0 & 0.0003 & -0.0001 & 0.0204 \\ 0 & -0.0001 & 0 & -0.0061 \\ 0.0051 & 0.0204 & -0.0061 & -2.9191 \end{bmatrix}.$$

Exercise 4.4.

Modify exercise 4.3 by assuming that the household seeks to maximize

$$-\sum_{t=1}^{\infty} \beta^t \left\{ (s_t - b)^2 + \gamma i_t^2 \right\}$$

Here s_t measures consumption services that are produced by durables or habits according to

$$\begin{aligned} lls_t &= \lambda h_t + \pi c_t \\ h_{t+1} &= \delta h_t + \theta c_t \end{aligned}$$

where h_t is the stock of the durable good or habit, $(\lambda, \pi, \delta, \theta)$ are parameters, and h_0 is an initial condition.

a. Map this problem into a linear regulator problem.

b. For the same parameter values as in exercise 4.3 and $(\lambda, \pi, \delta, \theta) = (1, .05, .95, 1)$, compute the optimal policy for the household.

c. For the same parameter values as in exercise 4.3 and $(\lambda, \pi, \delta, \theta) = (-1, 1, .95, 1)$, compute the optimal policy.

d. Interpret the parameter settings in part b as capturing a model of durable consumption goods, and the settings in part c as giving a model of habit persistence.

Solution

a. The key to thing to notice is that we have to include h_t in the state space to keep track of the stock of durables, while s_t itself does not have to be included. To see why, note that s_t can simply be written as a function of the current state vector. Hence, s_t does not contain any additional information not in the state vector when x_t is appropriately defined as $x_t' = \begin{bmatrix} a_t & y_t & y_{t-1} & h_t & 1 \end{bmatrix}$. Having defined the state space vector, the rest of the derivation is purely mechanical. Consider the first-order difference equation:

$$\begin{bmatrix} a_{t+1} \\ y_{t+1} \\ y_t \\ h_{t+1} \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \rho_1 & \rho_2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \theta r & \theta & 0 & \delta & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_t \\ y_t \\ y_{t-1} \\ h_t \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ -\theta \\ 0 \end{bmatrix} i_t,$$

where I have used $c_t = ra_t + y_t - i_t$, which implies that:

$$h_{t+1} = \delta h_t + \theta \left(ra_t + y_t - i_t \right).$$

Next, we recast the one-period return function in the generic linear quadratic setup by choosing matrices R, Q and H such that:

$$-\left(s_t - b\right)^2 - \gamma i_t^2 = x_t' R x_t + u_t' Q u_t + 2u_t' H x_t$$

$$R = -\begin{bmatrix} \pi^2 r^2 & \pi^2 r & 0 & \lambda \pi r & -b\pi r \\ \pi^2 r & \pi^2 & 0 & \lambda \pi & -b\pi \\ 0 & 0 & 0 & 0 & 0 \\ \lambda \pi r & \lambda \pi & 0 & \lambda^2 & -b\lambda \\ -b\pi r & -b\pi & 0 & -b\lambda & b^2 \end{bmatrix}$$

$$Q = -\left(\pi^2 + \gamma\right)$$

$$H = -\begin{bmatrix} -2\pi^2 r & -2\pi^2 & 0 & -2\lambda\pi & 2b\pi \end{bmatrix}'.$$

b. For the parameter values $(\beta, (1+r), b, \gamma, \rho_1, \rho_2) = (.95, .95^{-1}, 30, 1, 2., -.3)$ and $(\lambda, \pi, \delta, \theta) = (1, .05, .95, 1)$, we get a policy function $u = -Fx$, where F is given by:

$$F = \begin{bmatrix} .3038 & 1.5710 & -.6232 & -.2917 & .0928 \end{bmatrix},$$

and a value function $v(x) = x'Px$ with

$$P = 1.0e+4 \begin{bmatrix} -.0006 & .0041 & .0012 & -.0005 & .0310 \\ -.0041 & -.0299 & .0086 & -.0036 & .2244 \\ .0012 & .0086 & -.0025 & .0010 & -.0640 \\ -.0005 & -.0036 & .0010 & -.0006 & .0307 \\ .0310 & .2244 & -.0640 & .0307 & -1.799 \end{bmatrix}.$$

c. For the parameter values $(\beta, (1+r), b, \gamma, \rho_1, \rho_2) = (.95, .95^{-1}, 30, 1, 2., -.3)$ and $(\lambda, \pi, \delta, \theta) = (-1, 1, .95, 1)$, we get a policy function $u = -Fx$, where F is given by:

$$F = \begin{bmatrix} .1541 & .2258 & -.3550 & -.1055 & -1055 & 1.2887 \end{bmatrix},$$

and a value function $v(x) = x'Px$ with

$$P = 1.0e+4 \begin{bmatrix} -.0005 & -.0037 & .0011 & -.0004 & -.0261 \\ -.0037 & -.0282 & .0081 & -.003 & -.2073 \\ .0011 & .0081 & -.0023 & .0008 & .0588 \\ -.0004 & -.0030 & .0008 & -.0007 & -.0316 \\ -.0261 & -.2073 & .0588 & -.0316 & -1.7803 \end{bmatrix}.$$

d. First, consider the calibration in part b: $(\lambda, \pi, \delta, \theta) = (1, .05, .95, 1)$. This implies that the stock of durables h_t depreciates at a rate of (1-δ) = .05 and, since $\theta = 1$, c_t can be interpreted as investment in durable goods at time t. The stock of durables evelolves according to:

$$h_t = .95h_{t-1} + c_t.$$

The process s_t simply measures the stream of consumption services produced by the stock of durables h_t and by newly acquired durables:

$$s_t = h_t + .05c_t.$$

Durables produce a lower service stream during the period of purchase. This could capture adjustment costs.

Second, consider the calibration in part c: $(\lambda, \pi, \delta, \theta) = (-1, 1, .95, 1)$. Here it is natural to interpret the model as capturing habit peristence where h_t is the habit, which evolves according to a law of motion:

$$h_{t+1} = .95h_t + c_t,$$

where c_t is today's consumption. The utility of consumption is determined by today's consumption relative to today's 'habit':

$$s_t = c_t - h_t.$$

The utility flow is determined by the surplus of consumption relative to the state of today's habit.

Exercise 4.5.

A household's labor income follows the stochastic process

$$y_{t+1} = \rho_1 y_t + \rho_2 y_{t-1} + w_{t+1} + \gamma w_t,$$

where w_{t+1} is a Gaussian martingale difference sequence with unit variance. Calculate

$$E\sum_{j=0}^{\infty} \beta^j [y_{t+j}|y^t, w^t],$$

where y^t, w^t denotes the history of y, w up to t.

a. Write a Matlab program to compute expression 1.

b. Use your program to evaluate expression 1 for the parameter values $(\beta, \rho_1, \rho_2, \gamma) = (.95, 1.2, -.4, .5)$.

Solution

Rewrite the ARMA(2,1) labor income process in lag-notation:

$$y_t = \frac{1+\gamma L}{1-\rho_1 L - \rho_2 L^2} w_t = c(L) w_t.$$

To predict the geometrically distributed lag, we use a formula from Sargent (1987, Chapter 11, p.303-304)

$$E_t\left[\sum_{j=0}^{\infty} \beta^j y_{t+j}\right] = \left[\frac{c(L) - \beta c(\beta) L^{-1}}{1-\beta L^{-1}}\right] w_t.$$

Manipulation of the term in brackets yields

$$\begin{aligned}
\frac{c(L) - \beta c(\beta) L^{-1}}{1-\beta L^{-1}} &= \frac{1+\gamma L - \beta(1+\gamma\beta)L^{-1}}{(1-\rho_1 L - \rho_2 L^2)(1-\beta L^{-1})} \\
&= \frac{L + \gamma L^2 - \beta(1+\gamma\beta)}{(1-\rho_1 L - \rho_2 L^2)(L-\beta)} \\
&= \frac{\beta(1+\gamma\beta) - L - \gamma L^2}{\beta - (1+\rho_1\beta)L + (\rho_1 - \rho_2\beta)L^2 + \rho_2 L^3}.
\end{aligned}$$

a. Using the matlab program `show.m` we compute the impulse-response function, the covariance generating function, the spectral density and a simulated time path for $E_t\left[\sum_{j=0}^{\infty} \beta^j y_{t+j}\right]$

b. For parameters $(.95, 1.2, -.4, .5)$, see figure 1.

Exercise 4.6. *Dynamic Laffer curves*

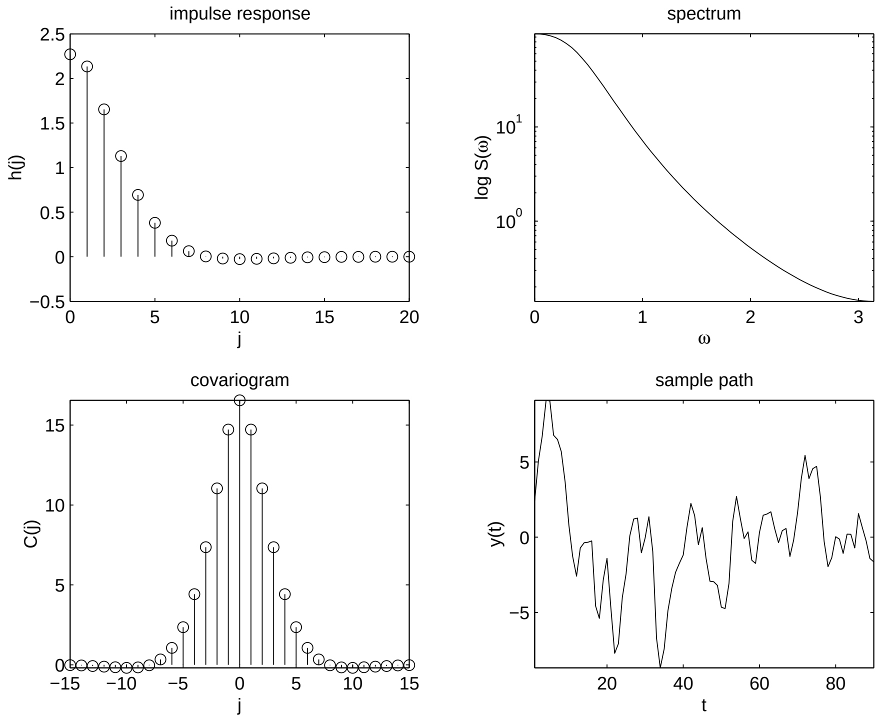


FIGURE 1. Exercise 4.5

The demand for currency in a small country is described by

$$M_t/p_t = \gamma_1 - \gamma_2 p_{t+1}/p_t, \tag{25}$$

where $\gamma_1 > \gamma_2 > 0$, M_t is the stock of currency held by the public at the end of period t, and p_t is the price level at time t. There is no randomness in the country, so that there is perfect foresight. Equation (25) is a Cagan-like demand function for currency, expressing real balances as an inverse function of the expected gross rate of inflation.

Speaking of Cagan, the government is running a permanent real deficit of g per period, measured in goods, all of which it finances by currency creation. The government's budget constraint at t is

$$(M_t - M_{t-1})/p_t = g, \tag{26}$$

where the left side is the real value of the new currency printed at time t. The economy starts at time $t = 0$, with the initial level of nominal currency stock $M_{-1} = 100$ being given.

For this model, define an *equilibrium* as a pair of *positive* sequences $\{p_t > 0, M_t > 0\}_{t=0}^{\infty}$ that satisfy equations (25) and (26) (portfolio balance and the government budget constraint, respectively) for $t \geq 0$, and the initial condition assigned for M_{-1}.

a. Let $\gamma_1 = 100, \gamma_2 = 50, g = .05$. Write a computer program to compute equilibria for this economy. Describe your approach and display the program.

b. Argue that there exists a continuum of equilibria. Find the *lowest* value of the initial price level p_0 for which there exists an equilibrium. (*Hint Number 1:* Notice the positivity condition that is part of the definition of equilibrium. *Hint Number 2:* Try using the general approach to solving difference equations described in the section "A Lagrangian formulation."

c. Show that for all of these equilibria except the one that is associated with the minimal p_0 that you calculated in part b, the gross inflation rate and the gross money creation rate both eventually converge to the *same* value. Compute this value.

d. Show that there is a unique equilibrium with a lower inflation rate than the one that you computed in part c. Compute this inflation rate.

e. Increase the level of g to .075. Compare the (eventual or asymptotic) inflation rate that you computed in part b and the inflation rate that you computed in part c. Are your results consistent with the view that "larger permanent deficits cause larger inflation rates"?

f. Discuss your results from the standpoint of the "Laffer curve."

Hint: A Matlab program `dlqrmon.m` performs the calculations. It is available from the web site for the book.

Solution

a. See explanation in part b and matlab program `ex0406b.m` and `dlqrmon.m`

b. Write the supply and demand equations as the following system

$$\begin{bmatrix} 1 & 0 \\ 1 & \gamma_2 \end{bmatrix} \begin{bmatrix} M_t \\ p_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & g \\ 0 & \gamma_1 \end{bmatrix} \begin{bmatrix} M_{t-1} \\ p_t \end{bmatrix}.$$

Upon inversion of the matrix on the left-hand side

$$\begin{aligned} \begin{bmatrix} M_t \\ p_{t+1} \end{bmatrix} &= A \begin{bmatrix} M_{t-1} \\ p_t \end{bmatrix} \\ A &= \begin{bmatrix} 1 & g \\ \frac{-1}{\gamma_2} & \frac{\gamma_1 - g}{\gamma_2} \end{bmatrix}. \end{aligned}$$

For the parameters in question, the matrix A is

$$A = \begin{bmatrix} 1 & 0.05 \\ -0.02 & 1.999 \end{bmatrix}.$$

The eigenvalues of the matrix A are $\lambda_1 = 1.0010$ and $\lambda_2 = 1.998$, both greater than one and thus unstable roots. The associated eigenvectors are $V_1 = [0.9998, 0.0200]'$ and $V_2 = [0.0500, 0.9987]'$. Since the eigenvectors are linearly independent, we know that

$$\begin{bmatrix} M_{-1} \\ p_0 \end{bmatrix} = \alpha \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} + \beta \begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix}.$$

The solution is then

$$\begin{bmatrix} M_{t-1} \\ p_t \end{bmatrix} = A^t \begin{bmatrix} M_{-1} \\ p_0 \end{bmatrix} = \alpha\lambda_1^t V_1 + \beta\lambda_2^t V_2.$$

Solving for α in $M_{-1} = \alpha V_{11} + \beta V_{12}$ and substituting in the other equation, we obtain an expression for $p_0 = \left[\frac{M_{-1}-\beta V_{12}}{V_{11}}\right] V_{21} + \beta V_{22}$ which is indexed by β. We obtain a continuum of equilibria indexed by β. The lowest feasible β is 0. For negative β prices are negative along the solution path, thereby violating the definition of an equilibrium. the corresponding initial price level is $p_0 = 2.0040$. Alternatively, we can use a Schur decomposition of the matrix $A = VWV^{-1}$. To attain stability of the system, we impose that $p_0 = V_{21}V_{11}^{-1}M_{-1}$. Using the result from the schur decomposition (using `schur.m`) and $M_{-1} = 100$, we compute that $p_0 = 2.0040$. The general solution to the problem is

$$p_{t+1} = V_{21}V_{11}^{-1}M_t.$$

c. No matter what the exact value of β is, $\lambda_2 > \lambda_1$, which implies that for large t, the dynamics of M and p are dominated by the second eigenvalue λ_2. As a consequence, the gross money growth rate $\frac{M_t}{M_{t-1}}$ and the gross inflation rate $\frac{p_{t+1}}{p_t}$ are converging to $\lambda_2 = 1.998$.

d. The equilibrium that is associate with $\beta = 0$ is unique and features $\frac{M_t}{M_{t-1}} = \frac{p_{t+1}}{p_t} = \lambda_1 = 1.001$.

e. Increasing government spending to $g = 0.075$ changes the eigenvalues of matrix A: $\lambda_1 = 1.0015$ and $\lambda_2 = 1.997$. The analysis is unchanged, so that the stable equilibrium features a lower inflation rate than before $\frac{M_t}{M_{t-1}} = \frac{p_{t+1}}{p_t} = 1.997$ and the unstable equilibrium has a higher inflation rate than before $\frac{M_t}{M_{t-1}} = \frac{p_{t+1}}{p_t} = 1.0015$. An increase of government expenditures shifts the graph of the characteristic polynomial inwards. The high-inflation equilibrium is now at a lower level than before. The low-inflation equilibrium is at a higher level the higher government spending, in line with neoclassical theories of the effects of government spending. However, this is the unstable equilibrium because an epsilon departure from it will lead us to the other high-inflation equilibrium.

f. Just as in the Laffer curve analysis in figure 8.5 in Chapter 8, we can plot the inverse of the inflation rate against the seigniorage earnings. With increasing inflation the revenue from the inflation tax first rises and later falls as the high inflation level discourages households to hold money. For a given level of g there are two equilibria, associated with λ_1 and λ_2. Raising the (constant) level of government expenditures identifies two new equilibria associated with a higher λ_1 and a lower λ_2, just as in figure 8.5.

CHAPTER 5

Search, matching, and unemployment

Exercise 5.1. *Being unemployed with only a chance of an offer*

An unemployed worker samples wage offers on the following terms. Each period, with probability ϕ, $1 > \phi > 0$, she receives no offer (we may regard this as a wage offer of zero forever). With probability $(1-\phi)$ she receives an offer to work for w forever, where w is drawn from a cumulative distribution function $F(w)$. Successive drawings across periods are independently and identically distributed. The worker chooses a strategy to maximize

$$E \sum_{t=0}^{\infty} \beta^t y_t, \qquad \text{where } 0 < \beta < 1,$$

$y_t = w$ is the worker is employed, and $y_t = c$ is the worker is unemployed. Here c is unemployment compensation, and w is the wage at which the worker is employed. Assume that, having once accepted a job offer at wage w, the worker stays in the job forever.
Let $v(w)$ be the expected value of $\sum_{t=0}^{\infty} \beta^t y_t$ for an unemployed worker who has offer w in hand and who behaves optimally. Write Bellman's functional equation for the worker's problem.

Solution

Let $v(w)$ be the expected value of $\sum_{t=0}^{\infty} \beta^t y_t$ for an unemployed worker who has offer w in hand and who behaves optimally.

$$v(w) = \max_{A,R} \left\{ \frac{w}{1-\beta}, c + \phi\beta v(0) + (1-\phi)\beta \int v(w')dF(w') \right\}. \tag{27}$$

Here the maximization is over the two actions: accept the offer to work forever at wage w, or reject the current offer and take a chance on drawing a new offer next period.

Exercise 5.2. *Two offers per period*

Consider an unemployed worker who each period can draw *two* independently and identically distributed wage offers from the cumulative probability distribution function $F(w)$. The worker will work forever at the same wage after having once accepted an offer. In the event of unemployment during a period, the worker receives unemployment compensation c. The worker derives a decision rule to maximize $E\sum_{t=0}^{\infty} \beta^t y_t$, where $y_t = w$ or $y_t = c$, depending on whether she is employed or unemployed. Let $v(w)$ be the value of $E\sum_{t=0}^{\infty} \beta^t y_t$ for a currently unemployed worker who has best offer w in hand.

a. Formulate Bellman's equation for the worker's problem.

b. Prove that the worker's reservation wage is *higher* than it would be had the worker faced the same c and been drawing only *one* offer from the same distribution $F(w)$ each period.

Solution

a. Note that the event $\max\{w_1, w_2\} < w$ is the event $(w_1 < w) \cap (w_2 < w)$. Therefore $\text{prob}\{\max(w_1, w_2) < w\} = F(w)^2$. The worker will evidently limit his choice to the larger of the two offers each period. Bellman's equation is therefore

$$v(w) = \max\left\{\frac{w}{1-\beta}, c + \beta \int v(w')d(F^2)(w')\right\},$$

where w is the best offer in hand.

b. The reservation wage obeys the following equation:

$$(\bar{w}_2 - c) = \frac{\beta}{1-\beta}\int_{\bar{w}_2}^{\infty}(w' - \bar{w}_2)d(F^2)(w').$$

Using the usual integration by part argument, one obtains the equation:

$$h_2(\bar{w}_2) \equiv (1-\beta)\bar{w}_2 - \beta\int_{\bar{w}_2}^{B}(1 - F(w')^2)dw' = 0.$$

Observe that h_2 is an increasing function. When the worker is given only one offer, the reservation wage solves :

$$h_1(\bar{w}_1) \equiv (1-\beta)\bar{w}_1 - \beta\int_{\bar{w}_1}^{B}(1 - F(w'))dw' = 0.$$

Since $F(w)^2 \leq F(w)$, we have $h_2(w) \leq h_1(w)$. Therefore:

$$0 = h_1(\bar{w}_1) = h_2(\bar{w}_2) \leq h_1(\bar{w}_2).$$

Since h_2 is increasing if follows that

$$\bar{w}_1 \leq \bar{w}_2.$$

The intuition underlying this result is as follows: the worker could choose always to ignore the second offer. This policy, possibly suboptimal, would leave the worker with a decision problem that is formally identical to the standard one-offer problem. The value of the objective function of the true problem is at least as high as the value of the objective function under the artificially restricted problem. Because the reservation wage has the property of equating the value of accepting a job, $w/(1-\beta)$, with the value of rejecting, $c + \beta Ev(w')$, a higher value of $Ev(w')$, which results in the two-offer case, requires a higher reservation wage.

Exercise 5.3. *A random number of offers per period*

An unemployed worker is confronted with a random number, n, of job offers each period. With probability π_n, the worker receives n offers in a given period, where $\pi_n \geq 0$ for $n \geq 1$, and $\sum_{n=1}^{\infty}\pi_n = 1$ for $N < +\infty$. Each offer is drawn independently from the same distribution $F(w)$. Assume that the number of offers n is independently distributed across time. The worker works forever at wage w after having accepted a job and receives unemployment compensation

of c during each period of unemployment. He chooses a strategy to maximize $E\sum_{t=0}^{\infty}\beta^t y_t$ where $y_t = c$ if he is unemployed, $y_t = w$ if he is employed.
Let $v(w)$ be the value of the objective function of an unemployed worker who has best offer w in hand and who proceeds optimally. Formulate Bellman's equation for this worker.

Solution

$$v(w) = \max\left\{\frac{w}{1-\beta}, c + \sum_{n=1}^{N}\pi_n \int v(w')d(F^n)(w')\right\}.$$

In effect, the worker is confronted with a lottery with probabilities π_n over distributions $F^n(w)$, from which he will sample next period. As in Exercise 2.1, w is the highest offer in hand.

Exercise 5.4. *Cyclical fluctuations in number of job offers*

Modify Exercise 5.3 as follows. Let the number of job offers n follow a Markov process, with

$$\begin{aligned} &\text{prob}\,\{\text{number of offers next period} = m|\text{number of offers this} \\ &\quad \text{period} = n\} = \pi_{mn}, \qquad m = 1,\ldots,N, \quad n = 1,\ldots,N \\ &\textstyle\sum_{m=1}^{N}\pi_{mn} = 1 \qquad \text{for} \quad n = 1,\ldots,N. \end{aligned} \tag{28}$$

Here $[\pi_{mn}]$ is a "stochastic matrix" generating a Markov chain. Keep all other features of the problem as in Exercise 2.3. The worker gets n offers per period, where n is now generated by a Markov chain so that thenumber of offers is possibly correlated over time.

a. Let $v(w,n)$ be the value of $E\sum_{t=0}^{\infty}\beta^t y_t$ for an unemployed worker who has received n offers this period, the best of which is w. Formulate Bellman's equation for the worker's problem.

b. Show that the optimal policy is to set a reservation wage $\bar{w}(n)$ that depends on the number of offers received this period.

Solution

a. The Bellman equation for the worker's problem is

$$v(w,n) = \max_{\text{accept,reject}}\left\{\frac{w}{1-\beta}, c + \sum_{m=1}^{N}\pi_{m,n}\int v(w',m)d(F^m)(w')\right\}. \tag{29}$$

b. From equation (29), we see that the right branch of the right side of the functional equation is evidently a function only of n. The argument in the text applies for each n and implies a reservation wage that is a function of n.

Exercise 5.5. *Choosing the number of offers*

An unemployed worker must choose the number of offers n to solicit. At a cost of $k(n)$ the worker receives n offers this period. Here $k(n+1) > k(n)$ for $n \geq 1$. The number of offers n must be chosen in advance at the beginning of the period and cannot be revised during the period. The worker wants to maximize $E\sum_{t=0}^{\infty}\beta^t y_t$. Here y_t consists of w each period she is employed but not searching, $[w - k(n)]$ the first period she is employed but searches for n offers, and $[c - k(n)]$ each period she is unemployed but solicits and rejects n offers. The offers are each independently drawn from $F(w)$. The worker who accepts an offer works forever at wage w.
Let Q be the value of the problem for an unemployed worker who has not yet chosen the number of offers to solicit. Formulate Bellman's equation for this worker.

Solution

$$Q = \max_n \int \max_{\substack{\text{accept}\\ \text{reject}}} \left\{ \frac{w}{1-b} - k(n), -k(n) + \beta Q \right\} d(F^n)(w).$$

The worker proceeds sequentially each period, first choosing n, then deciding whether to accept or reject the best offer.

Exercise 5.6. *Mortensen externality*

Two parties to a match (say, worker and firm) jointly draw a match parameter θ from a c.d.f. $F(\theta)$. Once matched, they stay matched forever, each one deriving a benefit of θ per period from the match. Each unmatched pair of agents can influence the number of offers received in a period in the following way. The worker receives n offers per period, with $n = f(c_1 + c_2)$, where c_1 is the resources the worker devotes to searching and c_2 is the resources the typical firm devotes to searching. Symmetrically, the representative firm receives n offers per period where $n = f(c_1 + c_2)$. (We shall define the situation so that firms and workers have the same reservation θ so that there is never unrequited love.) Both c_1 and c_2 must be chosen at the beginning of the period, prior to searching during the period. Firms and workers have the same preferences, given by the expected present value of the match parameter θ, net of search costs. The discount factor β is the same for worker and firm.

a. Consider a Nash equilibrium in which party i chooses c_i, taking c_j, $j \neq i$, as given. Let Q_i be the value for an unmatched agent of type i before the level of c_i has been chosen. Formulate Bellman's equation for agents of type 1 and 2.

b. Consider the social planning problem of choosing c_1 and c_2 sequentially so as to maximize the criterion of λ times the utility of agent 1 plus $(1-\lambda)$ times the utility of agent 2, $0 < \lambda < 1$. Let $Q(\lambda)$ be the value for this problem for two unmatched agents before c_1 and c_2 have been chosen. Formulate Bellman's equation for this problem.

c. Comparing the results in (a) and (b), argue that, in the Nash equilibrium, the optimal amount of resources has not been devoted to search.

Solution

a.

$$Q_1 = \max_{c_1} \int \max_{\text{accept,reject}} \left\{ \frac{\theta}{1-\beta} - c_1, -c_1 + \beta Q_1 d(F^n)(\theta) \right\},$$

subject to $n = f(c_1 + c_2)$, c_2 given

$$Q_2 = \max_{c_2} \int \max_{\text{accept,reject}} \left\{ \frac{\theta}{1-\beta} - c_2, -c_2 + \beta Q_2 \right\} d(F^n)(\theta)$$

subject to $n = f(c_1 + c_2)$, c_1 given.

b.

$$\begin{aligned} Q(\lambda) = \quad & \max_{c_1,c_2} \Big\{ \int \max_{\text{accept,reject}} \Big\{ \lambda \tfrac{\theta}{1-\beta} - \lambda c_1 + (1-\lambda)\Big(\tfrac{\theta}{1-\beta} - c_2\Big), \\ & -\lambda c_1 - (1-\lambda) c_2 + \beta Q(\lambda) \Big\} d(F^n)(\theta) \Big\} \end{aligned}$$

subject to $n = f(c_1 + c_2)$.

c. The Nash equilibrium is a (c_1, c_2) pair that solves the two functional equations in (a). In general, this (c_1, c_2) pair will not solve the functional equation in (b) because each agent in (a) neglects the effects of his choice of c_j on the welfare of the other agent. In general, there will be too little search in the Nash equilibrium if $f(c_1 + c_2)$ is increasing in $(c_1 + c_2)$.

Exercise 5.7. *Variable labor supply*

An unemployed worker receives each period a wage offer w drawn from the distribution $F(w)$. The worker has to choose whether to accept the job – and therefore to work forever – or to search for another offer and collect c in unemployment compensation. The worker who decides to accept the job must choose the number of hours to work in each period. The worker chooses a strategy to maximize

$$E \sum_{t=0}^{\infty} \beta^t u(y_t, l_t), \qquad \text{where } 0 < \beta < 1,$$

and $y_t = c$ if the worker is unemployed, and $y_t = w(1 - l_t)$ if the worker is employed and works $(1 - l_t)$ hours; l_t is leisure with $0 \leq l_t \leq 1$.
Analyze the worker's problem. Argue that the optimal strategy has the reservation wage property. Show that the number of hours worked is the same in every period.

Solution

Let s be the state variable. We choose $s = (w, 0)$, where w is the wage offer and $0 = E$ if the worker is employed, and $0 = U$ if she is unemployed. Consider first the situation of an employed worker. Bellman's equation is

$$v(w, E) = \max_{l} \{u[w(1-l), l] + \beta v(w, E)\}.$$

Then it follows that

$$v(w, E) = \frac{u\left(w(1 - l(w)), l(w)\right)}{1 - \beta},$$

where $l(w) \equiv \underset{l}{\operatorname{argmax}}\ u(w(1-l), l)$.

Let's show that $v(w, E)$ is increasing in w. Consider $w_1 < w_2$. We have :

$$\begin{aligned} u\left(w_1(1 - l(w_1)), l(w_1)\right) &\le u\left(w_2(1 - l(w_1)), l(w_1)\right) \\ &\le \max_l u\left(w_2(1 - l), l\right) \\ &\equiv u\left(w_2(1 - l(w_2)), l(w_2)\right). \end{aligned}$$

Intuitively, a worker receiving $w_2 > w_1$ has the option work $1 - l(w_1)$ hours paid w_2, that yields a higher utility than working $1 - l(w_1)$ hours paid w_1. Its optimal choice $1 - l(w_2)$ necessarily yields an even higher utility.

Now consider an unemployed worker. Bellman's equation is

$$v(w, U) = \max_{\text{accept,reject}} \left\{ V(w, E), u(c, 1) + \beta \int v(w', U) dF(w') \right\}.$$

The outside maximization is over two actions: accept the offer (in which case the worker chooses l optimally) or reject the offer, collect unemployment compensation, and wait for a new offer next period. The first term is incresing in w and the second is independent of w. Therefore the optimal policy is to accept offers offers that are at least equal to some $\bar{w}$. Once an offer has been accepted, hours worked are constant and equal to $l(w)$.

Exercise 5.8. *Wage growth rate and the reservation wage*

An unemployed worker receives each period an offer to work for wage w_t forever, where $w_t = w$ in the first period and $w_t = \phi^t w$ after t periods in the job. Assume $\phi > 1$, that is, wages increase with tenure. The initial wage offer is drawn from a distribution $F(w)$ that is constant over time (entry-level wages are stationary); successive drawings across periods are independently and identically distributed. The worker's objective function is to maximize

$$E \sum_{t=0}^{\infty} \beta^t y_t, \qquad \text{where } 0 < \beta < 1,$$

and $y_t = w_t$ if the worker is employed and $y_t = c$ if the worker is unemployed, where c is unemployment compensation. Let $v(w)$ be the optimal value of the objective function for an unemployed worker who has offer w in hand. Write Bellman's equation for this problem. Argue that, if two economies differ only in the growth rate of wages of employed workers, say $\phi_1 > \phi_2$, the economy with the higher growth rate has the smaller reservation wage.
Note. Assume that $\phi_i \beta < 1,\ i = 1, 2$.

Solution

If the worker accepts employment at wage w, the sequence $\{y_t\}$ is given by $y_t = w$, $y_{t+1} = \phi w \ldots, y_{t+j} = \phi^j w \ldots$. Therefore the value of the objective function if the worker accepts is $\sum_{j=0}^{\infty} \beta^j y_{t+j} = w/(1-\beta\phi)$. Bellman's equation for the worker's problem is

$$v(w) = \max\left\{\frac{w}{1-\beta\phi}, c + \beta \int v(w')dF(w')\right\}.$$

Using the same argument as when studying McCall's model, one shows that that the optimal policy is to accept all offers to work with an initial wage higher than a reservation wage $\bar{w}$.

$$v(w) = \begin{cases} \frac{\bar{w}}{1-\phi\beta} & w \leq \bar{w} \\ \frac{w}{1-\phi\beta} & w \geq \bar{w}. \end{cases}$$

Because, at $w = \bar{w}$, we have

$$\frac{\bar{w}}{1-\phi\beta} = c + \beta \int_0^B v(w')dF(w'),$$

we get, after substituting for $v(w)$ its expression,

$$\frac{\bar{w}}{(1-\phi\beta)} = \frac{c+\beta}{1-\phi\beta}\bar{w}\int_0^{\bar{w}} dF(w') + \frac{\beta}{1-\phi\beta}\int_{\bar{w}}^B w'dF(w').$$

This equation can be rearranged to give

$$(1-\beta)\bar{w} - \beta\int_{\bar{w}}^B (w'-\bar{w})dF(w') = (1-\beta\phi)c.$$

It is easy to see (using the Leibniz rule) that the left-hand side is increasing in $\bar{w}$. Therefore, if $\phi_1 > \phi_2$, that is, $(1-\beta\phi_1)c < (1-\beta\phi_2)c$, it must be that $\bar{w}_1 < \bar{w}_2$. The intuition behind this result is simple: for any given offer w, the value of accepting the offer is higher, the higher the growth rate of wages ϕ. Therefore, the sooner an offer is accepted, the sooner the benefits of the growth in wages are realized. This pattern makes some job offers more attractive even though the initial wage is not very high.

Exercise 5.9. *Search with a finite horizon*

Consider a worker who lives two periods. In each period the worker, if unemployed, receives an offer of lifetime work at wage w, where w is drawn from a distribution F. Wage offers are identically and independently distributed over time. The worker's objective is to maximize $E\{y_1 + \beta y_2\}$, where $y_t = w$ if the worker is employed and is equal to c – unemployment compensation – if the worker is not employed.
Analyze the worker's optimal decision rule. In particular, establish that the optimal strategy is to choose a reservation wage in each period and to accept any offer with a wage at least as high as the reservation wage and to reject offers below that level. Show that the reservation wage decreases over time.

Solution

We first analyze the worker's problem in the second period of life. We consider an unemployed worker; an employed worker does not have to solve any decision problem. Let $v_2(w)$ be the optimal value of the problem for an unemployed worker with offer w in hand. Then $v_2(w) = \max\{w, c\}$. It follows that the optimal strategy is to accept offers that are at least c and to reject all others. The second-period reservation wage, $\bar{w}_2$, is equal to c. In the first period if the worker is faced with a wage w and accepts the offer, the value of the objective function is $w(1+\beta)$. If the worker rejects he gets c in the first period and $v_2(w')$, a random variable, in the following period. The expected value of rejecting the offer is thus $c + \beta \int_0^\infty v_2(w')dF(w')$.
Therefore the optimal value of the objective function for a worker with offer w in hand is given by

$$v_1(w) = \max\left\{w(1+\beta), c + \beta \int_0^B v_2(w')dF(w')\right\}.$$

Notice that the second term in brackets is constant, whereas the first is increasing in w. It follows that the optimal policy is of the reservation wage form. There exists a $\bar{w}_1$ such that, for $w \leq \bar{w}_1$, the second term is higher, and therefore the optimal strategy is to reject the job offer and to remain unemployed. Similarly, when $w > \bar{w}_1$, the first term is higher and the optimal strategy is to accept the job. As usual $\bar{w}_1$ satisfies :

$$w_1(1+\beta) = c + \beta \int_0^B v_2(w')dF(w').$$

Observe that $v_2(w) = \max\{w, c\} \geq c$. In words, a worker who is unemployed in the second period get at least c, the unemployement compensation. Thus $\int_0^B v_2(w')dF(w') = E(v_2(w')) \geq c$, with a strict inequality if $\Pr(w' \geq c) > 0$. This inequality implies :

$$w_1(1+\beta) = c + \beta \int_0^B v_2(w')dF(w') \geq (1+\beta)c.$$

The reservation wage decreases as the retirement date approaches. The intuition underlying this result is that, the shorter the horizon, the smaller the benefits of "waiting to see if next period the wage offer is really high" (the option value of waiting) because those benefits cannot be enjoyed for a long period. The implication is hence that the alternative to waiting – that is, accepting a job – becomes more attractive. This aspect is reflected in the model by a decrease in the reservation wage, which in fact corresponds to an increase in the percentage of job offers that are accepted.

Exercise 5.10. *Finite horizon and mean-preserving spread*

Consider a worker who draws every period a job offer to work forever at wage w. Successive offers are independently and identically distributed drawings from a distribution $F_i(w),\ i = 1, 2$. Assume that F_1 has been obtained from F_2 by a mean-preserving spread (see Section 2.4). The worker's objective is to maximize

$$E \sum_{t=0}^{T} \beta^t y_t, \qquad 0 < \beta < 1,$$

where $y_t = w$ is the worker has accepted employment at wage w and is zero otherwise. Assume that both distributions, F_1 and F_2, share a common upper bound, B.

a. Show that the reservation wages of workers drawing from F_1 and F_2 coincide at $t = T$ and $t = T - 1$.

b. Argue that for $t \leq T - 2$ the reservation wage of the workers that sample wage offers from the distribution F_1 is higher than the reservation wage of the workers that sample from F_2. **c.** Now introduce unemployment compensation: the worker who is unemployed collects c dollars. Prove that the result in (a) no longer holds, that is, the reservation wage of the workers that sample from F_1 is higher than the one corresponding to workers that sample from F_2 for $t = T - 1$.

Solution

a. Let $v_t^i(w)$ be the optimal value of the objective function of an unemployed worker at time t who has offer w in hand and draws wage offers from the distribution $F_i,\ i = 1, 2$. Then it is clear that $v_T^i(w) = \max\{0, w\} = w$. Therefore $\int_0^B v_T^i(w) dF_i(w) = \int_0^B w dF_i(w) = Ew,\ i = 1, 2$. Clearly the reservation wage at time T is zero: the worker accepts every offer. At time $(T - 1)$, Bellmans' equation for the worker's problem is

$$\begin{aligned} v_{T-1}^i(w) &= \max\left\{ w(1+\beta), \beta \textstyle\int_0^B v_T^i(w') F_i(dw') \right\} \\ &\max\left\{ w(1+\beta), \beta E w \right\}. \end{aligned}$$

It is then clear that the worker will accept the offer if $w(1+\beta) \geq \beta E w$ and will reject it otherwise. Therefore the reservation wage $\bar{w}_{T-1}$ is $\beta E w/(1+\beta)$. Because the expectation of w is the same no matter whether w is drawn from F_1 or F_2, it follows that both types of workers have the same reservation wage.

b. We prove this point by induction. Assume that at $t+1$ the optimal policy under both distribution is of the reservation wage form. Also, assume that $w_{t+1}(1)$, the reservation wage under c.d.f. F_1, is greater than $w_{t+1}(2)$, the reservation wage under c.d.f. F_2. Observe that those two assumptions are true at time T. The Bellman equation at time t is:

$$v_t^i(w) = \max\left\{ w \frac{1 - \beta^{T-t+1}}{1 - \beta}, \beta \int_0^B v_{t+1}^i(w') dF_i(w') \right\},$$

where $w\frac{1-\beta^{T-t+1}}{1-\beta}$ is the value of working at wage w in periods $t, t+1, \dots T$. The first term is increasing in w while the second one is constant. It follows that, at time t, the optimal policy is also of the reservation wage form. Furthermore, the time t reservation wage $w_t(i)$ solves the usual indifference condition:

$$\begin{aligned}
w_t(i)\tfrac{1-\beta^{T-t+1}}{1-\beta} &= \beta \int_0^B v_{t+1}^i(w')dF_i(w') \\
w_t(i)\tfrac{1-\beta^{T-t+1}}{1-\beta} &= \beta \int_0^{w_{t+1}(i)} \tfrac{1-\beta^{T-t}}{1-\beta} w_{t+1}(i)dF_i(w') + \beta \int_{w_{t+1}(i)}^B \tfrac{1-\beta^{T-t}}{1-\beta} w'dF_i(w') \\
w_t(i)\tfrac{1-\beta^{T-t+1}}{1-\beta} &= \beta\tfrac{1-\beta^{T-t}}{1-\beta}\left(\int_0^{w_{t+1}(i)} \left(w_{t+1}(i) - w'\right) dF_i(w') + \int_0^B w'dF_i(w')\right).
\end{aligned}$$

Integrating the first term by part and rearanging yields :

$$w_t(i) = \frac{\beta - \beta^{T-t+1}}{1-\beta^{T-t+1}}\left(\int_0^{w_{t+1}(i)} F_i(w')dw' + E_i(w)\right).$$

Observe that $E_1(w) = E_2(w)$ by assumption. Also, by definition of a mean preserving spread and since $w_{t+1}(1) \geq w_{t+1}(2)$, we have:

$$\begin{aligned}
\int_0^{w_{t+1}(1)} F_1(w')dw' &\geq \int_0^{w_{t+1}(2)} F_1(w')dw' + \int_{w_{t+1}(2)}^{w_{t+1}(1)} F_2(w')dw' \\
&\geq \int_0^{w_{t+1}(2)} F_2(w')dw' + \int_{w_{t+1}(2)}^{w_{t+1}(1)} F_2(w')dw'.
\end{aligned}$$

Therefore $w_t(1) \geq w_t(2)$.

c. The value of the problem at $t = T$ is $v_T^i(w) = \max\{w, c\},\ i = 1, 2$. Then $\bar{w}_T^1 = \bar{w}_T^2 = c$. If we use the same argument as in (b), however, it follows directly that $\int_0^B \max\{w, c\}dF_1(w) \geq \int_0^B \max\{w, c\}dF_2(w)$, or $Ev_T^1 \geq Ev_T^2$. On the other hand, the reservation wage at $(T-1)$ satisfies $\bar{w}_{T-1}^i = \beta/(1+\beta)Ev_T^i$. Therefore $\bar{w}_{T-1}^1 \geq \bar{w}_{T-1}^2$.

Exercise 5.11. *Pissarides' Analysis of Taxation and Variable Search Intensity*

An unemployed worker receives each period a zero offer (or no offer) with probability $[1-\pi(e)]$. With probability $\pi(e)$ the worker draws an offer w from the distribution F. Here e stands for effort – a measure of search intensity – and $\pi(e)$ is increasing in e. A worker who accepts a job offer can be fired with probability α, $0 < \alpha < 1$. The worker chooses a strategy, that is, whether to accept an offer or not and how much effort to put into search when unemployed, to maximize

$$E\sum_{t=0}^{\infty} \beta^t y_t, \qquad 0 < \beta < 1,$$

where $y_t = w$ if the worker is employed with wage w and $y_t = 1 - e + z$ if the worker spends e units of leisure searching and does not accept a job. Here z is unemployment compensation. For the worker who searched and accepted a job, $y_t = w - e - T(w)$; that is, in the first period the wage is net of search costs. Throughout, $T(w)$ is the amount paid in taxes when the worker is employed. We assume that $w - T(w)$ is increasing in w. Assume that $w - T(w) = 0$ for $w = 0$, that, if $e = 0$, $\pi(e) = 0$ – that is, the worker gets no offers – and that $\pi'(e) > 0$, $\pi''(e) < 0$.

a. Analyze the worker's problem. Establish that the optimal strategy is to choose a reservation wage. Display the condition that describes the optimal choice of e, and show that the reservation wage is independent of e.

b. Assume that $T(w) = t(w - a)$ where $0 < t < 1$ and $a > 0$. Show that an increase in a decreases the reservation wage and increases the level of effort, increasing the probability of accepting employment.

c. Show under what conditions a change in t has the opposite effect.

Solution

a. Let the state variable that completely summarizes current and future opportunities be $x = (w, e, s)$, where w is the wage, e is the effort, and $s = E$ if the worker is employed and $s = U$ if he is unemployed. Recall that, if the worker is employed, then $e = 0$. Let Q be the expected value of the objective function for an unemployed worker who behaves optimally before getting an offer. Then if the worker is employed, the value of the objective function is given by

$$v(w, 0, E) = w - T(w) + \beta(1-\alpha)v(w, 0, E) + \beta\alpha Q,$$

or

$$v(w, 0, E) = \frac{w - T(w)}{1 - \beta(1-\alpha)} + \frac{\beta\alpha Q}{1 - \beta(1-\alpha)}.$$

If the worker is unemployed, has an offer w in hand, and spent $e > 0$ units of leisure searching this period, the value of the objective function is

$$\begin{aligned} v(w, e, U) = \ & \max\{w - T(w) - e + \beta(1-\alpha)v(w, 0, E) \\ & +\beta\alpha Q, 1 - e + z + \beta Q\}, \end{aligned}$$

where the first term reflects the value of accepting employment and the second the value of rejecting the offer. Using the expression we found for $v(w, 0, E)$, we get

$$\begin{aligned} v(w, e, U) = \ & \max\left\{\tfrac{w-T(w)}{1-\beta(1-\alpha)} - e \right. \\ & \left. +\tfrac{\beta\alpha Q}{1-\beta(1-\alpha)}, 1 - e + z + \beta Q\right\}. \end{aligned}$$

Then, using a standard argument, we see from the above equation that the optimal strategy is to accept offers greater than or equal to $\bar{w}$ and to reject all others; $\bar{w}$ is such that it makes the worker indifferent between accepting or rejecting the job offer; that is, $\bar{w}$ solves

$$\frac{\bar{w} - T(\bar{w})}{1 - \beta(1-\alpha)} - e + \frac{\beta\alpha Q}{1 - \beta(1-\alpha)} = 1 - e + z + \beta Q,$$

or

$$\bar{w} - T(\bar{w}) = [1 - \beta(1-\alpha)](1 + z + \beta Q) - \beta\alpha Q. \tag{30}$$

Notice that we cannot use this expression for $\bar{w}$ to compute the reservation wage, because Q must be determined endogenously. It is clear, however, that, if Q is independent of e (as we will show that it is), then $\bar{w}$ does not depend on e.
Because we established that the optimal policy is of the reservation wage variety, we can compute $v(w,e,U)$. This function is given by

$$v(w,e,U)=\begin{cases}\frac{w-T(w)}{1-\beta(1-\alpha)}-e+\frac{\beta\alpha Q}{1-\beta(1-\alpha)} & w\geq\bar{w}\\ 1-e+z+\beta Q & w\leq\bar{w}.\end{cases}$$

Let $\Phi(e)=Ev(w,e,U)=\int_0^\infty v(w,e,U)F(dw)$,

$$\begin{aligned}\Phi(e)=\ & (1+z+\beta Q)F(\bar{w})\\ & +\tfrac{1}{1-\beta(1-\alpha)}\int_{\bar{w}}^\infty[w-T(w)]F(dw)-e\\ & +[1-F(\bar{w})]\tfrac{\beta\alpha Q}{1-\beta(1-\alpha)}.\end{aligned}$$

Because we have shown that

$$\frac{\beta\alpha Q}{1-\beta(1-\alpha)}=(1+z+\beta Q)-\frac{\bar{w}-T(\bar{w})}{1-\beta(1-\alpha)},$$

we have, after some substitution, that

$$\begin{aligned}\Phi(e)=\ & \tfrac{1}{1-\beta(1-\alpha)}\\ & \cdot\int_{\bar{w}}^\infty([w-T(w)]-[\bar{w}-T(\bar{w})])F(dw)+1+z+\beta Q-e.\end{aligned}$$

Now consider $\Phi(0)$. Recall that, if $e=0$, the worker gets no offers, and hence $v(w,0,U)=1+z+\beta Q$. This expression is independent of w, and so $\Phi(0)=1+z+\beta Q$. Therefore

$$\begin{aligned}\Phi(e)=\ & \tfrac{1}{1-\beta(1-\alpha)}\\ & \cdot\int_{\bar{w}}^\infty([w-T(w)]-[\bar{w}-T(\bar{w})])F(dw)+\Phi(0)-e.\end{aligned}$$

To simplify notation let $(w-T(w))-(\bar{w}-T(\bar{w}))\equiv\Delta Y(w)$. Then the above expression becomes

$$\Phi(e)=\frac{1}{1-\beta(1-\alpha)}\int_{\bar{w}}^\infty\Delta Y(w)F(dw)+\Phi(0)-e.$$

If the worker chooses to spend e units of effort, he gets an offer with probability $\pi(e)$ and expected value $\Phi(e)$. With probability $[1-\pi(e)]$ he gets no offers. This alternative has value $\Phi(0)-e$.
Then the value of the problem for an unemployed worker who behaves optimally is given by Q, where Q satisfies

$$\begin{aligned}Q\ &\equiv\max\nolimits_{0\leq e\leq1}\{\pi(e)\Phi(e)+[1-\pi(e)][\Phi(0)-e]\}\\ Q\ &\equiv\max\nolimits_{0\leq e\leq1}\{\pi(e)[\Phi(e)-\Phi(0)+e]+\Phi(0)-e\}\\ Q\ &=\max\nolimits_{0\leq e\leq1}\left\{\tfrac{\pi(e)}{1-\beta(1-\alpha)}\int_{\bar{w}}^\infty\Delta Y(w)F(dw)+1-e+z+\beta Q\right\}\end{aligned}\tag{31}$$

The right-hand side defines a mapping from Q into the reals. To guarantee that the problem is well behaved, we want to show that one such Q exists. This is not a trivial problem: Q affects $\bar{w}$ and $\Delta Y(w)$, so that the mapping is highly nonlinear. In any case, it is clear that Q, and therefore $\bar{w}$, are independent of e.

Let H be the mapping defined by the right-hand side of (31). Because $\pi(e)$ is increasing in e, we have that

$$\begin{aligned} HQ &\leq \tfrac{1}{1-\beta(1-\alpha)} \int_{\bar{w}}^{\infty} \Delta Y(w)F(dw) + 1 + z + \beta Q \\ &\leq \bar{H}Q \equiv \tfrac{1}{1-\beta(1-\alpha)} \int_{0}^{\infty} [w - T(w)]F(dw) + 1 + z + \beta Q. \end{aligned}$$

Therefore, if Q_1 is such that $Q_1 = \bar{H}Q_1$ (such a Q_1 is easy to compute directly), it follows that, for all $Q \geq Q_1$, $Q \geq \bar{H}Q$. Thus $\forall Q \geq Q_1$, $HQ \leq Q$. On the other hand,

$$\begin{aligned} HQ &\geq \left\{ \tfrac{\pi(0)}{1-\beta(1-\alpha)} \int_{\bar{w}}^{\infty} \Delta Y(w)F(dw) + 1 + z + \beta Q \right\} \\ &= 1 + z + \beta Q \equiv \underline{\mathrm{H}}Q. \end{aligned}$$

Then we have that, for all $Q \geq 0$, $\underline{\mathrm{H}}Q \leq HQ \leq \bar{H}Q$ and $\underline{\mathrm{H}}0 > 0$. Hence we have established that $H0 > 0$ and that there exists $Q_1 < \infty$ such that $HQ \leq Q$ for $Q \geq Q_1$.

Inasmuch as H is a continuous function of Q [this follows because $\bar{w}$ is continuous in Q, as is $\Delta Y(w)$], we establish that there exists a $\bar{Q}$ such that $H\bar{Q} = \bar{Q}$.

We next prove that $\bar{Q}$ is unique. To do so it suffices to show that the mapping H is monotone in Q. A sufficient condition is that

$$0 \leq \frac{\partial}{\partial \bar{w}} \left[\int_{\bar{w}}^{\infty} \Delta Y(w)F(dw) \right] \frac{\partial \bar{w}}{\partial Q} + \beta < 1.$$

Still, $(\partial/\partial\bar{w}) \int_{\bar{w}}^{\infty} \Delta Y(w)F(dw)$ is (using the Leibniz rule) equal to $-[1-(\partial T/\partial w)(\bar{w})]$ $[1-F(\bar{w})]$. From the equation determining $\bar{w}$, we get that $[1-(\partial T/\partial w)(\bar{w})](\partial\bar{w}/\partial Q) = \beta(1-\beta)(1-\alpha)$. Because $-[1-F(\bar{w})]\beta(1-\beta)(1-\alpha) + \beta \in (0,1)$, however, H is increasing. Next we use (31) to characterize the optimal choice of e. It is clear that it satisfies

$$\pi'(e) \frac{1}{1-\beta(1-\alpha)} \int_{\bar{w}}^{\infty} \Delta Y(w)F(dw) = 1, \tag{32}$$

if the solution is interior. We assume that the distribution of w has sufficient mass in the tail to make search attractive – that is, we assume that the solution is interior. It is being claimed that it is possible to make assumptions about the deep parameters of the model, $F(w), \alpha, \beta, z, \pi(e)$, that will guarantee that the optimal choice of e is $e > 0$. We focus on this case only because the other is trivial.

From (31) it is clear that the optimal Q satisfies

$$\bar{Q} = (1-\beta)^{-1} \left[\frac{\pi(\bar{e})}{1-\beta(1-\alpha)} \int_{\bar{w}}^{\infty} \Delta Y(w)F(dw) + 1 - \bar{e} + z \right].$$

Using this equation in equation (30), we obtain another, more familiar characterization of the optimal reservation wage,

$$\begin{aligned} \bar{w} - T(\bar{w}) = \; & (1+z) - \beta(1-\alpha)\bar{e} + \tfrac{\beta(1-\alpha)\pi(\bar{e})}{1-\beta(1-\alpha)} \int_{\bar{w}}^{\infty} \{[w - T(w)] \\ & - [\bar{w} - T(\bar{w})]\} F(dw). \end{aligned} \tag{33}$$

Then equations (33) and (32) summarize the determination of the endogenous variables, e and $\bar{w}$.

b. Assume that $T(w) = t(w - a)$. To explore the effect of a change in a, we differentiate completely (33) and (32) with respect to a. We start with (33).

$$\begin{aligned}(1-t)\tfrac{\partial \bar{w}}{\partial a} + t = \ & -\beta(1-\alpha)\tfrac{\partial \bar{e}}{\partial a} + \tfrac{\beta(1-\alpha)}{1-\beta(1-\alpha)} \\ & \cdot \textstyle\int_{\bar{w}}^{\infty} \Delta Y(w) F(dw) \pi'(\bar{e}) \tfrac{\partial e}{\partial a} \\ & - \tfrac{\beta(1-\alpha)\pi(\bar{e})}{1-\beta(1-\alpha)}(1-t)[1-F(\bar{w})]\tfrac{\partial \bar{w}}{\partial a}.\end{aligned}$$

Using equation (32) to eliminate $1/[1-\beta(1-\alpha)]\int_{\bar{w}}^{\infty} \Delta Y(w) F(dw)\pi'(\bar{e}) = 1$, we get

$$(1-t)\left[1 + \frac{\beta(1-\alpha)\pi(\bar{e})[1-F(\bar{w})]}{1-\beta(1-\alpha)}\right]\frac{\partial \bar{w}}{\partial a} = -t.$$

Then $(\partial \bar{w}/\partial a) < 0$.

The intuition underlying this result is that an increase in a makes the income tax more progressive, as it increases the subsidy to low-income workers. Because taxes are paid (and the subsidy is received) only if the worker is employed, the increased attractiveness of low-income jobs is reflected by a reduction in the minimum wage at which an unemployed worker is willing to accept an offer. Notice that the term $(\partial e/\partial a)$ disappears in the above equation. This is just another consequence of the property that e does not affect the choice of the reservation wage.

We next explore the effect on e. From (32) we get

$$\begin{aligned}& \tfrac{\pi''(\bar{e})}{1-\beta(1-\alpha)} \textstyle\int_{\bar{w}}^{\infty} \Delta Y(w) F(dw) \tfrac{\partial e}{\partial a} \\ & \quad = \tfrac{\pi'(\bar{e})}{1-\beta(1-\alpha)}(1-t)\tfrac{\partial \bar{w}}{\partial a}[1-F(\bar{w})] \\ \text{or} \quad & \tfrac{\pi''(\bar{e})}{\pi'(\bar{e})^2}\tfrac{\partial e}{\partial a} = \tfrac{(1-t)[1-F(\bar{w})]}{1-\beta(1-\alpha)}\tfrac{\partial \bar{w}}{\partial a}.\end{aligned}$$

Because $\pi''(e) < 0$, we have that $(\partial e/\partial a) > 0$, that is, effort is increased. Notice that the increase in e increases $\pi(\bar{e})$, and hence the probability of getting an acceptable offer $\pi(\bar{e})[1-F(\bar{w})]$ rises. To fix the notation, let $p = \pi(e)[1-F(\bar{w})]$. Then

$$\frac{\partial p}{\partial a} = [1-F(\bar{w})]\pi'(\bar{e})\frac{\partial e}{\partial a} - F'(\bar{w})\pi(e)\frac{\partial \bar{w}}{\partial a},$$

and our results show that$(\partial p/\partial a) > 0$.

c. Next we analyze the effects of changing the marginal tax rate t. We follow exactly the same method of totally differentiating (33) and (32) to get, from (33),

$$\begin{aligned}& (1-t)\tfrac{\partial \bar{w}}{\partial t}\left\{1 + \tfrac{\beta(1-\alpha)\pi(\bar{e})[1-F(\bar{w})]}{1-\beta(1-\alpha)}\right\} \\ & \quad = \bar{w} - a - \tfrac{\beta(1-\alpha)\pi(\bar{e})}{1-\beta(1-\alpha)}\textstyle\int_{\bar{w}}^{\infty}(w-\bar{w})F(dw).\end{aligned}$$

From (33), however, we got that

$$\begin{aligned}& \bar{w} - \tfrac{\beta(1-\alpha)\pi(\bar{e})}{1-\beta(1-\alpha)}\textstyle\int_{\bar{w}}^{\infty}(w-\bar{w})F(dw) \\ & \quad = (1-t)^{-1}[(1+z) - a - \beta(1-\alpha)\bar{e}].\end{aligned}$$

Then

$$\text{sign}\ \frac{\partial \bar{w}}{\partial t} = \text{sign}\ [(1+z) - a - \beta(1-\alpha)\bar{e}].$$

From (32), after we substitute into the expression for $(\partial \bar{w}/\partial t)$, we get

$$\frac{\pi''(\bar{e})}{\pi'(\bar{e})}\frac{\partial e}{\partial t} = \frac{\pi'(\bar{e})}{[1-\beta(1-\alpha)+\beta(1-\alpha)\pi(\bar{e})[1-F(\bar{w})]} \cdot \left\{\bar{w} + \int_{\bar{w}}^{\infty}(w-\bar{w})F(dw)\right\}.$$

Therefore $(\partial e/\partial t) < 0$ unambiguously.
Notice that, in this case, an increase in t reduces the returns of being employed and therefore makes working less attractive. Consequently, it is optimal for the unemployed worker to reduce the level of effort, decreasing the probability of finding a job. On the other hand, it is possible for the reservation wage to decrease, that is, for some wage offers to be acceptable to the worker after the increase in the tax rate. Such a decrease becomes more likely as a grows larger. In this case, the increase in the marginal rate can actually increase payments to the worker when $w - a < 0$. This higher subsidy makes working more attractive, consequently reducing the reservation wage.

Exercise 5.12. *Search and nonhuman wealth*

An unemployed worker receives every period an offer to work forever at wage w, where w is drawn from the distribution $F(w)$. Offers are independently and identically distributed. Every agent has another source of income, which we denote ϵ_t, and that may be regarded as nonhuman wealth. In every period all agents get a realization of ϵ_t, which is independently and identically distributed over time, with distribution function $G(\epsilon)$. We also assume that w_t and ϵ_t are independent. The objective of a worker is to maximize

$$E\sum_{t=0}^{\infty}\beta^t y_t, \qquad 0<\beta<1,$$

where $y_t = w + \phi\epsilon_t$ if the worker has accepted a job that pays w, and $y_t = c + \epsilon_t$ if the worker remains unemployed. We assume that $0 < \phi < 1$ to reflect the fact that an employed worker has less time to engage in the collection of nonhuman wealth. Assume $1 > \text{prob}\{w \geq c + (1-\phi)\epsilon\} > 0$.
Analyze the worker's problem. Write down Bellman's equation and show that the reservation wage increases with the level of nonhuman wealth.

Solution

If the worker accepts a job that pays w, her total utility is given by

$$w + \theta\epsilon_t + E\sum_{j=1}^{\infty}\beta^j(w+\phi\epsilon_{t+j}) = w + \phi\epsilon + \frac{\beta}{1-\beta}(w + \phi E\epsilon).$$

Then let $v(w, \epsilon)$ be the optimal value of the objective function for an unemployed worker who has an offer w in hand and nonhuman wealth equal to ϵ. Then

$$v(w, \epsilon) = \max \left\{ w + \phi\epsilon + \tfrac{\beta}{1-\beta}(w + \phi E\epsilon) \right. \\ \left. c + \epsilon + \beta \int \int v(w', \epsilon') dF(w') dG(\epsilon') \right\}.$$

The second term in the bracketed expression does not depend on w. Therefore, for each ϵ, the optimal strategy is to choose a reservation wage. To see how the reservation wage $\bar{w}(\epsilon)$ varies with ϵ, write the indifference condition :

$$\bar{w}(\epsilon) + \phi\epsilon + \frac{\beta}{1-\beta}\left(\bar{w}(\epsilon) + \phi E(\epsilon)\right) = c + \epsilon + \beta Q,$$

where $Q \equiv \beta \int \int v(w', \epsilon') dF(dw') dG(\epsilon')$. Rearanging gives :

$$\frac{\bar{w}(\epsilon)}{1-\beta} = c + (1-\phi)\epsilon + \beta Q - \frac{\beta}{1-\beta}\phi E(\epsilon).$$

Since $0 < \phi < 1$, the above equation implies that $\bar{w}(\epsilon)$ is an increasing function of ϵ.

Exercise 5.13. *Search and asset accumulation*

A worker receives, when unemployed, an offer to work forever at wage w, where w is drawn from the distribution $F(w)$. Wage offers are identically and independently distributed over time. The worker maximizes

$$E \sum_{t=0}^{\infty} \beta^t u(c_t, l_t), \qquad 0 < \beta < 1,$$

where c_t is consumption and l_t is leisure. Assume R_t is i.i.d. with distribution $H(R)$. The budget constraint is given by

$$a_{t+1} \leq R_t(a_t + w_t n_t - c_t)$$

and $l_t + n_t \leq 1$ if the worker has a job that pays w_t. If the worker is unemployed, the budget constraint is $a_{t+1} \leq R_t(a_t + z - c_t)$ and $l_t = 1$. Here z is unemployment compensation. It is assumed that $u(\cdot)$ is bounded and that a_t, the worker's asset position, cannot be negative. This corresponds to a no borrowing assumption. Write down Bellman's equation for this problem.

Solution

A natural choice for the state variable in this problem is the vector (w, a, R, s), where $s = E$ if the worker is employed and $s = U$ if the worker is unemployed.

We first analyze the problem faced by an employed worker. This problem is

$$v(w, a, R, E) = \max_{c,l,n,a'} \left\{ u(c, l) + \beta \int v(w, a', R', E) dH(R') \right\},$$

subject to $a' \leq R(a + wn - c)$, $l + n \leq 1$.
If the worker is unemployed, the value function is given by

$$\begin{aligned} v(w, a, R, U) = \max\Big\{ \quad & v(w, a, R, E), \\ & \max\Big[u(c, 1) + \beta \int \int v(w', a', R', U) F(dw') dH(R')\Big]\Big\}, \end{aligned}$$

subject to $a' \leq R(a + z - c)$, where the first term in brackets reflects the value of accepting the job, whereas the second represents the value of remaining unemployed. In each case the asset position is chosen optimally. It is possible to argue that the optimal strategy is to set a reservation wage $\bar{w}(a, R)$ that depends on both the asset position and the rate of interest R.

Exercise 5.14. *Temporary unemployment compensation*

Each period an unemployed worker draws one, and only one, offer to work forever at wage w. Wages are i.i.d. draws from the c.d.f. F, where $F(0) = 0$ and $F(B) = 1$. The worker seeks to maximize $E \sum_{t=0}^{\infty} \beta^t y_t$, where y_t is the sum of the worker's wage and unemployment compensation, if any. The worker is entitled to unemployment compensation in the amount $\gamma > 0$ only during the *first* period that she is unemployed. After one period on unemployment compensation, the worker receives none.

a. Write the Bellman equations for this problem. Prove that the worker's optimal policy is a time-varying reservation wage strategy.

b. Show how the worker's reservation wage varies with the duration of unemployment.

c. Show how the worker's "hazard of leaving unemployment" (i.e., the probability of accepting a job offer) varies with the duration of unemployment.

Now assume that the worker is also entitled to unemployment compensation if she quits a job. As before, the worker receives unemployment compensation in the amount of γ during the first period of an unemployment spell, and zero during the remaining part of an unemployment spell. (To requalify for unemployment compensation, the worker must find a job and work for at least one period.)

The timing of events is as follows. At the very beginning of a period, a worker who was employed in the previous period must decide whether or not to quit. The decision is irreversible; that is, a quitter cannot return to an old job. If the worker quits, she draws a new wage offer as described previously, and if she accepts the offer she immediately starts earning that wage without suffering any period of unemployment.

d. Write the Bellman equations for this problem. [*Hint*: At the very beginning of a period, let $v^e(w)$ denote the value of a worker who was employed in the previous period with wage w (before any wage draw in the current period). Let $v_1^u(w')$ be the value of an unemployed worker who has drawn wage offer w' and who is entitled to unemployment compensation, if she rejects the offer. Similarly, let $v_+^u(w')$ be the value of an unemployed worker who has drawn wage offer w' but who is not eligible for unemployment compensation.]

e. Characterize the three reservation wages, $\bar{w}^e$, $\bar{w}_1^u$, and $\bar{w}_+^u$, associated with the value functions in part d. How are they related to γ? (*Hint*: Two of the reservation wages are straightforward to characterize, while the remaining one depends on the actual parameterization of the model.)

Solution

a. Let $v_u^1(w)$ ($v_+^u(w)$) be the value function of an unemployed worker with wage w in hand in the first (after the first) period of unemployment and who behaves optimally. The Bellman equation are :

$$
\begin{aligned}
v_1^u(w) &= \max_{\{A,R\}} \left\{ \frac{w}{1-\beta}, \gamma + \beta \int_0^B v^+(w')dF(w') \right\} \\
v_+^u(w) &= \max_{\{A,R\}} \left\{ \frac{w}{1-\beta}, \beta \int_0^B v^+(w')dF(w') \right\}.
\end{aligned}
$$

In each of the two periods the problem is a standard one, leading to a reservation wage policy. If the unemployed is in her fist period of unemployment then the optimal policy is accept for $w \geq w^1$ and reject otherwise. The associated value function is $v_1^u(w) = \frac{w}{1-\beta}$ for $w \geq w^1$ and $v_1^u(w) = \frac{w^1}{1-\beta} = \gamma + \beta \int_0^B v_1^u(w')dF(w')$ for $w < w^1$. After one (or more) period(s) of unemployment the optimal policy is to accept when $w \geq w^+$ and to reject otherwise.

b. To show that $w^1 > w^+$, just write the two indifference conditions satisfied by the two reservation wages :

$$
\begin{array}{rcl}
\frac{w^1}{1-\beta} & = & \gamma + \beta \int_0^B v_+^u(w')dF(w') \\
\frac{w^+}{1-\beta} & = & \beta \int_0^B v_+^u(w')dF(w').
\end{array}
$$

Clearly, $w^1 > w^+$. Note that $w^1 - w^+ = (1-\beta)\gamma$. This equality has the following interpretation. Suppose that an unemployed worker in the first period of unemployment receive an offer w. If he accepts it, her payoff is :

$$\frac{w}{1-\beta}.$$

If, on the other hand, she rejects it, then her payoff is made of two terms. The first term is the unemployment compensation, γ. The second term is the option value of waiting, which is equal to $\frac{w^+}{1-\beta}$. Thus, the worker accepts whenever :

$$w \geq \gamma(1-\beta) + w^+.$$

in term of "average payoff" per period, the worker accepts whenever the wage exceed the reservation wage w^+ plus the annuity value of receiving unemployment compensation today.

c. The workers probability of finding a job is determined by $P[w > w^i]$, $i = +, 1$. Since $w^1 > w^+$, the probability of accepting a job is higher after one period of unemployment: $P[w > w^+] \geq P[w > w^1]$.

d. $v_1^u(w)$ and v_+^u are defined as in question a. Let $v^e(w)$ be the value of an employed worker with wage w in hand and who behaves optimally. The three value functions are solution of the following system of Bellman equations :

$$v^e(w) = \max_{\text{stay,quit}} \left\{ w + \beta v^e(w), \int_0^B v_1^u(w')dF(w') \right\} \tag{34}$$

$$v_1^u(w) = \max_{\text{accept,reject}} \left\{ w + \beta v^e(w), \gamma + \beta \int_0^B v_+^u(w')dF(w') \right\} \tag{35}$$

$$v_+^u(w) = \max_{\text{accept,reject}} \left\{ w + \beta v^e(w), \beta \int_0^B v_+^u(w')dF(w') \right\}. \tag{36}$$

e. To simplify notations, define first $Q_1 \equiv \int_0^B v_1^u(w')dF(w')$ and $Q_+ \equiv \int_0^B v_+^u(w')dF(w')$. The characterization goes in several steps.

Step 1 : Characterizing $v^e(w)$

From equation (34) it is clear that if an employed worker decides to stay in a given period, he will decide to stay in all the subsequent periods. Thus we can rewrite equation (34) as :

$$v^e(w) = \max_{\text{stay,quit}} \left\{ \frac{w}{1-\beta}, Q_1 \right\}.$$

Furthermore, the above expression shows that the optimal policy of an employed worker is of the reservation wage form. Specificaly, there exists w_e such that for all $w \leq w_e$ the worker quits her job and $v^e(w) = Q_1$. For all $w > w_e$ the worker stays at her job and $v^e(w) = \frac{w}{1-\beta}$. Lastly, w_e solves the following indifference condition :

$$\frac{w_e}{1-\beta} = Q_1.$$

Step 2 : Unemployed workers have reservation wage policies

Since $v^e(w)$ is increasing, it follows from the Bellman equations (35) and (36) that unemployed workers have reservation wage policies. Let w^1 and w^+ be the corresponding reservation wages.

Step 3 : $Q_1 \geq Q_+$

Equations (35) and (36) imply that $v_1^u(w) \geq v_+^u(w)$ for all w. In word, unemployed workers are better off in the first period of unemployment because they receive the benefit γ. Integrating with respect to $dF(w')$ implies that $Q_1 \geq Q_+$.

Step 4 : $w^+ = 0$

This is an intuitive fact. After the first period of unemployment a worker does not receive any benefit. Accepting an offer and quitting is as least as good as rejecting an offer and drawing again next period. Let's prove it formally. Assume $w^+ > 0$. Then $w^+ + \beta v_+^u(w^+) = \beta Q_+ < Q_1 = w_e + \beta v^e(w^e)$. Since $w + v^e(w)$ is weakly increasing this implies that $w^+ < w^e$. Thus $v^e(w^+) = Q_1$. Thus $w^+ + \beta Q_1 = \beta Q_+$, implying that $w^+ = \beta(Q_+ - Q_1) < 0$. A contradiction.

Step 5 : $\gamma > 0 \Rightarrow w^1 > 0$

This is also an intuitive result. Since an unemployed workers receives benefits in its first period of unemployment and, she surely refuses to work when the wage offer is small enough. Formally, assume that $w_1 = 0$. Then $v_1^u(w) = v_+^u(w)$ for all w. This implies that $Q_1 = Q_+$. Also, since $w_1 = 0$, accepting wage offer 0 is at least as good as rejecting it. Thus :

$$0 + \beta v^e(0) = \beta Q_1 \geq \gamma + \beta Q_+ = \gamma + \beta Q_1.$$

When we use the fact that $v^e(0) = Q_1$. The above implies $\gamma < 0$. A contradiction.

Step 6 : $w_1 \leq w_e$ *and* $w_1 = \gamma - \beta(Q_1 - Q_+)$

The Bellman equation (35) implies that $v_1^u(w) \geq \gamma + \beta Q_+$ for all w. Integrating with respect to $dF(w')$ gives $Q_1 \geq \gamma + \beta Q_+$. From the indifference conditions defining w^e and w^1, this is equivalent to :

$$w^e + \beta v^e(w^e) \geq w^1 + \beta v^e(w^1).$$

Since v^e is weakly increasing, it implies that $w^e \geq w^1$. Thus $v^e(w^1) = Q_1$. Using this equality to rewrite the indifference condition defining w^1 gives :

$$w^1 + \beta Q_1 = \gamma + \beta Q_+ \Rightarrow w^1 = \gamma + \beta(Q_+ - Q_1).$$

The above manipulations show that $0 = w^+ \leq w^1 \leq w^e$ and $w^1 = \gamma + \beta(Q_+ - Q_1)$.

Note that w_1 is strictly less than γ. This reflect the fact that, when an agent reject, she receives unemployment compensation this period but also loose the right to receive it next period. On the other hand,if she accepts, she keeps the right to receive unemployment compensation next period.
Lastly, we cannot tell whether or not w^e is smaller or greater than gamma.

Step 6 : Dependence on γ

First note that the value functions are weakly increasing in γ. To see why this is the case consider the optimization problem when the compensation is $\gamma' = \gamma + \Delta\gamma > \gamma$. A posible decision rule for the agents is to use the same reservation wage policy as when the compensation is γ. Payoffs are the same as before except for an additional $\Delta\gamma$ in the first period of unemployment. Therefore, the value of using this decision rule has increased. Now, under the *optimal* decision rule, the value is necessarily even larger.

Since $v_1^u(w,\gamma)$ is weakly increasing in γ, $Q_1 = \int_0^B v_1^u(w',\gamma)dF(w')$ is also weakly increasing in γ. Thus $w_e = (1-\beta)Q_1$ is weakly increasing in γ.

Exercise 5.15. *Seasons, I*

An unemployed worker seeks to maximize $E\sum_{t=0}^{\infty}\beta^t y_t$, where $\beta \in (0,1)$, y_t is her income at time t, and E is the mathematical expectation operator. The person's income consists of one of two parts: unemployment compensation of c that she receives each period she remains unemployed, or a fixed wage w that the worker receives if employed. Once employed, the worker is employed forever with no chance of being fired. Every odd period (i.e., $t = 1, 3, 5, \ldots$) the worker receives one offer to work forever at a wage drawn from the c.d.f. $F(W) = \text{prob}(w \leq W)$. Assume that $F(0) = 0$ and $F(B) = 1$ for some $B > 0$. Successive draws from F are independent. Every even period (i.e., $t = 0, 2, 4, \ldots$), the unemployed worker receives two offers to work forever at a wage drawn from F. Each of the two offers is drawn independently from F.

a. Formulate the Bellman equations for the unemployed person's problem.

b. Describe the form of the worker's optimal policy.

Solution

a. The Bellman equations for the even periods $V^e(w)$ and for the odd periods $V^o(w)$ are:

$$\begin{aligned} V^o(w) &= \max_{\{A,R\}}\left\{\frac{w}{1-\beta}, c + \beta\int_0^B V^e(w')dF^2(w')\right\} \\ V^e(w) &= \max_{\{A,R\}}\left\{\frac{w}{1-\beta}, c + \beta\int_0^B V^o(w')dF(w')\right\}. \end{aligned}$$

b. The workers optimal policy will be a reservation wage in each period below which the worker refuses the best offer outstanding and above which she accepts. For the odd periods, this reservation wage obeys the equation

$$
\begin{aligned}
\frac{\bar{w}^o}{1-\beta} &= c + \beta \int_0^{\bar{w}^e} \frac{\bar{w}^e}{1-\beta} dF^2(w') + \beta \int_{\bar{w}^e}^B \frac{w'}{1-\beta} dF^2(w') \\
c &= \int_0^{\bar{w}^e} \frac{\bar{w}^o - \beta \bar{w}^e}{1-\beta} dF^2(w') + \int_{\bar{w}^e}^B \frac{(\bar{w}^o - \beta w')}{1-\beta} dF^2(w') \\
c(1-\beta) &= (\bar{w}^o - \beta \bar{w}^e) F^2(\bar{w}^e) + \bar{w}^o \left(1 - F^2(\bar{w}^e)\right) - \beta \int_{\bar{w}^e}^B w' dF^2(w'),
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\bar{w}^e}{1-\beta} &= c + \beta \int_0^{\bar{w}^o} \frac{\bar{w}^o}{1-\beta} dF(w') + \beta \int_{\bar{w}^o}^B \frac{w'}{1-\beta} dF(w') \\
c &= \int_0^{\bar{w}^o} \frac{\bar{w}^e - \beta \bar{w}^o}{1-\beta} dF(w') + \int_{\bar{w}^o}^B \frac{\bar{w}^e - \beta w'}{1-\beta} dF(w') \\
c(1-\beta) &= (\bar{w}^e - \beta \bar{w}^o) F(\bar{w}^o) + \bar{w}^e \left(1 - F^2(\bar{w}^o)\right) - \beta \int_{\bar{w}^o}^B w' dF(w').
\end{aligned}
$$

Equating the two expressions for $c(1-\beta)$:

$$
\begin{aligned}
\bar{w}^o - \beta \bar{w}^e F^2(\bar{w}^e) - \beta \int_{\bar{w}^e}^B w' dF^2(w') &= \bar{w}^e - \beta \bar{w}^o F(\bar{w}^o) - \beta \int_{\bar{w}^o}^B w' dF(w') \\
\bar{w}^o + \beta \left[\bar{w}^o F(\bar{w}^o) + \int_{\bar{w}^o}^B w' dF(w') \right] &= \bar{w}^e + \beta \left[\bar{w}^e F^2(\bar{w}^e) + \int_{\bar{w}^e}^B w' dF^2(w') \right].
\end{aligned}
$$

For a given $\bar{w}$, we know that $\bar{w}F^2(\bar{w}) + \int_{\bar{w}}^B w'dF^2(w') \geq \bar{w}F(\bar{w}) + \int_{\bar{w}}^B w'dF(w')$. Furthermore, using Leibnitz rule, we know that $\bar{w}F(\bar{w}) + \int_{\bar{w}}^B w'dF(w')$ and $\bar{w}F^2(\bar{w}) + \int_{\bar{w}}^B w'dF^2(w')$ are increasing in $\bar{w}$. Using these two facts, the above equality cannot hold for $\bar{w}^o < \bar{w}^e$, because both terms on the left hand side would be less than the corresponding terms on the right hand side. We conclude that $\bar{w}^o \geq \bar{w}^e$. The intuition is that in odd periods, the unemployed worker's outside option (reject and two draws next period) is better than his outside option in even periods (reject and sample once next period). That makes him want a higher reservation wage in odd periods.

Exercise 5.16. *Seasons, II*

Consider the following problem confronting an unemployed worker. The worker wants to maximize

$$E_0 \sum_0^\infty \beta^t y_t, \quad \beta \in (0,1),$$

where $y_t = w_t$ in periods in which the worker is employed and $y_t = c$ in periods in which the worker is unemployed, where w_t is a wage rate and c is a constant level of unemployment compensation. At the start of each period, an unemployed worker receives one and only one offer to work at a wage w drawn from a c.d.f. $F(W)$, where $F(0) = 0, F(B) = 1$ for some $B > 0$. Successive draws from F

are identically and independently distributed. There is no recall of past offers. Only unemployed workers receive wage offers. The wage is fixed as long as the worker remains in the job. The only way a worker can leave a job is if she is fired. At the *beginning* of each odd period ($t = 1, 3, \ldots$), a previously employed worker faces the probability of $\pi \in (0,1)$ of being fired. If a worker is fired, she immediately receives a new draw of an offer to work at wage w. At each even period ($t = 0, 2, \ldots$), there is no chance of being fired.

a. Formulate a Bellman equation for the worker's problem.

b. Describe the form of the worker's optimal policy.

Solution

a. Let $v_U^e(w)$, ($v_U^o(w)$) be the value an uemployed worker who has just received an offer w at the start of an even (odd) period and proceeds optimally. Similarly, let $v_E^e(w)$, ($v_E^o(w)$) be the value an employed with wage w the beginning of an even (odd) period. The Bellman equation for v_U^e is :

(37)

$$v_U^e(w) = \max\left\{w + \beta\left[\pi\int_0^B v_U^o(w')dF(w') + (1-\pi)v_E^o(w)\right], c + \beta\int_0^B v_U^o(w')dF(w')\right\}.$$

Similarly, the Bellman equation for v_U^o is :

$$v_U^o(w) = \max\left\{w + v_E^e(w), c + \beta\int_0^B v_U^e(w')dF(w')\right\}. \tag{38}$$

Bellman equations for employed workers are :

$$v_E^o(w) = w + \beta v_E^e(w) \tag{39}$$

$$v_E^e(w) = w + \beta\pi\int_0^B v_U^o(w')dF(w') + (1-\pi)v_E^o(w). \tag{40}$$

There is no "max" in the two Bellman equations because it was assumed that the only way an employed worker can leave a job is by being fired. These 4 Bellman equations fully describe the worker's dynamic choice problem.

b. First, solve for $v_E^o(w)$ and $v_E^e(w)$ in terms of the optimum value functions. This produces:

$$v_E^e(w) = \frac{w\left(1 + (1-\pi)\beta\right) + \beta\pi\int v_U^o(w')dF(w')}{1-(1-\pi)\beta^2}$$

$$v_E^o(w) = \frac{w\left(1+\beta\right) + \beta^2\pi\int v^o(w')dF(w')}{1-(1-\pi)\beta^2}.$$

Replacing those expressions in (37) and (38) shows that a reservation wage policy is optimal in this setting. Let the w^e (w^o) be the reservation wage in even (odd) periods. The indifference conditions are :

$$w^e + \beta \left[\pi \int_0^B v_U^o(w')dF(w') + (1-\pi)v_E^o(w^e) \right] = c + \beta \int_0^B v_U^o(w')dF(w'),$$

which implies that:

$$w^e \frac{1+\beta(1-\pi)}{1-(1-\pi)\beta^2} + \frac{\beta^2\pi}{1-(1-\pi)\beta^2} Q^o = c + \beta Q^o,$$

where $Q^o =\equiv \int_0^B v_U^o(w')dF(w')$. Some simple algebra yields:

$$\text{(41)} \qquad w^e = \frac{1-(1-\pi)\beta^2}{1+\beta(1-\pi)} c + \frac{\beta\left[1-(1-\pi)\beta^2\right] Q^o - \beta^2\pi Q^o}{1+\beta(1-\pi)}$$

$$\text{(42)} \qquad = \frac{1-(1-\pi)\beta^2}{1+\beta(1-\pi)} c + \frac{\beta\left[1-(\pi+\beta(1-\pi))\beta^2\right]}{1+\beta(1-\pi)} Q^o.$$

Similarly, we know from (38) that, at the reservation wage w^o:

$$\text{(43)} \qquad w^o + \beta\left[v_E^e(w^o)\right] = c + \beta Q^e,$$

which in turn implies that:

$$w^o \frac{(1-\beta)}{1-(1-\pi)\beta^2} + \frac{\beta^2\pi}{1-(1-\pi)\beta^2} Q^o = c + \beta Q^e.$$

Some simple algebra yields:

$$\text{(44)} \qquad w^o = \frac{1-(1-\pi)\beta^2}{1-\beta} c + \frac{\beta\left[1-(1-\pi)\beta^2\right] Q^e - \beta^2\pi Q^o}{1-\beta}.$$

Now, we can compare the reservation wages in even and odd periods by comparing eqs. (41) and (44). It is easy to verify that $w^o \geq w^e$ whenever $Q^e \geq Q^o$, since $\beta \in (0,1)$ and $\pi \in (0,1)$. This means that, if the optimum value of the unemployed (or, equivalently, just fired) worker is higher at the start of an even period than at the start of an odd period, then the worker sets a higher reservation wage in the odd period, because he is quite willing to wait another period while being unemployed (in order to receive Q^e). Also note that the reverse statement is not necessarily true.

Exercise 5.17. *Gittins indices for beginners*

At the end of each period, a worker can switch between two jobs, A and B, to begin the following period at a wage that will be drawn at the beginning of next period from a wage distribution specific to job A or B, and to the worker's history of past wage draws from jobs of either type A or type B. The worker must decide to stay or leave a job at the end of a period after his wage for this period on his current job has been received, but before knowing what his wage would be next period in either job. The wage at either job is described by a job-specific

n-state Markov chain. Each period the worker works at either job A or job B. At the end of the period, before observing next period's wage on either job, he chooses which job to go to next period. We use lowercase letters $(i, j = 1, \ldots, n)$ to denote states for job A, and uppercase letters $(I, J = 1, \ldots n)$ for job B. There is no option of being unemployed.

Let $w_a(i)$ be the wage on job A when state i occurs and $w_b(I)$ be the wage on job B when state I occurs. Let $A = [A_{ij}]$ be the matrix of one-step transition probabilities between the states on job A, and let $B = [B_{ij}]$ be the matrix for job B. If the worker leaves a job and later decides to returns to it, he draws the wage for his first new period on the job from the conditional distribution determined by his last wage working at that job.

The worker's objective is to maximize the expected discounted value of his lifetime earnings, $E_0 \sum_{t=0}^{\infty} \beta^t y_t$, where $\beta \in (0,1)$ is the discount factor, and where y_t is his wage from whichever job he is working at in period t.

a. Consider a worker who has worked at both jobs before. Suppose that $w_a(i)$ was the last wage the worker receives on job A and $w_b(I)$ the last wage on job B. Write the Bellman equation for the worker.

b. Suppose that the worker is just entering the labor force. The first time he works at job A, the probability distribution for his initial wage is $\pi_a = (\pi_{a1}, \ldots, \pi_{an})$. Similarly, the probability distribution for his initial wage on job B is $\pi_b = (\pi_{b1}, \ldots, \pi_{bn})$ Formulate the decision problem for a new worker, who must decide which job to take initially. [*Hint*: Let $v_a(i)$ be the expected discounted present value of lifetime earnings for a worker who was last in state i on job A and has never worked on job B; define $v_b(I)$ symmetrically.]

Solution

a. First we consider a worker who has worked at both jobs before. Suppose that $w_a(i)$ was the last wage the worker receives at job A and $w_b(I)$ was the last wage he received at job B.

Let $v(i, I)$ be the optimum value, starting from next period, of a worker currently active in job A at wage $w_a(i)$ who has also worked at job B (at some point in the past) at a wage $w_B(I)$. Again, this worker is at the end of the current period and has to decide where to go in the next period **before** having observed next period's wage on either job. Similarly, let $v(I, i)$ be the optimum value, starting from next period, of a worker currently active in job B at wage $w_b(I)$ who has also worked at job A (at some point in the past) at a wage $w_A(i)$.

The Bellman equation for the first worker is given by:

$$v(i, I) = \max_{A,B} \left\{ \sum_{j=1}^{n} A_{ij} \left[w_A(j) + \beta v(j, I) \right], \sum_{J=1}^{n} B_{IJ} \left[w_B(J) + \beta v(J, i) \right] \right\}, \tag{45}$$

while the Bellman equation of the second worker is given by:

$$v(I,i) = \max_{A,B} \left\{ \sum_{j=1}^{n} A_{ij} \left[w_A(j) + \beta v(j,I) \right], \sum_{J=1}^{n} B_{IJ} \left[w_B(J) + \beta v(J,i) \right] \right\}. \tag{46}$$

Notice how $v(i,I) = v(I,i)$ by comparing the r.h.s of eq. (45) and eq. (46). This implies we can let $v(i,I)$ denote the optimum value of a worker whose last wage at job A was $w_a(i)$ and at job B was $w_b(I)$. (Let's agree on making the first argument of the value function the last wage at A).

$$v(i,I) = \max_{A,B} \left\{ \sum_{j=1}^{n} A_{ij} \left[w_A(j) + \beta v(j,I) \right], \sum_{J=1}^{n} B_{IJ} \left[w_B(J) + \beta v(i,J) \right] \right\}. \tag{47}$$

b. Next, we turn to consider the problem facing a worker who is just about to enter the labor force. Working backwards, we first examine the case of a worker who has only worked on one job. Let $v_a(i)$ denote the optimum value of a worker at job A, making a wage $w_a(i)$, who has never worked on job B before and let $v_b(I)$ denote the same value for a worker earning a wage $w_b(I)$ at B, who has never worked job A before. Then we know that the Bellman equation of a worker who has only worked at A is:

$$v_a(i) = \max_{A,B} \left\{ \sum_{j=1}^{n} A_{ij} \left[w_a(j) + \beta v_a(j) \right], \sum_{J=1}^{n} \pi_b(J) \left[w_b(J) + \beta v(i,J) \right] \right\},$$

while the Bellman equationof a worker who has only worked at B is given by:

$$v_b(I) = \max_{A,B} \left\{ \sum_{j=1}^{n} \pi_a(j) \left[w_a(j) + \beta v(j,J) \right], \sum_{J=1}^{n} B_{IJ} \left[w_b(J) + \beta v_b(J) \right] \right\},$$

Finally, consider the problem facing a worker who is about to enter the labor force; naturally, she starts at the job that yields the highest expected lifetime utility:

$$Q = \max_{A,B} \left\{ \sum_{i=1}^{n} \pi_a(i) \left[w_a(i) + \beta v_a(i) \right], \sum_{I=1}^{n} \pi_a(I) \left[w_b(I) + \beta v_b(I) \right] \right\}.$$

Now we have exhaustively described the worker's problem, proceeding **backwards**, which is the only way to solve this type of problem.

Exercise 5.18. *Jovanovic (1979b)*

An employed worker in the tth period of tenure on the current job receives a wage $w_t = x_t(1 - \phi_t - s_t)$ where x_t is job-specific human capital, $\phi_t \in (0,1)$ is the fraction of time that the worker spends investing in job-specific human capital, and $s_t \in (0,1)$ is the fraction of time that the worker spends searching for a new job offer. If the worker devotes s_t to searching at t, then with probability $\pi(s_t) \in (0,1)$ at the beginning of $t+1$ the worker receives a new job offer to begin working at new job-specific capital level μ' drawn from the c. d. f. $F(\cdot)$. That is, searching for a new job offer promises the prospect of instantaneously reinitializing job-specific human capital at μ'. Assume that $\pi'(s) > 0, \pi''(s) < 0$. While on a given job, job-specific human capital evolves according to

$$x_{t+1} = G(x_t, \phi_t) = g(x_t \phi_t) - \delta x_t,$$

where $g'(\cdot) > 0, g''(\cdot) < 0$, $\delta \in (0,1)$ is a depreciation rate, and $x_0 = \mu$ where t is tenure on the job, and μ is the value of the "match" parameter drawn at the start of the current job. The worker is risk neutral and seeks to maximize $E_0 \sum_{\tau=0}^{\infty} \beta^\tau y_\tau$, where y_τ is his wage in period τ.

a. Formulate the worker's Bellman equation.

b. Describe the worker's decision rule for deciding whether to accept an offer μ' at the beginning of next period.

c. Assume that $g(x\phi) = A(x\phi)^\alpha$ for $A > 0, \alpha \in (0,1)$. Assume that $\pi(s) = s^{.5}$. Assume that F is a discrete n-valued distribution with probabilities f_i; for example, let $f_i = n^{-1}$. Write a Matlab program to solve the Bellman equation. Compute the optimal policies for ϕ, s and display them.

Solution

a. Let $v(x)$ be the optimum value at the start of the current period of an employed worker who has accumulated a total amount x of job-specific capital and who proceeds optimally. We know the worker will accept the new draw μ' at the start of next period whenever μ' exceeds next period's capital on the old job x'. This means her Bellman equation is given by:

$$v(x) = \max_{\phi,s} x(1-\phi-s) + \beta \left[((1-\pi(s))v(x')) + \pi(s) \int \max(v(\mu'), v(x')) dF(\mu') \right]$$

$$v(x) = \max_{\phi,s} x(1-\phi-s) + \beta \left[((1-\pi(s))v(x')) + \pi(s) \left[\int_{x'} v(\mu') dF(\mu') + F(x')v(x') \right] \right]$$

,
where $x' = G(x, \phi) = g(x\phi) - \delta x$

b. The question is answered in part a.

c. The matlab code is in zia.stanford.edu/public/sarg/webdocs/teaching/econ210/ in files `jova.m` and `readjova.txt`

CHAPTER 6

Recursive (partial) equilibrium

Exercise 6.1. *A competitive firm*

A competitive firm seeks to maximize

$$\sum_{t=0}^{\infty} \beta^t R_t, \tag{48}$$

where $\beta \in (0, 1)$, and time-t revenue R_t is

$$R_t = p_t y_t - .5d(y_{t+1} - y_t)^2, \quad d > 0, \tag{49}$$

where p_t is the price of output, and y_t is the time-t output of the firm. Here $.5d(y_{t+1} - y_t)^2$ measures the firm's cost of adjusting its rate of output. The firm starts with a given initial level of output y_0. The price lies on the market demand curve

$$p_t = A_0 - A_1 Y_t, A_0, A_1 > 0, \tag{50}$$

where Y_t is the market level of output, which the firm takes as exogenous, and which the firm believes follows the law of motion

$$Y_{t+1} = H_0 + H_1 Y_t, \tag{51}$$

with Y_0 as a fixed initial condition.

a. Formulate the Bellman equation for the firm's problem.

b. Formulate the firm's problem as a discounted optimal linear regulator problem, being careful to describe all of the objects needed. What is the *state* for the firm's problem?

c. Use the Matlab program `olrp.m` to solve the firm's problem for the following parameter values: $A_0 = 100, A_1 = .05, \beta = .95, d = 10, H_0 = 95.5$, and $H_1 = .95$. Express the solution of the firm's problem in the form

$$y_{t+1} = h_0 + h_1 y_t + h_2 Y_t, \tag{52}$$

giving values for the h_j's.

d. If there were n identical competitive firms all behaving according to equation (52), what would equation (52) imply for the *actual* law of motion (51) for the market supply Y?

e. Formulate the Euler equation for the firm's problem.

Solution

a. The Bellman equation corresponding to the firm's problem is given by:

$$v(y, Y) = \max_{y'} (A_0 - A_1 Y) y - .5d (y' - y)^2 + \beta v(y', Y'), \tag{53}$$

where Y evolves according to the farmer's beliefs about the aggregate law of motion:

$$Y' = H_0 + H_1 Y.$$

b. The discounted optimal linear regulator:
Consider the first order difference equation fot the state vector:

$$\begin{bmatrix} y_{t+1} \\ Y_{t+1} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & H_1 & H_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_t \\ Y_t \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} (u_t),$$

where the control u_t is given by $(y_{t+1} - y_t)$. The state for the firm's problem is given by $x_t' = \begin{bmatrix} y_t & Y_t & 1 \end{bmatrix}$. Having defined A and B, we only need to define Q and R (note that the interaction between u and x can be left out here: no H matrix is needed!).

$$Q = -.5d; R = \begin{bmatrix} 0 & -A_1/2 & A_0/2 \\ -A_1/2 & 0 & 0 \\ A_0/2 & 0 & 0 \end{bmatrix}.$$

c. olrp To compute the solution, type $[f, p] = olrp[\beta, A, B, R, Q]$ or $A_0 = 100, A_1 = .05, \beta = .95, d = 10, H_0 = 95.5$ and $H_1 = .95$, this yields:

$$y_{t+1} - y_t = 96.9487 - 0.0463 Y_t,$$

or

$$y_{t+1} = 96.9487 + y_t - 0.0463 Y_t,$$

which means $h_0 = 96.94, h_1 = 1$ and $h_2 = -.0463$ in $Y_{t+1} = h_0 + h_1 y_t + h_2 Y_t$.

d. In equilibrium the actual law of motion for the market supply Y_t would be :

$$\begin{aligned} Y_{t+1} &= n y_{t+1} \\ &= n h_0 + (h_1 + n h_2) Y_t \\ &= n(96.9487) + (1 - n 0.0463) Y_t. \end{aligned}$$

e. First derive the r.h.s of (53) w.r.t. y' to get the first order condition:

$$-d(y' - y) + \beta v_1(y', Y') = 0,$$

where v_i denotes the partial derivative w.r.t. i-th argument.
In the infinite horizon case, there is an additional transversality condition that needs to be satisfied:

$$\lim_{t \to \infty} \beta^t v(y_t, Y_t) = 0.$$

Applying the Benveniste-Scheinkman formula yields:

$$v_1(y, Y) = A_0 - A_1 Y + d(y' - y).$$

Substitution in the first order condition yields the Euler equation:

$$-d(y' - y) + \beta(A_0 - A_1 Y' + d(y'' - y'))) = 0.$$

Exercise 6.2. *Rational Expectations*

Now assume that the firm in exercise 6.1 is "representative." We implement this idea by setting $n = 1$. In equilibrium, we will require that $y_t = Y_t$, but we don't want to impose this condition at the stage that the firm is optimizing (because we want to retain competitive behavior). Define a rational expectations equilibrium to be a pair of numbers H_0, H_1 such that if the representative firm solves the problem ascribed to it in exercise 6.1, then the firm's optimal behavior given by equation (52) implies that $y_t = Y_t \ \forall \ t \geq 0$.

a. Use the program that you wrote for exercise 6.1 to determine which if any of the following pairs (H_0, H_1) is a rational expectations equilibrium: (i) (94.0888, .9211); (ii) (93.22, .9433), and (iii) (95.08187459215024, .95245906270392)?

b. Describe an iterative algorithm by which the program that you wrote for exercise 6.1 might be used to compute a rational expectations equilibrium. You are not being asked actually to use the algorithm you are suggesting.

Solution

a. Recall that a rational expectations equilibrium is defined to be a pair H_0, H_1 such that the representative firm's optimal decision rule, given these beliefs, implies that $y_t = Y_t$ for all t

(i).

$$F = \begin{bmatrix} 0 & 0.0350 & -118.4668 \end{bmatrix}.$$

Check that this does not produce an RE equilibrium.

(ii).

$$F = \begin{bmatrix} 0 & 0.0431 & -104.7364 \end{bmatrix}.$$

Check that this does not produce an RE equilibrium.

(iii).

$$F = \begin{bmatrix} 0 & 0.0475 & -95.0819 \end{bmatrix}.$$

The third feedback function implies that:

$$y_{t+1} - y_t = -Fx_t = .950819 - .0475Y_t,$$

which implies in equilibrium ($y_t = Y_t$ for all t) that:

$$Y_{t+1} = .950819 + .95245Y_t.$$

The beliefs are such that the law of motion for aggregate output $\mathcal{M}(H)$ implied by the optimal policy -given these beliefs H- equals these beliefs. This is an **RE equilibrium.**

b. Recall that the farmer's optimization maps his beliefs into a law of motion. Let H denote his beliefs. Then we can define $H' = \mathcal{M}(H)$ where $\mathcal{M}$ is the operator mapping beliefs into a law of motion. Since a RE equilibrium is a fixed point of the $\mathcal{M}$ operator, it seems natural to try an **iterative** approach where, starting with initial beliefs H_0, we iterate on the $\mathcal{M}$ operator until convergence is achieved: $H = \mathcal{M}(H)$.

Exercise 6.3. *Maximizing Welfare*

A planner seeks to maximize the welfare criterion

$$\sum_{t=0}^{\infty} \beta^t S_t,$$

where S_t is "consumer surplus plus producer surplus" defined to be

$$S_t = S(Y_t, Y_{t+1}) = \int_0^{Y_t} (A_0 - A_1 x) d\, x - .5d(Y_{t+1} - Y_t)^2.$$

a. Formulate the planner's Bellman equation.

b. Formulate the planner's problem as an optimal linear regulator, and solve it using the Matlab program `olrp.m` . Represent the solution in the form $Y_{t+1} = s_0 + s_1 Y_t$.

c. Compare your answer in part b with your answer to part a of exercise 6.2.

Solution

a. We look for an optimal policy function and an optimal value function by solving the Bellman equation:

$$v(Y) = \max_{Y'} A_0 Y - \frac{A_1}{2} Y^2 - .5d\,(Y' - Y)^2 + \beta v(Y').$$

Derive the r.h.s. with respect to Y' to derive the Euler equation:

$$-d\,(Y' - Y) + \beta v(Y') = 0.$$

In the infinite horizon case, there is an additional transversality condition that needs to be satisfied:

$$\lim_{t \to \infty} \beta^t v(Y_t) = 0.$$

Applying the Benveniste-Scheinkman formula yields:

$$v'(Y) = A_0 - A_1 Y + d(Y' - Y).$$

To understand this last result, note that the law of motion for Y is such that the next period's value of Y, *i.e.* Y', is independent of Y. Or, using the notation in Chapter 2, $\frac{\partial g(u,x)}{x} = 0$ when $x_{t+1} = g(u_t) = u_t$. In this case, the Benveniste Sheinkman condition becomes:

$$\begin{aligned} V'(x) &= \frac{\partial}{\partial x} r(x, h(x)) + \beta \frac{\partial}{\partial x} g(x, h(x)) V'(g(x, h(x)) \\ &= \frac{\partial}{\partial x} r(x, h(x)). \end{aligned}$$

b. The discounted optimal linear regulator for the planner.
Consider the first order difference equation for the state vector:

$$\begin{bmatrix} Y_{t+1} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Y_t \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} (u_t),$$

where the control u_t is given by $(Y_{t+1} - Y_t)$. The state for the planner's problem is given by $x_t' = \begin{bmatrix} Y_t & 1 \end{bmatrix}$. Having defined A and B, we only need to define Q and R (note that the interaction between u and x can be left out here: no H matrix is needed!).

$$Q = -.5d; R = \begin{bmatrix} -A_1/2 & A_0/2 \\ A_0/2 & 0 \end{bmatrix}.$$

The feedback rule that results for $A_0 = 100, A_1 = .05, \beta = .95, d = 10$:

$$F = \begin{matrix} 0.0475 & -95.0819 \end{matrix},$$

which implies:

$$Y_{t+1} = .950819 + .95245 Y_t. \tag{54}$$

c. This confirms that the planner's problem yields a law of motion for aggregate output that corresponds to a rational expectations competitive equilibrium (identical to *6.2.a.(iii)*). To see why, recall from the previous ex. that if we take eq. (54) to be the farmer's initial beliefs, the law of motion that is implied by these beliefs is given by the same equation.

Exercise 6.4. *Monopoly*

A monopolist faces the industry demand curve (50) and chooses Y_t to maximize

$$\sum_{t=0}^{\infty} \beta^t R_t \tag{55}$$

where $R_t = p_t Y_t - .5d(Y_{t+1} - Y_t)^2$ and where Y_0 is given.

a. Formulate the firm's Bellman equation.

b. For the parameter values listed in exercise 6.1, formulate and solve the firm's problem using `olrp.m` .

c. Compare your answer in part b with the answer you obtained to part b of exercise 6.3.

Solution

a. We look for an optimal policy function and an optimal value function by solving the Bellman equation:

$$v(Y) = \max_{Y'} A_0 Y - A_1 Y^2 - .5d\left(Y' - Y\right)^2 + \beta v(Y')$$

.

b. Consider the first order difference equation for the state vector:

$$\begin{bmatrix} Y_{t+1} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Y_t \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} (u_t),$$

where the control u_t is given by $(Y_{t+1} - Y_t)$. The state for the monopolist's problem is given by $x_t' = \begin{bmatrix} Y_t & 1 \end{bmatrix}$. Having defined A and B, we only need to define Q and R (note that the interaction between u and x can be left out here: no H matrix is needed!).

$$Q = -.5d; R = \begin{bmatrix} -A_1 & A_0/2 \\ A_0/2 & 0 \end{bmatrix}.$$

The feedback rule that results for $A_0 = 100, A_1 = .05, \beta = .95, d = 10:$

$$F = \begin{array}{cc} 0.0735 & -73.4729 \end{array},$$

which implies that Y_t follows the following law of motion:

$$Y_{t+1} - Y_t = -.0735 Y_t + 73.4729,$$

or

$$Y_{t+1} = .9265 Y_t + 73.4729,$$

which clearly shows that the monopolist restricts total output to increase profits. In fact, you can easily verify that the unconditional mean of Y_t in the monopolist's case is exactly half of that in the competitive equilibrium.

Exercise 6.5. *Duopoly*

An industry consists of two firms that jointly face the industry-wide demand curve (50), where now $Y_t = y_{1t} + y_{2t}$. Firm $i = 1, 2$ maximizes

$$\sum_{t=0}^{\infty} \beta^t R_{it}, \tag{56}$$

where $R_{it} = p_t y_{it} - .5d(y_{i,t+1} - y_{it})^2$.

a. Define a Markov perfect equilibrium for this industry.

b. Formulate the Bellman equation for each firm.

c. Use the Matlab program `nash.m` to compute an equilibrium, assuming the parameter values listed in exercise 6.1.

Solution

a. Consider firm $i's$ Bellman equation:

$$v_i(y_{it}, y_{-i,t}) = \max_{y_{i,t+1}} \{R_{it} + \beta v_i(y_{it+1}, y_{-i,t+1})\}, \tag{57}$$

where the maximization is subject to:

$$y_{-i,t+1} = f_{-i}(y_{-i,t}, y_{i,t}),$$

and $R_{i,t} = p_t y_{it} - .5d(y_{it+1} - y_{it})^2$ where $p_t = A_0 - A_1(y_{1t} + y_{2t})$.

Definition 1. *A Markov perfect equilibrium is a pair of value functions v_i and a pair of policy functions f_i for $i = 1, 2$ such that*
a. Given f_{-i}, v_i satisfies the Bellman equation.
b. The policy function f_i attains the r.h.s. of the Bellman equation.

The crucial thing to note is that firm i optimizes taking the other firm's policy function as **given**.

b. Consider firm $1's$ Bellman equation:

$$v_1(y_{1t}, y_{2,t}) = \max_{y_{1,t+1}} \{R_{1t} + \beta v_1(y_{1t+1}, y_{2,t+1})\}, \tag{58}$$

where the maximization is subject to:

$$y_{2,t+1} = f_2(y_{2,t}, y_{1,t}).$$

Similarly, firm 2's Bellman equation:

$$v_2(y_{2t}, y_{1,t}) = \max_{y_{1,t+1}} \{R_{2t} + \beta v_2(y_{2t+1}, y_{1,t+1})\}, \tag{59}$$

where the maximization is subject to:

$$y_{1,t+1} = f_1(y_{1,t}, y_{2,t}).$$

c. For computational reasons, we'll **assume** $\beta = 1$. Consider the state vector dynamics, represented in the usual way:

$$x_{t+1} = A_t x_t + B_1 u_{1t} + B_2 u_{2t},$$

where:

$$\begin{matrix} y_{1t+1} \\ y_{2t+1} \\ 1 \end{matrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{1t} \\ y_{2t} \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_{1,t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_{2,t},$$

where $u_{1t} = y_{1,t+1} - y_{1,t}$ and $u_{2,t} = y_{2,t+1} - y_{2,t}$ are the control variables of firms 1 and 2 resp.
Now, let the return function $R_{i,t}$ be given by:

$$R_{i,t} = x_t' R_i x_t + u_{i,t}' Q_i u_{i,t} + u_{-i,t}' S_i u_{-i,t} \text{ for } i = 1, 2.$$

Then it is easy to verify that:

$$Q_1 = -.5d; S_1 = 0; R_1 = \begin{bmatrix} -A_1 & -A_1/2 & A_0/2 \\ -A_1/2 & 0 & 0 \\ A_0/2 & 0 & 0 \end{bmatrix},$$

$$Q_2 = -.5d; S_2 = 0; R_1 = \begin{bmatrix} 0 & -A_1/2 & 0 \\ -A_1/2 & -A_1 & A_0/2 \\ 0 & A_0/2 & 0 \end{bmatrix}.$$

The equilibrium feedback rule for firm 1 is given by:

$$y_{1,t+1} - y_{1,t} = -\begin{bmatrix} .0878 & 0.02520 & -83.3556 \end{bmatrix} x_t.$$

The equilibrium feedback rule for firm 2 is given by:

$$y_{2,t+1} - y_{2,t} = -\begin{bmatrix} 0.02520 & .0878 & -83.3556 \end{bmatrix} x_t.$$

This implies that in equilibrium:

$$y_{i,t+1} = 83.3556 + (1 - .0878) y_{i,t} + 0.02520 y_{-i,t}.$$

Exercise 6.6. *Self-Control*

This is a model of a human who has time-inconsistent preferences, of a type proposed by Phelps and Pollak (1968) and used by Laibson (1994). The human lives from $t = 0, \dots, T$. Think of the human as actually consisting of $T+1$ personalities, one for each period. Each personality is a distinct agent (i.e., a distinct utility function and constraint set). Personality T has preferences ordered by $u(c_T)$ and personality $t < T$ has preferences that are ordered by

$$u(c_t) + \delta \sum_{j=1}^{T-t} \beta^j u(c_{t+j}),$$

where $u(\cdot)$ is a twice continuously differentiable, increasing and strictly concave function of consumption of a single good; $\beta \in (0,1)$, and $\delta \in (0,1]$. When $\delta < 1$, preferences of the sequence of personalities are time-inconsistent (that is, not recursive). At each t, let there be a savings technology described by

$$k_{t+1} + c_t \leq f(k_t),$$

where f is a production function with $f' > 0, f'' \leq 0$.

a. Define a Markov perfect equilibrium for the $T+1$ personalities.

b. Argue that the Nash-Markov equilibrium can be computed by iterating on the following functional equations:

$$\begin{array}{l} V_{j+1}(k) = \max_c \{u(c) + \beta\delta W_j(k')\}, \\ W_{j+1}(k) = u[c_{j+1}(k)] + \beta W_j[f(k) - c_{j+1}(k)]. \end{array} \tag{60}$$

where $c_{j+1}(k)$ is the maximizer of the right side of the first equation for $j+1$, starting from $W_0(k) = u[f(k)]$. Here $W_j(k)$ is the value of $u(c_{T-j})+\beta u(c_{T-j+1})+\ldots+\beta^{T-j}u(c_T)$, taking the decision rules $c_h(k)$ as given for $h = 0, 1, \ldots, j$.

c. State the optimization problem of the time-0 person who is given the power to dictate the choices of all subsequent persons. Write the Bellman equations for this problem. The time zero person is said to have a commitment technology for "self-control" in this problem.

Solution

a.

DEFINITION 2. *A Markov perfect equilibrium is a set of $T+1$ value functions $\{V_t\}_{t=0}^T$ and policy functions $\{c_t, k_{t+1}\}_{t=0}^T$ such that:*
a. $\forall t \in \Upsilon = \{0, 1, ..., T\}$, the value function for agent t, V_t, satisfies the Bellman equation given the policy functions for all other agents s in $\Upsilon, s \neq t$.
b. $\forall t \in \Upsilon = \{0, 1, ..., T\}$, the policy functions $\{c_t, k_{t+1}\}$ attains the right hand side of the Bellman equation.

b. The person living at time t values current consumption and the consumption of future selves living after period t, summarized in the value function W_{T-t}. This 'equilibrium' value function W_{T-t} consists of the utility derived from consumption derived in period t and in the future. It imposes market clearing (hence 'equilibrium' value function). The original problem formulation is not recursive because of the different time horizon of each of the agents. The introduction of the value function W makes the problem recursive again.

c. Working backwards from period T with $W_0(k') = u(f(k'))$ we have

$$\begin{aligned} V_1(k) &= \max_{\{c\}} \{u(c) + \beta\delta u(f(k'))\} \Longrightarrow c_1^*(k) \\ W_1(k) &= u\left(c_1^*(k)\right) + \delta W_0(k') \\ &= u\left(c_1^*(k)\right) + \delta u(f(f\left(k\right) - c_1^*(k))). \end{aligned}$$

The person who lives 2 periods before time T has the follwoing value function

$$\begin{aligned} V_2(k) &= \max_{\{c\}} \{u(c) + \beta\delta W_1(k')\} \\ &= \max_{\{c\}} \left\{u(c) + \beta\delta u\left(c_1^*(k')\right) + \beta\delta^2 u(f(f\left(k'\right) - c_1^*(k')))\right\} \\ &\Longrightarrow c_2^*(k). \end{aligned}$$

This person takes into account the optimal consumption decisions that the people after him will make. Each person takes into account that future persons will face a shorter decision horizon. Updating the equilibrium value function:

$$\begin{aligned} W_2(k) &= u\left(c_2^*(k)\right) + \delta W_1(f(k) - c_2^*(k)) \\ &= u\left(c_2^*(k)\right) + \delta u\left(c_1^*(f(k) - c_2^*(k))\right) + \delta^2 u(f(f\left(f(k) - c_2^*(k)\right) - c_1^*(f(k) - c_2^*(k)))). \end{aligned}$$

Working backwards until time zero, we find

$$\begin{aligned} V_T(k) &= \max_{\{c\}} \left\{u(c) + \beta\delta W_{T-1}(k')\right\} \Longrightarrow c_T^*(k) \\ W_T(k) &= u\left(c_T^*(k)\right) + \delta W_{T-1}(f(k) - c_T^*(k)), k \text{ is given.} \end{aligned}$$

Person zero has the ability to commit to self control by choosing $c_T^*(k)$ descibed in the manner above. By doing so he will leave the right amount of resources for future persons to make the consumption decisions in the same 'controlled' fashion.

CHAPTER 7

Competitive equilibrium with complete markets

Exercise 7.1. *Existence of a representative consumer*

Suppose households 1 and 2 have one-period utility functions $u(c^1)$ and $w(c^2)$, respectively, where u and w are both increasing, strictly concave, twice-differentiable functions of a scalar consumption rate. Consider the Pareto problem:

$$\max_{\{c^1,c^2\}} \left[\theta u(c^1) + (1-\theta)w(c^2)\right]$$

subject to the constraint $c^1 + c^2 = c$. Show that the solution of this problem has the form of a concave utility function $v_\theta(c)$, which depends on the Pareto weight θ.

The function $v_\theta(c)$ is the utility function of the *representative consumer*. Such a representative consumer always lurks within a complete markets competitive equilibrium even with heterogeneous preferences.

Solution

Let $x = (c^1, c^2)$. Define the set

$$B(c) \equiv \left\{x = (c^1, c^2) :, c^1 \geq 0, c^2 \geq 0, c^1 + c^2 \leq c\right\}$$

Also define the continuous, strictly concave and increasing function $\tilde{v}(x) = \theta u(c^1) + (1-\theta)w(c^2)$. We are interested in the program :

$$v(c) = \max_{x \in B(c)} \tilde{v}(x).$$

$B(c)$ is compact and $\tilde{v}(x)$ is continuous. Thus $\tilde{v}(x)$ achieves its maximum on B. Since furthermore $\tilde{v}$ is strictly concave, this maximum is unique. Call it $x^*(c)$. We now have for $(c, c') \geq 0$ and $\lambda \in [0,1]$:

$$\begin{aligned}
\lambda v(c) + (1-\lambda)v(c') &= \lambda\tilde{v}(x^*(c)) + (1-\lambda)\tilde{v}(x^*(c')) && (61)\\
&\leq \tilde{v}\left(\lambda x^*(c) + (1-\lambda)x^*(c')\right) && (62)\\
&\leq v(\lambda c + (1-\lambda)c'), && (63)
\end{aligned}$$

where the first equality is definitional, the second line holds because of concavity of $\tilde{v}(x)$ and the third line holds because $\lambda x^*(c) + (1-\lambda)x^*(c')$ is $B\left(\lambda c + (1-\lambda)c'\right)$ and less than the maximum attainable value.

Exercise 7.2. *Term structure of interest rates*

Consider an economy with a single consumer. There is one good in the economy, which arrives in the form of an exogenous endowment obeying

$$y_{t+1} = \lambda_{t+1} y_t,$$

where y_t is the endowment at time t and $\{\lambda_{t+1}\}$ is governed by a two-state Markov chain with transition matrix

$$P = \begin{bmatrix} p_{11} & 1-p_{11} \\ 1-p_{22} & p_{22} \end{bmatrix},$$

and initial distribution $\pi_\lambda = \begin{bmatrix}\pi_0 & 1-\pi_0\end{bmatrix}$. The value of λ_t is given by $\bar{\lambda}_1 = .98$ in state 1 and $\bar{\lambda}_2 = 1.03$ in state 2. Assume that the history of y_s, λ_s up to t is observed at time t. The consumer has endowment process $\{y_t\}$ and has preferences over consumption streams that are ordered by

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where $\beta \in (0,1)$ and $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, where $\gamma \geq 1$.

a. Define a competitive equilibrium, being careful to name all of the objects of which it consists.

b. Tell how to compute a competitive equilibrium. For the remainder of this

problem, suppose that $p_{11} = .8, p_{22} = .85, \pi_0 = .5$, $\beta = .96$, and $\gamma = 2$. Suppose that the economy begins with $\lambda_0 = .98$ and $y_0 = 1$.

c. Compute the (unconditional) average growth rate of consumption, computed before having observed λ_0.

d. Compute the time-0 prices of three risk-free discount bonds, in particular, those promising to pay one unit of time-j consumption for $j = 0, 1, 2$, respectively.

e. Compute the time-0 prices of three bonds, in particular, ones promising to pay one unit of time-j consumption contingent on $\lambda_j = \bar{\lambda}_1$ for $j = 0, 1, 2$, respectively.

f. Compute the time-0 prices of three bonds, in particular, ones promising to pay one unit of time-j consumption contingent on $\lambda_j = \bar{\lambda}_2$ for $j = 0, 1, 2$, respectively.
g. Compare the prices that you computed in parts d, e, and f.

Solution

The program associated with the exercise is `ex0702.m` .
a. The household's problem is to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t(\lambda^t)),$$

subject to the time zero budget constraint

$$\sum_{t=0}^{\infty} q_t^0(\lambda^t) c_t(\lambda^t) \leq \sum_{t=0}^{\infty} q_t^0(\lambda^t) y_t(\lambda^t).$$

Definition 3. *A competitive equilibrium is an allocation* $\{c_t(\lambda^t)\}_{t=0}^{\infty}$ *and a price system* $\{q_t^0(\lambda^t)\}_{t=0}^{\infty}$ *such that the allocation solves the household problem and markets clear.*

Observe that, in a representative agent economy, market clearing imposes that $c_t(\lambda^t) = y_t(\lambda^t)$.

b. After normalization ($q_0^0 = 1$) this leads to the familiar first order condition

$$q_t^0(\lambda^t|\lambda_0) = \beta^t \pi(\lambda^t|\lambda_0) \left[\frac{c_t(\lambda^t)}{c_0(\lambda_0)}\right]^{-\gamma}.$$

Imposing market clearing ($c_t(\lambda^t) = y_t(\lambda^t), \forall t, \lambda^t$), this leads to

$$q_t^0(\lambda^t|\lambda_0) = \beta^t \pi(\lambda^t|\lambda_0) \left[\frac{y_t(\lambda^t)}{y_0(\lambda_0)}\right]^{-\gamma}.$$

Using $y_{t+1} = \lambda_{t+1} y_t$, we obtain

$$\begin{aligned} q_t^0(\lambda^t|\lambda_0) &= \beta^t \pi(\lambda^t|\lambda_0) \left[\lambda_t \lambda_{t-1} ... \lambda_1\right]^{-\gamma} && (64) \\ &= \beta^t \pi(\lambda_t|\lambda_{t-1}) ... \pi(\lambda_1|\lambda_0) \left[\lambda_t \lambda_{t-1} ... \lambda_1\right]^{-\gamma}. && (65) \end{aligned}$$

c. Think of an econometrician computing time average of the growth rate :

$$\frac{1}{T} \sum_{t=1}^{T} \frac{y_t}{y_{t-1}} = \frac{1}{T} \sum_{t=1}^{T} \lambda_t$$

The above time average will converge almost surely towards the expectation of λ under the stationary probability [1] , that is towards

$$\frac{1-p_{22}}{1-p_{11}+1-p_{22}} \bar{\lambda}_1 + \frac{1-p_{11}}{1-p_{11}+1-p_{22}} \bar{\lambda}_2.$$

e. f. and **g.** Assume as in the text that the economy starts at $\lambda_0 = .98$. We use the formula (64).

A bond promising to pay 1 unit of consumption at time 0 has a time zero price of $Q_0 = 1$.

A bond promising to pay 1 unit of consumption at time 1 in state $\bar{\lambda}_1$ has a time zero price of $Q_{11} = q_1^0(\bar{\lambda}_1) = .7997$. A bond promising to pay 1 unit of consumption at time 1 in state $\bar{\lambda}_2$ has a time zero price of $Q_{11} = q_1^0(\bar{\lambda}_2) = .1810$. Lastly, a "risk free" bond promising one unit for sure at time 1 has time zero price $Q_1 = Q_{11} + Q_{12} = .9806$.

A bond promising to pay 1 unit of consumption at time 2 in state $\bar{\lambda}_1$ has a time zero price of

$$Q_{21} = q_2^0(\bar{\lambda}_1, \bar{\lambda}_1) + q_2^0(\bar{\lambda}_1, \bar{\lambda}_2) = .666.$$

A bond promising to pay 1 unit of consumption at time 2 in state $\bar{\lambda}_2$ has a time zero price of

$$Q_{22} = q_2^0(\bar{\lambda}_2, \bar{\lambda}_1) + q_2^0(\bar{\lambda}_2, \bar{\lambda}_2) = .2839.$$

[1] We know that the Markov chain λ_t is ergodic (all coefficients of P are positive is sufficient).

Lastly, a bond promising to pay 1 unit for sure at time 2 has time zero price

$$Q_2 = Q_{21} + Q_{22} = .9505.$$

The prices of risk free bonds decrease as the pay-out period is further into the future (i.e. with their maturity). Furthermore, consumption continent on the bad state is more expansive than consumption continent on the good state.

Exercise 7.3.

An economy consists of two infinitely lived consumers named $i = 1, 2$. There is one nonstorable consumption good. Consumer i consumes c_t^i at time t. Consumer i ranks consumption streams by

$$\sum_{t=0}^{\infty} \beta^t u(c_t^i),$$

where $\beta \in (0, 1)$ and $u(c)$ is increasing, strictly concave, and twice continuously differentiable. Consumer 1 is endowed with a stream of the consumption good $y_t^i = 1, 0, 0, 1, 0, 0, 1, \ldots$. Consumer 2 is endowed with a stream of the consumption good $0, 1, 1, 0, 1, 1, 0, \ldots$. Assume that there are complete markets with time-0 trading.

a. Define a competitive equilibrium.

b. Compute a competitive equilibrium.

c. Suppose that one of the consumers markets a derivative asset that promises to pay .05 units of consumption each period. What would the price of that asset be?

Solution

a.

Definition 4. *A competitive equilibrium is a feasible allocation $\{c_t^i\}_{t=0}^{\infty}$ for each agent $i = 1, 2$ and a price sequence $\{q_t^0\}_{t=0}^{\infty}$ such that*
(i) Given price, the allocation solves the household's problem, $\forall i$
The allocation is feasible: $\sum_i c_t^i = \sum_i y_t^i$.

b. The first order conditions for optimality of the household problem imply for each agent $i = 1, 2$: $q_t^0 = \frac{\beta^t u'(c_t^i)}{u'(c_0^i)}$.
Guess and verify a competitive equilibrium in which the consumption of each agent is constant across time. The first order condition implies then that $q_t^0 = \beta^t$. To find c^i, use agent i's budget constraint :

$$(1 - \beta) \sum_{t=0}^{\infty} \beta^t y_t^i = c^i.$$

For agent 1 this gives :

$$c^1 = (1-\beta)(1+\beta^3+\beta^6+...) \tag{66}$$
$$= \frac{1-\beta}{1-\beta^3} \tag{67}$$
$$= \frac{1}{1+\beta+\beta^2}. \tag{68}$$

For agent 2 market clearing implies that $c^2 = 1 - c^1$.

c. Since markets are complete markets, the derivative security is redundant: it can be priced off the Arrow-Debreu priced determined in part b. The time zero price of the derivative is

$$P_0 = \sum_{t=0}^{\infty} 0.05 Q_t = \frac{0.05}{1-\beta}.$$

Exercise 7.4.

Consider a pure endowment economy with a single representative consumer; $\{c_t, d_t\}_{t=0}^{\infty}$ are the consumption and endowment processes, respectively. Feasible allocations satisfy

$$c_t \leq d_t.$$

The endowment process is described by

$$d_{t+1} = \lambda_{t+1} d_t.$$

The growth rate λ_{t+1} is described by a two-state Markov process with transition probabilities

$$P_{ij} = \text{Prob}(\lambda_{t+1} = \bar{\lambda}_j | \lambda_t = \bar{\lambda}_i).$$

Assume that

$$P = \begin{bmatrix} .8 & .2 \\ .1 & .9 \end{bmatrix},$$

and that

$$\bar{\lambda} = \begin{bmatrix} .97 \\ 1.03 \end{bmatrix}.$$

In addition, $\lambda_0 = .97$ and $d_0 = 1$ are both known at date 0. The consumer has preferences over consumption ordered by

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma},$$

where E_0 is the mathematical expectation operator, conditioned on information known at time 0, $\gamma = 2, \beta = .95$.

Part I

At time 0, after d_0 and λ_0 are known, there are complete markets in date- and state-contingent claims. The market prices are denominated in units of time-0 consumption goods.

a. Define a competitive equilibrium, being careful to specify all the objects composing an equilibrium.

b. Compute the equilibrium price of a claim to one unit of consumption at date 5, denominated in units of time-0 consumption, contingent on the following history of growth rates: $(\lambda_1, \lambda_2, \ldots, \lambda_5) = (.97, .97, 1.03, .97, 1.03)$. Please give a numerical answer.

c. Compute the equilibrium price of a claim to one unit of consumption at date 5, denominated in units of time-0 consumption, contingent on the following history of growth rates: $(\lambda_1, \lambda_2, \ldots, \lambda_5) = (1.03, 1.03, 1.03, 1.03, .97)$.

d. Give a formula for the price at time 0 of a claim on the entire endowment sequence.

e. Give a formula for the price at time 0 of a claim on consumption in period 5, contingent on the growth rate λ_5 being .97 (regardless of the intervening growth rates).

Part II

Now assume a different market structure. Assume that at each date $t \geq 0$ there is a complete set of one-period forward Arrow securities.

f. Define a (recursive) competitive equilibrium with Arrow securities, being careful to define all of the objects that compose such an equilibrium.

g. For the representative consumer in this economy, for each state compute the "natural debt limits" that constrain state-contingent borrowing.

h. Compute a competitive equilibrium with Arrow securities. In particular, compute both the pricing kernel and the allocation.

i. An entrepreneur enters this economy and proposes to issue a new security each period, namely, a risk-free two-period bond. Such a bond issued in period t promises to pay one unit of consumption at time $t+1$ for sure. Find the price of this new security in period t, contingent on λ_t.

Solution

Part I

a. First, the household's problem is to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t(\lambda^t))$$

subject to the time zero budget constraint

$$\sum_{t=0}^{\infty} q_t^0(\lambda^t) c_t(\lambda^t) \leq \sum_{t=0}^{\infty} q_t^0(\lambda^t) d_t(\lambda^t).$$

We now define an equilibrium:

DEFINITION 5. *A competitive equilibrium is an allocation* $\{c_t(\lambda^t)\}_{t=0}^{\infty}$ *and a price system* $\{q_t^0(\lambda^t)\}_{t=0}^{\infty}$ *such that the allocation solves the household problem and markets clear.*

After normalizing $q_t^0 = 1$, the first order condition gives:

$$q_t^0(\lambda^t) = \beta^t \pi(\lambda^t|\lambda_0) \left[\frac{c_t(\lambda^t)}{c_0(\lambda_0)}\right]^{-\gamma}.$$

Imposing market clearing $c_t(\lambda^t) = d_t(\lambda^t), \forall t, \lambda^t$, this leads to

$$q_t^0(\lambda^t) = \beta^t \pi(\lambda^t|\lambda_0) \left[\frac{d_t(\lambda^t)}{d_0(\lambda_0)}\right]^{-\gamma}.$$

When $d_0 = 1$ and using $d_{t+1} = \lambda_{t+1} d_t$, we obtain

$$q_t^0(\lambda^t) = \beta^t \pi(\lambda^t|\lambda_0) \left[\lambda_t \lambda_{t-1} ... \lambda_1\right]^{-\gamma}.$$

b. A claim to one unit of consumption at date 5 contingent on the history $\lambda_1^5 = (0.97, 0.97, 1.03, 0.97, 1.03)$ has a time zero price q of

$$\begin{aligned} q_5^0(\lambda_1^5) &= \beta^5 \pi(\lambda_1^5|\lambda_0) \left[\lambda_5 \lambda_4 ... \lambda_1\right]^{-\gamma}. && (69) \\ &= 0.0025 && (70) \end{aligned}$$

c. A claim to one unit of consumption at date 5 contingent on the history $\lambda_1^5 = (1.03, 1.03, 1.03, 1.03, 0.97)$ has a time zero price q of

$$\begin{aligned} q_5^0(\lambda_1^5) &= \beta^5 \pi(\lambda_1^5|\lambda_0) \left[\lambda_5 \lambda_4 ... \lambda_1\right]^{-\gamma} \\ &= 0.0111. \end{aligned}$$

d. The (cum-dividend) price at time zero of a claim on the entire endowment is

$$\begin{aligned} P(0) &= \sum_{t=0}^{\infty} \sum_{\lambda^t} \beta^t \pi(\lambda^t|\lambda_0) \left(\frac{d(\lambda^t)}{d(0)}\right)^{-\gamma} d(\lambda^t) && (71) \\ &= d(0) E_0 \sum_{t=0}^{\infty} \beta^t \left(\lambda_1 \lambda_2 \ldots \lambda_t\right)^{1-\gamma}. && (72) \\ & && (73) \end{aligned}$$

Note that the price is proportional to the initial dividend. We are going to derive a recursive formula to compute $P(0)/d(0)$, the price dividend ratio. Let $p(\bar{\lambda}_i))$ be the time zero price-dividend ratio of a claim on the entire endowment when the initial state is $\bar{\lambda}_i$. The above formula can be written recursively as :

$$p(\bar{\lambda}_i) = 1 + \beta \left(\pi(\bar{\lambda}_1|\bar{\lambda}_i) \bar{\lambda}_1^{1-\gamma} p(\bar{\lambda}_1) + \pi(\bar{\lambda}_2|\bar{\lambda}_i) \bar{\lambda}_2^{1-\gamma} p(\bar{\lambda}_2)\right). \quad (74)$$

Define

$$R \equiv \begin{pmatrix} \pi_{11}\bar{\lambda}_1^{1-\gamma} & \pi_{12}\bar{\lambda}_2^{1-\gamma} \\ \pi_{21}\bar{\lambda}_1^{1-\gamma} & \pi_{22}\bar{\lambda}_2^{1-\gamma} \end{pmatrix}.$$

Formula (74) is, in matrix form :

$$p = I + \beta R p.$$

Which gives

$$p = (I - \beta R)^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus $P(0) =$.
e. A claim to one unit of consumption at date 5 contingent on the period 5 growth rate $\lambda_5 = 0.97$ has a time zero price q of

$$\begin{aligned} q &= 0.95^5 \left[Q^5\right]_{11} \\ &= 0.4402, \end{aligned}$$

where Q is

$$Q = \begin{pmatrix} \pi_{11}\bar{\lambda}_1^{-\gamma} & \pi_{12}\bar{\lambda}_2^{-\gamma} \\ \pi_{21}\bar{\lambda}_1^{-\gamma} & \pi_{22}\bar{\lambda}_2^{-\gamma}. \end{pmatrix}$$

Part II

f.

Definition 6. *A recursive competitive equilibrium is a pricing kernel $Q(\lambda'|\lambda)$, a pair of decision rules $\{c(\theta,\lambda), \theta'(\theta,\lambda,\lambda')\}$ and a value function $v(\theta,\lambda)$ such that*

- *the decision rules solve the households problem*

$$v(\theta,\lambda) = \max_{\{c,\theta'\}} \{u(c) + \beta E[v(\theta',\lambda')]\},$$

  subject to the budget constraint

$$c + \sum_{\lambda'} \theta'(\lambda') Q(\lambda'|\lambda) \leq d + \theta,$$

  a non negativity constraint on consumption $c \geq 0$ and the natural debt limits (see part g) $-\theta'(\lambda') \leq \bar{A}(\lambda')$.
- *For each realization $\{\lambda_t\}_{t=0}^{\infty}$, the allocation induced by the decision rule clears the markets:$c = d$ and $\theta = 0$*

g. The natural debt limit can be recursively computed as

$$\bar{A}(\lambda) = d + \beta \sum_{\lambda'} Q(\lambda'|\lambda) \bar{A}(\lambda').$$

It states that the borrowing limit is the present discounted value of future income. Anticipating the results of the next part we can rewrite $\bar{A}(\lambda)$ as

$$\bar{A}(\lambda_t = \bar{\lambda}_1) = \sum_{j=0}^{\infty} \beta^j \pi(\lambda^j|\bar{\lambda}_1) \left[\lambda_{t+j}\lambda_{t+j-1}...\lambda_t\right]^{-\gamma} d_j.$$

h. Using the first order condition from the household's problem

$$v(\theta,\lambda) = \max_{\{\theta'\}} \left\{ u\left(d + \theta - \sum_{\lambda'} \theta'(\lambda') Q(\lambda'|\lambda) \right) + \beta E\left[v(\theta',\lambda')\right] \right\}.$$

The first order condition w.r.t. θ' is:

$$u_c\left(d + \theta - \sum_{\lambda'} \theta'(\lambda') Q(\lambda'|\lambda) \right) Q(\lambda'|\lambda) = \beta\pi(\lambda'|\lambda) v_\theta(\theta',\lambda'),$$

and the envelope theorem

$$v_\theta(\theta,\lambda) = u_c\left(d + \theta - \sum_{\lambda'} \theta'(\lambda') Q(\lambda'|\lambda) \right).$$

This gives the familiar expression for the pricing kernel

$$Q(\lambda'|\lambda) = \beta\pi(\lambda'|\lambda)\frac{u_c\left(c(\lambda')\right)}{u_c\left(c(\lambda)\right)} = \beta\pi(\lambda'|\lambda)\left[\frac{c(\lambda')}{c(\lambda)}\right]^{-\gamma}.$$

In this representative agent economy aggregate endowments equal aggregate consumption ($c = d$), so that

$$Q(\lambda'|\lambda) = \beta\pi(\lambda'|\lambda)\left[\lambda'\right]^{-\gamma},$$

and $\theta'(\lambda') = 0$, $\forall \lambda'$

i. The price of the security promising to pay 1 unit of consumption at time $t+2$, when the state of the economy at time t is $\lambda_t = \bar{\lambda}_1$ is given by

$$Q(\lambda'' = \bar{\lambda}_1|\lambda = \bar{\lambda}_1) + Q(\lambda'' = \bar{\lambda}_2|\lambda = \bar{\lambda}_1) = 0.5615 + 0.3166 = 0.8781,$$

and if the state of the economy at time t is $\lambda_t = \bar{\lambda}_2$ it is

$$Q(\lambda'' = \bar{\lambda}_1|\lambda = \bar{\lambda}_2) + Q(\lambda'' = \bar{\lambda}_2|\lambda = \bar{\lambda}_2) = 0.1533 + 0.7936 = 0.9469.$$

Exercise 7.5. *A periodic economy*

An economy consists of two consumers, named $i = 1, 2$. The economy exists in discrete time for periods $t \geq 0$. There is one good in the economy, which is not storable and arrives in the form of an endowment stream owned by each consumer. The endowments to consumers $i = 1, 2$ are

$$\begin{aligned} y_t^1 &= s_t, \\ y_t^2 &= 1. \end{aligned} \tag{75}$$

where s_t is a random variable governed by a two-state Markov chain with values $s_t = \bar{s}_1 = 0$ or $s_t = \bar{s}_2 = 1$. The Markov chain has time-invariant transition probabilities denoted by $\pi(s_{t+1} = s'|s_t = s) = \pi(s'|s)$, and the probability distribution over the initial state is $\pi_0(s)$. The *aggregate endowment* at t is $Y(s_t) = y_t^1 + y_t^2$.

Let c^i denote the stochastic process of consumption for agent i. Household i orders consumption streams according to

$$U(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \ln[c_t^i(s^t)]\pi(s^t), \tag{2}$$

where $\pi(s^t)$ is the probability of the history $s^t = (s_0, s_1, \ldots, s_t)$.

a. Give a formula for $\pi(s^t)$.

Let $\theta \in (0,1)$ be a Pareto weight on household 1. Consider the planning problem

$$\max_{c^1, c^2} \left\{\theta \ln(c^1) + (1-\theta)\ln(c^2)\right\} \tag{3}$$

where the maximization is subject to

$$c_t^1(s^t) + c_t^2(s^t) \le Y(s_t). \tag{4}$$

Solve the Pareto problem, taking θ as a parameter.

b. Define a *competitive equilibrium* with history-dependent Arrow-Debreu securities traded once and for all at time 0. Be careful to define all of the objects that compose a competitive equilibrium.

c. Compute the competitive equilibrium price system (i.e., find the prices of all of the Arrow-Debreu securities).

d. Tell the relationship between the solutions (indexed by θ) of the Pareto problem and the competitive equilibrium allocation. If you wish, refer to the two welfare theorems.

e. Briefly tell how you can compute the competitive equilibrium price system *before* you have figured out the competitive equilibrium allocation.

f. Now define a recursive competitive equilibrium with trading every period in one-period Arrow securities only. Describe all of the objects of which such an equilibrium is composed. (Please denominate the prices of one-period time–$t+1$ state-contingent Arrow securities in units of time-t consumption.) Define the "natural borrowing limits" for each consumer in each state. Tell how to compute these natural borrowing limits.

g. Tell how to compute the prices of one-period Arrow securities. How many prices are there (i.e., how many numbers do you have to compute)? Compute all of these prices in the special case that $\beta = .95$ and $\pi(s_j|s_i) = P_{ij}$ where $P = \begin{bmatrix} .8 & .2 \\ .3 & .7 \end{bmatrix}$.

h. Within the one-period Arrow securities economy, a new asset is introduced. One of the households decides to market a one-period-ahead riskless claim to one unit of consumption (a one-period real bill). Compute the equilibrium prices of this security when $s_t = 0$ and when $s_t = 1$. Justify your formula for these prices in terms of first principles.

i. Within the one-period Arrow securities equilibrium, a new asset is introduced. One of the households decides to market a two-period-ahead riskless claim to one unit consumption (a two-period real bill). Compute the equilibrium prices of this security when $s_t = 0$ and when $s_t = 1$.

j. Within the one-period Arrow securities equilibrium, a new asset is introduced. One of the households decides at time t to market five-period-ahead claims to consumption at $t+5$ contingent on the value of s_{t+5}. Compute the equilibrium prices of these securities when $s_t = 0$ and $s_t = 1$ and $s_{t+5} = 0$ and $s_{t+5} = 1$.

Solution

a. The chain $s^t = (s_0, s_1, ..., s_{t-1}, s_t)$ has probability $\pi(s^t) = \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2})...\pi(s_1|s_0)\pi_0(s_0)$.

b. The first order condition for the planner problem is:

$$\frac{\theta}{c^1} = \frac{1-\theta}{Y(s^t) - c^1}.$$

This implies the following optimal allocation among agents 1 and 2: $c^1(s^t) = \theta Y(s^t)$ and $c^2(s^t) = (1-\theta)Y(s^t)$. For later reference, this rule implies

$$\frac{c^2(s^t)}{c^1(s^t)} = \frac{1-\theta}{\theta}.$$

c.

Definition 7. *A competitive equilibrium is a feasible allocation $\{c_t^i, \theta_{t+1}^i\}_{t=0}^{\infty}$ for each agent $i = 1, 2$ and a pricing kernel $\{Q_t\}_{t=0}^{\infty}$ such that*

- *Given the pricing kernel, the allocation solves the household's problem, $\forall i$*
- *The decision rules satisfy market clearing conditions: $\sum_i c_t^i = \sum_i y_t^i$ and $\sum_i \theta_{t+1}^i = 0, \forall t$*

d. The first order conditions for optimality of the household problem imply for agent 1 and 2:

$$Q(s^t|s_0) = \frac{\beta^t \pi(s^t|s_0) c_0^1(s_0)}{c_t^1(s^t|s_0)} = \frac{\beta^t \pi(s^t|s_0) c_0^2(s_0)}{c_t^2(s^t|s_0)}.$$

This leads to

$$\frac{c_t^2(s^t|s_0)}{c_t^1(s^t|s_0)} = \frac{c_0^2(s_0)}{c_0^1(s_0)}.$$

Imposing the market clearing condition at time zero, we find

$$\frac{c_t^2(s^t|s_0)}{c_t^1(s^t|s_0)} = \frac{Y_0(s_0) - c_0^1(s_0)}{c_0^1(s_0)}.$$

Furthermore $c_0^1(s_0)$ will be some fraction ξ of $Y_0(s_0)$. Therefore, the competitive equilibrium is characterized by

$$\frac{c_t^2(s^t|s_0)}{c_t^1(s^t|s_0)} = \frac{1-\xi}{\xi},$$

imposing market clearing at time t we obtain the optimal consumption rules $c_t^2(s^t|s_0) = (1-\xi)Y(s^t)$ and $c_t^1(s^t|s_0) = \xi Y(s^t)$. It follows that the Arrow-Debreu prices are given by

$$Q(s^t|s_0) = \frac{\beta^t \pi(s^t|s_0)Y(s_0)}{Y(s^t)}.$$

e. In part b we computed the Pareto optimal allocation and in part c the competitive equilibrium. The two welfare theorems state conditions under which the Pareto optimal allocation can be implemented as a competitive equilibrium (supported by prices $Q(s^t|s_0)$) and vice versa. The planner's weight θ on agent 1 has its counterpart in ξ, the inverse of agent 1's marginal utility of consumption at time zero as a fraction of aggregate income.

f. Competitive equilibrium prices can be priced off the IMRS for a representative agent whose consumption equals the aggregate endowment $Y(s^t)$. This is possible because there is no ex-ante heterogeneity between the agents so that there exists a representative consumer.

g.

DEFINITION 8. *A recursive competitive equilibrium is an initial wealth distribution* Θ_0*, decision rules* $\{c^i(\theta, s), \theta'^i(\theta, s, s')\}_{i=1}^2$ *a pricing kernel* $Q(s'|s)$ *a pair of value functions* $\{v^i(\theta, s)\}_{i=1}^2$ *such that*

- *Given the pricing kernel and the initial wealth distribution, the decision rules solve each household's problem*
- *For all realizations* $\{s_t\}_{t=0}^{\infty}$*, the allocations implied decision rules satisfy market clearing conditions:* $\sum_i c_t^i = \sum_i y^i(s_t)$ *and* $\sum_i \theta_{t+1}^i(s') = 0, \forall t, s'$

The natural borrowing limit rules out Ponzi schemes by restricting short positions in Arrow securities to be less than the present discounted value of all future income in each state tomorrow.

$$-\theta_{t+1}^i(s') \leq \bar{A}^i(s'),$$

where

$$\bar{A}^i(s_t) = \sum_{\tau \geq t} \sum_{s^\tau | s^t} Q(s^\tau|s^t) y^i(s^\tau|s^t).$$

As noted in the text, this natural debt limit can be recursively computed using

$$\bar{A}^i(s) = y^i(s) + \beta\pi(s'|s)\frac{Y(s)}{Y(s')}\bar{A}^i(s').$$

h. There are 2 Arrow securities for each state today: one pays off 1 unit of consumption in the bad state $\bar{s}_1$, the other pays off one unit of consumption in the good state $\bar{s}_2$. Their price p can be determined from the Arrow-Debreu price;

$$\frac{Q(s^{t+1}|s_0)}{Q(s^t|s_0)} = p(s^{t+1}|s^t) = \frac{\beta\pi(s^{t+1}|s^t)Y(s^t)}{Y(s^{t+1})}.$$

We find: $p_{11} = 0.76, p_{12} = 0.095, p_{21} = 0.57, p_{22} = 0.665$.

i. Since markets are complete, they span the payoff of this new asset. The price of this redundant security is the sum of the Arrow securities in each of the states tomorrow. In state $\bar{s}_1$ the price of this asset is $p_{11} + p_{12} = 0.885$ and in state $\bar{s}_2$ it is $p_{21} + p_{22} = 1.235$.

j. First compute the price of two-period Arrow securities:

$$p(s^{t+2}|s^t) = \sum_{s'} \frac{\beta^2\pi(s^{t+2}|s')\pi(s'|s^t)Y(s^t)}{Y(s^{t+2})}.$$

These Arrow prices are $\tilde{p}_{11} = 0.63175$, $\tilde{p}_{12} = 0.135375, \tilde{p}_{21} = 0.81225, \tilde{p}_{22} = 0.496375$. In state $\bar{s}_1$ the price of this new asset is $\tilde{p}_{11} + \tilde{p}_{12} = 0.767125$ and in state $\bar{s}_2$ it is $\tilde{p}_{21} + \tilde{p}_{22} = 1.308625$.

k. The method is the same. The transition matrix is P^5. The answer is $\hat{p}_{11} = 0.4739, \hat{p}_{12} = 0.1499, \hat{p}_{21} = 0.8995, \hat{p}_{22} = 0.3240$.

CHAPTER 8

Overlapping generation models

Exercise 8.1.

At each date $t \geq 1$, an economy consists of overlapping generations of a constant number N of two-period-lived agents. Young agents born in t have preferences over consumption streams of a single good that are ordered by $u(c_t^t) + u(c_{t+1}^t)$, where $u(c) = c^{1-\gamma}/(1-\gamma)$, and where c_t^i is the consumption of an agent born at i in time t. It is understood that $\gamma > 0$, and that when $\gamma = 1$, $u(c) = \ln c$. Each young agent born at $t \geq 1$ has identical preferences and endowment pattern (w_1, w_2), where w_1 is the endowment when young and w_2 is the endowment when old. Assume $0 < w_2 < w_1$. In addition, there are some initial old agents at time 1 who are endowed with w_2 of the time-1 consumption good, and who order consumption streams by c_1^0. The initial old (i.e., the old at $t = 1$) are also endowed with M units of unbacked fiat currency. The stock of currency is constant over time.

a. Find the saving function of a young agent.

b. Define an equilibrium with valued fiat currency.

c. Define a stationary equilibrium with valued fiat currency.

d. Compute a stationary equilibrium with valued fiat currency.

e. Describe how many equilibria with valued fiat currency there are. (You are not being asked to compute them.)

f. Compute the limiting value as $t \to +\infty$ of the rate of return on currency in each of the non stationary equilibria with valued fiat currency. Justify your calculations.

Solution

We focus on the case $0 < \gamma \leq 1$. The case $\gamma > 1$ exhibits more complicated dynamics due to a non monotonic saving function.
a. The saving function of a young agent is:

$$s(R) = \operatorname*{argmax}_s \frac{(w_1 - s)^{1-\gamma}}{1-\gamma} + \frac{(w_2 + Rs)^{1-\gamma}}{1-\gamma}.$$

The first order necessary and sufficient condition of this program is:

$$(w_1 - s)^{-\gamma} = R(w_2 + Rs)^{-\gamma}.$$

Which gives:

$$s(R) = \frac{w_1 - w_2 R^{-\frac{1}{\gamma}}}{1 + R^{1-\frac{1}{\gamma}}}$$

Note that the derivative of this function with respect to R is:

$$\frac{R^{-\frac{1}{\gamma}}}{(1 + R^{1-\frac{1}{\gamma}})^2}\left[(\frac{1}{\gamma} - 1)w_1 + w_2 R^{-\frac{1}{\gamma}} + \frac{w_2}{\gamma} R^{-1}\right].$$

Which is strictly positive for all $0 < \gamma \leq 1$. This proves that the saving function is increasing, a fact that is going to be crucial to characterize unambiguously the dynamic of the system.

b. The problem faced by the young of generation t is

$$\begin{array}{rl} & \max_{c_t^t, c_{t+1}^t, M_t} u(c_t^t) + u(c_{t+1}^t) \\ \text{subject to} & c_t^t + \frac{M_t}{p_t} \leq w_1 \\ & c_{t+1}^t \leq w_2(t+1) + \frac{M_t}{p_{t+1}}, \\ & [c_t^t, c_{t+1}^t, M_t] \geq 0. \end{array}$$

The problem of the initial old is simply

$$\begin{array}{rl} & \max_{c_1^0} c_1^0 \\ \text{subject to} & 0 \leq c_1^0 \leq \frac{M}{p_1} \end{array}$$

We can now define an equilibrium:

DEFINITION 9. *An equilibrium with valued fiat currency is a consumption plan for the initial old c_1^0, consumption decisions for the young born at time $t \geq 1$, $\{c_t^t, c_{t+1}^t\}_{t=1}^{+\infty}$, money demand $\{M_t\}_{t=1}^{+\infty}$, and a positive price sequence $\{p_t\}_{t=1}^{+\infty}$, such that the two following conditions are satisfied:*

(i) *Optimality: given p_1, c_1^0 solves the initial old problem. Given $\{p_t, p_{t+1}\}$, $\{c_t^t, c_{t+1}^t\}$ solves agent of generation t problem, for all $t \geq 1$.*

(ii) *Feasibility: the market for good and the market for money clear for all $t \geq 1$:*

$$\begin{array}{l} c_t^t + c_t^{t-1} = w_1 + w_2 \\ M_1 = M \quad \text{and} \quad M_{t+1} = M_t. \end{array}$$

c. In a stationary equilibrium the rate of return on currency is constant. More formally a stationary equilibrium with valued fiat currency is an equilibrium with valued fiat currency for which there is $R > 0$ such that

$$\frac{p_t}{p_{t+1}} = R.$$

d. We simplify the list of conditions describing an equilibrium. First, Walras law allows to eliminate one market clearing condition at each $t \geq 1$. We eliminate the market clearing condition for good and keep the one for money. Second, going back to the agent problem, we note that it can be reduced to a saving problem, as described in question (a), with $R_t = \frac{p_t}{p_{t+1}}$. The optimal money holding are $\frac{M_t}{p_t} = s(R_t)$ if $s(R_t) > 0$ and 0 otherwise. The optimal consumption stream is $c_t^t = w_1 - \frac{M_t}{p_t}$ and $c_{t+1}^t = w_2 + \frac{M_t}{p_{t+1}}$. These remarks allow us to define an equilibrium as a positive price sequence $\{p_t\}_{t=1}^{+\infty}$ such that, for all t:

$$\frac{M}{p_t} = s\left(\frac{p_t}{p_{t+1}}\right).$$

Looking for a stationary equilibrium, we write $\frac{p_t}{p_{t+1}} = R$ so that

$$\frac{M}{p_t} = s(R) = \text{constant}.$$

Therefore the price level is constant and p_1 solves

$$\frac{M}{p_1} = s(1) = \frac{w_1 - w_2}{2} \Rightarrow p_1 = p_t = \frac{2M}{w_1 - w_2}.$$

Note here how the condition $w_2 < w_1$ is necessary to ensure existence of a stationary equilibrium with valued fiat currency.

e. In order to describe the set of equilibria, we rewrite the equilibrium condition using the auxiliary variable $R_t = \frac{p_t}{p_{t+1}}$:

$$\begin{array}{lll} \frac{M}{p_1} & = & s(R_1) \\ s(R_{t+1}) & = & R_t s(R_t), \quad t \geq 1 \\ R_t & = & \frac{p_t}{p_{t+1}}. \end{array}$$

The first equation says that the saving of the initial young must equal the supply of real money balance. The second equation is found by expressing that nominal money balances stay constant over time. An equilibrium is constructed the following way:

(i) Choose a positive sequence $\{R_t\}_{t=1}^{+\infty}$ solving the difference equation $s(R_{t+1}) = R_t s(R_t)$. We will see that there are infinitely many.
(ii) Once R_1 is chosen, find a solution $p_1 < +\infty$ to the first equation.
(iii) Construct the sequence of price using $p_{t+1} = \frac{p_t}{R_t}$.

The assumptions made on the utility function put some structure on the set of positive sequence solving $s(R_{t+1}) = R_t s(R_t)$. We are going to show in particular that all non-stationary equilibria are associated with decreasing sequences of rate of return on currency, converging towards $R^* = \left[\frac{w_2}{w_1}\right]^\gamma$.

Let's first gain some intuition from a simple graphical analysis. On figure 1 we plot the function $R_t s(R_t)$ and the function $s(R_{t+1})$. A candidate interest rate sequence is constructed as follows. Choose R_0 on the x-axis. Go on the first curve to "compute" $R_0 s(R_0)$. Then go on the second curve to "solve" $s(R_1) = R_0 s(R_0)$. Go back on the x-axis. Iterate. It is clear from this exercise that $R = 1$ is an unstable stationary point. Also, all sequences starting on the left of $R = 1$ and on the right of $R^* = \left[\frac{w_2}{w_1}\right]^\gamma$ converges to R^*. It is not possible to construct a sequence starting far on the right of $R = 1$. Lastly, sequence starting on the left of R^* are not admissible as they are associated with negative savings.

Let's make the previous arguments more formal. We go in several steps.

(1) There is no equilibrium such that $R_1 \leq \left[\frac{w_2}{w_1}\right]^\gamma$.

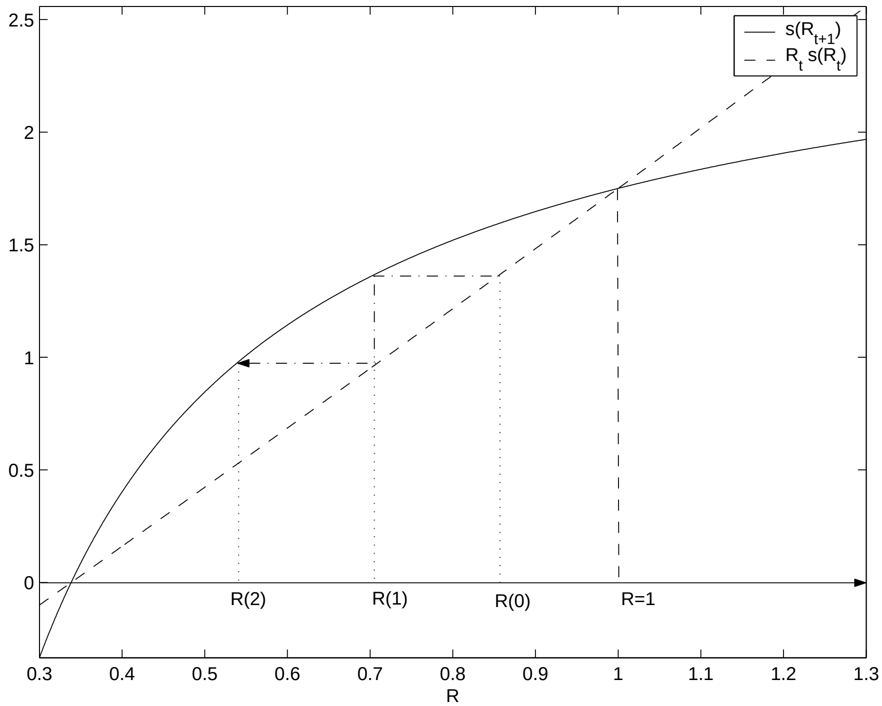


FIGURE 1. Exercise 8.1

This follows from the fact that for all such R_1, $s(R_1)$ is non-positive and that equilibrium imposes that $s(R_1) = \frac{M}{p_1} > 0$.

(2) There is no equilibrium such that $R_1 > 1$.

Assume that there is one. First note that $s(R_{t+1}) - s(R_t) = (R_t - 1)s(R_t)$. Therefore, if $R_t > 1$, then $s(R_{t+1}) > s(R_t)$. And since $s(.)$ is increasing, this implies that $R_{t+1} > R_t$. Thus, if $R_0 > 1$, the sequence $\{R_t\}_{t=1}^{+\infty}$ is increasing. As any increasing sequence, it has a limit, finite or infinite. If it has a finite limit R^*, it must satisfy, by continuity of $s(.)$:

$$s(R^*) = R^* s(R^*).$$

It is easy to see that this equation has only two solutions $R^* = 1$ and $R^* = \left[\frac{w_2}{w_1}\right]^{\gamma}$, both less than 1. Since $R_t > R_0 > 1$, the sequence R_t cannot have such a finite limit. Therefore R_t goes to infinity. This also implies that $R_t s(R_t) > R_t s(1)$ goes to infinity. But $s(R_{t+1})$ is bounded above by w_1 since an agent cannot save more than her young period endowment. This means that, for t large enough, this sequence violates the equality

$$s(R_{t+1}) = R_t s(R_t).$$

(3) All equilibria are such that $\left[\frac{w_2}{w_1}\right]^\gamma < R_1 \leq 1$

We already know that $R_1 = 1$ is associated with the stationary equilibrium. For $\left[\frac{w_2}{w_1}\right]^\gamma < R_1 < 1$, write as before that $s(R_{t+1}) - s(R_t) = (R_t - 1)s(R_t)$. If $R_t < 1$, then $s(R_{t+1}) < s(R_t)$. Since $s(.)$, is increasing, this implies that $R_{t+1} < R_t$. Furthermore, $R_t > \left[\frac{w_2}{w_1}\right]^\gamma$ implies that $s(R_t) > 0$ so that $s(R_{t+1}) > 0$ and $R_{t+1} > \left[\frac{w_2}{w_1}\right]^\gamma$. Thus the sequence $\{R_t\}_{t=1}^{+\infty}$ is decreasing and bounded below. It thus converges towards a finite limit R^* such that

$$s(R^*) = R^* s(R^*) \quad \text{and} \quad \left[\frac{w_2}{w_1}\right]^\gamma \leq R^* < 1.$$

We already know the solutions of this equation. It must be that

$$R^* = \left[\frac{w_2}{w_1}\right]^\gamma.$$

f. All the non stationary equilibria are inflationary path with limiting rate of return R^*. See question (e) for the proof.

Exercise 8.2.

Consider an economy with overlapping generations of a constant population of an even number N of two-period-lived agents. New young agents are born at each date $t \geq 1$. Half of the young agents are endowed with w_1 when young and 0 when old. The other half are endowed with 0 when young and w_2 when old. Assume $0 < w_2 < w_1$. Preferences of all young agents are as in problem 1, with $\gamma = 1$. Half of the N initial old are endowed with w_2 units of the consumption good and half are endowed with nothing. Each old person orders consumption streams by c_1^0. Each old person at $t = 1$ is endowed with M units of unbacked fiat currency. No other generation is endowed with fiat currency. The stock of fiat currency is fixed over time.

a. Find the saving function of each of the two types of young person for $t \geq 1$.

b. Define an equilibrium without valued fiat currency. Compute all such equilibria.

c. Define an equilibrium with valued fiat currency.

d. Compute all the (nonstochastic) equilibria with valued fiat currency.

e. Argue that there is a unique stationary equilibrium with valued fiat currency.

f. How are the various equilibria with valued fiat currency ranked by the Pareto criterion?

Solution

a. Let (w^y, w^o) be the endowment of the agent. The saving function of a young agent is:

$$s(R) = \underset{s}{\operatorname{argmax}} \log(w^y - s) + \log(w^o + Rs).$$

$$\frac{1}{w^y - s} = \frac{R}{w^o + Rs}.$$

Which gives:

$$s(R) = \frac{w^o}{2} - \frac{w^y}{2R}.$$

Thus, a consumer of type 1 has saving function

$$s_1(R) = \frac{w_1}{2}.$$

And a consumer of type 2 has saving function

$$s_2(R) = -\frac{w_2}{2R}.$$

b. The problem faced by a young of generation t of type $h = 1, 2$ is

$$\begin{array}{rl} & \max_{c_t^{ht}, c_{t+1}^{ht}, b_t} u(c_t^{ht}) + u(c_{t+1}^{ht}) \\ \text{subject to} & c_t^{ht} + b_t^h \leq w_t^h \\ & c_{t+1}^{ht} \leq w_{t+1}^h + R_t b_t^h \\ & [c_t^{ht}, c_{t+1}^{ht}] \geq 0. \end{array}$$

The problem of the initial old is simply

$$\begin{array}{rl} & \max_{c_1^{h0}} c_1^{h0} \\ \text{subject to} & 0 \leq c_1^{h0} \leq w_0^h. \end{array}$$

We can now define an equilibrium:

DEFINITION 10. *An equilibrium without valued fiat currency is a consumption decision for the initial old c_1^{h0}, $h = 1, 2$, consumption decisions for the young born at time $t \geq 1$, $\{c_t^{ht}, c_{t+1}^{ht}\}_{t=1}^{+\infty}$, $h = 1, 2$, lending/borrowing decisions $\{b_t^h\}_{t=1}^{+\infty}$, and a positive return sequence $\{R_t\}_{t=1}^{+\infty}$, such that the two following conditions are satisfied:*

(i) *Optimality: Given p_1, c_1^{h0} solves the initial old problem. $h = 1, 2$. Given R_t, $\{c_t^{ht}, c_{t+1}^{ht}\}$ solves agent of generation t problem for all $t \geq 1$, $h = 1, 2$.*

(ii) *Feasibility: the market for good, the market for private lending clear for all t clear for all $t \geq 1$:*

$$\begin{array}{l} \frac{N}{2} \sum_{h=1,2} c_t^{ht} + \frac{N}{2} \sum_{h=1,2} c_t^{ht-1} = \frac{N}{2} w_1 + \frac{N}{2} w_2 \\ \frac{N}{2} \sum_{h=1,2} b_t^h = 0 \end{array}$$

To characterize such an equilibrium we first notice that Walras Law allows to restrict attention to the market for private lending. Using the saving function derived above, market clearing can be written:

$$s_1(R_t) + s_2(R_t) = 0.$$

And the only solution of this equation is $R_t = \frac{w_2}{w_1}$. Thus there is only one equilibrium without valued fiat currency, it is stationary and the interest rate is lower than the inverse of the discount factor $R_t = \frac{w_2}{w_1} < 1$.

c. The problem faced by a young of generation t of type $h = 1, 2$ is

$$\begin{array}{rl} & \max_{c_t^{ht}, c_{t+1}^{ht}, b_t, M_t^h} u(c_t^{ht}) + u(c_{t+1}^{ht}) \\ \text{subject to} & c_t^{ht} + b_t^h + \frac{M_t^h}{p_t} \leq w_t^h \\ & c_{t+1}^{ht} \leq w_{t+1}^h + R_t b_t^h + \frac{M_t^h}{p_{t+1}} \\ & [c_t^{ht}, c_{t+1}^{ht}, M_t^h] \geq 0. \end{array}$$

The problem of the initial old is simply

$$\begin{array}{rl} & \max_{c_1^{h0}} c_1^{h0} \\ \text{subject to} & 0 \leq c_1^{h0} \leq w_0^h + \frac{M}{p_1}. \end{array}$$

DEFINITION 11. *An equilibrium with valued fiat currency is a consumption plan for the initial old c_1^{h0}, $h = 1, 2$, consumption plans for the young born at time $t \geq 1$, $\{c_t^{ht}, c_{t+1}^{ht}\}_{t=1}^{+\infty}$, $h = 1, 2$, lending/borrowing decisions $\{b_t^h\}_{t=1}^{+\infty}$, money holding decisions $\{M_t^h\}_{t=1}^{+\infty}$, a positive price sequence $\{p_t\}_{t=1}^{+\infty}$, and a positive return sequence $\{R_t\}_{t=1}^{+\infty}$ such that the two following conditions are satisfied:*

(i) *Optimality: Given p_1, c_1^{h0} solves the initial old problem. $h = 1, 2$. Given R_t and $\{p_t, p_{t+1}\}$, $\{c_t^{ht}, c_{t+1}^{ht}\}$ solves agent of generation t problem for all $t \geq 1$, $h = 1, 2$.*

(ii) *Feasibility: the market for good and the market for private lending and the market for money clear for all $t \geq 1$: clear for all $t \geq 1$:*

$$\begin{array}{l} \frac{N}{2} \sum_{h=1,2} c_t^{ht} + \frac{N}{2} \sum_{h=1,2} c_t^{ht-1} = \frac{N}{2} w_1 + \frac{N}{2} w_2 \\ \frac{N}{2} \sum_{h=1,2} b_t^h = 0 \\ \frac{N}{2} \sum_{h=1,2} M_t^h = NM. \end{array}$$

d. Walras law allows to restrict attention to the last two market clearing conditions. Note also that, since agents can borrow ar rate R_t, no arbitrage imposes that the return on money, $\frac{p_t}{p_{t+1}}$, is lower than R_t. Furthermore, in an equilibrium with valued fiat currency, agents are holding positive money balance. Thus $\frac{p_t}{p_{t+1}} = R_t$. Given this equality, savers (agents of type 1) are indifferent between holding money and lending:

$$b_t^1 + \frac{M_t^1}{p_t} = s_1(R_t).$$

This implies that the last two market clearing conditions can be replaced by their sum:

$$\frac{N}{2} s_1(R_t) + \frac{N}{2} s_2(R_t) = \frac{NM}{p_t}.$$

This equation says that all the money demand must equal the aggregate saving of type 1 agents minus the borrowing of the type 2 agents. Using the expressions derived in part (a) and $R_t = \frac{p_t}{p_{t+1}}$, this equations can be reduced to the following linear difference equation in p_t:

$$p_{t+1} = -\frac{4M}{w_2} + \frac{w_1}{w_2} p_t.$$

We first solve for a stationary point

$$p^* = -\frac{4M}{w_2} + \frac{w_1}{w_2} p^*.$$

We find $p^* = \frac{4M}{w_1 - w_2}$. We subtract the equation defining p^* to the difference equation for p_t:

$$(p_{t+1} - p^*) = \frac{w_1}{w_2}(p_t - p^*).$$

We then iterate on this equation to find:

$$p_{t+1} = p^* + \left[\frac{w_1}{w_2}\right]^t (p_1 - p^*).$$

Note that $\frac{w_1}{w_2} > 1$ and $p_t > 0$ imposes that $p_1 \geq p^*$ – otherwise the price level would be negative for t large enough. It is apparent from this formula that there are a continuum of equilibria with valued fiat currency indexed by $p_1 \geq p^*$.

e. Stationary equilibria are such that the rate of return on currency is constant. We can compute this rate of return explicitly using the solution we derived in part (d):

$$R_t = \frac{p_t}{p_{t+1}} = \frac{w_2}{w_1} + (1 - \frac{w_2}{w_1}) \frac{p^*}{p^* + \left[\frac{w_1}{w_2}\right]^t (p_1 - p^*)}.$$

This is a constant if and only if $p^* + \left[\frac{w_1}{w_2}\right]^t (p_1 - p^*)$ is constant, that is if and only if $p_1 = p^*$.

There is an unique stationary equilibria, associated with the lowest price level p_1 and a rate of return on currency equal to the discount factor, $R = 1$.

f. Along an equilibrium path, the utility of an agent of type 1 is:

$$\log\left[\frac{w_1}{2}\right] + \log\left[\frac{R_t w_1}{2}\right].$$

And is increasing in R_t. Similarly the utility of an agent of type 2 is:

$$\log\left[\frac{w_2}{2R_t}\right] + \log\left[\frac{w_2}{2}\right].$$

And is decreasing in R_t. Now, from the formula that we derived in part (e), it is clear that R_t is a decreasing function of p_1. Note that this property is quite strong: higher p_1 will correspond to uniformly lower rate of return on currency, formally:

$$p_1' > p_1 \Rightarrow \forall t, R_t(p_1') < R_t(p_1).$$

Therefore, type 1 agents will be worse off in a higher p_1 economy – their saving will earn lower interest. Conversely type 2 agents will be better off – they'll borrow at a lower rate.

Lastly, the initial old are worse off for higher p_1 because their real money balance M/p_1 is lower.

Exercise 8.3.

Take the economy of Exercise 8.1, but make one change. Endow the initial old with a tree that yields a constant dividend of $d > 0$ units of the consumption good for each $t \geq 1$.

a. Compute all the equilibria with valued fiat currency.

b. Compute all the equilibria without valued fiat currency.

c. If you want, you can answer both parts of this question in the context of the following particular numerical example: $w_1 = 10, w_2 = 5, d = .000001$.

Solution

We first define an equilibrium. The problem faced by the young of generation t is

$$\begin{array}{rl} & \max_{c_t^t, c_{t+1}^t, M_t, \alpha_t} u(c_t^t) + u(c_{t+1}^t) \\ \text{subject to} & c_t^t + \frac{M_t}{p_t} + \alpha_t q_t \leq w_1 \\ & c_{t+1}^t \leq w_2(t+1) + \frac{M_t}{p_{t+1}} + \alpha_t(q_{t+1} + d), \\ & [c_t^t, c_{t+1}^t, M_t, \alpha_t] \geq 0. \end{array}$$

The problem of the initial old is simply

$$\begin{array}{rl} & \max_{c_1^0} c_1^0 \\ \text{subject to} & 0 \leq c_1^0 \leq w_2(1) + \frac{M}{p_1} + q + d. \end{array}$$

Definition 12. *An equilibrium with valued fiat currency is a consumption plan for the initial old c_1^0, consumption plans for the young born at time $t \geq 1$, $\{c_t^t, c_{t+1}^t\}_{t=1}^{+\infty}$, money demand $\{M_t\}_{t=1}^{+\infty}$, demand for shares of the tree $\{\alpha_t\}_{t=1}^{+\infty}$, a positive price sequence $\{p_t, q_t\}_{t=1}^{+\infty}$, such that the two following conditions are satisfied:*

(i) *Optimality: given p_1, c_1^0 solves the initial old problem. Given $\{p_t, p_{t+1}, q_t\}$, $\{c_t^t, c_{t+1}^t\}$ solves agent of generation t problem, for all $t \geq 1$.*
(ii) *Feasibility: the market for good, the market for money and the market for shares of the tree clear for all $t \geq 1$:*

$$\begin{array}{l} c_t^t + c_t^{t-1} = w_1 + w_2 + d \\ M_1 = M \quad \text{and} \quad M_{t+1} = M_t \\ \alpha_t = 1. \end{array}$$

a. We now solve for equilibria. Walras Law allows to restrict attention to the market for money and the market for shares of the tree. Also, since both money and tree are held in equilibrium, they must earn the same return, therefore:

$$R_t \equiv \frac{p_t}{p_{t+1}} = \frac{d + q_{t+1}}{q_t}.$$

Writing this equation $q_t = \frac{d+q_{t+1}}{R_t}$ and iterating forward, we obtain:

$$q_t = \sum_{k=0}^{+\infty} \left[\prod_{i=0}^{k} \frac{1}{R_{t+i}} \right] d < +\infty.$$

This last equation imposes a strong restriction on equilibrium R_t: R_t cannot be uniformly less than 1, otherwise the sum would diverge.
Equilibrium on the market for shares of the tree and on the market for money can be written using the agent saving function as in Exercise 1: $s(R_t) = \frac{M}{p_t} + q_t$. Expressing that nominal money balances stay constant over time, we can rewrite the equilibrium conditions imply

$$\begin{array}{lcl} \frac{M}{p_1} + q_1 & = & s(R_1) \\ s(R_{t+1}) & = & R_t s(R_t) - d, \quad t \geq 1 \\ R_t & = & \frac{p_t}{p_{t+1}} \\ q_t & = & \sum_{k=0}^{+\infty} \left[\prod_{i=0}^{k} \frac{1}{R_{t+i}} \right] d < +\infty. \end{array}$$

Conversely, suppose that we have solved for an initial price $p_1 < +\infty$ and for sequences $\{R_t, q_t\}_{t=1}^{+\infty}$ solving the above three equations. Then, one can show that an equilibrium is $p_{t+1} = p_t/R_t$, $c_1^0 = w_2 + M/p_1 + q_1$, $c_t^t = w_1 - s(R_t)$, $c_{t+1}^t = w_2 + R_t s(R_t)$, $M_t = M$, $\alpha_t = 1$.

We construct equilibria in three steps. First we choose a positive sequences $\{R_t\}_{t=1}^{+\infty}$ solving the difference equation $s(R_{t+1}) = R_t s(R_t) - d$. Second, we check if the q_t it implies are finite. Third, given R_1 and q_1, we solve the first equation for $p_1 < +\infty$.
Towards a characterization of equilibria, we now prove the following results:

(1) There is a unique R^* such that $s(R^*) = R^* s(R^*) - d$ and $R^* > 1$.

This follows from the following facts. The function $(R-1)s(R) - d$ is continuous and increasing for $R \geq 1$, its value is $-d < 0$ at $R = 1$ and $(R-1)s(R) - d > (R-1)s(1) - d \to +\infty$ when $R \to +\infty$.

(2) There is no equilibrium such that $R_1 > R^*$.

The reasoning is the same as in exercise 8.1, we only sketch it here. For any $R_t > R^*$, we show that $R_{t+1} > R_t$. So R_t has a limit, which must be $+\infty$ since $R_t > R^*$. But this means that, for t large enough, R_t necessarily violates the difference equation $s(R_{t+1}) = R_t s(R_t) - d$.

(3) There is no equilibrium with $R_1 \leq \left[\frac{w_2}{w_1}\right]^\gamma$.

As in question 8.1, this follows from the fact that, for such R_1, $s(R_1) < 0$, and that saving must be positive in an equilibrium with valued fiat currency.

(4) There is no equilibrium such that $R_1 < 1$.

Assume $R_t < 1$. We want to show that it implies that $R_{t+1} < 1$. There are two cases. (i) If $R_t \leq \left[\frac{w_2}{w_1}\right]^\gamma$, then $s(R_{t+1}) < 0$ so that $R_{t+1} \leq \left[\frac{w_2}{w_1}\right]^\gamma < 1$. (ii) If $\left[\frac{w_2}{w_1}\right]^\gamma < R_t < 1$, then $s(R_{t+1}) - s(R_t) = (R_t - 1)s(R_t) - d < 0$, so that $R_{t+1} < R_t < 1$. This shows that $R_1 < 1 \Rightarrow R_t < R_1 < 1 \quad \forall t$. But then q_t cannot be finite.

(5) There is no equilibrium such that $1 \leq R_1 < R^*$.

Consider $1 \leq R_t < R^*$. Then $s(R_{t+1}) - s(R_t) = (R_t - 1)s(R_t) - d < 0$. This follows from the fact that the function $(R-1)s(R) - d$ is negative at $R = 1$, is zero at $R = R^*$, and is continuous and increasing for $R \geq 1$. Hence, $R_{t+1} < R_t$. Therefore the first terms of the sequence are decreasing. The sequence cannot stay bounded below by 1, however, since otherwise it would have a limit greater than 1 and smaller than R^*, which is impossible from point (1). Therefore there must be a T such that $R_T < 1$. And, from point (3), we know that for all $t \geq T$, $R_t \leq R_T < 1$. But then, as before, q_t cannot be finite.

(6) There is no equilibrium with valued fiat money.

The only candidate remaining is the stationary one $R_t = R^* > 1$, for all t. In this candidate equilibrium, we have $(R^* - 1)s(R^*) = d$ and:

$$q^* = \sum_{t=1}^{+\infty} \frac{d}{(R^*)^t} = \frac{d}{R^* - 1}.$$

so that $q^* = s(R^*)$. Replacing it in the first of our four equations defining an equilibrium we find:

$$\frac{M}{p_1} = s(R^*) - q^* = 0,$$

which imply that p_1 cannot be finite. Thus, there are no equilibrium with valued fiat currency.

b. All the previous analysis can be made in the context of equilibria without valued fiat currency. One can show that there exists a unique equilibrium, that it is stationary and that the tree has value:

$$q^* = \frac{d}{R^* - 1}.$$

Exercise 8.4.

Take the economy of exercise 8.1 and make the following changes. First, assume that $\gamma = 1$. Second, assume that the number of agents born at t is $N(t) = nN(t-1)$, where $N(0) > 0$ and is given and $n \geq 1$. Everything else about the economy remains the same.

a. Compute an equilibrium without valued fiat money.

b. Compute a stationary equilibrium with valued fiat money.

Solution

See exercise 8.5.

Exercise 8.5.

Consider an economy consisting of overlapping generations if two period-lived consumers. At each date $t \geq 1$, there are born $N(t)$ identical young people each of whom is endowed with $w_1 > 0$ units of a single consumption good when young and $w_2 > 0$ units of the consumption good when old. Assume that $w_2 < w_1$. The consumption good is non storable. The population of young people is described by $N(t) = nN(t-1)$ where $n > 0$. Young people born at t rank utility streams according to $\ln(c_t^t) + \ln(c_{t+1}^t)$, where c_t^i is the consumption of the time t good of the agents born at time i. In addition, there are $N(0)$ old people at time 1, each of whom is endowed with w_2 units of the time-1 consumption good. The old at $t = 1$ are also endowed with one unit of unbacked pieces of infinitely durable but intrisically worthless pieces of paper called fiat money.

a. Define an equilibrium without valued fiat currency. Compute such an equilibrium.

b. Define an equilibrium with valued fiat currency.

c. Compute all equilibra with valued fiat currency.

d. Find the limiting rates of return on currency as $t \to +\infty$ in each of the equilibria found in part c. Compare them with the one period interes rate in the equilibrium in part a.

e. Are the equilibria in part c ranked according to the Pareto criterion ?

Solution

a. The problem faced by a young of generation t is to choose consumption c_t^t, c_{t+1}^t and IOU holdings b_t so as to maximize

$$\begin{array}{rl} & \ln(c_t^t) + \ln(c_{t+1}^t) \\ \text{subject to} & c_t^t + b_t \leq w^1 \\ & c_{t+1}^t \leq w_2 + R_t b_t \\ & [c_t^t, c_{t+1}^t] \geq 0. \end{array}$$

The problem of the initial old is simply to choose consumption c_1^0 so as to maximize c_1^0 subject to $0 \leq c_1^0 \leq w^2$. We define

Definition 13. *An equilibrium without valued fiat currency is a consumption plan for the initial old c_1^0, consumption plan for the young born at time $t \geq 1$, $\{c_t^t, c_{t+1}^t\}_{t=1}^{+\infty}$, IOU holding plan $\{b_t\}_{t=1}^{+\infty}$, and a positive return sequence $\{R_t\}_{t=1}^{+\infty}$ such that the two following conditions are satisfied:*

(i) *Optimality: c_1^{h0} solves the initial old problem. Given R_t and $\{c_t^t, c_{t+1}^t\}$ solves agent of generation t problem for all $t \geq 1$.*

(ii) *Feasibility: the market for good and the market for private IOU clear for all $t \geq 1$:*

$$\begin{array}{l} N(t)c_t^t + N(t-1)c_t^{t-1} = N(t)w_1 + N(t-1)w_2 \\ b_t = 0. \end{array}$$

We know that the saving function of a young agents is $s(R) = w_1/2 - w_2/(2R)$. In an equilibrium without valued fiat currency, young agents can only write IOU between themselves. Since all young are identical it must be that $b_t = s(R_t) = 0$, for all t. Therefore $R_t = \frac{w_2}{w_1} < 1$. Furthermore $c_t^t = w_1$ and $c_t^{t-1} = w_2$.

b. A young of generation t chooses a consumption plan c_t^t, c_{t+1}^t, a money holding m_t and an IOU holding b_t so as to maximize

$$\begin{array}{rl} & \ln(c_t^t) + \ln(c_{t+1}^t) \\ \text{subject to} & c_t^t + b_t + \frac{m_t}{p_t} \leq w_1 \\ & c_{t+1}^t \leq w_2 + R_t b_t + \frac{m_t}{p_{t+1}} \\ & [c_t^t, c_{t+1}^t, m_t] \geq 0. \end{array}$$

The problem of the initial old is simply to maximize c_1^0 subject to $c_1^0 \leq \frac{M}{p_1}$. We have

Definition 14. *An equilibrium with valued fiat currency is a consumption plan for the initial old c_1^0, consumption plans for the young born at time $t \geq 1$, $\{c_t^t, c_{t+1}^t\}_{t=1}^{+\infty}$, money holdings $\{m_t\}_{t=1}^{+\infty}$, and bond holdings $\{b_t\}_{t=1}^{+\infty}$, a positive price sequence and a positive interest rate sequence $\{p_t, R_t\}_{t=1}^{+\infty}$, such that the following conditions are satisfied:*

(i) *Optimality: given p_1, c_1^0 solves the initial old problem. Given $\{p_t, p_{t+1}, R_t\}$, $\{c_t^t, c_{t+1}^t, m_t, b_t\}$ solves agent of generation t problem, for all $t \geq 1$.*

(ii) *Feasibility: the market for good, the market for money and the market for IOU clear for all $t \geq 1$:*

$$\begin{aligned}&N(t)c_t^t + N(t)c_t^{t-1} = N(t)w_1 + N(t-1)w_2\\&N(t)m_t = N(0)\\&N(t)b_t = 0.\end{aligned}$$

c. Walras Law allows to restrict attention to the market for money and the market for IOU. Also, by no arbitrage, it must be that both money and IOU earn the same return, that is $R_t = p_t/p_{t+1}$. The savings of the young agents are

$$s(R_t) = \frac{w_1}{2} - \frac{w_2}{2R_t} = b_t + \frac{m_t}{p_t}.$$

Using $b_t = 0$, the equilibrium equations reduce to

$$N(t)\left(\frac{w_1}{2} - \frac{w_2}{2R_t}\right) = \frac{N(0)}{p_t} \tag{76}$$

$$R_t = \frac{p_t}{p_{t+1}}. \tag{77}$$

Substitutiong the second equation into the first and rearanging, we obtain that all equilibria are the positive solutions of

$$w_1 p_t - w_2 p_{t+1} = 2n^{-t}. \tag{78}$$

When $nw_1 = w_2$, the the previous equation can be written

$$p_{t+1} = -2w_1\left(\frac{w_1}{w_2}\right)^{t+1} + \frac{w_1}{w_2}p_t.$$

Iterating backwards we find that

$$p_{t+1} = -2t\left(\frac{w_1}{w_2}\right)^{t+1} + \left(\frac{w_1}{w_2}\right)^t p_1.$$

Which is negative for t large enough. Thus, when $nw_1 = w_2$, there is no equilibrium with valued fiat currency.
When $nw_1 \neq w_2$, a particular solution of this equation is $p^* n^{-t}$, where $p^* = 2n/(nw_1 - w_2)$. Substracting this particular solution from equation (78), we find that the general solution takes the form

$$p_{t+1} = p^* n^{-t} + \left(\frac{w_1}{w_2}\right)^t (p_1 - p^*). \tag{79}$$

If $nw_1 - w_2 < 0$, then since $p^* < 0$ and $1/n > \frac{w_1}{w_2}$, it is clear that all solutions are negative for t large enough. Thus, there do not exists an equilibrium with valued fiat currency.

If, on the other hand, $nw_1 - w_2 > 0$, then all solution such that $p_1 \geq p^*$ are positive for all t.

d. When $nw_1 > w_2$, then

$$
\begin{aligned}
R_t &= \frac{p^* n^{-(t-1)} + \left(\frac{w_1}{w_2}\right)^{t-1} (p_1 - p^*)}{p^* n^{-t} + \left(\frac{w_1}{w_2}\right)^{t} (p_1 - p^*)} \\
R_t &= \frac{w_2}{w_1} + \frac{\left(1 - \frac{w_2}{nw_1}\right) n^{-t+1} p^*}{p^* n^{-t} + \left(\frac{w_2}{w_1}\right)^{t} (p_1 - p^*)}.
\end{aligned}
$$

Therefore, if $p_1 = p^*$, then the limiting rate of return on curruncy is n. Otherwise if $p_1 > p^*$, the limiting rate of return on currency is $\frac{w_2}{w_1}$.

e. To rank the various equilibria we first note that

$$\ln(w_1 - s(R)) + \ln(w_2 + Rs(R)).$$

is increasing in R whenever $R > \frac{w_2}{w_1}$, which is true in all our equilibria. Furthermore, as it is clear from the last expression for R_t, R_t is a decreasing function of p_1. Since the initial old are better off with lower p_1, this implies that the p_1-equilibrium pareto dominate all p_1'-equilibria with $p_1' > p_1$.

Exercise 8.6. *Exchange rate determinacy* $\diamond$

The world consists of two economies, named $i = 1, 2$, which except for their governments' policies are "copies" of one another. At each date $t \geq 1$, there is a single consumption good, which is storable, but only for rich people. Each economy consists of overlapping generations of two-period-lived agents. For each $t \geq 1$, in economy i, N poor people and N rich people are born. Let $c_t^h(s), y_t^h(s)$ be the time s (consumption, endowment) of a type-h agent born at t. Poor agents are endowed $[y_t^h(t), y_t^h(t+1)] = (\alpha, 0)$; Rich agents are endowed $[y_t^h(t), y_t^h(t+1)] = (\beta, 0)$, where $\beta >> \alpha$. In each country, there are $2N$ initial old who are endowed in the aggregate with $H_i(0)$ units of an unbacked currency, and with $2N\epsilon$ units of the time-1 consumption good. For the rich people, storing k units of the time-t consumption good produces Rk units of the time–$t+1$ consumption good, where $R > 1$ is a fixed gross rate of return on storage. Rich people can earn the rate of return R either by storing goods or lending to either government by means of indexed bonds. We assume that poor people are prevented from storing capital or holding indexed government debt by the sort of denomination and intermediation restrictions described by Sargent and Wallace (1982).
For each $t \geq 1$, all young agents order consumption streams according to $\ln c_t^h(t) + \ln c_t^h(t+1)$.

For $t \geq 1$, the government of country i finances a stream of purchases (to be thrown into the ocean) of $G_i(t)$ subject to the following budget constraint:

$$G_i(t) + RB_i(t-1) = B_i(t) + \frac{H_i(t) - H_i(t-1)}{p_i(t)} + T_i(t), \tag{1}$$

where $B_i(0) = 0$; $p_i(t)$ is the price level in country i; $T_i(t)$ are lump-sum taxes levied by the government on the *rich* young people at time t; $H_i(t)$ is the stock of i's fiat currency at the end of period t; $B_i(t)$ is the stock of indexed government interest-bearing debt (held by the rich of either country). The government does not explicitly tax poor people, but might tax through an inflation tax. Each government levies a lump-sum tax of $T_i(t)/N$ on each young rich citizen of its own country.
Poor people in both countries are free to hold whichever currency they prefer. Rich people can hold debt of either government and can also store; storage and both government debts bear a constant gross rate of return R.

a. Define an *equilibrium* with valued fiat currencies (in both countries). **b.** In

a nonstochastic equilibrium, verify the following proposition: if an equilibrium exists in which both fiat currencies are valued, the exchange rate between the two currencies must be constant over time.

c. Suppose that government policy in each country is characterized by specified (exogenous) levels $G_i(t) = G_i, T_i(t) = T_i$, $B_i(t) = 0, \forall t \geq 1$. (The remaining elements of government policy adjust to satisfy the government budget constraints.) Assume that the exogenous components of policy have been set so that an equilibrium with two valued fiat currencies exists. Under this description of policy, show that the equilibrium exchange rate is indeterminate.

d. Suppose that government policy in each country is described as follows: $G_i(t) = G_i, T_i(t) = T_i, H_i(t+1) = H_i(1), B_i(t) = B_i(1)\ \forall t \geq 1$. Show that if there exists an equilibrium with two valued fiat currencies, the exchange rate is indeterminate.

e. Suppose that government policy in country 1 is specified in terms of exogenous levels of $s_1 = [H_1(t) - H_1(t-1)]/p_1(t)\ \forall t \geq 2$, and $G_1(t) = G_1\ \forall t \geq 1$. For country 2, government policy consists of exogenous levels of $B_2(t) = B_2(1), G_2(t) = G_2 \forall t \geq 1$. Show that if there exists an equilibrium with two valued fiat currencies, then the exchange rate is determinate.

Solution

a. In order to simplify the analysis we assume that the rate of return on money is strictly less than the rate of return on storage R. This implies that only poor agents hold money and therefore simplify considerably the demand for money. At the end of the exercise, we provide restrictions on exogenous parameters ensuring that this assumption hold.

We first state the choice problem faced by the agents. Agents are indexed by $h \in \{1, \dots 4N\}$. Agents $h \in \{1, \dots 2N\}$ are from country 1 and agents $h \in$

$\{2N+1,\dots 4N\}$ are from country 2. A young poor agent of generation t chooses some positive consumption plan $c_t^h(t), c_t^h(t+1)$, some positive money holdings $H_1^h(t), H_2^h(t)$, so as to maximize $\ln(c_t^h(t)) + \ln(c_t^h(t+1))$ subject to

$$\begin{aligned} c_t^h(t) + \frac{H_1^h(t)}{p_1(t)} + \frac{H_2^h(t)}{p_2(t)} &\leq \alpha \\ c_t^h(t+1) \leq \frac{H_1^h(t)}{p_1(t+1)} &+ \frac{H_2^h(t)}{p_2(t+1)}. \end{aligned}$$

A young rich agent of generation t chooses some positive consumption plan $c_t^h(t), c_t^h(t+1)$, some bond holdings $B_1^h(t), B_2^h(t)$, and some positive storage $A^h(t)$, so as to maximize $\ln(c_t^h(t)) + \ln(c_t^h(t+1))$ subject to

$$\begin{aligned} &c_t^h(t) + B_1^h(t) + B_2^h(t) \leq \beta - T^h(t) \\ &c_t^h(t+1) \leq R(B_1^h(t) + B_2^h(t) + A^h(t)). \end{aligned} \tag{81}$$

The initial old in country i maximizes $c_1^h(0)$ subject to $0 \leq c_0^h(1) \leq \frac{H_i(0)}{2Np_i(1)} + \varepsilon$. We define an equilibrium

Definition 15. *An equilibrium with valued fiat currency is a consumption plan for the initial old $c_0^h(1)$, consumption plans for the young agents born at time $t \geq 1$, $\{c_t^h(t), c_t^h(t+1)\}_{t=1}^{+\infty}$, money demand $\{H_i^h(t)\}_{t=1}^{+\infty}$, bond demands $\{B_i^h(t)\}_{t=1}^{+\infty}$, government fiscal and monetary policies $\{T_i(t), H_i(t), B_i(t)\}_{t=1}^{+\infty}$, and positive price sequences $\{p_1(t), p_2(t)\}_{t=1}^{+\infty}$, such that the three following conditions are satisfied:*

(i) *Optimality: given $p_i(t)$, $c_0^h(1)$ solves the initial old h problem. Given prices $p_i(t)$ and interest rate R, the consumption plans, the money holdings, the bond holdings and the storage plans solve the young agents' problems.*

(ii) *Feasibility: the markets for good, the market for money and the market for bonds clear for all $t \geq 1$:*

$$\begin{array}{llll} t=1 & i=1,2 & \sum_{h\in i}\left(c_t^h(t) + c_{t-1}^h(t) + A^h(t)\right) + G_i(t) & = N\alpha + N\beta + 2N\varepsilon \\ t\geq 2 & i=1,2 & \sum_{h\in i}\left(c_t^h(t) + c_{t-1}^h(t) + A^h(t)\right) + G_i(t) & = N\alpha + N\beta + R\sum_{h\in i} A^h(t-1) \\ & i=1,2 & \sum_{h=1}^{4N} \frac{H_i^h(t)}{p_1(t)} & = \frac{H_i^h(t)}{p_i(t)} \\ & i=1,2 & \sum_{h=1}^{4N} B_i^h(t) & = B_i(t). \end{array}$$

(iii) *The governments' policies satisfy their budget constraints:*

$$t \geq 1 \quad i=1,2 \qquad G_i(t) + RB_i(t-1) = B_i(t) + \frac{H_i(t) - H_i(t-1)}{p_i(t-1)} + T_i(t).$$

b. In an equilibrium with valued fiat currency, both currencies are held in equilibrium. Therefore, they must earn the same rate of return:

$$\frac{p_1(t)}{p_1(t+1)} = \frac{p_2(t)}{p_2(t+1)} \quad \forall t \geq 1.$$

This is equivalent to

$$\frac{p_1(t)}{p_2(t)} = \frac{p_1(t+1)}{p_2(t+1)} \equiv e \quad \forall t \geq 1.$$

In other words, the exchange rate must be constant over time. In all what follows, we'll thus write $p_2(t) = \frac{p_1(t)}{e}$.

The equilibrium equations

We derive the equilibrium equations and we show that they are recursive. Namely, we show that one can solve first for prices, bond and money holding using the money demand and the government budget constraints. Then, one chooses storage so that the market clearing condition for good holds at each date.

With log utility, the saving function of a young agents is of the form $s(R_t) = \frac{y_t}{2} - \frac{y_{t+1}}{2R_t}$. We let $R_m(t) = \frac{p_i(t)}{p_i(t+1)}$ be the rate of return on money. We assume that $R_m(t) < R$ for all t, so that the world demand for money is equal to $N\alpha$. At the end of the solution, we present sufficient conditions ensuring that $R_m(t) < R$, at each time. The market clearing condition for the good markets are

$$t = 1, i = 1, 2 \qquad \frac{N\alpha}{2} + \frac{N\beta - T_i(1)}{2} + \frac{H_i(0)}{p_i(1)} + 2N\varepsilon + NA_i(1) + G_i(1) \tag{82}$$

$$= N\alpha + N\beta + 2N\varepsilon. \tag{83}$$

$$\begin{aligned} t \geq 2, i = 1, 2 \qquad & \frac{N\alpha}{2} + \frac{N\beta - T_i(t)}{2} + R_m(t-1)\frac{N\alpha}{2} + R\frac{N\beta - T_i(t-1)}{2} \\ & + NA_i(t) + G_i(t) \\ & = N\alpha + N\beta + RNA_i(t-1). \end{aligned} \tag{84}$$

The market clearing condition for the money market and the market for bond and storage are, for all $t \geq 1$

$$N\alpha = \frac{H_1(t)}{p_1(t)} + \frac{H_2(t)}{p_2(t)} \tag{85}$$

$$N\beta = B_1(t) + B_2(t) + A_1(t) + \frac{T_1(t)}{2} + A_2(t) + \frac{T_2(t)}{2}. \tag{86}$$

Lastly, the government budget constraints are for all $t \geq 1$

$$G_i(t) + RB_i(t-1) = \frac{H_i(t) - H_i(t-1)}{p_i(t)} + T_i(t) + B_i(t). \tag{87}$$

We observe that, by Walras Law, one equation can be ignored at each t. Let's ignore the market clearing condition for bonds and storage (86). Also, it is convenient to substitute the market clearing condition for money (85) into the sum of government budget constraints (87). Then, we can ignore (85) and keep the following equations

$$\begin{aligned}
&(88) \qquad t = 1 \quad && G_1(1) + G_2(1) \\
& && = T_1(1) + T_2(1) + B_1(1) + B_2(1) + N\alpha - \frac{H_1(0)}{p_1(1)} - \frac{H_2(0)}{p_2(1)} \\
&(89) \qquad t \geq 2 \quad && G_1(t) + G_2(t) + R(B_1(t-1) + B_2(t-1)) \\
& && = T_1(t) + T_2(t) + N\alpha(1 - R_m(t-1)) + B_1(t) + B_2(t).
\end{aligned}$$

Equilibria are solution of (82), (84), (87) (88) and (89). Furthermore, since we ignore (86), storage appears only in (82) and (84). This shows that the equilibrium equations can be solved recursively. First, we solves equations (87), (88), and (89). Then, we choose a sequence of storage such that (82) and (84) hold for all t.

c. and d. In these questions, we assume that the governments chooses $B_i(t)$, $G_i(t)$ and $T_i(t)$ for all t. With equations (89), we solve for the rate of return on money $R_m(t), t \geq 1$. Given $p_1(1)$ and $p_2(1)$, this allows to solve for the entire sequence of price $p_i(t)$, $t \geq 2$. Given a sequence of price, we use the government budget constraints to solve for the the money supply $H_i(t)$, $t \geq 1$.

We need to determine the initial price $p_1(1)$ and $p_2(1)$. We have only one equation left, the worldwide government budget constraint at time 1, (88). The exchange rate $p_1(1)/p_2(1)$ is thus indeterminate.

Sufficient Conditions for $R_m(t) < R$. Consider the world wide government budget constraint $t \geq 2$

$$\begin{aligned}
& G_1(t) + G_2(t) - T_1(t) - T_2(t) + RB_1(t-1) - B_1(t) + RB_2(t-1) - B_2(t) \\
= \;& \frac{H_1(t)}{p_1(t)} + \frac{H_2(t)}{p_2(t)} - \frac{H_1(t-1)}{p_1(t)} - \frac{H_2(t-1)}{p_2(t)} \\
\equiv \;& h(t) - R_m(t-1)h(t-1).
\end{aligned}$$

Assume that the governments are running a global deficit for all t

$$(90) \quad G_1(t) + G_2(t) - T_1(t) - T_2(t) + RB_1(t-1) - B_1(t) + RB_2(t-1) - B_2(t) > 0.$$

This ensures that the government needs to raise seignoriage revenue and forces $R_m(t)$ to be less then R: assume, towards a contradiction, that $R_m(\tau - 1) \geq R > 1$ for some $\tau \geq 2$. Then, necessarily $h(\tau) > h(\tau - 1) \geq N\alpha$. Since the demand for real balance is greater than $N\alpha$, it must be that the rate of return on money weakly dominate the rate of return on storage. In other words, $R_m(\tau) \geq R$. Thus $h(\tau + 1) > h(\tau)$. By induction, one can show that $h(t)$ is an increasing sequence and $R_m(t) \geq R$ for $t \geq \tau$. Since $h(t)$ is bounded above by $N\alpha + N\beta$, it has a finite limit. In particular $h(t) - h(t-1)$ goes to zero. Therefore

$$h(t) - R_m(t-1)h(t-1) = h(t) - h(t-1) + h(t-1)(1 - R_m(t-1)). \tag{91}$$

is negative for t large enough, which contradicts (90). Assumptions on exogenous parameters that garantee (90) are

Question c.	$G_1 + G_2 - T_1 - T_2 > 0$
Question d.	$G_1 + G_2 - T_1 - T_2 - B_1(1-R) - B_2(1-R) > 0$
Question e.	$s_1 + G_2 - T_2 - B_2(1-R) > 0$

Exercise 8.7. *Credit Controls*

Consider the following overlapping-generations model. At each date $t \geq 1$ there appear N two-period-lived young people, said to be of generation t, who live and consume during periods t and $(t+1)$. At time $t = 1$ there exist N old people who are endowed with $H(0)$ units of paper "dollars," which they offer to supply inelastically to the young of generation 1 in exchange for goods. Let $p(t)$ be the price of the one good in the model, measured in dollars per time t good. For each $t \geq 1$, $N/2$ members of generation t are endowed with $y > 0$ units of the good at t and 0 units at $(t+1)$, whereas the remaining $N/2$ members of generation t are endowed with 0 units of the good at t and $y > 0$ units when they are old. All members of all generations have the same utility function:

$$u[c_t^h(t), c_t^h(t+1)] = \ln c_t^h(t) + \ln c_t^h(t+1),$$

where $c_t^h(s)$ is the consumption of agent h of generation t in period s. The old at $t = 1$ simply maximize $c_0^h(1)$. The consumption good is nonstorable. The currency supply is constant through time, so $H(t) = H(0),\ t \geq 1$.

a. Define a competitive equilibrium without valued currency for this model. Who trades what with whom?

b. Compute the nonvalued-currency competitive equilibrium values of the gross return on consumption loans, the consumption allocation of the old at $t = 1$, and that of the "borrowers" and "lenders" for $t \geq 1$.

c. Define a competitive equilibrium with valued currency. Who trades what with whom?

d. Prove that for this economy there does not exist a competitive equilibrium with valued currency.

e. Now suppose that the government imposes the restriction that $l_t^h(t)[1+r(t)] \geq -y/4$, where $l_t^h(t)[1+r(t)]$ represents claims on $(t+1)$-period consumption purchased (if positive) or sold (if negative) by household h of generation t. This is a restriction on the amount of borrowing. For an equilibrium without valued currency, compute the consumption allocation and the gross rate of return on consumption loans.

f. In the setup of (e), show that there exists an equilibrium with valued currency in which the price level obeys the quantity theory equation $p(t) = qH(0)/N$. Find a formula for the undetermined coefficient q. Compute the consumption allocation and the equilibrium rate of return on consumption loans.

g. Are lenders better off in economy (b) or economy (f)? What about borrowers? What about the old of period 1 (generation 0)?

Solution

a. We first describe the problem faced by the young of generation t. This problem is:

$$\begin{aligned} & \max_{c_t^h(t), c_t^h(t+1), l_t^h(t), m_t^h(t)} u^h(c_t^h(t), c_t^h(t+1)) \\ \text{subject to} \quad & c_t^h(t) + l_t^h(t) + \tfrac{m_t^h(t)}{p(t)} \leq w_t^h(t) \\ & c_t^h(t+1) \leq w_t^h(t+1) + [1 + r(t)] l_t^h(t) + \tfrac{m_t^h(t)}{p(t+1)}, \\ & [c_t^h(t), c_t^h(t+1), m_t^h(t)] \geq 0. \end{aligned}$$

Let $c_t^h = [c_t^h(t), c_t^h(t+1)]$ and denote by $c_t = (c_t^1, \ldots, c_t^N)$ the consumption vector of generation t. We use c_0 to denote second-period consumption of the generation that is old at $t = 1$. A sequence $c = \{c_t\}_{t=0}^{\infty}$ is called an allocation. We are now ready to define an equilibrium. A competitive equilibrium without valued fiat money is a sequence $\{1/p(t)\}_{t=1}^{\infty}$ identically equal to zero, a sequence $\{r(t)\}_{t=1}^{\infty}$, and an allocation c that satisfies two conditions.

(i) Given $r(t)$, c_t^h solves the agents' maximization problem for every h and $t \geq 1$.

(ii) Given c_t^h, we know that $l_t^h(t) = w_t^h(t) - c_t^h(t)$. Market clearing requires that

$$\sum_{h=1}^{N} l_t^h(t) = 0, \qquad t = 1, 2, \ldots$$

In this economy with only one good at each date, the only possible trades are intertemporal ones, that is, exchanges of consumption in one period for consumption in some other period. No intergenerational trades are possible. At any time t, an "old" agent cares only about consumption. This agent would be willing to buy the good but has nothing to offer to a young agent in exchange. Therefore no exchanges can be made. Intragenerational trades will occur in equilibrium. The utility function is such that agents want consumption over time to be smooth. Endowments vary across time, however, and are asymmetric across agents, making room for exchange of loans. Agents who are well endowed during their first period of life will be willing to give up some consumption when they are young in exchange for goods in their second period of life. Agents who are well endowed when they are old will be willing to accept those trades.

b. Solving the competitive problem for the Cobb-Douglas utility function we find that

$$s_t^h(t) \equiv w_t^h(t) - c_t^h(t) = \frac{1}{2}\left(w_t^h(t) - \frac{w_t^h(t+1)}{1+r(t)}\right)$$

In the nonvalued-currency equilibrium, $s_t^h(t) = l_t^h(t)$. To compute the rate of interest that clears the market for consumption loans, we need to determine the aggregate savings function. For an agent endowed $(y, 0)$, savings are $y/2$. As there are $N/2$ agents of this type, their aggregate demand is given by $Ny/4$. For an agent endowed $(0, y)$, the savings are $-y/(2[1 + r(t)])$. Total savings for this group are $-(Ny/4)[1 + r(t)]^{-1}$. The aggregate per capita savings function of the economy is

$$f[1 + r(t)] \equiv \frac{1}{4}\left(y - \frac{y}{1+r(t)}\right).$$

We defined an equilibrium as a sequence $\{r(t)\}$ and c, satisfying utility maximization and market clearing. Then the first part – utility maximization – is embedded already in $f(\cdot)$, whereas market clearing requires that

$$\frac{1}{N}\sum_{h=1}^{N} l_t^h(t) = f[1 + r(t)] = 0.$$

The unique solution to this condition is $r(t) = 0$, for all t, corresponding to a gross rate of return of one. To compute the equilibrium allocation, recall that it can be obtained from

$$c_t^h(t) = w_t^h(t) - l_t^h(t), \qquad c_t^h(t+1) = w_t^h(t+1) + [1 + r(t)]l_t^h(t).$$

For a lender – an agent endowed $(y, 0)$ – we obtain $l_t^h(t) = y/2$. Consequently, $c_t^h(t) = y/2$, $c_t^h(t+1) = y/2$, $h = 1, \ldots, N/2$, and $t \geq 1$. In the case of a borrower – an individual endowed $(0, y)$ – we obtain $l_t^h(t) = -y/2$. Therefore $c_t^h(t) = y/2$, $c_t^h(t+1) = y/2$, $h = N/2+1, \ldots, N$, and $t \geq 1$. The old at $t = 1$, that is, the members of generation zero, consume their endowments of the one good.

c. Notice that our definition of the competitive problem faced by the young is general enough to incorporate the maximization exercise that is solved in an equilibrium with valued currency. As in (a), define $m_t = [m_t^1(t), \ldots, m_t^N(t)]$, $m = \{m_t\}_{t=1}^{\infty}$. Then a competitive equilibrium with valued fiat currency is a pair of sequences $\{r(t)\}_{t=1}^{\infty}$ and $\{p(t)\}_{t=1}^{\infty}$ with $p(t)$ finite and greater than zero $\forall t$, an allocation c, and a sequence m such that

(i) Given $r(t)$ and $p(t)$, c_t^h and $m_t^h(t)$ solve the maximization problem defined in (a), for $h = 1, \ldots, N, t \geq 1$.

(ii) Given the choices of individual agents, markets clear, that is,

$$\sum_{h=1}^{N} l_t^h(t) + \sum_{h=1}^{N} \frac{m_t^h(t)}{p(t)} = \frac{H(t)}{p(t)}, \qquad t \geq 1.$$

This last condition is equivalent to the condition that the market for the consumption good clears.

In this equilibrium, there occur the same kinds of trades as in the equilibrium without valued fiat currency, because no markets have been shut. The fact that there is a "new" market, however – the market for currency – permits additional exchanges to be made. In this equilibrium, the "old" at each t have something that is valuable to the young – currency. The "young" are willing to give up some of the good at t in exchange for currency, because they know that next period – when they are the "old" – they will be able to exchange currency for goods. It is still true that agents engage in trade for the sole purpose of making the time pattern of consumption different (in general, also smoother) than the time pattern of endowments.

d. To prove the nonexistence results, we proceed by contradiction. From the first-order condition of the utility maximization problem we obtain

$$\begin{array}{ll} c_t^h(t): u_1^h[c_t^h(t), c_t^h(t+1)] - \lambda_{1t}^h \leq 0, & = 0 \quad \text{if } c_t^h(t) > 0 \\ c_t^h(t+1): u_2^h[c_t^h(t), c_t^h(t+1)] - \lambda_{2t}^h \leq 0, & = 0 \quad \text{if } c_t^h(t+1) > 0 \\ l_t^n(t): -\lambda_{1t}^h + [1+r(t)]\lambda_{2t}^h = 0 & \\ m_t^h(t): -\frac{1}{p(t)}\lambda_{1t}^h + \lambda_{2t}^h \frac{1}{p(t+1)} \leq 0, & = 0 \quad \text{if } m_t^h(t) > 0, \end{array}$$

where λ_{1t}^h and λ_{2t}^h are nonnegative Lagrange multipliers.
Because the definition of an equilibrium with valued currency requires that $m_t^h(t) > 0$ for some h and for every t, we have that, for that h [assuming that $c_t^h(t)$ and $c_t^h(t+1)$ are strictly positive, which is true in any equilibrium],

$$1 + r(t) = \frac{p(t)}{p(t+1)}.$$

This arbitrage condition must hold in equilibrium.
Notice that as both assets – loans and currency – have the same rate of return and are equally safe, individuals are indifferent about the composition of their portfolios, because assets are held to profit only from the intertemporal shifts of consumption that they allow, and because, if a valued currency equilibrium exists, both assets must offer exactly the same intertemporal terms of trade. Consequently, agents should not care which asset they hold. We can therefore view each agent as choosing "savings": $s_t^h(t) = l_t^h(t) + m_t^h(t)/p(t)$. The equilibrium aggregate composition of "savings" is determined not from the asset demand side but from the restriction that markets clear. Formally the choice of $s_t^h(t)$ is no different from the choice of $l_t^h(t)$ that we analyzed in (b).
We first derive a contradiction from the assumption that an equilibrium exists in a somewhat more general setup. Subsequently we analyze the particular case that constitutes the present exercise.
Given $1 + r(t) = p(t)/p(t+1)$, it is clear that $s_t^h(t)$ depends on $1 + r(t)$ [or $p(t)/p(t+1)$]. If we denote this function as $s_t^h(t) = f^h[1 + r(t)]$, and

$$\frac{1}{N}\sum_{h=1}^{N} s_t^h(t) = \frac{1}{N}\sum_{h=1}^{N} f^h[1 + r(t)] = f[1 + r(t)],$$

then the equilibrium condition (ii) instructs us to set $r(t)$ such that

$$f[1+r(t)] = \frac{H(t)}{p(t)N}$$

or

$$\begin{aligned} f[1+r(t)]p(t) = \tfrac{H(t)}{N} &= \tfrac{H(t+1)}{N} \\ &= p(t+1)f[1+r(t+1)]. \end{aligned}$$

Hence

$$f[1+r(t+1)] = [1+r(t)]f[1+r(t)].$$

We have shown that, in an equilibrium where currency is not valued, $r(t) = 0$, that is, $f(1) = 0$. Moreover, if $f(\cdot)$ is increasing in $r(t)$ (an assumption satisfied in this exercise), we have that $H(t)/p(t)N > 0$ implies $r(t) > 0$. If we set $r(t) > 0$, then $[1 + r(t)]f[1 + r(t)] > f[1 + r(t)] > 0$. Hence $r(t+1) > r(t)$ because $f[1 + r(t+1)] = [1 + r(t)]f[1 + r(t)]$. Thus we see that, if we start with any $r(1) > 0$, the sequence $\{r(t)\}$ generated under the assumption that an equilibrium with valued fiat currency exists is increasing. Now at every t, $f[1 + r(t)] = H(t)/p(t)N$ has a natural interpretation as the amount of time t good that the "old" at t consume in excess of their endowment. In this economy total resources are finite (actually they are constant), so $f[1 + r(t)]$ must be bounded, that is,

$$f[1+r(t)] \leq B, \quad \text{some } B > 0.$$

Because $f(\cdot)$ is an increasing function of $r(t)$, however, we have that $[1+r(t)]f[1+r(t)]$ grows without bound. Inasmuch as $f[1 + r(t+1)] = [1 + r(t)]f[1 + r(t)]$, this is a contradiction.

Notice that the "key" element in the argument is that $r(1) > 0$. Without this inequality we would not have been able to show that $\{r(t)\}$ is increasing. Yet $r(1) > 0$ is an implication of $p(1) > 0$ and the requirement that the per capita excess saving function be equal to $H(1)/p(1)N$.

In our case, we have

$$f[1+r(t)] = \frac{1}{4}\left(y - \frac{y}{1+r(t)}\right)$$

Using $1 + r(t) = p(t)/p(t+1)$ and $f[1 + r(t)] = H(0)/p(t)N$, we have

$$\frac{y}{4}[p(t) - p(t+1)] = H(0)N \quad \text{or} \quad p(t+1) = p(t) - \frac{4H(0)}{Ny}.$$

Hence for any finite $p(1) > 0$, the sequence $p(t)$ is decreasing with constant decrements of size $4H(0)/Ny$. Consequently, no matter how high $p(1)$ is, there exists a finite T such that $p(t) < 0, \forall t \geq T$. This series of statements contradicts our definition of equilibrium.

e. To analyze this form of credit limit, we solve the same problem as in (a) supplemented by the constraint $l_t^h(t)[1 + r(t)] \geq -y/4$. Clearly for the first group – the natural lenders – the new constraint is not going to be binding, and $f^h[1 + r(t)] = y/2,\ h = 1, \ldots, N/2$. For the borrowers, the constraint is binding. Recall that the "unconstrained" problem for this group gives $m_t^h(t) =$

0 and $l_t^h(t) = -y/(2[1+r(t)])$. Then no matter what $r(t)$ is, we have $[1+r(t)]l_t^h(t) = -y/2 < -y/4$. Hence borrowers will be effectively constrained, and $[1+r(t)]l_t^h(t) = -y/4$. This equation gives a new savings schedule equal to

$$f^h[1+r(t)] = \frac{-y}{4[1+r(t)]}, \qquad h = \frac{N}{2}+1, \ldots, N.$$

In an equilibrium where currency is not valued, $1/p(t) = 0 \forall t$, and the relevant market-clearing condition is (ii) in (a),

$$\frac{1}{N}\sum_{h=1}^{N} f^h[1+r(t)] \equiv f[1+r(t)] = 0.$$

Substituting the "new" functions $f^h(\cdot)$, we have

$$f[1+r(t)] = \frac{y}{4}\left\{1 - \frac{1}{2[1+r(t)]}\right\}.$$

Then the equilibrium rate is $r(t) = -1/2$. As in (b), we can compute c_t^h, given $s_t^h(t)$ and the endowment to get

$$\begin{aligned} c_t^h &= \left(\tfrac{y}{2}, \tfrac{y}{4}\right), \qquad h = 1, \ldots, \tfrac{N}{2}, \\ c_t^h &= \left(\tfrac{y}{2}, \tfrac{3y}{4}\right), \qquad h = \tfrac{N}{2}+1, \ldots, N. \end{aligned}$$

It is clear that as expected, because the interest rate decreased, borrowers are better off and lenders worse off. The welfare of the old at $t = 1$ remains unchanged, as they do not trade with any generation born at $t = 1$ or later.

f. The savings function that we derived in (e) remains unchanged. The relevant equilibrium conditions are

$$\begin{aligned} f[1+r(t)] &= H(t)/[p(t)N], \qquad t = 1, 2, \ldots, \\ 1+r(t) &= p(t)/p(t+1), \qquad t = 1, 2, \ldots \end{aligned}$$

Given $H(t) = H(0)$ and the particular form of $f(\cdot)$, the equilibrium price sequence must satisfy the difference equation

$$p(t+1) = 2p(t) - \frac{8H(0)}{Ny}.$$

One possible solution to this difference equation (for which we do not have "initial conditions") is a constant $p(t)$. Then $p(t) = p^* = 8H(0)/Ny$ is a solution. If we write the quantity equation as

$$p(t) = qH/N$$

it follows that $q = 8/y$.

It is not hard to see that, if $p(1) < p^*$, the corresponding $\{p(t)\}$ sequence cannot be an equilibrium, because there is a finite T_1 such that $\forall t \geq T_1$, $p(t) < 0$. On the other hand, if $p(1) > p^*$, we can establish – given the linearity of the difference equation – that $p(t) > p(t-1) \forall t \geq 2$ and that $p(t) \to \infty$. This is still an equilibrium with valued fiat currency, but as $H(t)/p(t)N \to 0$ in equilibrium, we must have that $f[1+r(t)] \to 0$ and consequently that the equilibrium allocation converges to the allocation of the equilibrium in which currency is not valued. For the "stationary" equilibrium $p(t) = p^*$, we compute c_t^h as in (b) to get

$$\begin{aligned} c_t^h &= \left(\tfrac{y}{2}, \tfrac{y}{2}\right), \qquad h = 1, \ldots, \tfrac{N}{2}, \\ c_t^h &= \left(\tfrac{y}{4}, \tfrac{3y}{4}\right), \qquad h = \tfrac{N}{2} + 1, \ldots, N. \end{aligned}$$

Because $1 + r(t) = p(t)/p(t+1) = p^*/p^* = 1$, we get $r(t) = 0$.

g. Lenders face the same constraints in economies (b) and (f), because the rate of return is in both cases zero, and the constraint on borrowing is not effective for them. Consequently, their welfare level must be the same.
The initial old are better off in economy (f), because their endowment is more highly valued. In economy (b) the value of their endowment of $H(0)$ is zero, whereas in economy (f) this value is $H(0)/p^* > 0$. Their consumption can therefore be higher. Because for the old the ranking according to consumption and welfare is the same, we conclude that the old at $t = 1$ are better off in (f). Finally borrowers are worse off in economy (f). They cannot be better off, because in both cases the rate of return is the same, but they face an additional constraint in economy (f), which can only shrink their choice set. [This argument depends heavily on the fact that the rate of return is the same. If that were not the case, the conclusion would not follow. For a counter-example, consider economy (e). In its environment borrowers are more constrained than in (b), but the rate of interest is sufficiently low to allow them to achieve a higher level of welfare.] To establish that borrowers in economy (f) are actually worse off, we use strict convexity and symmetry of preferences. By strict convexity we mean that if $u(x_1) = u(x_2)$, $x_1 \neq x_2$, then

$$u(x^\lambda) > u(x_1) = u(x_2),$$

where $x^\lambda = \lambda x_1 + (1 - \lambda)x_2$, $0 < \lambda < 1$.
By "symmetry" we mean

$$u(c_1, c_2) = u(c_2, c_1), \qquad \forall (c_1, c_2) > 0.$$

In economy (b) borrowers completely smooth out consumption over their life span. The consumption bundle is $c_t^h = (\hat{c}, \hat{c})$, where $\hat{c} = y/2$. In economy (f) they consume $c_t^h = (c_1, c_2)$, where $c_1 = y/4$ and $c_2 = 3y/4$. Then we have $u(c_1, c_2) = u(c_2, c_1)$. Define

$$\tilde{c}_1 = \lambda c_1 + (1 - \lambda)c_2, \qquad \tilde{c}_2 = \lambda c_2 + (1 - \lambda)c_1.$$

By strictly convexity of preferences

$$u(\tilde{c}_1, \tilde{c}_2) > u(c_1, c_2) = u(c_2, c_1)$$

for any $0 < \lambda < 1$. Choose $\lambda = 1/2$ to get $\tilde{c}_1 = \hat{c}$ and $\tilde{c}_2 = \hat{c}$. This statement completes the proof.

Exercise 8.8. *Inside Money and Real Bills*

Consider the following overlapping-generations model of two-period-lived people. At each date $t \geq 1$ there are born N_1 individuals of type 1 who are endowed with $y > 0$ units of the consumption good when they are young and zero units when they are old; there are also born N_2 individuals of type 2 who are endowed with zero units of the consumption good when they are young and $Y > 0$ units when they are old. The consumption good is nonstorable. At time $t = 1$, there are N old people, all of the same type, each endowed with zero units of the consumption good and H_0/N units of unbacked paper called "fiat currency." The populations of type 1 and 2 individuals, N_1 and N_2, remain constant for all $t \geq 1$. The young of each generation are identical in preferences and maximize the utility function $\ln c_t^h(t) + \ln c_t^h(t+1)$ where $c_t^h(s)$ is consumption in the sth period of a member h of generation t.

a. Consider the equilibrium without valued currency (that is, the equilibrium in which there is no trade between generations). Let $[1 + r(t)]$ be the gross rate of return on consumption loans. Find a formula for $[1 + r(t)]$ as a function of N_1, N_2, y, and Y.

b. Suppose that N_1, N_2, y, and Y are such that $[1 + r(t)] > 1$ in the equilibrium without valued currency. Then prove that there can exist no quantity-theory-style equilibrium where fiat currency is valued and where the price level $p(t)$ obeys the quantity theory equation $p(t) = q \cdot H_0$, where q is a positive constant and $p(t)$ is measured in units of currency per unit good.

c. Suppose that N_1, N_2, y, and Y are such that in the nonvalued-currency equilibrium, $1 + r(t) < 1$. Prove that there exists an equilibrium in which fiat currency is valued and that there obtains the quantity theory equation $p(t) = q \cdot H_0$, where q is a constant. Construct an argument to show that the equilibrium with valued currency is not Pareto superior to the nonvalued-currency equilibrium.

d. Suppose that N_1, N_2, y, and Y are such that, in the above nonvalued-currency economy, $[1 + r(t)] < 1$, so that there exists an equilibrium in which fiat currency is valued. Let $\bar{p}$ be the stationary equilibrium price level in that economy. Now consider an alternative economy, identical with the preceding one in all respects except for the following feature: a government each period purchases a constant amount L_g of consumption loans and pays for them by issuing debt on itself, called "inside money" M_I, in the amount $M_I(t) = L_g \cdot p(t)$. The government never retires the inside money, using the proceeds of the loans to finance new purchases of consumption loans in subsequent periods. The quantity of outside money, or currency, remains H_0, whereas the "total high-power money" is now $H_0 + M_I(t)$.

(i) Show that in this economy there exists a valued-currency equilibrium in which the price level is constant over time at $p(t) = \bar{p}$, or equivalently, as in the economy in (c), $p(t) = qH_0$.

(ii) Explain why government purchases of private debt are not inflationary in this economy.

(iii) In standard macroeconomic models, once-and-for-all government open-market operations in private debt normally affect real variables and/or the price level. What accounts for the difference between those models and the one in this problem?

Solution

a. The problem solved by the young of generation $t \geq 1$ is

$$\begin{array}{ll} \max_{c_t^h(t), c_t^h(t+1), s_t^h(t)} & u^h[c_t^h(t), c_t^h(t+1)] \\ \text{subject to} & c_t^h(t) + s_t^h(t) \leq w_t^h(t), \\ & c_t^h(t+1) \leq w_t^h(t+1) + [1+r(t)]s_t^h(t), \end{array}$$

where $s_t^h(t)$ is interpreted as savings, measured in time t consumption good, and $[1 + r(t)]$ is the rate of return on savings. Given any desired amount of savings, an agent has to choose the composition of his portfolio. If two assets are available – loans and currency – we have that $s_t^h(t) = l_t^h(t) + m_t^h(t)/p(t)$. Let the rate of return on consumption loans be $[1 + r(t)]$. Next period the value of the portfolio $l_t^h(t) + m_t^h(t)/p(t)$ will be $[1 + r(t)]l_t^h(t) + m_t^h(t)p(t)/[p(t+1)p(t)]$. In an equilibrium with valued currency $1/p(t) > 0$ and $m_t^h(t) > 0$ for some h; consequently, currency cannot be dominated by loans – otherwise no agent would hold currency – and both assets have the same rate of return $p(t)/p(t+1) = 1 + r(t)$. Therefore, the value of the portfolio in terms of $(t+1)$ good can be written as $[1+r(t)][l_t^h(t)+m_t^h(t)/p(t)] = [1+r(t)]s_t^h(t)$. Clearly, in an equilibrium where currency is not valued, $1/p(t) = 0$, and the same formulation in terms of $s_t^h(t)$ is applicable.

For the logarithmic utility function, the first-order conditions yield

$$s_t^h(t) = f^h[1 + r(t)] = \frac{1}{2}\left[w_t^h(t) - \frac{w_t^h(t+1)}{1 + r(t)}\right].$$

Let

$$\begin{array}{rl} f[1 + r(t)] & \equiv \frac{1}{N_1+N_2} \sum_{h=1}^{N_1+N_2} f^h[1 + r(t)] \\ & = \frac{1}{2}\left[(1-\alpha)y - \frac{\alpha Y}{1+r(t)}\right], \\ \text{where} \quad 1-\alpha & = \frac{N_1}{N_1+N_2} \quad \text{and} \quad \alpha = \frac{N_2}{N_1+N_2}. \end{array}$$

In an equilibrium in which currency is not valued, the relevant equilibrium condition is that the market for (intragenerational) loans clears, that is,

$$f[1 + r(t)] = 0.$$

Using the particular form of $f[1 + r(t)]$, we get that the unique rate of return that clears the market is given by

$$1 + r(t) = \frac{N_2 Y}{N_1 y}.$$

b. We give a proof of a more general result, namely, that no equilibrium with valued currency can exist – either quantity-theory style or not – for an even larger class of economies. To do so we assume that $f[1 + r(t)]$ is monotone increasing and continuous, an assumption that is clearly satisfied by the $f(\cdot)$ function we derived.

If the rate of return that clears the market in the nonmonetary equilibrium, say r_1, is greater than zero, we have

$$f(1 + r_1) = 0.$$

In an equilibrium such that currency is valued, we have

$$f[1 + r(t)] = \frac{H_0}{\bar{N}p(t)} > 0, \quad \text{all } t, \quad \text{where } \bar{N} = N_1 + N_2.$$

Given

$$1 + r(t) = \frac{p(t)}{p(t+1)},$$

the implication is that

$$f[1 + r(t+1)] = [1 + r(t)]f[1 + r(t)], \qquad r(t) > r_1 > 0.$$

Then notice that $r(1) > r_1$ implies $f[1+r(2)] > f[1+r(1)]$, which in turn implies $r(2) > r(1)$. Proceeding in this manner, we establish that the sequence of rates of return that can potentially be equilibrium rates of return is monotone increasing. Then it either converges to some $\bar{r}$ or diverges. We now prove that it cannot converge. By continuity of $f(\cdot)$, we have that if $r(t) \to \bar{r}$ then $f[1 + r(t)] \to f(1 + \bar{r})$. Then as t goes to infinity, we have

$$f(1 + \bar{r}) = (1 + \bar{r})f(1 + \bar{r}).$$

Yet $\bar{r} \geq r(t) > r_1 > 0$, which yields a contradiction. Therefore the sequence $\{r(t)\}$ goes to infinity. But this contradicts the assumption that an equilibrium exists, because $f[1 + r(t)]$ is bounded by, say, total endowments, whereas $[1 + r(t)]f[1 + r(t)]$ is going to infinity as $t \to \infty$. Consequently, we cannot have equality for all t.

For the example at hand, we can show this result by simply establishing that no matter how large $p(1)$ is, the sequence $\{p(t)\}$ generated by the equilibrium condition contains negative terms for all $t \geq T$, T finite.

The condition $f[1 + r(t)] = H_0/\bar{N}p(t)$ corresponds to $(1/2)\{(1 - \alpha)y - \alpha Y/[1 + r(t)]\} = H_0/\bar{N}p(t)$, whereas $1 + r(t) > 1$ in a nonvalued-currency equilibrium is simply $N_2Y/N_1y > 1$. Using $1 + r(t) = p(t)/p(t + 1)$, we get

$$(1 - \alpha)yp(t) - \alpha Yp(t + 1) = \frac{2H_0}{\bar{N}}$$

or

$$p(t + 1) = \frac{N_1y}{N_2Y}p(t) - \frac{2H_0}{N_2Y}$$

Let

$$\frac{N_1y}{N_2Y} = \phi.$$

Then

$$p(t+1) = \phi^t p(1) - \frac{2H_0}{N_2 Y} \sum_{j=0}^{t-1} \phi^j.$$

Because $0 < \phi < 1$, however, for t large the first term becomes negligible, whereas the second term converges to $-2H_0/[N_2 Y(1-\phi)]$, a negative number. Therefore, for large t, $p(t)$ must be negative.

c. As in (b), we can first show that this result obtains in greater generality. Assume, as before, that $f(\cdot)$ is increasing and continuous. We are given that

$$f(1+r_1) = 0 \quad \text{and} \quad r_1 < 0.$$

By continuity and monotonicity, we have that $f(1) > 0$. Then let $p(t) = \bar{p} = H_0/f(1)\bar{N}$. It is easy to verify that the conditions for existence of an equilibrium with valued currency are satisfied by construction. For the example, we want a constant solution to

$$p(t+1) = \frac{N_1 y}{N_2 Y} p(t) - \frac{2H_0}{N_2 Y}$$

where

$$\frac{N_1 y}{N_2 Y} > 1.$$

Such a solution is

$$p(t) = p = \frac{2H_0}{N_1 y - N_2 Y} > 0, \quad \text{and} \quad 1 + r(t) = 1.$$

To argue that the equilibrium with valued currency does not Pareto dominate the equilibrium where currency is valueless, it suffices to show that at least one agent is worse off. We do so for the type 2 agents, that is, for borrowers. The basic idea is that when the rate of interest increases – and the rate of return in the equilibrium with valued currency is higher than the rate of return in the equilibrium without valued currency – borrowers are worse off.
Given a rate of interest r, we say that an agent is a borrower if argmax $u[w_1 - s, w_2 + (1+r)s] < 0$. Let s_i be the solution to the maximization problem when the rate of interest is r_i. Assume that $r_1 > r_2$ and that s_1 and s_2 are negative (the agent is a borrower). Then

$$u[w_1 - s_1, w_2 + (1+r_2)s_1] \leq u[w_1 - s_2, w_2 + (1+r_2)s_2],$$

given that the optimal choice is s_2 when $r = r_2$. Still, $u(w_1 - s_1, w_2 + [1+r_2]s_1) < u[w_1 - s_1, w_2 + (1+r_2)s_1]$, given $s_1 < 0$, $r_1 > r_2$, and monotonicity of $u(\cdot)$. This statement proves the proposition. In terms of the example, pick an agent of type 2 endowed $(0, Y)$. This agent will always borrow, and the logarithmic utility function is monotone increasing. Consequently the previous result applies.

d. i. The basic argument that explains this sort of irrelevance result is that the gross composition of individual portfolios is not determinate. Consider, for example, an individual who, at the going rate of return, wants to "save" ten dollars. This agent should be indifferent among portfolios that consist of $(10 + l)$ assets or loans acquired and l debts or loans granted, for any l. This result implies that, if the government wants to increase its demand for "loans," private agents will be willing to supply those "loans" and simultaneously to "borrow" in another market – the market for currency, for example – so that their net asset position remains unchanged.

Given the government policy, the "new" per capita aggregate demand is $L_g/N + f[1 + r(t)]$. Equilibrium requires that excess demand for loans, that is, excess savings, be equal to net supply of assets, which is $M_I(t)/p(t) + H/p(t)$. Let $\bar{p}$ be the price we found in (c), that is,

$$f(1) = \frac{H}{N\bar{p}}.$$

If $p(t) = \bar{p}$ is an equilibrium, we must have $M_I(t) = M_I = Lg\bar{p}$, and

$$\frac{L_g}{N} + f(1) = \frac{H}{N\bar{p}} + \frac{M_I}{N\bar{p}}.$$

This is the case if and only if $L_g = M_I/\bar{p}$ – exactly the condition we are given. Hence the conjecture is verified.

ii. Notice that in this setup it is a little bit arbitrary to call M_I money. We can think of M_I as liabilities of a financial intermediary that are fully backed by real loans. In this sense there is no creation of new currency, because the "new" asset inherits all the characteristics of the real loans by which it is backed. Therefore such a trivial operation – an exchange of names – cannot have any effects.

iii. In standard macro models we usually assume that money and bonds are "different" assets, different enough for us to start out with well-defined demands for each. In the model we are analyzing, we have a well-defined demand for savings but *not* for each asset that can potentially be part of those savings. Actually, in our model, in an equilibrium with valued currency, agents are completely indifferent between portfolios that contain only bonds and portfolios that contain only currency, because the rate of return is the same.

We could alter our model in at least two ways to obtain well-defined demands for each asset. First, we could suppose that, even though bonds have a higher return, currency is held because of some legal restrictions. Second, in the absence of legal restrictions there are perhaps sources of demand other than the stream of goods that assets can buy. Currency-in-the-utility-function and cash-in-advance models are examples of such approaches.

It seems possible to imagine a transactions technology that makes money and bonds "different." This technology must be rich enough to rule out private intermediation that produces "moneylike" assets backed by bonds. In some sense, these situations can be interpreted either as some form of legal restriction or as arising from the assumption that the government and the private sector have different technologies for producing "currencylike" assets. In either case, we would also have obtained nonneutrality of an open-market operation.

Exercise 8.9. *Social Security and the Price Level*

Consider an economy ("economy I") that consists of overlapping generations of two-period-lived people. At each date $t \geq 1$ there are born a constant number N of young people, who desire to consume both when they are young, at t, and when they are old, at $(t+1)$. Each young person has the utility function $\ln c_t(t) + \ln c_t(t+1)$, where $c_s(t)$ is time t consumption of an agent born at s. For all dates $t \geq 1$, young people are endowed with $y > 0$ units of a single nonstorable consumption good when they are young and zero units when they are old. In addition, at time $t = 1$ there are N old people endowed in the aggregate with H units of unbacked fiat currency. Let $p(t)$ be the nominal price level at t, denominated in dollars per time t good.

a. Define and compute an equilibrium with valued fiat currency for this economy. Argue that it exists and is unique. Now consider a second economy ("economy II") that is identical to the above economy except that economy II possesses a social security system. In particular, at each date $t \geq 1$, the government taxes $\tau > 0$ units of the time t consumption good away from each young person and at the same time gives τ units of the time t consumption good to each old person then alive.

b. Does economy II possess an equilibrium with valued fiat currency? Describe the restrictions on the parameter τ, if any, that are needed to ensure the existence of such an equilibrium.

c. If an equilibrium with valued fiat currency exists, is it unique?

d. Consider the *stationary* equilibrium with valued fiat currency. Is it unique? Describe how the value of currency or price level would vary across economies with differences in the size of the social security system, as measured by τ.

Solution

a. We first define an equilibrium with valued currency as a pair of sequences $\{p(t)\}_{t=1}^{\infty}$ and $\{r(t)\}_{t=1}^{\infty}$, with $p(t) > 0$ and finite, all t; and an allocation $c = \{c_t^h\}_{t=1}^{\infty}$ such that

(i) $1 + r(t) = p(t)/p(t+1)$.

(ii) Given $\{p(t)\}$ and $\{r(t)\}$, each agent h of generation t chooses savings, $s_t^h(t)$, and lifetime consumption $c_t^h = [c_t^h(t), c_t^h(t+1)]$ to solve

$$\begin{array}{ll} \max_{c_t^h(t), c_t^h(t+1), s_t^h(t)} & u^h[c_t^h(t), c_t^h(t+1)] \\ \text{subject to} & c_t^h(t) + s_t^h(t) \leq w_t^h(t), \\ & c_t^h(t+1) \leq w_t^h(t+1) + [1 + r(t)] s_t^h(t). \end{array}$$

Denote

$$f^h[1 + r(t)] \equiv w_t^h(t) - c_t^h(t).$$

(iii) Market clearing requires that

$$\frac{1}{N} \sum_{h=1}^{N} f^h[1 + r(t)] \equiv f[1 + r(t)] = \frac{H(t)}{p(t)N}, \qquad t = 1, 2, \ldots$$

Notice that condition (i) requires that neither currency nor private loans dominate the other in rate of return. This is a consequence of utility maximization. If an agent holds both loans and currency, then the two must have equal rates of return. If not, the budget set can clearly be enlarged by holding only one asset, namely the one with the higher rate of return.

It is easy to derive $f^h(\cdot)$ for the Cobb-Douglas utility function. It turns out to be

$$f^h[1+r(t)] = \frac{1}{2}\left[w_t^h(t) - \frac{w_t^h(t+1)}{1+r(t)}\right].$$

For economy I, we have $w_t^h(t) = y$ and $w_t^h(t+1) = 0$. Then

$$f[1+r(t)] = \frac{y}{2}.$$

Still, (iii) requires $f[1+r(t)] = H/p(t)N$. Hence the unique solution is $p(t) = 2H/Ny$, and $r(t) = 0$.

This condition establishes existence – because $p(t) > 0$ – and uniqueness.

b. To compute an equilibrium for economy II, we note that this second eco nomy shares the characteristics of economy I except for one, namely the endowment pattern. In this economy, part of the first-period endowment is taxed away by the government. The effective or disposable first-period endowment is then $w_t^h(t) = y - \tau$. On the other hand, agents receive a transfer in their second period of life that can formally be considered an endowment. Consequently we set $w_t^h(t+1) = \tau$.

The definition of equilibrium remains unchanged. For this "new" economy, we can compute that $f[1+r(t)] = (1/2)(y - \tau - \tau/[1+r(t)])$.

We know that a valued-currency equilibrium with a gross rate of interest strictly greater than one cannot exist. To see this point, suppose that $1+r(t) = 1+r_1 > 1$. We have $p(t)/p(t+1) = 1+r(t)$, however, or $p(t+1) = (1+r_1)^{-1}p(t)$. Hence prices decrease in this economy. On the other hand, aggregate real currency balances $H/p(t)$, are growing without bound. [It is easy to see that they grow exponentially at the rate $(1+r_1)$.] This statement violates condition (iii) in the definition of equilibrium, because the left-hand side – savings – is bounded by the level of first-period endowments. Then it is clear that, if moving τ affects the "admissible" values $r(t)$, we will probably have to rule out some τ to guarantee the existence of an equilibrium.

In any equilibrium with valued currency, $f[1+r(t)] > 0$. For this economy that condition is

$$\frac{1}{2}\left[(y-\tau) - \frac{\tau}{1+r(t)}\right] > 0,$$

or equivalently

$$1 + r(t) > \frac{\tau}{y-\tau}.$$

We have argued, however, that in an equilibrium with valued currency the rate of return cannot be bounded below by a number strictly greater than one. Therefore, we cannot have $\tau/(y-\tau) > 1$. This situation requires that $\tau \leq y/2$. The case

$\tau = y/2$ can be handled similarly, as we now see. Notice that, if $1 + r(t) = 1$, then $f^h[1 + r(t)] = 0$. Hence we need $1 + r(t) > 1$. We also have to rule out the possibility of a sequence of terms $r(t)$ that has strictly positive components but that converges to zero. To do so, we show that if $r(1) > 0$ – a necessary condition for $H/p(1) > 0$ – the sequence $r(t)$ goes to infinity. We have that

$$\begin{aligned} \tfrac{y}{2}\left[1 - \tfrac{1}{1+r(t)}\right] &= \tfrac{H}{p(t)N} = \tfrac{H}{p(t+1)N}[1 + r(t)] \\ &= \tfrac{y}{2[1+r(t)]}\left[1 - \tfrac{1}{1+r(t+1)}\right]. \end{aligned}$$

Rearranging terms we get

$$r(t+1) = \frac{r(t)}{1 - r(t)}.$$

It is easy to see that, if $r(1) > 0$, then $r(2) > r(1)$. The sequence $\{r(t)\}$ is increasing (actually it diverges, but we do not need such a strong result) and hence $\{1 + r(t)\}$ cannot converge to one.

To recapitulate, we learned that if $\tau \geq y/2$ there cannot be an equilibrium with valued fiat currency. Now it is simple to establish that for any $\tau < y/2$ there exists at least one such equilibrium. We prove this claim by simply constructing one. Notice that if $\tau < y/2$, then $f(1) > 0$. Pick $p(t) = \bar{p} = f(1)^{-1}H/N$, and this is an equilibrium.

The restrictions on the parameters of the economy – basically the relative size of first- and second-period endowments – that are needed for an equilibrium with valued fiat currency to exist have a natural economic interpretation. Recall that condition (iii) of the definition of equilibrium requires that the economy save a positive amount. Moreover we know that aggregate average savings must occur at rates of return that are less than or equal to one. Young individuals – the only potential savers – will save positive amounts only to increase their consumption in the second period of their life. If their lifetime endowments are tilted toward their second period (in other words, if τ is large), then at "low" interest rates there will be no positive excess savings and hence no valued fiat currency.

This interpretation readily suggests that a model to generate nonexistence of an equilibrium with valued fiat money can be devised by increasing the number of borrowers to the point where the "average" agent does not want to have positive savings at rates of return smaller than or equal to one.

c. In this section, we argue that there are many equilibria with valued currency. The conditions for existence are

$$\begin{aligned} \tfrac{1}{2}\left(w_1 - \tfrac{w_2}{1+r(t)}\right) &= \tfrac{H}{p(t)N}, \\ 1 + r(t) &= \tfrac{p(t)}{p(t+1)}, \qquad p(t) > 0, \quad \text{all } t, \end{aligned}$$

where $w_1 = y - \tau$, $w_2 = \tau$, and $w_1 > w_2$, because $\tau < y/2$. Making the appropriate substitutions, we can reduce the condition for existence to finding a solution to the following difference equation

$$p(t)w_1 - w_2 p(t+1) = \frac{2H}{N} \quad \text{or} \quad p(t+1) = \frac{w_1}{w_2}p(t) - \frac{2H}{Nw_2}.$$

Notice that we are not given an initial condition. Therefore many different functions mapping the positive integers into real numbers can be solutions of that difference equation. One of them is precisely that found in (b), that is, a constant price.

$$p(t) = \bar{p} = \frac{2H}{N(w_1 - w_2)}.$$

We can now parameterize the set of solutions by the initial value $p(1)$. Basically, we care about two sets: (1) the set of paths $\{p(t)\}$ such that $p(1) < \bar{p}$ and (2) the set of paths $\{p(t)\}$ such that $p(1) > \bar{p}$. We want to argue that no element of the first set can be an equilibrium and that any element of the second is, that is, that there is a continuum of equilibria.

First, given the definition of $\bar{p}$, we can rearrange the difference equation to read

$$p(t+1) - \bar{p} = \frac{w_1}{w_2}[p(t) - \bar{p}].$$

This is a first-order homogeneous difference equation in the variable $Z_t \equiv p(t) - \bar{p}$. For any initial condition Z_1, the solution is

$$Z_{t+1} = \left(\frac{w_1}{w_2}\right)^t Z_1, \qquad \frac{w_1}{w_2} > 1.$$

Then if $Z_1 < 0$ – that is, $p(1) < \bar{p} - Z_t$ decreases without bound, and for some finite T, $Z_t < -\bar{p}$ for all $t \geq T$. The implication is that $p(t) < 0$, which contradicts the definition of equilibrium.

If we pick $p(1)$ in the second set, $Z_1 = p(1) - \bar{p} > 0$. Then as t grows, Z_t also grows, that is, $p(t)$ diverges. Yet this *is* an equilibrium. Nothing prevents prices from going to infinity in our definition of equilibrium with valued currency. Clearly, "real" currency balances, $H/p(t)$, are converging to zero. The rate of return is also converging to the rate of return of the nonvalued-currency equilibrium, and the consumption allocation converges to the equilibrium allocation of the economy without currency. The equilibrium is not "stationary," in the sense that the allocations depend on time even in a purely stationary physical endowment.

d. We have already argued that there is only one equilibrium with constant rate of return, namely the one that obtains when $p(t) = \bar{p}$, all t.

Now if we analyze different economies indexed by τ, it is clear from our findings in (b) that, if $\tau_1 > \tau_2$, which corresponds to $w_1^1 < w_1^2$ and $w_2^1 > w_2^2$, we have

$$\bar{p}_1 = \frac{2H}{N(w_1^1 - w_2^1)} > \bar{p}_2 = \frac{2H}{N(w_1^2 - w_2^2)}.$$

One interpretation of this result is that, the more "important" the social security system ($\tau_1 > \tau_2$), the less important are private savings as a way of providing consumption in the second period of life. Consequently the value of those savings must be smaller ($H/\bar{p}_1 < H/\bar{p}_2$).

For these economies a social security system is a perfect substitute for private savings in the sense that per capita second-period consumption is $y/2$ regardless of τ.

Exercise 8.10. *Seignorage*

Consider an economy consisting of overlapping generations of two-period-lived agents. At each date $t \geq 1$, there are born N_1 "lenders" who are endowed with $\alpha > 0$ units of the single consumption good when they are young and zero units when they are old. At each date $t \geq 1$, there are also born N_2 "borrowers" who are endowed with zero units of the consumption good when they are young and $\beta > 0$ units when they are old. The good is nonstorable, and N_1 and N_2 are constant through time. The economy starts at time 1, at which time there are N old people who are in the aggregate endowed with $H(0)$ units of unbacked, intrinsically worthless pieces of paper called dollars. Assume that α, β, N_1, and N_2 are such that

$$\frac{N_2 \beta}{N_1 \alpha} < 1.$$

Assume that everyone has preferences

$$u[c_t^h(t), c_t^h(t+1)] = \ln c_t^h(t) + \ln c_t^h(t+1),$$

where $c_t^h(s)$ is consumption of time s good of agent h born at time t.

a. Compute the equilibrium interest rate on consumption loans in the equilibrium without valued currency.

b. Construct a *brief* argument to establish whether or not the equilibrium without valued currency is Pareto optimal.

The economy also contains a government that purchases and destroys G_t units of the good in period t, $t \geq 1$. The government finances its purchases entirely by currency creation. That is, at time t,

$$G_t = \frac{H(t) - H(t-1)}{p(t)},$$

where $[H(t) - H(t-1)]$ is the additional dollars printed by the government at t and $p(t)$ is the price level at t. The government is assumed to increase $H(t)$ according to

$$H(t) = zH(t-1), \qquad z \geq 1,$$

where z is a constant for all time $t \geq 1$.

At time t, old people who carried over $H(t-1)$ dollars between $(t-1)$ and t offer these $H(t-1)$ dollars in exchange for time t goods. Also at t the government offers $H(t) - H(t-1)$ dollars for goods, so that $H(t)$ is the total supply of dollars at time t, to be carried over by the young into time $(t+1)$.

c. Assume that $1/z > N_2\beta/N_1\alpha$. Show that under this assumption there exists a continuum of equilibria with valued currency.

d. Display the unique stationary equilibrium with valued currency in the form of a "quantity theory" equation. Compute the equilibrium rate of return on currency and consumption loans.

e. Argue that if $1/z < N_2\beta/N_1\alpha$, then there exists no valued-currency equilibrium. Interpret this result. (Hint: Look at the rate of return on consumption loans in the equilibrium without valued currency.)

f. Find the value of z that *maximizes* the government's G_t in a stationary equilibrium. Compare this with the largest value of z that is compatible with the existence of a valued-currency equilibrium.

Solution

a. Given the logarithmic structure of preferences, it is easy to show that the solution to the problem

$$\begin{array}{ll} \max_{c_t^h(t),c_t^h(t+1),s_t^h(t)} & u^h[c_t^h(t), c_t^h(t+1)] \\ \text{subject to} & c_t^h(t) + s_t^h(t) \leq w_t^h(t), \\ & c_t^h(t+1) \leq w_t^h(t+1) + [1+r(t)]s_t^h(t) \end{array}$$

is a savings function of the form

$$s_t^h(t) \equiv f^h[1+r(t)] = \frac{1}{2}\left[w_t^h(t) - \frac{w_t^h(t+1)}{1+r(t)}\right]$$

where, as usual, savings is to be understood as the sum of loans $l_t^h(t)$ and "real" currency holdings $m_t^h(t)/p(t)$. It has also been proved in the text that, if $m_t^h(t) > 0$, then $p(t)/p(t+1) = 1+r(t)$, and hence there is no loss of generality in assuming a single rate of return on savings.
In an equilibrium without valued currency, $1/p(t) = 0$, all t. Then an equilibrium is a sequence $\{r(t)\}$, $t = 1, 2, \ldots$, and an allocation $[\{c_{t-1}^h(t)\}, h = 1, \ldots, N, t = 1, 2, \ldots]$ such that

$$f[1+r(t)] = \frac{1}{N_1+N_2}\sum_{h=1}^{N_1+N_2} f^h[1+r(t)] = 0.$$

In this environment there are N_1 agents with savings function $\alpha/2$, and N_2 agents with savings function $-\beta/2[1+r(t)]$. Denoting $k = N_1/(N_1+N_2)$, average aggregate savings are

$$f[1+r(t)] = \frac{1}{2}\left[k\alpha - \frac{(1-k)\beta}{1+r(t)}\right], \quad \text{and} \quad f[1+r(t)] = 0$$

implies

$$1+r(t) = \frac{(1-k)\beta}{k\alpha} = \frac{N_2\beta}{N_1\alpha} < 1.$$

b. We want to argue that, because the rate of return is lower than the rate of growth, the equilibrium allocation is not Pareto optimal. To prove this point it suffices to display another allocation that is Pareto superior, that is, that is feasible and increases the utility level of at least one agent in the economy without decreasing the utility of others.

A typical "lender" consumes $c^L = (\alpha/2, N_2\beta\alpha/N_1\alpha 2)$, whereas a "borrower" consumes $c^B = (\beta N_1\alpha/2N_2\beta, \beta/2)$. The new allocation that we are going to construct gives borrowers exactly the same lifetime consumption as the competitive equilibrium. In the competitive allocation, lenders' total consumption is

$$N_1\frac{\alpha}{2} + N_1\left(\frac{N_2\beta}{N_1\alpha}\frac{\alpha}{2}\right) = \frac{1}{2}(N_1\alpha + N_2\beta).$$

An allocation that is stationary (all generations indexed $t \geq 1$ get the same lifetime consumption) and treats all lenders symmetrically must satisfy

$$N_1\hat{c}_1 + N_1\hat{c}_2 \leq \frac{1}{2}(N_1\alpha + N_2\beta).$$

In particular, we can write $\hat{c}_1 = c_1^L - \delta$ and $\hat{c}_2 = c_2^L + \delta$. This guarantees that feasibility is satisfied. The utility derived from the competitive bundle is $u(c_1^L, c_2^L)$. The maximal utility that is consistent with feasibility and keeping the consumption of the initial old at least at the original level can be obtained by setting δ so that

$$\delta = \operatorname*{argmax}_{\delta \geq 0}\ u(c_1^L - \delta, c_2^L + \delta).$$

If the solution is $\delta > 0$, strict quasi-concavity of $u(\cdot)$ implies that $u(\hat{c}_1, \hat{c}_2) > u(c_1^L, c_2^L)$. That the solution is indeed positive can be established from the first-order condition of the maximization problem that requires

$$\frac{u_2(c_1^L - \delta, c_2^L + \delta)}{u_1(c_1^L - \delta, c_2^L + \delta)} = 1.$$

In a competitive equilibrium

$$\frac{u_2(c_1^L, c_2^L)}{u_1(c_1^L, c_2^L)} = \frac{1}{1 + r(t)} = \frac{N_1\alpha}{N_2\beta} > 1.$$

Still,

$$\frac{u_2(c_1^L - \delta, c_2^L + \delta)}{u_1(c_1^L - \delta, c_2^L + \delta)} \to 0 \quad \text{as} \quad \delta \to c_1^L$$

and goes to infinity as $\delta \to -c_2^L$. Moreover, u_2/u_1 is monotone decreasing. Then if

$$\frac{u_2(c_1^L - \delta, c_2^L + \delta)}{u_1(c_1^L - \delta, c_2^L + \delta)} = 1,$$

it must be that $\delta > 0$. This conclusion shows that the "new" allocation makes all lenders better off, increases the consumption of the old at $t = 1$, and gives the borrowers the same consumption that they get in the competitive equilibrium. Consequently we have found an allocation that is Pareto superior (although *not* Pareto optimal) to the competitive allocation. Therefore the latter cannot be Pareto optimal.

c. An equilibrium with valued fiat currency is a pair of sequences $\{r(t)\}$ and $\{p(t)\}$, $p(t) > 0$, all t; and an allocation $\left[\{c_{t-1}^h(t)\}, h = 1, \ldots, N_1 + N_2, t = 1, 2, \ldots\right]$

such that

$$\begin{aligned} f[1+r(t)] &= \tfrac{H(t)}{p(t)(N_1+N_2)}, \quad t=1,2,\dots && (1)\\ 1+r(t) &= \tfrac{p(t)}{p(t+1)}. && (2) \end{aligned}$$

We can reduce the two equations to

$$\frac{1}{2}\left[k\alpha-(1-k)\beta\frac{p(t+1)}{p(t)}\right]=\frac{H(t)}{p(t)(N_1+N_2)}$$

or

$$p(t+1)=\frac{k\alpha}{(1-k)\beta}p(t)-\frac{2H(0)}{(1-k)\beta}\frac{z^t}{(N_1+N_2)}.$$

Define

$$\hat{p}(t)=\frac{p(t)}{z^t}.$$

Then

$$\hat{p}(t+1)=\frac{N_1\alpha}{N_2\beta z}\hat{p}(t)-\frac{2H(0)}{(1-k)\beta z(N_1+N_2)}. \tag{3}$$

Notice that there is a one-to-one correspondence between solutions $\{\hat{p}(t)\}$ and $\{p(t)\}$. In particular, if for some parameter values $\hat{p}(t)<0$ for some t, for those values $p(t)<0$ in the same set of values t.

Assume that $1/z > N_2\beta/N_1\alpha$. Then define

$$\bar{p}=\frac{2H(0)}{N_1\alpha-N_2\beta z}.$$

Clearly, $\hat{p}(t)=\bar{p}$ is a solution to (3), which means that $p(t)=z^t\bar{p}$ is an equilibrium price sequence. By simple iteration it follows that, if $\hat{p}(1)>\bar{p}$, then the sequence $\{\hat{p}(t)\}$ is positive and diverges. Correspondingly, the sequence $\{p(t)\}$, with initial condition $p(1)=\hat{p}(1)z$ and given by $p(t)=z^t\hat{p}(t)$, also diverges, but this in no way contradicts our definition of equilibrium. In this equilibrium, "real" currency balances $H(t)/p(t)$ are converging to zero. The rate of return on loans and the consumption allocation converge to the values we computed in (a).

d. We have already done much of the work. Notice that an equilibrium is stationary if the consumption allocation does not depend on time. For the environment of this exercise, such an equilibrium requires that the interest rate be constant. In such an equilibrium

$$f[1+r(t)]=f(1+r)=\frac{H(t)}{p(t)}, \quad \text{all } t.$$

Then

$$\frac{H(t)}{p(t)}=\frac{H(t+1)}{p(t+1)}=\frac{zH(t)}{p(t+1)}.$$

This equality requires $p(t+1)=zp(t)$ and $1+r=p(t)/p(t+1)=1/z$. Recall, however, that we have already found an equilibrium where prices grow at the rate z. This is given by $p(t)=z^t\bar{p}$. This is the unique path $\{p(t)\}$ that satisfies the difference equation and gives $1+r(t)=1/z$, all t.

e. If $1/z < N_2\beta/N_1\alpha$, we can write (3) as

$$\hat{p}(t+1) = \phi\hat{p}(t) - \frac{2H(0)}{N_2\beta z}, \quad \phi = N_1\alpha/N_2\beta z < 1.$$

Iterating on this equation, we get

$$\hat{p}(t) = \phi^{t-1}\hat{p}(1) - \frac{2H(0)}{N_2\beta z}\frac{1-\phi^{t-1}}{1-\phi}.$$

Then, because $0 < \phi < 1$, it is clear that, no matter how high $\hat{p}(1)$ is, there exists a finite T such that for every $t \geq T$, $\hat{p}(t) < 0$. This statement in turn implies that $p(t) < 0$, which contradicts the definition of equilibrium. This nonexistence result clearly puts a bound to z, that is, currency supply cannot grow too fast. It shows that, for an equilibrium with valued fiat currency to exist, it is necessary that the "stationary rate of return" $1/z$ be greater than the rate of return that obtains when currency is not valued, $N_2\beta/N_1\alpha$. The result illustrated here holds for this class of models, namely that, if an equilibrium with valued currency exists, then the rate of return is greater than the rate that clears the market for loans when currency has no value.

f. In this section, we compare different stationary equilibria. In any of these equilibria, average real currency balances are constant, and we can write

$$\begin{aligned}\tfrac{G(z)}{N} &= \tfrac{H(t+1)-H(t)}{Np(t+1)} \\ &= \tfrac{H(t+1)}{Np(t+1)} - \tfrac{p(t)}{p(t+1)}\tfrac{H(t)}{Np(t)} = f\left(\tfrac{1}{z}\right)\left(1-\tfrac{1}{z}\right).\end{aligned}$$

For the economy of this exercise, we have

$$\frac{G(z)}{N} = \frac{1}{2}[k\alpha - (1-k)\beta z]\left(1 - \frac{1}{z}\right).$$

It is clear that, if $G(z) > 0$, we need $z > 1$. On the other hand, to guarantee existence of an equilibrium, $z < N_1\alpha/N_2\beta$. The value of z that maximizes G solves

$$\max_{1<z<N_1\alpha/N_2\beta} \frac{1}{2}[k\alpha - (1-k)\beta z]\left(1 - \frac{1}{z}\right).$$

This is a concave program. The solution is given by any z within the feasible set that satisfies the first-order condition. Such a z is $(N_1\alpha/N_2\beta)^{\frac{1}{2}}$. Notice that the value of z, or steady-state inflation, that maximizes government revenue from inflation is not the largest z for which an equilibrium with valued currency exists. The underlying intuition is simple: this largest feasible z maximizes the rate at which real currency balances are taxed. A higher z, however, reduces real money balances or the "base" of the inflation tax. The optimal choice balances the effects of the higher tax rate against the lower base on which the inflation tax is levied.

Exercise 8.11. *Unpleasant monetarist arithmetic*

Consider an economy in which the aggregate demand for government currency for $t \geq 1$ is given by $[M(t)p(t)]^d = g[R_1(t)]$, where $R_1(t)$ is the gross rate of return on currency between t and $(t+1)$, $M(t)$ is the stock of currency at t, and $p(t)$ is the value of currency in terms of goods at t (the reciprocal of the price level). The function $g(R)$ satisfies

$$
\begin{array}{ll}
g(R)(1-R) = h(R) > 0 & \text{for } \ R \in (\underline{R}, 1), \\
h(R) \leq 0 \quad \text{for } \ R < \underline{R}, & R \geq 1, \qquad \underline{R} > 0. \\
h'(R) < 0 & \text{for } \ R > R_m \\
h'(R) > 0 & \text{for } \ R < R_m \\
h(R_m) > D, & \text{where } D \text{ is a positive number to be defined shortly.}
\end{array}
$$

The government faces an infinitely elastic demand for its interest-bearing bonds at a constant-over-time gross rate of return $R_2 > 1$. The government finances a budget deficit D, defined as government purchases minus explicit taxes, that is constant over time. The government's budget constraint is

$$
D = p(t)[M(t) - M(t-1)] + B(t) - B(t-1)R_2, \qquad t \geq 1 \tag{1}
$$

subject to $B(0) = 0, M(0) > 0$. In equilibrium,

$$
M(t)p(t) = g[R_1(t)] \tag{2}
$$

The government is free to choose paths of $M(t)$ and $B(t)$, subject to equations (1) and (2).

a. Prove that, for $B(t) = 0$, for all $t > 0$, there exist two stationary equilibria for this model.

b. Show that there exist values of $B > 0$, such that there exist stationary equilibria with $B(t) = B$, $M(t)p(t) = Mp$.

c. Prove a version of the following proposition: among stationary equilibria, the lower the value of B, the lower the stationary rate of inflation consistent with equilibrium. (You will have to make an assumption about Laffer curve effects to obtain such a proposition.)

This problem displays some of the ideas used by Sargent and Wallace (1981). They argue that, under assumptions like those leading to the proposition stated in part c, the "looser" money is today [that is, the higher $M(1)$ and the lower $B(1)$], the lower the stationary inflation rate.

Solution

Let's recall the properties of the function $h(R)$. Those property are illustrated in figure 8.5 in the book.

Solutions of $h(R) = 0$: The function has two zeros, $\underline{R} < 1$ and 1. It is non negative for $R \in [\underline{R}, 1]$ and negative otherwise.

Maximum: The function is increasing for $R < R_m$ and decreasing for $R > R_m$. Clearly, R_m achieves its maximum.

In the economy we consider, the government finances its deficit either by printing money $M_t - M_{t-1}$, or by issuing interest bearing bonds $B(t)$.

a. Assume first that the government does not issue bond. Let's call $R_{t-1} \equiv \frac{p_t}{p_{t-1}}$ the rate of return on currency. The government budget constraint can be conveniently written:

$$D = p(t)M(t) - \frac{p(t)}{p(t-1)}p(t-1)M(t-1)$$

Now use the equilibrium condition on the market for money:

$$D = g(R_t) - R_{t-1}g(R_{t-1})$$

In a stationary equilibrium, the return on money is constant:

$$D = g(R)(1-R) = h(R)$$

$h(R)$ is the amount of real resources raised by the government by printing money when inflation is constant and equal to $\frac{1}{R}$. It is a "Laffer curve". For high inflation $\frac{1}{\underline{R}}$, nobody is willing to hold money and the government does not raise any inflation tax. Also, when there is no inflation $R = 1$, then the government obviously does not raise any inflation tax. And there is an "optimal" rate of inflation $1 < \frac{1}{R_m} < \frac{1}{\underline{R}}$ which maximizes government revenue.
$h(R)$ increases from 0 to $h(R_m) > D$ for $\underline{R} < R < R_m$, so there is a unique stationary equilibrium $R_{low} \in [\underline{R}, R_m]$. Similarly, $h(R)$ decreases from $h(R_m) > D$ to 0 for $R_m < R < 1$, so that there is a unique stationary equilibrium $R_{high} \in [R_m, 1]$. Since $h(R)$ is negative otherwise, those are the only stationary equilibria.

b. If $B(t) = B$, the government budget constraint can be rewritten:

$$D + (R_2 - 1)B = g(R)(1-R) = h(R)$$

Deficit is augmented by constant interest payment on government debt. The reasoning of part (a) applies with D being replaced by $D + (R_2 - 1)B$. So there are two stationary equilibria provided $D + (R_2 - 1)B < h(R_m)$.
c. We know that there are two stationary equilibria, one associated with a l except that now R_{high} belongs to the decreasing side of $h(R)$. Take $B' > B$. We have

$$h(R'_{high}) = D + (R_2 - 1)B' > D + (R_2 - 1)B > h(R_{high})$$

So that $h(R'_{low}) > h(R_{low})$. But this time h is decreasing in the range we consider, so this implies that $R'_{high} < R_{high}$.
The unpleasant monetarist arithmetic is thus associated with the low interest rate equilibrium.

Exercise 8.12. *Grandmont-Hall*

Consider a nonstochastic, one-good overlapping-generations model consisting of two-period-lived young people born in each $t \geq 1$ and an initial group of old people at $t = 1$ who are endowed with $H(0) > 0$ units of unbacked currency at the beginning of period 1. The one good in the model is not storable. Let the aggregate first-period saving function of the young be time invariant and be denoted $f[1+r(t)]$ where $[1+r(t)]$ is the gross rate of return on consumption loans between t and $(t+1)$. The saving function is assumed to satisfy $f(0) = -\infty$, $f'(1+r) > 0$, $f(1) > 0$.
Let the government pay interest on currency, starting in period 2 (to holders of currency between periods 1 and 2). The government pays interest on currency at a nominal rate of $[1+r(t)]p(t+1)/\bar{p}$, where $[1+r(t)]$ is the real gross rate of return on consumption loans, $p(t)$ is the price level at t, and $\bar{p}$ is a target price level chosen to satisfy

$$\bar{p} = H(0)/f(1). \tag{1}$$

The government finances its interest payments by printing new money, so that the government's budget constraint is:

$$H(t+1) - H(t) = \left\{[1+r(t)]\frac{p(t+1)}{\bar{p}} - 1\right\}H(t), \qquad t \geq 1, \tag{2}$$

given $H(1) = H(0) > 0$. The gross rate of return on consumption loans in this economy is $1 + r(t)$. In equilibrium, we have that $[1 + r(t)]$ must be at least as great as the real rate of return on currency

$$[1+r(t)]p(t)/\bar{p} = [1+r(t)]\frac{p(t+1)}{\bar{p}}\frac{p(t)}{p(t+1)}$$

w with equality if currency is valued,

$$1 + r(t) \geq [1+r(t)]p(t)/\bar{p}, \qquad 0 < p(t) < \infty. \tag{3}$$

The loan market-clearing condition in this economy is

$$f[1+r(t)] = H(t)/p(t). \tag{4}$$

a. Define an equilibrium.

b. Prove that there exists a unique monetary equilibrium in this economy and compute it.

Solution

a. We define an equilibrium as sequences $\{r(t)\}$, $\{p(t)\}$, and $\{H(t)\}$ and an allocation associated with the savings function $f(\cdot)$ such that

$$\begin{aligned}
&H(t+1) = [1+r(t)]\tfrac{p(t+1)}{\bar{p}}H(t), \qquad t \geq 1, \quad (1)\\
&\quad H(1) = H(0) > 0\\
&f[1+r(t)] = \tfrac{H(t)}{p(t)} \qquad (2)\\
&1+r(t) \geq [1+r(t)]\tfrac{p(t)}{\bar{p}} \qquad (3)\\
\text{and} \quad &\left\{1+r(t) - [1+r(t)]\tfrac{p(t)}{\bar{p}}\right\}\tfrac{H(t)}{p(t)} = 0.
\end{aligned}$$

b. We want to argue that the unique equilibrium is given by $H(t) = H(0)$, $p(t) = \bar{p}$, and $r(t) = 0$, all t. That this is in fact an equilibrium can be verified by checking conditions (1)–(3). Given $1 + r(t) = 1$ and $p(t) = \bar{p}$, (1) implies that $H(t) = H(1) = H(0)$, all t. Because $f(1) > 0$, we have $f(1) = H(0)/\bar{p} > 0$. Finally condition (3) is satisfied with equality.
To prove that the equilibrium just discussed is unique within the class of valued-currency equilibria, notice that, in any such equilibrium, (3) must be met with equality. The implication is that $p(t) = \bar{p}$, all $t \geq 1$. At $t = 1$, we have from (2)

$$f[1+r(1)] = \frac{H(1)}{\bar{p}} = \frac{H(0)}{\bar{p}} = f(1)$$

or

$$r(1) = 0.$$

Hence

$$H(2) = [1+r(1)]H(1) = H(0),$$

and consequently

$$r(2) = r(1) = 0.$$

Iterating upon this argument, it follows that $r(t) = 0$, all t, which establishes uniqueness.

Exercise 8.13. *Bryant-Keynes-Wallace*

Consider an economy consisting of overlapping generations of two-period-lived agents. There is a constant population of N young agents born at each date $t \geq 1$. There is a single consumption good that is not storable. Each agent born in $t \geq 1$ is endowed with w_1 units of the consumption good when young and with w_2 units when old, where $0 < w_2 < w_1$. Each agent born at $t \geq 1$ has identical preferences $\ln c_t^h(t) + \ln c_t^h(t+1)$, where $c_t^h(s)$ is time s consumption of agent h born at time t. In addition, at time 1, there are alive N old people who are endowed with $H(0)$ units of unbacked paper currency and who want to maximize their consumption of the time 1 good.
A government attempts to finance a constant level of government purchases $G(t) = G > 0$ for $t \geq 1$ by printing new base money. The government's budget constraint is

$$G = [H(t) - H(t-1)]/p(t),$$

where $p(t)$ is the price level at t, and $H(t)$ is the stock of currency carried over from t to $(t+1)$ by agents born in t. Let $g = G/N$ be government purchases per young person. Assume that purchases $G(t)$ yield no utility to private agents. **a.**

Define a stationary equilibrium with valued fiat currency.

b. Prove that, for g sufficiently small, there exists a stationary equilibrium with valued fiat currency.

c. Prove that, in general, if there exists one stationary equilibrium with valued fiat currency, with rate of return on currency $1 + r(t) = 1 + r_1$, then there exists at least one other stationary equilibrium with valued currency with $1 + r(t) = 1 + r_2 \neq 1 + r_1$.

d. Tell whether the equilibria described in (b) and (c) are Pareto optimal, among those allocations that allocate among private agents what is left after the government takes $G(t) = G$ each period. (A proof is not required here: an informal argument will suffice.)
Now let the government institute a forced saving program of the following form. At time 1, the government redeems the outstanding stock of currency $H(0)$, exchanging it for government bonds. For $t \geq 1$, the government offers each young consumer the option of saving at least F worth of time t goods in the form of bonds bearing a constant rate of return $(1 + r_2)$. A legal prohibition against private intermediation is instituted that prevents two or more private agents from sharing one of these bonds. The government's budget constraint for $t \geq 2$ is

$$G/N = B(t) - B(t-1)(1 + r_2),$$

where $B(t) \geq F$. Here $B(t)$ is the saving of a young agent at t. At time $t = 1$, the government's budget constraint is

$$G/N = B(1) - \frac{H(0)}{Np(1)},$$

where $p(1)$ is the price level at which the initial currency stock is redeemed at $t = 1$. The government sets F and r_2.
Consider stationary equilibria with $B(t) = B$ for $t \geq 1$ and r_2 and F constant.

e. Prove that if g is small enough for an equilibrium of type (a) to exist, then a stationary equilibrium with forced saving exists. (Either a graphic argument or an algebraic argument is sufficient.)

f. Given g, find the values of F and r_2 that maximize the utility of a representative young agent for $t \geq 1$.

g. Is the equilibrium allocation associated with the values of F and $(1+r_2)$ found in (f) optimal among those allocations that give $G(t) = G$ to the government for all $t \geq 1$? (Here an informal argument will suffice.)

Solution

a. Consider the problem faced by agent h of generation $t \geq 1$.

$$\begin{array}{ll} \max_{c_t^h(t), c_t^h(t+1), s_t^h(t)} & u^h[c_t^h(t), c_t^h(t+1)] \\ \text{subject to} & c_t^h(t) + s_t^h(t) \leq w_t^h(t), \\ & c_t^h(t+1) \leq w_t^h(t+1) + [1+r(t)]s_t^h(t), \end{array}$$

where $s_t^h(t)$ is interpreted as savings at time t, and $1+r(t)$ is the rate of return on savings. The solution to this maximization problem is a function $s_t^h(t) = f^h[1+r(t)]$.

In this economy two assets can be used to transfer wealth between the first and second period of life, namely privately issued bonds and currency. Because there is no randomness, an arbitrage argument establishes that, if currency is held, $1+r(t) = p(t)/p(t+1)$.

We can now define equilibrium with valued fiat currency. It is a set of sequences $[\{r(t)\}, \{p(t)\}, \{H(t)\}]$ and an allocation $\{c_{t-1}^h(t)\}$, $h = 1, \ldots, N$, and $t \geq 1$, such that

(i) $1 + r(t) = p(t)/p(t+1)$ all t. (Both assets are held.)

(ii) $G = [H(t) - H(t-1)]/p(t)$. (The government budget constraint is satisfied.)

(iii) $\sum_{h=1}^N f^h[1+r(t)] \equiv Nf[1+r(t)] = H(t)/p(t)$, all $t \geq 1$. (This condition incorporates utility maximization and imposes market clearing.)

(iv) $c_t^h(t) = w_t^h(t) - f^h[1+r(t)]$
$c_t^h(t+1) = w_t^h(t+1) + [1+r(t)]f^h[1+r(t)]$
for all $h = 1, \ldots, N$ and all $t \geq 1$.

Consumption of the old at $t = 1$ is given by the value of the currency they hold, $H(0)$, plus whatever endowment they have.

We say that an equilibrium is stationary if $c_t^h(t) = c_1^h$ and $c_t^h(t+1) = c_2^h$, all $t \geq 1$. In this particular setup, in which there is no heterogeneity, consumption will not be indexed by h. Moreover, given the logarithmic utility function, it is easy to show that the individual and the average aggregate savings function take the form

$$f[1+r(t)] = f^h[1+r(t)] = \frac{1}{2}\left[w_1 - \frac{w_2}{1+r(t)}\right].$$

In our setup stationary equilibria are necessarily associated with constant interest rates. This is the type of equilibrium we seek.

b. In any stationary equilibrium we have that

$$f(1+r) = \frac{H(t)}{Np(t)}.$$

Then $H(t)/p(t)$ must be constant. The government budget constraint requires that

$$\begin{array}{ll} eg & = \frac{H(t)}{Np(t)} - \frac{p(t-1)}{p(t)} \frac{H(t-1)}{Np(t-1)} \\ \text{or} & \\ g & = f(1+r) - (1+r)f(1+r). \end{array}$$

We want to claim that, if there exists a rate of return $(1+r)$ such that $f(1+r) > 0$ and $1 + r < 1$, then there exists a range of g values that can be financed. To see this point, let $(1 + r)$ satisfy the assumptions. Because $(1 + r) < 1$, we have

$$(1 + r)f(1 + r) < f(1 + r).$$

Define $g = f(1 + r) - (1 + r)f(1 + r)$, and we have our result. For our particular economy it is easy to see that for any $w_2/w_1 < 1+r < 1$, $f(1+r) > 0$. Moreover, any g given by

$$0 < g = f(1 + r) - (1 + r)f(1 + r), \qquad \frac{w_2}{w_1} < 1 + r < 1$$

can be financed.
To sum up, to describe the set of g that is feasible to finance we can follow these steps: first pick any gross interest rate in the interval $(w_2/w_1, 1)$. Then compute $f(1+r)-(1+r)f(1+r)$. By construction, this quantity is positive. Set g equal to this value. Then the set of feasible g corresponds to the image of that expression for values of $(1 + r)$ in the interval $(w_2/w_1, 1)$. The key step in demonstrating existence is to establish that the interest rate at which aggregate savings equal zero (in this case w_2/w_1) is less than one.

c. This problem can be posed in the following alternative way: fix any g in the feasible interval. Then, in general, there are at least two r's, r_1 and r_2, such that

$$g = f(1 + r_1) - (1 + r_1)f(1 + r_1) = f(1 + r_2) - (1 + r_2)f(1 + r_2).$$

Let

$$h(1 + r) = f(1 + r) - (1 + r)f(1 + r).$$

We know $h(w_2/w_1) = h(1) = 0$. If $f(\cdot)$ is continuous and differentiable, these properties are inherited by $h(\cdot)$. Now if it can be established that $h'(w_2/w_1) > 0$ and $h'(1) < 0$, then it follows that any $g = h(1 + r)$ can be financed with at least two r's except possibly for $\bar{g} = \max_r h(1 + r)$. Now

$$h'(1 + r) = f'(1 + r) - (1 + r)f'(1 + r) - f(1 + r).$$

It follows that $f(1) > 0$ implies that $h'(1) < 0$. Also $h'(w_2/w_1) = (1-w_2/w_1)f'(w_2/w_1)-f(w_2/w_1) = (1 - w_2/w_1)f'(w_2/w_1)$, because $f(w_2/w_1) = 0$. Thus, whenever savings increase with the rate of interest – a condition satisfied for the savings function derived from the logarithmic utility – we get the desired result.

d. We want to argue that the equilibrium is not Pareto optimal. To do so we show that there exist feasible allocations that improve the welfare of at least one individual, without reducing the utility level of the others. We first note that any allocation consistent with the government purchasing $G(t) = G > 0$ each period satisfies

$$\sum_{h=1}^{N} c_t^h(t) + \sum_{h=1}^{N} c_{t-1}^h(t) \leq Nw_2 + Nw_1 - G.$$

Any feasible allocation that treats all agents symmetrically and is stationary – in the sense of (b) – satisfies

$$c_1 + c_2 \leq w_1 + w_2 - g.$$

The allocation corresponding to a stationary equilibrium with valued currency $(\hat{c}_1, \hat{c}_2)$ also satisfies that constraint. Let (c_1^*, c_2^*) be the solution to

$$\begin{array}{ll} \max & u(c_1, c_2) \\ \text{subject to} & c_1 + c_2 \leq w_1 + w_2 - g, \qquad c_1 \geq 0,\ c_2 \geq 0. \end{array}$$

Such an allocation is feasible. We want to claim that it is Pareto superior to $(\hat{c}_1, \hat{c}_2)$. Suppose that $c_1^* > 0$ and $c_2^* > 0$. Then (c_1^*, c_2^*) satisfy

$$\frac{u_1(c_1^*, c_2^*)}{u_2(c_1^*, c_2^*)} = 1.$$

Denote

$$v(c_1, c_2) \equiv \frac{u_1(c_1, c_2)}{u_2(c_1, c_2)}.$$

We make the following assumptions about $v(c_1, c_2)$:

(i) $\forall\ c_2 > 0\ \lim_{c_1 \to 0} v(c_1, c_2) = \infty$

(ii) $\forall\ c_1 > 0\ \lim_{c_2 \to 0} v(c_1, c_2) = 0$

(iii) $v_1 < 0,\ v_2 > 0$.

The assumptions are satisfied by the logarithmic utility function. Recall that in a stationary equilibrium, utility maximization requires (for an interior solution) that $v(c_1, c_2) = (1 + r)$. Because $g > 0$ implies that $w_2/w_1 < (1 + r) < 1$, it follows that

$$v(\hat{c}_1, \hat{c}_2) < v(c_1^*, c_2^*) = 1.$$

Given that the feasibility constraint is satisfied with equality for both allocations, we can write

$$v(y - \hat{c}_2, \hat{c}_2) < v(y - c_2^*, c_2^*),$$

where

$$y \equiv w_1 + w_2 - g.$$

Our assumptions on $v(\cdot)$ imply that the above inequality holds if and only if $\hat{c}_2 < c_2^*$. Then $c_1^* < \hat{c}_1$, and the old at $t = 1$ consume – on a per capita basis – $y - c_1^* > y - \hat{c}_1$. Consequently the old are better off. We now have to argue that the young born at $t \geq 1$ are not worse off, but this follows by construction, because $(\hat{c}_1, \hat{c}_2)$ satisfies the feasibility constraint and (c_1^*, c_2^*) is chosen to maximize utility subject to that constraint. Therefore

$$u(c_1^*, c_2^*) \geq u(\hat{c}_1, \hat{c}_2).$$

e. Consider the competitive problem faced by the young born at $t \geq 1$. Notice that the legal restriction permits no borrowing and lending among agents of the same generation. If that were not the case, a given agent would be able to share a bond by issuing private IOUs. The choice problem can be formulated as

$$\begin{array}{ll} \max & u^h[c_t^h(t), c_t^h(t+1)] \\ \text{subject to} & c_t^h(t) + b_t^h(t) \leq w_t^h(t), \\ & c_t^h(t+1) \leq w_t^h(t+1) + (1+r_2)b_t^h(t) \end{array}$$

$F \leq b_t^h(t)$ and $b_t^h(t) = 0$ if $b_t^h(t) < F$.
The optimal decision rule gives $b_t^h(t)$ as a function of $(1+r_2)$ and F. Let

$$b(1+r;F) \equiv \frac{1}{N}\sum_{h=1}^{N} b_t^h(t)$$

be the aggregate demand for bonds.
We can define a stationary equilibrium as a sequence $\{B(t)\}$, an allocation $\{c_{t-1}^h(t)\}$, and a vector of numbers $[F, r_2, p(1)]$ such that

$$\frac{G}{N} = B(1) - \frac{H(0)}{Np(1)}, \qquad \frac{G}{N} = B(t) - B(t-1)(1+r_2), \qquad t \geq 2 \tag{1}$$

where $B(t) = b(1+r_2, F)$

$$\begin{array}{ll} c_t^h(t) & = w_t^h(t) - b_t^h(t), \qquad t \geq 1, \quad (2) \\ c_t^h(t+1) & = w_t^h(t+1) + (1+r_2)b_t^h(t). \end{array}$$

Notice that, because r_2 and F are constant, $B(t) = B$, all t, and $B = b(1+r_2, F)$. Then to show existence we need to find numbers F, N_2, g such that, $b(1+r_2, F)$ is given by

$$b(1+r_2, F) \equiv \underset{b}{\operatorname{argmax}}\ u(w_1 - b, w_2 + (1+r_2)b)$$

subject to $b \geq F$ and $b = 0$ if $b < F$,
g satisfies the government budget constraint, that is,

$$g = -r_2 b(1+r_2, F) \qquad \text{and finally} \qquad -1 < r_2 < 0.$$

Given such values, $g = b - (1+r_2)b > 0$ and $H(0)/Np(1) = (1+r_2)b$ determines $p(1)$. The problem is to establish existence of the vector (r_2, F, g). This can be done analytically, but the argument turns out to be complicated. Instead we give a diagrammatic proof.
We fix g at the same level as in (a). [Clearly, we could pick $F = f(1+r_1)$ and $1+r_2 = 1+r_1$ and trivially we have existence.] In Figure 7.2, the (a) equilibrium is shown. The budget constraint is represented by the line that passes through the endowment point w. Tangency of that line [which has slope $-(1+r_1)^{-1}$] and an indifference curve occurs at the point A. Notice that, for this to be an equilibrium, the resulting allocation $(\hat{c}_1, \hat{c}_2)$ has to satisfy market clearing, that is, $\hat{c}_1 + \hat{c}_2 \leq w_1 + w_2 - g$; therefore, the tangency occurs on the feasibility locus denoted by TT'.
Consider now point B on TT', which also allows the government to consume g per capita. That this point exists and lies southwest of A is a key element of the argument. Notice that at A the indifference curve has slope $-1/(1+r_1) < -1$, making it steeper than TT'. Moreover, as they have been drawn, the indifference

curves are very flat near the axis. Therefore the same indifference curve $\hat{u}$ must cross also the locus TT' at a point like that labeled A'. Strict convexity implies that the slope at A' is, in absolute value, less than one. These two facts guarantee the existence of an indifference curve that yields a level of utility $u^* > \hat{u}$, which is tangent to TT' somewhere between A and A'.

If we set $F = w_1 - c_1^*$ and choose r_2 such that the slope of the line BD is $-(1+r_2)^{-1}$, the budget constraint faced by a representative agent is the shaded area c_1^*BD and the endowment point. Utility maximization occurs at B.

Notice that we apparently have some freedom in choosing r_2 in the sense that many different values would have resulted in the same choice of the same utility-maximizing bundle (c_1^*, c_2^*). This freedom is illusory, because it must also be true that $g = -r_2 F$, that is, $g = -r_2(w_1 - c_1^*)$, or

$$r_2 = \frac{g}{c_1^* - w_1}.$$

This equality satisfies $-1 < r_2 < 0$, because $c_1^* - w_1 < 0$ and $c_1^* + g < w_1$ as $c_2^* > w_2$.

f. We now find F and r_2 for our economy. First we want to determine (c_1^*, c_2^*). We have already argued in (d) what program this vector solves, namely,

$$\begin{array}{ll} \max_{c_1,c_2} & \ln c_1 + \ln c_2 \\ \text{subject to} & c_1 + c_2 \leq w_1 + w_2 - g. \end{array}$$

The solution is $c_1^* = (w_1 + w_2 - g)/2$, $c_2^* = (w_1 + w_2 - g)/2$. We choose

$$F^* = w_1 - c_1^* = \frac{w_1 - w_2 + g}{2} \quad \text{and} \quad r_2^* = \frac{2g}{w_2} - w_1 - g.$$

Then we verify that (c_1^*, c_2^*) is also the solution to

$$\text{argmax}\left\{\max_{[c_1,c_2]}[\ln c_1 + \ln c_2], \ln w_1 + \ln w_2\right\},$$

where the maximization inside the braces is subject to

$$c_1 + b \leq w_1, \qquad c_2 \leq w_2 + (1 + r_2^*)b, \qquad b \geq F^*.$$

To verify this point, notice that c_1^*, c_2^* are the optimizing values, given the less restrictive constraint for the previous programming problem. Then if they are feasible, they must also be the solution to this problem. Feasibility is guaranteed by construction. The last step is to make sure that the utility is higher than in autarky, that is,

$$2\ln(w_1 + w_2 - g) - 2\ln 2 \geq \ln w_1 + \ln w_2.$$

This requirement is clearly met for small g.

g. We can show directly that the allocation (c_1^*, c_2^*) would also be the equilibrium allocation for another economy with endowment patterns $\tilde{w}_1 = (w_1 + w_2 - g)/2$ and $\tilde{w}_2 = (w_1 + w_2 - g)/2$, which when we use the Balasko-Shell criterion is Pareto optimal (the gross interest rate is one). In terms of *feasible allocations* (though *not* in terms of individual endowments), however, both economies are identical. Because optimality is defined only in terms of preferences and aggregate endowments – or feasibility constraints – it must be the case that the allocation for our original economy is also Pareto optimal.

CHAPTER 9

Ricardian equivalence

CHAPTER 10

Asset pricing

Exercise 10.1. *Hansen-Jagannathan bounds*

Consider the following annual data for annual gross returns on U.S. stocks and U.S. Treasury bills from 1890 to 1979. These are the data used by Mehra and Prescott. The mean returns are $\mu = \begin{bmatrix} 1.07 & 1.02 \end{bmatrix}$ and the covariance matrix of returns is $\begin{bmatrix} 0274 & .00104 \\ .00104 & .00308 \end{bmatrix}$.

a. For data on the excess return of stocks over bonds, compute Hansen and Jagannathan's bound on the stochastic discount factor y. Plot the bound for $E(y)$ on the interval $[.9, 1.02]$.

b. Using data on both returns, compute and plot the bound for $E(y)$ on the interval $[.9, 1.02]$. Plot this bound on the same figure as you used in part a.

c. On the textbook's web page `ftp://zia.stanford.edu/pub/sargent/webdocs/matlab`, there is a Matlab file `epdata.m` with Kydland and Prescott's time series. The series epdata(:,4) is the annual growth rate of aggregate consumption c_t/c_{t-1}. Assume that $\beta = .99$ and that $m_t = \beta u'(c_t)/u'(c_{t-1})$, where $u(\cdot)$ is the CRRA utility function. For the three values of $\gamma = 0, 5, 10$, compute the standard deviation and mean of m_t and plot them on the same figure as in part b. What do you infer from where the points lie?

Solution

The matlab program associated with this exercise is .

a. Denote the excess return by z. Following the text, the Hansen-Jagannathan bound is given by

$$\sigma(y) \geq \left(\frac{E(z)}{\sigma(z)} \right) E(y).$$

From the data we compute $E(z) = 1.07 - 1.02 = 0.05$ and $\sigma^2(z) = 0.0274 + 0.00308 - 2 * 0.00104$. This gives a HJ bound of 0.297. Figure 10.1 plots the bound on $\sigma(y)$ for $E(y) \in [0.9, 1.02]$. the bound in the straight line.

b. For returns, we compute the bounds using

$$\begin{aligned} b &= [cov(x,x)]^{-1}[1 - E(y)E(x)] \\ \sigma(y) &\geq \sqrt{b' cov(x,x) b}. \end{aligned}$$

Here, x is the vector containing both returns on stocks and bonds. Figure 10.1 plots the bound on $\sigma(y)$ for $E(y) \in [0.9, 1.02]$. The bound is the convex shaped curve.

c. From the annual consumption growth data we find that the mean and standard deviation of the discount factor are given by

$$\begin{aligned} E(y) &= \frac{1}{T}\sum_{t=0}^{T}\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} \\ \sigma(y) &= \frac{1}{T^2-1}\sum_{t=0}^{T}\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}-E(y)\right]^2. \end{aligned}$$

We find for $\gamma = 0$, $E(y) = 1, \sigma(y) = 0$. for $\gamma = 5$, $E(y) = 0.9307, \sigma(y) = 0.1739$ and for for $\gamma = 10$, $E(y) = 0.8961, \sigma(y) = 0.3635$. These three points are plotted into 10.

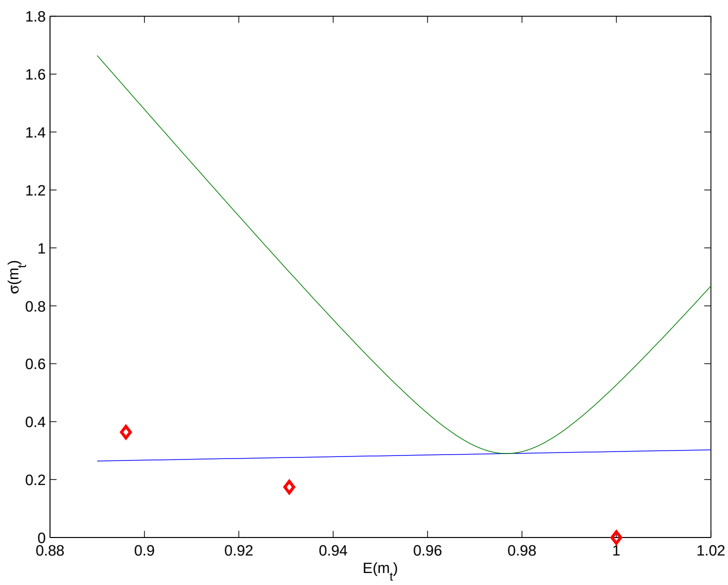


FIGURE 1. Exercise 10.1 : Hansen-Jagannathan bounds

Exercise 10.2. *The term structure and regime switching, donated by Rodolfo Manuelli*

Consider a pure exchange economy where the stochastic process for consumption is given by,

$$c_{t+1} = c_t \exp[\alpha_0 - \alpha_1 s_t + \varepsilon_{t+1}],$$

where

(i) $\alpha_0 > 0$, $\alpha_1 > 0$, and $\alpha_0 - \alpha_1 > 0$.

(ii) ε_t is a sequence of i.i.d. random variables distributed $N(\mu, \tau^2)$. Note: Given this specification, it follows that $E[e^{\varepsilon}] = \exp[\mu + \tau^2/2]$.

(iii) s_t is a Markov process independent from ε_t that can take only two values, $\{0,1\}$. The transition probability matrix is completely summarized by

$$\begin{aligned}\text{Prob}[s_{t+1} &= 1|s_t = 1] = \pi(1),\\ \text{Prob}[s_{t+1} &= 0|s_t = 0] = \pi(0).\end{aligned}$$

(iv) The information set at time t,Ω_t, contains $\{c_{t-j}, s_{t-j}, \varepsilon_{t-j}; j \geq 0\}$.

There is a large number of individuals with the following utility function

$$U = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $u(c) = c^{(1-\sigma)}/(1-\sigma)$. Assume that $\sigma > 0$ and $0 < \beta < 1$. As usual, $\sigma = 1$ corresponds to the log utility function.

a. Compute the "short-term" (one-period) interest rate.

b. Compute the "long-term" (two-period) interest rate measured in the same time units as the rate you computed in a. (That is, take the appropriate square root.)

c. Note that the log of the rate of growth of consumption is given by

$$\log(c_{t+1}) - \log(c_t) = \alpha_0 - \alpha_1 s_t + \varepsilon_{t+1}.$$

Thus, the conditional expectation of this growth rate is just $\alpha_0 - \alpha_1 s_t + \mu$. Note that when $s_t = 0$, growth is high and, when $s_t = 1$, growth is low. Thus, loosely speaking, we can identify $s_t = 0$ with the peak of the cycle (or good times) and $s_t = 1$ with the trough of the cycle (or bad times). Assume $\mu > 0$. Go as far as you can describing the implications of this model for the cyclical behavior of the term structure of interest rates.

d. Are short term rates pro- or countercyclical?

e. Are long rates pro- or countercyclical? If you cannot give a definite answer to this question, find conditions under which they are either pro- or countercyclical, and interpret your conditions in terms of the "permanence" (you get to define this) of the cycle.

Solution

a. We use the formula derived in chapter 10. Specifically:

$$\frac{1}{R_{1t}} = E_t\left(\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}\right) \tag{92}$$

$$\frac{1}{R_{1t}} = \beta\exp(-\sigma\alpha_0 + \sigma\alpha_1 s_t)E_t\left(\exp(-\sigma\varepsilon_{t+1})\right) \tag{93}$$

$$\frac{1}{R_{1t}} = \beta\exp(-\sigma\alpha_0 + \sigma\alpha_1 s_t - \sigma\mu + \sigma^2\tau^2/2), \tag{94}$$

where the last equality follows from the fact that $-\sigma\varepsilon_{t+1}$ is normal with mean $-\sigma\mu$ and variance $\sigma^2\tau^2$.

b. Using again the formula of chapter 10 we have :

$$
\begin{aligned}
(95)\frac{1}{R_{2t}^2} &= E_t\left(\beta^2\left(\frac{c_{t+2}}{c_{t+1}}\frac{c_{t+1}}{c_t}\right)^{-\sigma}\right)\\
(96)\frac{1}{R_{2t}^2} &= \beta^2 E_t\left(\exp(-\sigma\alpha_0+\sigma\alpha_1 s_{t+1}+\sigma\varepsilon_{t+2})\right)\exp(-\sigma\alpha_0+\sigma\alpha_1 s_t+\sigma\varepsilon_{t+1})\\
(97)\frac{1}{R_{2t}^2} &= \beta^2\exp\left(2(-\sigma\alpha_0-\sigma\mu+\sigma^2\tau^2/2)\right)E_t\left(\exp(\sigma\alpha_1(s_t+s_{t+1}))\right).
\end{aligned}
$$

Observe that either $s_{t+1}=s_t$, or $s_{t+1}=1-s_t$. Therefore, we can write:

$$E_t\left(\exp(\sigma\alpha_1 s_{t+1})\right)=\exp(\sigma\alpha_1 s_t)\times\left[\pi(s_t|s_t)+\pi(1-s_t|s_t)\exp(\sigma\alpha_1(1-2s_t))\right].$$

This yields to the following two expressions for the long rate :

$$
\begin{aligned}
(98)\qquad \frac{1}{R_{2t}} &= \frac{1}{R_{1t}}\left[\pi(s_t|s_t)+\pi(1-s_t|s_t)\exp(\sigma\alpha_1(1-2s_t))\right]^{1/2}\\
(99)\qquad \frac{1}{R_{2t}} &= \beta\exp(-\sigma\alpha_0-\sigma\mu+\sigma^2\tau^2/2)\\
(100)\qquad &\quad\times\left[\pi(s_t|s_t)\exp(2\sigma\alpha_1 s_t)+\pi(1-s_t|s_t)\exp(\sigma\alpha_1)\right]^{1/2}.
\end{aligned}
$$

c.,d. and e. Equation (98) implies that, at the peak $s_t=0$, the long rate is smaller than the sort rate : the term structure of interest rates is downwards slopping. The intuition goes as follows. In two periods, there is a positive probability of low growth. Therefore, "long term consumption" is relatively scarcer than "short term consumption". Its price should be higher. In other words, the long term interest rate is lower than the short term interest rate.
Conversely, at the trough $s_t=1$, the long term interest rate is higher than the short term interest rate : the term structure of interest rates is upwards slopping.

Short term interest rates are low when $s_t=1$ (trough) and high when $s_t=0$ (peak). Again, this is because when $s_t=1$, the growth rate of consumption is low. Tomorrow's good is relatively scarcer than if $s_t=1$. Therefore, tomorrow's good should have a higher price when $s_t=1$ than when $s_t=0$. In other words, the short term interest rate is low at a trough and high at a peak. In this precise sense, the short term interest rate is procyclical

Examination of equation (100) shows that long term interest rate is procyclical. Also, procyclicality is stronger if $\pi(s_t|s_t)$ is closer to one, i.e. if shocks are persistent.

Exercise 10.3. *Growth slowdowns and stock market crashes, donated by Rodolfo Manuelli*

Consider a simple one-tree pure exchange economy. The only source of consumption is the fruit that grows on the tree. This fruit is called dividends by the tribe inhabiting this island. The stochastic process for dividend d_t is described as follows: If d_t is not equal to d_{t-1}, then $d_{t+1} = \gamma d_t$ with probability π, and $d_{t+1} = d_t$ with probability $(1-\pi)$. If in any pair of periods j and $j+1$, $d_j = d_{j+1}$, then for all $t > j$, $d_t = d_j$. In words, the process – if not stopped – grows at a rate γ in every period. However, once it stops growing for one period, it remains constant forever on. Let d_0 equal one. Preferences over stochastic processes for consumption are given by

$$U = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $u(c) = c^{(1-\sigma)}/(1-\sigma)$. Assume that $\sigma > 0$, $0 < \beta < 1$, $\gamma > 1$, and $\beta\gamma^{(1-\sigma)} < 1$.

a. Define a competitive equilibrium in which shares to this tree are traded.

b. Display the equilibrium process for the price of shares in this tree p_t as a function of the history of dividends. Is the price process a Markov process in the sense that it depends just on the last period's dividends?

c. Let T be the first time in which $d_{T-1} = d_T = \gamma^{(T-1)}$. Is $p_{T-1} > p_T$? Show conditions under which this is true. What is the economic intuition for this result? What does it say about stock market declines or crashes?

d. If this model is correct, what does it say about the behavior of the aggregate value of the stock market in economies that switched from high to low growth (e.g., Japan)?

Solution

a. First define the household's problem :

$$\max_{\{c_t(d^t)\}} \sum_{t\geq 0} \sum_{d^t} \beta^t u\left(c_t(d^t)\right), \tag{101}$$

subject to $c_t(d^t) + s_t(d^t)p_t(d^t) = s_t(d^t)d_t + s_{t-1}(d^{t-1})p_t(d^t)$ and with $s_{-1} = 1$. Observe that we assume that the tree price is "cum-dividend".

DEFINITION 16. *An equilibrium is an allocation $\{c_t(d^t), s_t(d^t)\}_{t\geq 0}$ and a price process $\{p_t(d^t)\}_{t\geq 0}$ such that:*

(i) *Optimality: given price, the allocation solves the household's problem*

(ii) *Feasibility: markets clear, i.e. $c_t(d^t) \leq d_t$ for all d^t.*

Let's derive the first order conditions of the household problem. Attach multiplier $\mu_t(d^t)$ to node d^t budget constraint. The first order conditions are :

$$\begin{array}{lllll} c_t(d^t): & \beta^t\pi(d^t)u'(c_t(d^t)) & = & \mu_t(d^t) \\ s_t(d^t): & \mu_t(d^t)p_t(d^t) & = & d_t + \sum_{d_{t+1}} \mu_{t+1}(d^t, d_{t+1})p_{t+1}(d^t, d_{t+1}). \end{array}$$

Substituting the first equation into the second one gives the familiar Euler equation:

$$p_t = d_t + \beta E_t\left(\frac{u'(c_{t+1})}{u'(c_t)}p_{t+1}\right). \tag{102}$$

Imposing market clearing $c_t = d_t$ gives the pricing formula :

$$p_t = d_t + \beta E_t\left(\frac{u'(d_{t+1})}{u'(d_t)}p_{t+1}\right). \tag{103}$$

b. We guess and verify that the price of the tree is of the form $p_i d_t$, where $i = g$ if the growth process is not stopped and $i = s$ if the growth process is stopped.

If the growth process is stopped, then $c_{t+k} = d_t$ for all $k \geq 0$. Therefore $\beta^k \frac{u'(c_{t+k})}{u'(c_t)} = \beta^k$ and the (cum dividend) price of the tree is $\frac{d_t}{1-\beta}$. Thus :

$$p_s = 1/(1-\beta).$$

If the growth process is not stopped then two things can happen tomorrow. First, with probability π the economy grows. In this event $c_{t+1}/c_t = \gamma$ and the dividend of the tree is $d_{t+1} = \gamma d_t$. Second, with probability $1-\pi$ the economy stops growing. In this event $c_{t+1}/c_t = 1$ and the dividend of the tree is $d_{t+1} = d_t$. Thus, the (cum dividend) price of the tree at time t is

$$p_g d_t = d_t + \beta\left(\pi\gamma^{-\sigma}p_g\gamma d_t + (1-\pi)d_t/(1-\beta)\right).$$

Solving for p_g gives :

$$p_g = \left(1-\beta\pi\gamma^{1-\sigma}\right)^{-1}\left[1+\frac{\beta(1-\pi)}{1-\beta}\right]. \tag{104}$$

The price of the tree is Markov provided we expand the state to (d_t, d_{t-1}). Specifically, if $d_t = d_{t-1}$, then the price is $p_s d_t$. If $d_t \neq d_{t-1}$, then $d_t = p_g d_t$.

c. and d. In term of our notations, we need to find conditions under which $p_g > p_s$. Using the above expressions shows that the inequality is equivalent to $\gamma > 1$. The value of the aggregate stock market is the value of a claim to the economy output. In the event of a growth slowdown, the economy is expected to grow at a lower rate and, thus, the value of the stock market declines from $p_g d_t$ to $p_s d_t$. This can be interpreted as a stock market crash.

Exercise 10.4. *The term structure and consumption, donated by Rodolfo Manuelli*

Consider an economy populated by a large number of identical households. The (common) utility function is

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $0 < \beta < 1$, and $u(x) = x^{1-\theta}/(1-\theta)$, for some $\theta > 0$. (If $\theta = 1$, the utility is logarithmic.) Each household owns one tree. Thus, the number of households and trees coincide. The amount of consumption that grows in a tree satisfies

$$c_{t+1} = c^* c_t^{\varphi} \varepsilon_{t+1},$$

where $0 < \varphi < 1$, and ε_t is a sequence of i.i.d. log normal random variables with mean one, and variance σ^2. Assume that, in addition to shares in trees, in this economy bonds of all maturities are traded.

a. Define a competitive equilibrium.

b. Go as far as you can calculating the term structure of interest rates, $\tilde{R}_{jt}$, for $j = 1, 2, \ldots$.

c. Economist A argues that economic theory predicts that the variance of the log of short-term interest rates (say one-period) is always lower than the variance of long-term interest rates, because short rates are "riskier." Do you agree? Justify your answer.

d. Economist B claims that short-term interest rates, i.e., $j = 1$, are "more responsive" to the state of the economy, i.e., c_t, than are long-term interest rates, i.e., j large. Do you agree? Justify your answer.

e. Economist C claims that the Fed should lower interest rates because whenever interest rates are low, consumption is high. Do you agree? Justify your answer.

f. Economist D claims that in economies in which output (consumption in our case) is very persistent ($\varphi \approx 1$), changes in output (consumption) do not affect interest rates. Do you agree? Justify your answer and, if possible, provide economic intuition for your argument.

Solution

a. We first describe the household's problem. To simplify it, we assume that bonds of maturities $k = j, \ldots J$ are traded, for a fix $J \geq 1$. In any period, the agent chooses bond holding of various maturities B_{jt}. The price at time t of a zero coupon bond maturing j periods from now is written q_{jt} [1]. The corresponding gross interest rate is $R_{jt} \equiv (1/q_{jt})^{1/j}$. The representative agent maximize:

---

[1] This bond pays 1 for sure at time $t + j$.

$$E_0\left[\sum_{t=0}^{+\infty}\beta^t u(c_t)\right],$$

subject to:

$$c_t + p_t s_t + \sum_{j=1}^{J} q_{jt}B_{jt} = s_t d_t + p_t s_{t-1} + B_{1t-1} + \sum_{j=1}^{J-1} q_{jt}B_{j+1,t-1},$$

and $B_{j,-1} = 0$ for all j. We have dropped the dependence on ε^t to simplify notations.

Definition 17. *An equilibrium is an allocation $\{c_t, s_t, B_{jt}\}_{t=0}^{+\infty}$, a price process $\{p_t, q_{jt}\}_{t=0}^{+\infty}$ such that given prices, the allocation solves the household's problem and markets clear, i.e. $c_t \leq$ dividend of the tree, $s_t = 1$ and $B_{jt} = 0$.*

After imposing market clearing, the first order conditions with respect to B_{jt} become :

$$q_{1t} = E_t\left(\beta\frac{u'(c_{t+1})}{u'(c_t)}\right) \tag{105}$$

$$q_{jt} = E_t\left(\beta\frac{u'(c_{t+1})}{u'(c_t)}q_{j-1,t+1}\right) \quad j = 2\dots J. \tag{106}$$

Iterating forward over the second equation and applying the law of iterated expectation gives the familiar :

$$q_{jt} = E_t\left(\beta^j\frac{u'(c_{t+j})}{u'(c_t)}\right). \tag{107}$$

b. Before computing R_{jt} we note that the log of consumption is an AR(1) process. Namely, we have :

$$\log(c_{t+1}) - \frac{\log(c^*)}{1-\phi} = \phi\left(\log(c_t) - \frac{\log(c^*)}{1-\phi}\right) + \log(\varepsilon_{t+1}).$$

Iterating on this equation, it is then easy to show that :

$$\log(c_{t+j}) - \frac{\log(c^*)}{1-\phi} = \phi^j\left(\log(c_t) - \frac{\log(c^*)}{1-\phi}\right) + \sum_{k=0}^{j-1}\phi^k\log(\varepsilon_{t+j-k}).$$

Consumption growth between period t and $t+j$ is :

$$\log(c_{t+j}) - \log(c_t) = (\phi^j - 1)\log(c_t) + \frac{1-\phi^j}{1-\phi}\log(c^*) + \sum_{k=0}^{j-1}\phi^k\log(\varepsilon_{t+j-k}).$$

Now observe that :

$$q_{jt} = \beta^j E_t \left[\left(\frac{c_{t+j}}{c_t} \right)^{-\theta} \right] \tag{108}$$

$$q_{jt} = \beta^j E_t \left[\exp\left(-\theta(\log(c_{t+j}) - \log(c_t) \right) \right] \tag{109}$$

$$q_{jt} = \beta^j \exp\left(-\theta(\phi^j - 1)\log(c_t) - \theta \frac{1-\phi^j}{1-\phi} \log(c^*) \right) \tag{110}$$

$$\times E_t \left[\exp\left(\sum_{k=0}^{j-1} -\theta\phi^k \log(\varepsilon_{t+j-k}) \right) \right]. \tag{111}$$

Remember that $\log(\varepsilon_t)$ is normal, with mean 1 and variance σ^2. In particular the $-\theta\phi^k \log(\varepsilon_{t+j-k})$ is normal, with mean $-\theta\phi^k$ and variance $\theta^2\phi^{2k}\sigma^2$. The expectation of $\exp\left((\phi^k \log(\varepsilon_{t+j-k})\right)$ is thus $-\theta\phi^k + \theta^2\phi^{2k}\sigma^2/2$. This gives :

$$\begin{aligned} q_{jt} &= \beta^j \exp\left(-\theta(\phi^j - 1)\log(c_t) + -\theta \tfrac{1-\phi^j}{1-\phi} \log(c^*) \right) \\ &\times \exp\left(-\theta \tfrac{1-\phi^j}{1-\phi} + 1/2\theta^2 \tfrac{1-\phi^{2j}}{1-\phi} \right). \end{aligned}$$

Using the definition $R_{jt} = (1/q_{jt})^{1/j}$, we obtain:

$$\log(R_{jt}) = a(j) + b(j)\log(c_t). \tag{112}$$

Where $b(j) = -\theta(1-\phi^j)/j$ and the constant $a(j)$ collects all the terms that do not depend on c_t. Importantly, $b(j)$ is negative and $|b(j)|$ decreases with maturity [2]. Equation (112) together with these two observations are the basis for answering all the following questions.

c.,d.,e.,f. Since the magnitude of $b(j)$ is decreasing with maturity, if follows that the variance of the interest rates is decreasing with maturity. Similarly, long term interest rates are less responsive than short term interest rates.

Economist C could not make his claim from studying our model. It is true that interest rates are countercyclical because $b(j)$ is negative. However, in our model, causation runs from consumption towards interest rates, not the converse. Specifically, properties of the exogenous consumption process pin down properties of the interest rates.

Lastly, if consumption is very persistent, it is easy to show that the term structure is flat and $b(j) = 0$. An intuition for this result is as follows : the interest rate reflects information about the growth rate of future consumption $\log(c_{t+j}) - \log(c_t)$. When $|\phi| < 1$, the log-consumption process is reverting to its long run value $\log(c^*)/(1-\phi)$. The current consumption level has thus some predictive content about consumption growth rate over subsequent periods: you know it is likely to revert to its mean. When $\phi = 1$ then the log-consumption process is a random walk (with drift). Therefore, the current consumption level has no

[2] to show this fact, study the function $1/x(1-\phi^x)$.

predictive content about future consumption growth rate. Interest rate should not depend on the current consumption level.

CHAPTER 11

Economic growth

Exercise 11.1.

Solution

a. The interior solution to the planning problem obeys the first order necessary conditions:

$$
\begin{aligned}
(113)\qquad u'(c_t) &= \beta u'(c_{t+1})\left[f_k(k_{t+1}, g_{t+1}) + 1 - \delta\right], \\
(114)\qquad f_g(k_t, g_t) &= 1,
\end{aligned}
$$

the resource constraint:

$$c_t + k_{t+1} + g_t = f(k_t, g_t) + (1-\delta)k_t,$$

the restriction $(c_t, k_t, g_t) > (0, 0, 0)$ and an appropriate transversality condition:

$$\lim_{T\to\infty} \beta^T u_c(c_T)k_{T+1} = 0$$

Together with the transversality condition, the first order conditions guarantee optimality.

b. The steady state is characterized by constant levels of consumption, capital stock and government spending. Then, the intertemporal marginal rate of substitution is 1 and equation (113) becomes

$$f_k(k^*, g^*) = \beta^{-1} - 1 + \delta, \tag{115}$$

and equation 114 becomes

$$f_g(k^*, g^*) = 1. \tag{116}$$

These are two equations in the two unknowns (k^*, g^*). The functions f_k and f_g are monotone, strictly decreasing (by virtue of the strict concavity of f) and converge to zero as k resp. g converge to ∞. Let $h(y) = f(y, y)$. Equations 115 and 116 have a unique solution (g^*, k^*) under the assumptions:

$$
\begin{aligned}
\lim_{y\to\infty} h'(y) &= 0, \\
h'(y) &> \beta^{-1} + \delta, \text{y small enough}, \\
f(0, g) &= f(k, 0) = 0.
\end{aligned}
$$

Proof: Consider the optimization problem

$$\max_{(k,g)\geq 0} f(k, g) - (\beta^{-1} - 1 + \delta)k - g. \tag{117}$$

The objective is strictly concave, implying that (117) has a unique solution. If $k = 0$ or $g = 0$, the objective is zero. If $k = g = y$, the objective is

$$h(y) - (\beta^{-1} + \delta)y \geq [h'(y) - \beta^{-1} - \delta]y > 0$$

for y small enough. Therefore, equation (117) has an interior solution that satisfies equations (115) and (116).

c. We use characterization (117) and monotone comparative statics analysis. Let $\theta = (\beta^{-1} - 1 + \delta)$, then we define $\tilde{f}$ and F:

$$\begin{aligned}\max_{(k,g)\geq 0}\left[f(k,g) - \theta k - g\right] &= \max_{(k)\geq 0}\left\{\max_{g\geq 0}\left[f(k,g) - g\right] - \theta k\right\} \\ &= \max_{k\geq 0}\left\{F(k,\theta)\right\}.\end{aligned}$$

$F(k,\theta)$ has increasing differences in $(k,-\theta)$ because

$$F(k',\theta) - F(k,\theta) = \tilde{f}(k') - \tilde{f}(k) - (k'-k)\theta.$$

If F has increasing differences in $(k,-\theta)$, then $k^*(-\theta)$ is non-decreasing. Therefore, $k^*(\theta)$ is non-increasing and $k^*(\beta)$ is non-decreasing. From $f_g(k,g) = 1$ and from $\frac{\partial^2 f}{\partial k \partial g} > 0$, $\frac{\partial^2 f}{\partial g^2} < 0$, we find that $g^*(k^*)$ is increasing. Therefore $g^*(\beta)$ is non-decreasing.

d. The optimality conditions at the steady state become:

$$z\alpha k^{\alpha-1}g^{\eta} = \beta^{-1} - 1 + \delta \tag{118}$$

$$z\eta k^{\alpha}g^{\eta-1} = 1. \tag{119}$$

and upon substitution of equation (119) in equation (118), we obtain an expression for the steady state government spending to capital ratio:

$$\frac{g^*}{k^*} = \left(\beta^{-1} - 1 + \delta\right)\frac{\eta}{\alpha}$$

From equation 119 and the steady state ratio, we get the expression for the steady state capital stock as a function of parameters:

$$k^* = z^{\frac{-1}{\alpha+\eta-1}}\left(\beta^{-1} - 1 + \delta\right)^{\frac{1-\eta}{\alpha+\eta-1}}\alpha^{\frac{\eta-1}{\alpha+\eta-1}}\eta^{\frac{-\eta}{\alpha+\eta-1}}.$$

The steady state value of government spending is obtained from the steady state capital stock and the steady state government spending to capital ratio:

$$g^* = z^{\frac{-1}{\alpha+\eta-1}}\left(\beta^{-1} - 1 + \delta\right)^{\frac{\alpha}{\alpha+\eta-1}}\alpha^{\frac{-\alpha}{\alpha+\eta-1}}\eta^{\frac{\alpha-1}{\alpha+\eta-1}}.$$

As long as there are decreasing returns to scale ($\alpha + \eta < 1$), it is easy to show that the partial derivatives of $k^*(\beta)$ and $g^*(\beta)$ are positive. This confirms our findings in part c.

Now we assume decreasing returns to scale. The investment in steady state is $x^* = \delta k^*$, whereas GDP is $zk^{\alpha}g^{\eta}$. The investment to GDP ratio in the steady state simplifies to:

$$\frac{\delta k}{zk^{\alpha}g^{\eta}} = \left(\frac{\delta\eta\alpha}{\beta^{-1} - 1 + \delta}\right)z$$

and the government spending to GDP ratio

$$\frac{g}{zk^{\alpha}g^{\eta}} = \left(\delta\eta\alpha(\beta^{-1} - 1 + \delta)\right)z$$

Because $\beta^{-1} - 1 + \delta > 0$, the partial derivatives w.r.t. z are strictly positive.

Exercise 11.2.

Solution

a. Let: $h(x) = f(x, 1)$ and $g(n) = \frac{u_n(an-b,1-n)}{u_c(an-b,1-n)}$ with $a > 0$ and $b < a$. We make the following assumptions:

$$
\begin{aligned}
f(\lambda k, \lambda n) &= \lambda f(k, n) \\
\lim_{x \to \infty} h_x(x) &= 0 \\
\lim_{x \to 0} h_x(x) &= \infty \\
\frac{\partial^2 u}{\partial c \partial n} &> 0 \\
\lim_{n \to 1} g(n) &= \infty \\
\lim_{n \to \frac{b}{a}} g(n) &= 0.
\end{aligned}
$$

The first equation says that the production function has constant returns to scale. The second and the fourth assumption are Inada conditions. The dynamics of this economy are characterized by the first order necessary conditions for optimality:

$$
u_c(c_t, 1 - n_t) = \beta u_c(c_{t+1}, 1 - n_{t+1}) \left[z f_k(k_{t+1}, n_{t+1}) + 1 - \delta \right], \tag{120}
$$

$$
\frac{u_n(c_t, 1 - n_t)}{u_c(c_t, 1 - n_t)} = z f_n(k_t, n_t), \tag{121}
$$

the resource constraint

$$
c_t + k_{t+1} + g_t = z f(k_t, n_t) + (1 - \delta) k_t,
$$

and the restriction $(c_t, k_t, n_t) > (0, 0, 0)$.
The steady state (k, n, c) is then implicitly defined by

$$
\begin{aligned}
1 &= \beta \left[z f_k(k, n) + 1 - \delta \right] \\
&= \beta \left[z h_x \left(\frac{k}{n} \right) + 1 - \delta \right], \\
\frac{u_n(c, 1 - n)}{u_c(c, 1 - n)} &= z f_n(k, n) \\
&= z h \left(\frac{k}{n} \right) - \frac{k}{n} z h_x \left(\frac{k}{n} \right), \\
c + g &= z f(k, n) - \delta k \\
&= n \left[z h \left(\frac{k}{n} \right) - \delta \frac{k}{n} \right].
\end{aligned}
$$

First, there exists a unique $x^* = \frac{k^*}{n^*}$ that solves the first equation. Second substitute the third into the second equation to get:

$$
\frac{u_n(n(zh(x^*) - \delta x^*) - g, 1 - n)}{u_c(n(zh(x^*) - \delta x^*) - g, 1 - n)} = z h\left(x^*\right) - z x^* h_x\left(x^*\right). \tag{122}
$$

This is one equation in one unknown n. By concavity of h:

$$zh(x^*) - \delta x^* \geq x^*(zh_x(x^*) - \delta) = x^*(\beta^{-1} - 1) > 0.$$

We then use the last assumption with $a = zh(x^*) - \delta x^*$, which is positive by the previous argument, and $b = g$. The right-hand side of equation (122) is constant. The left-hand side is increasing, goes to zero when n goes to b/a, and goes to infinity when n goes to 1. This establishes existence and uniqueness.

b. The steady state conditions become

$$1 = \beta\left[z\alpha k^{\alpha-1}n^{\eta} + 1 - \delta\right], \tag{123}$$

$$\frac{1-\mu}{\mu}\frac{c}{(1-n)} = z\eta k^{\alpha}n^{\eta-1}, \tag{124}$$

$$c + g = zk^{\alpha}n^{\eta} - \delta k. \tag{125}$$

After going through the steps described in part a, we obtain an equation with n as its only argument:

$$\eta z^{\frac{1}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}}\left(\beta^{-1} - 1 + \delta\right)^{\frac{-\alpha}{1-\alpha}}(1-n)n^{\frac{\alpha+\eta-1}{1-\alpha}} =$$

$$\frac{1-\mu}{\mu}\left[-g + z^{\frac{1}{1-\alpha}}\left\{\alpha^{\frac{\alpha}{1-\alpha}}\left(\beta^{-1} - 1 + \delta\right)^{\frac{-\alpha}{1-\alpha}} - \delta\alpha^{\frac{1}{1-\alpha}}\left(\beta^{-1} - 1 + \delta\right)^{\frac{-1}{1-\alpha}}\right\}n^{\frac{\eta}{1-\alpha}}\right],$$

For the CD technology of part b, $\eta = 1 - \alpha$ and the above expression simplifies to a linear expression in n:

$$(1-\alpha)z^{\frac{1}{1-\alpha}}A(1-n) = \frac{1-\mu}{\mu}\left[-g + z^{\frac{1}{1-\alpha}}\left\{A - \delta A^{\frac{1}{\alpha}}\right\}n\right],$$

where $A(\beta, \alpha, \delta)$ is given by

$$A = \alpha^{\frac{\alpha}{1-\alpha}}\left(\beta^{-1} - 1 + \delta\right)^{\frac{-\alpha}{1-\alpha}}.$$

Solving for the steady state labor level n, we get

$$n = \frac{(1-\alpha)A + \frac{1-\mu}{\mu}gz^{\frac{-1}{1-\alpha}}}{(1-\alpha)A + \frac{1-\mu}{\mu}\left(A - \delta A^{\frac{1}{\alpha}}\right)}$$

For any given, strictly positive level of government spending, the partial derivative of of employment w.r.t. the productivity level is strictly negative. Its magnitude depends on the level of government spending. Without government spending, the SS level of employment is independent of the technology level.
From equation (123) we solve for the capital labor ratio in steady state. The expression for output per capita is

$$y = z\left(\frac{k}{n}\right)^{\alpha}n = z^{\frac{1}{1-\alpha}}An$$

The partial derivative of output per capita w.r.t. the technology level is

$$\frac{\partial y}{\partial z} = (1-\alpha)z^{\frac{\alpha}{1-\alpha}}An + z^{\frac{1}{1-\alpha}}A\frac{\partial n}{\partial z}$$

It is strictly positive, even when $g = 0$.

c. For the Cobb-Douglas specification, the capital to labor ratio is a function of the parameters (β, δ, α) and the technology level z. It is independent of government spending. The partial derivative of output per capita w.r.t. the government spending is strictly positive:

$$\frac{\partial y}{\partial g} = z^{\frac{1}{1-\alpha}} A \frac{\partial n}{\partial g} = \frac{\frac{1-\mu}{\mu} A}{(1-\alpha)A + \frac{1-\mu}{\mu}\left\{A - \delta A^{\frac{1}{\alpha}}\right\}}$$

We also investigate these partial derivatives for a general production function $zk^{\alpha}n^{\eta}$; $0 < \alpha < 1$, $0 < \eta < 1$, which allows for decreasing $(\alpha + \eta < 1)$ and increasing returns to scale $(\alpha + \eta > 1)$. The capital to labor ratio in the steady state is

$$\frac{k}{n} = A^{\frac{1}{\alpha}} z^{\frac{1}{\alpha}} n^{\frac{\alpha+\eta-1}{1-\alpha}}$$

The partial derivative w.r.t. government spending is:

$$\frac{\partial\left(\frac{k}{n}\right)}{\partial g} = \left(\frac{\alpha+\eta-1}{1-\alpha}\right) A^{\frac{1}{\alpha}} z^{\frac{1}{1-\alpha}} n^{\frac{2\alpha+\eta-2}{1-\alpha}} \frac{\partial n}{\partial g}$$

Using the implicit function theorem, the partial derivative of n w.r.t. g can be shown to be

$$\frac{\partial n}{\partial g} = \frac{1-\mu}{\mu} z^{\frac{-1}{1-\alpha}} \frac{n^{\frac{1-\alpha-\eta}{1-\alpha}}}{\eta A + \frac{\eta}{1-\alpha}\left\{A - \delta A^{\frac{1}{\alpha}}\right\} + \eta A\left(\frac{1-\alpha-\eta}{1-\alpha}\right)\left(\frac{1-n}{n}\right)}.$$

Substituting this expression into the previous, we get

$$\frac{\partial\left(\frac{k}{n}\right)}{\partial g} = \frac{1-\mu}{\mu} A^{\frac{1}{\alpha}} \frac{\alpha+\eta-1}{(1-\alpha)\eta A n + \eta\left\{A - \delta A^{\frac{1}{\alpha}}\right\} n + \eta A(1-\alpha-\eta)(1-n)}.$$

When the production function displays IRS, the steady state capital to labor ratio increases with government spending if and only if the steady state employment level exceeds a threshold level C, where

$$C = \frac{\eta A(\alpha+\eta-1)}{\eta^2 A + \eta\left\{A - \delta A^{\frac{1}{\alpha}}\right\}}.$$

When the production function displays decreasing returns to scale, the steady state capital to labor ratio increases with government spending if and only if the steady state employment level is below C.
For output per capita we know that

$$\frac{\partial y}{\partial g} = \alpha z \left(\frac{k}{n}\right)^{\alpha-1} n^{\eta-\alpha} \frac{\partial\left(\frac{k}{n}\right)}{\partial g} + z\left(\frac{k}{n}\right)^{\alpha} (\eta-\alpha) A \frac{\partial n}{\partial g}$$

The above analysis for the partial derivatives of the capital to labor ratio and the employment w.r.t. government spending can be used to sign the derivative. The new element is the sign of $\eta - \alpha$.

Exercise 11.3.

Solution

a. The first order necessary conditions for optimality of the planner problem are described by a second order difference equation in (c, k):

$$u_c(c_t) = \beta\left[u_c(c_{t+1})zf'(k_{t+1}+k_t)+1\right] + \beta^2 u_c(c_{t+2})zf'(k_{t+2}+k_{t+1}),\ \forall t\ k_0 given$$

For sufficiency we impose some appropriate transversality conditions:

$$\lim_{T\to\infty} \beta^T u_c(c_T^*)k_{T+1}^* = 0,$$
$$\lim_{T\to\infty} \beta^{T+1} u_c(c_{T+1}^*)k_T^* f_k(k_T^* + k_{T-1}^*) = 0.$$

b. In steady state, the marginal utility of consumption and the capital stock are constant. The standard growth model has constant returns to scale technology, so that the production function is homogenous of degree 1. The steady state capital stock is uniquely determined by

$$1 = 2\beta z f'(k) + 2\beta^2 z f'(k),$$

or

$$f'(k) = \frac{1}{2z}\frac{1}{\beta+\beta^2}.$$

Existence and uniqueness follow from the monotonicity of f', the Inada condition $\lim_{k\to\infty} f'(k) = 0$, and the additional assumption $\lim_{k\to 0} f'(k) > \frac{1}{2z}\frac{1}{\beta+\beta^2}$.

c. We will show that a cyclical steady state cannot exist be assuming it does and deriving a contradiction. In odd and even periods resp. The SS equations become:

$$\begin{aligned} 1 &= \left[\frac{u_c(c^e)}{u_c(c^o)} + \beta\right]\beta z f'(k^0 + k^e), \\ 1 &= \left[\frac{u_c(c^o)}{u_c(c^e)} + \beta\right]\beta z f'(k^0 + k^e). \end{aligned}$$

Both equations can only hold simultaneously if and only if $u_c(c^e) = u_c(c^o)$ or $c^e = c^o$. This is a contradiction to the cyclical steady state.

Exercise 11.4.

Solution

a. The three constraints are:

$$\begin{aligned} c_t + k_{t+1} &\le zf(\kappa_t) - \delta\kappa_t + k_t, \\ \kappa_t &\le k_t, \\ c_t &\le zf(\kappa_t). \end{aligned}$$

Let λ_t, η_t and ν_t be the Lagrange multipliers on the time t resource constraint, capacity constraint non-negativity constraint on investment respectively.

The first order necessary conditions for optimality of the planner problem are described by:

$$u_c(c_t) = \lambda_t + \nu_t, \tag{126}$$

$$\lambda_t = \beta(\lambda_{t+1} + \eta_{t+1}), \tag{127}$$

$$\lambda_t(zf'(\kappa_t - \delta) + \nu_t zf'(\kappa_t) = \eta_t, \tag{128}$$

plus the complementary slackness conditions. For sufficiency, we impose the transversality condition:

$$\lim_{T\to\infty} \beta^T \lambda_T k_{T+1} = 0.$$

When the capacity constraint binds, the economy operates at full capacity, $\eta_t > 0$, and $k_t = \kappa_t$. The budget constraint and capacity utilization equations reduce to the standard ones. The intertemporal marginal rate of substitution is smaller than 1. For log utility, consumption at time t is higher than consumption at time $t-1$ multiplied by the discount factor.

When the capacity constraint does not bind, $\eta_t = 0$ $k_t > \kappa_t$. The intertemporal marginal rate of substitution equals 1. If investment is positive, $\nu_t = 0$ and $zf'(\kappa_t) = \delta$. The last equality says that the economy utilizes machinery up to the point where the marginal productivity equals the depreciation rate. If investment is zero, $\nu_t > 0$, and the marginal product of of utilized capital is lower than the depreciation rate.

b. and c. In steady state, the first order conditions become

$$u_c(c^*) = \lambda + \nu,$$
$$\lambda = \beta(\lambda + \eta),$$
$$\lambda(zf'(\kappa^* - \delta) + \nu zf'(\kappa^*) = \eta.$$

We show that in steady state the economy operates at full capacity: $k^* = \kappa^*$.

Proof: Because of the Inada conditions on u_c, the steady state consumption level is strictly positive. From the budget constraint $c^* \leq zf(\kappa^*) - \delta\kappa^*$. Because $c^* > 0$, $\kappa^* > 0$ and hence $c^* \leq zf(\kappa^*)$. Therefore $\nu = 0$. Second, $\eta > 0$. If η were zero, then $\lambda = \beta(\lambda + \eta)$ only holds for $\lambda = 0$. But that implies that $u_c(c) = 0$ which contradicts a finite consumption level. Therefore $k^* = \kappa^*$.
Combining the first order conditions, we then get $\beta(1 - \delta + zf'(k^*)) = 1$ and $c^* = zf(k^*) - \delta k^*$. The equations pins down a unique steady state capital stock and consumption level. This follows from the monotonicity of f', the Inada condition $\lim_{k\to\infty} f'(k) = 0$ and the additional assumption $\lim_{k\to 0} f'(k) > \frac{1-\beta+\delta}{z}$.

d. In the steady state all countries are operating at full capacity. Hence, in steady state, differences in output per capita are eliminated. In the long-run, the model is inconsistent with the statement. The reason is that there is a unique level of used machines κ^* that satisfies $f'(\kappa^*) = \frac{1-\beta+\delta}{z}$.

Along the path to the steady state, differences in output per capita can arise, and stem from differences in capacity utilization. There is a (weakly) negative relationship between capacity utilization and output per capita. Countries with low capacity utilization have high output per capita and vice versa.

Absent fluctuations in z, countries with a high initial capital stock ($k_0 > \hat{\kappa}$) will not use their capital stock fully, but choose a *constant* level of machines $\hat{\kappa}$ such that $f'(\hat{\kappa}) = \frac{\delta}{z}$. They use capital up to the point where its marginal product equals the depreciation. They do not invest. Note that $\hat{\kappa} > \kappa^* > k_0$. Because utilized capital depreciates, capacity utilization rises over time. Output per capita is constant and at a level $zf(\hat{\kappa})$. At some point τ, full capacity utilization is reached $k_\tau = \kappa_\tau = \hat{\kappa}$. From that point onwards, the capacity constraint starts to bind ($\eta_\tau > 0$), full capacity utilization is maintained ($k_t = \kappa_t,\ t > \tau$) and the number of machines employed, κ_t, gradually decreases to the steady state level κ^*.

Countries with a low initial capital stock ($k_0 < \kappa^*$) operate at full capacity along the transition path $k_t = \kappa_t$. They accumulate capital until $k_t = \kappa^*$.

Exercise 11.5.

Solution

a. Given a uniform initial wealth distribution $\{k_0^i\}_{i=0}^1$, a competitive equilibrium is a feasible allocation $\{c^i, k^i, n^i\}$ for each agent i and a price vector $\{w, r\}$ such that

- Given prices, households maximize the present discounted value of utility streams subject to their budget constraint $c_t^i + k_{t+1}^i \leq (1-\delta+r_t)k_t^i + w_t n_t$, given k_0^i.
- Firms maximize profits
- The labor market and goods market clear.

b. *i.* The first order necessary conditions for the household problem and firm problem imply:

$$u_c^i(c_t^i) = \beta u_c^i(c_{t+1}^i)(zf_k(k_{t+1}, n_{t+1}) + 1 - \delta).$$

Households supply their unit of labor inelastically. Capital and labor are paid their marginal products.
In steady state, consumption is constant for every agent, and so is marginal utility. Therefore, there is a unique steady state capital stock, determined by $zf_k(k, 1) = \beta^{-1} - 1 + \delta$. The existence and uniqueness follow from the monotonicity of $f_k(\cdot, 1)$, the Inada condition on f_k and the additional assumption $\lim_{k\to 0} f'(k) > \beta^{-1} - 1 + \delta$. The steady state interest rate $r(k^*)$ can be calculated without info on u_i. Optimal steady state consumption is: $c^* = f(k^*, 1) - \delta k^*$. Economist A is correct.

ii. Economist A is correct again. At any period t, optimality requires equalization of the IMRS of any pair of agents (i, j). Rewriting this equality, we get:

$$\frac{u_c^j\left(c_t^j\left(k_t^j\right)\right)}{u_c^i\left(c_t^i\left(k_t^i\right)\right)}=\frac{u_c^j\left(c_{t+1}^j\left(k_{t+1}^j\right)\right)}{u_c^i\left(c_{t+1}^i\left(k_{t+1}^i\right)\right)}$$

This condition implies that the marginal rate of substitution between consumers (i, j) is constant through time. The initial distribution of capital determines the initial marginal rate of substitution between consumers (i, j). By the equation above, the initial capital stock determines the consumption inequality and the latter stays constant over time.

c. In steady state, the capital stock is constant at k^*. The wage and rental price are $w = F_N(k^*, 1)$ and $r^* = F_K(k^*, 1) = \beta^{-1} - 1 + \delta$. The steady state consumption level in an economy without taxes is $c^i = w + (1-\beta)k^i$. In an economy with a tax z^i on individual i, we find $c^i = w + (\beta^{-1} - 1)k^i + (1-\beta)z^i$.

i. Since this is merely a redistribution of capital, the aggregate capital stock stays constant and the economy remains in the steady state. All aggregate variables are unchanged. Obviously, those taxed choose a lower level of consumption relative to those who receive the transfer. There may exist a tax and transfer scheme that generates $c^i = c^j$, $\forall\ (i, j)$. These allocations cannot be Pareto ranked with the original one.
More formally, we check that markets clear.

$$\begin{aligned}
\int_0^1 c^i di &= wN + (\beta^{-1} - 1)\int_0^1 k^i di + (1-\beta)\int_0^1 z^i di, \\
C &= F_N N + F_K K - \delta K + (1-\beta)\int_0^1 z^i di, \\
C &= F(K, N) - \delta K + (1-\beta)\int_0^1 z^i di.
\end{aligned}$$

In this question there is market clearing because $\int_0^1 z^i di = 0$.

ii. Same answer as in part *i.* because aggregate consumption does not change and neither does the aggregate capital stock. Again there is market clearing because $\int_0^1 z^i di = 0$.

iii. We consider two scenarios. In the first one $G_t = g = \int_0^1 z^i di\ \forall t$. The economy stays in the steady state. The steady state level of capital is unchanged (and likewise for the interest rate and the wage). The aggregate level of consumption is lower by the amount of government spending. The tax reduces the consumption of those taxed.

In a second scenario $G_0 = \int_0^1 z^i di > 0,\ G_t = 0\ \forall t \geq 1$. Then at $t = 1$, the market does not clear at the steady state capital level - consumption pair. The tax acts as an aggregate shock to the economy. The economy slowly returns to the consumption capital level steady state.

Exercise 11.6.

Solution

a. The first order necessary condition for an optimum of the second planner's problem is

$$u_c^i(c_t^i) = \beta u_c^i(c_{t+1}^i)((1-\tau)f_k(k_{t+1}) + 1 - \delta).$$

The steady state capital stock is the solution to

$$f_k(k) = \frac{\beta^{-1} - 1 + \delta}{(1-\tau)}.$$

The existence and uniqueness of the solution follow from the monotonicity of $f_k(\cdot)$, the Inada condition on f_k and the additional assumption $\lim_{k\to 0} f'(k) > \beta^{-1} - 1 + \delta$. Steady state consumption is $c = f(k) - \delta k - g$.

b. The new steady state is characterized by a lower tax rate and hence a higher steady state capital level. The transition dynamics are as follows. Upon impact investment jumps up and consumption jumps down. That follows from the fact that the marginal product of capital increased and from the resource constraint respectively. Because of the higher level of investment, the capital stock increases (but does not jump). Because of the higher marginal product of capital, it is more valuable to postpone consumption. That is, consumption grows after impact. It does so until it reaches its new steady state level $c^{**} > c^*$. Investment decreases gradually after its initial jump to its new steady state $x^{**} > x^*$. The investment level in the new steady state is higher because the capital stock is higher and so is the capital stock that depreciates each period. Capital monotonically increases to its new steady state level.

c. This economy decentralizes the planner problem in parts a and b. Labor is supplied inelastically: $n = 1$. The equilibrium interest rate equals the after-tax marginal product of capital. The new steady state interest rate is lower under the new, low tax regime. In the transition to the new steady state the interest rate decreases so as to make an increasing consumption profile optimal. Upon impact the interest rate jumps up so that consumers are induced to save and invest.

CHAPTER 12

Optimal taxation with commitment

Exercise 12.1. *A small open economy (Razin and Sadka, 1995)*

Consider the non stochastic model with capital and labor in this chapter, but assume that the economy is a small open economy that cannot affect the international rental rate on capital, r_t^*. Domestic firms can rent any amount of capital at this price, and the households and the government can choose to go short or long in the international capital market at this rental price. There is no labor mobility across countries. We retain the assumption that the government levies a tax τ_t^n on households' labor income but households no longer have to pay taxes on their capital income. Instead, the government levies a tax $\hat{\tau}_t^k$ on domestic firms rental payments to capital regardless of the capital origin (domestic or foreign). Thus, a domestic firm faces a total cost of $(1+\hat{\tau}_t^k)r_t^*$ on a unit of capital rented in period t.

a. Solve for the optimal capital tax $\hat{\tau}_t^k$.

b. Compare the optimal tax policy of this small open economy to that of the closed economy of this chapter.

Solution

a. and **b.** The household problem is to choose consumption, labor, capital and bonds holding $\{c_t, n_t, k_{t+1}^H, b_{t+1}^H\}_{t=0}^{+\infty}$, so as to maximize

$$\sum_{t=0}^{+\infty} \beta^t u(c_t, 1-n_t), \tag{129}$$

subject to

$$c_t + k_{t+1}^H + \frac{b_{t+1}^H}{R_t^*} = (1-\tau_t^n)w_t n_t + r_t^* k_t^H + (1-\delta)k_t^H + b_t^H, \tag{130}$$

and a transversality condition. No arbitrage imposes that $R_t^* = r_{t+1}^* + (1-\delta)$. The government has the budget constraint

$$g_t + b_t^G = \hat{\tau}_t^k r_t^* k_t + \tau_t^n w_t n_t + \frac{b_{t+1}^G}{R_t^*}, \tag{131}$$

where b_t^G denotes government debt and k_t is the total capital stock of the economy, that may not be entirely owned by domestic household. A transversality condition needs also to be added to the government budget constraint. Firm's profit maximization implies :

$$\begin{aligned} F_k(k_t, n_t) &= (1+\hat{\tau}_t^k)r_t^* && (132)\\ F_n(k_t, n_t) &= w_t. && (133) \end{aligned}$$

Add (130) and (131), and use the homogeneity of degree one of the production function to obtain

$$c_t + k_{t+1} + g_t + (k^H_{t+1} - k_{t+1}) + (b^G_t - b^H_t) = F(k_t, n_t) + \frac{b^G_{t+1} - b^H_{t+1}}{R^*_t} + (1 - \delta + r^*_t)(k^H_t - k_t). \tag{134}$$

Observe that, since both the government and the households can borrow and save in the international capital market, the resource constraint of the close economy does not necessarily hold. First, b^G_t may be different from b^H_t which means that all government bonds are not necessarily owned by domestic households. $b^G_t - b^H_t$ may be interpreted as the country deficit. Similarly k_t may be different from k^H_t. If $k_t > k^H_t$ then some domestic capital is owned by foreign investors. If on the other hand $k_t < k^H_t$, then domestic investors own foreign capital.

The primal approach to the Ramsey problem in this small economy context is to maximize the representative agent utility subject to the intertemporal budget constraint of the household and the intertemporal budget constraint of the government. Observe that those two intertemporal budget constraints do not imply the resource constraint of the closed economy. To write these constraints, we note that the first order conditions of the household's problem give

$$\beta^t u_c(t) = \lambda_0 q^*_t \tag{135}$$

$$\beta^t u_l(t) = \lambda_0 q^*_t (1 - \tau^n_t) w_t, \tag{136}$$

where λ_0 is the multiplier on the time zero budget constraint and $1/q^*_t = R^*_{t-1} R^*_{t-2} \dots R^*_0$. The intertemporal budget constraint of the household becomes

$$\sum_{t=0}^{+\infty} \left(q^*_t c_t - \beta^t \frac{u_l(t)}{u_c(0)} n_t \right) = \left((1 - \delta) + r_0 (1 - \tau^k_0) \right) k_0 + b_0. \tag{137}$$

To write the government intertemporal budget constraint, we note that

$$\begin{aligned} \hat{\tau}^k_t r^*_t k_t + \tau^n_t w_t n_t &= (1 + \hat{\tau}^k_t) r^*_t k_t + w_t n_t - r^*_t k_t - (1 - \tau^n_t) w_t n_t \\ &= F(k_t, n_t) - r^*_t k_t - (1 - \tau^n_t) w_t n_t. \end{aligned}$$

So that the intertemporal budget constraint of the government is

$$b^G_0 + \sum_{t=0}^{+\infty} q^*_t \left(g_t - \hat{\tau}^k_t r^*_t k_t + \tau^n_t w_t n_t \right) = \tag{138}$$

$$b^G_0 + \sum_{t=0}^{+\infty} \left(q^*_t g_t - \beta^t \frac{u_l(t)}{u_c(0)} n_t + q^*_t F(k_t, n_t) - q^*_t r^*_t k_t \right) = 0. \tag{139}$$

The Ramsey problem is to choose $\{c_t, n_t, k_{t+1}\}$ to maximize (129) subject to (137) and (139). Observe that the capital stock appears only in the government's budget constraint. Taking derivative in the Lagrangian with respect to k_t yields

$$F_k(t) = r_t^*. \tag{140}$$

which, together with (132) implies that $\hat{\tau}_t^k = 0$ for all $t \geq 1$.

Exercise 12.2. *Exercise 12.2 Consumption Taxes*

Consider the non stochastic model with capital and labor in this chapter, but instead of labor and capital taxation assume that the government sets labor and consumption taxes, $\{\tau_t^n, \tau_t^c\}$. Thus, the household's present-value budget constraint is now given by

$$\sum_{t=0}^{\infty} q_t^0 (1 + \tau_t^c) c_t = \sum_{t=0}^{\infty} q_t^0 (1 - \tau_t^n) w_t n_t + [r_0 + 1 - \delta]\, k_0 + b_0.$$

a. Solve for the Ramsey plan.

b. Suppose that the solution to the Ramsey problem converges to a steady state. Characterize the optimal limiting sequence of consumption taxes.

c. In the case of capital taxation, we imposed an exogenous upper bound on τ_0^k. Explain why a similar exogenous restriction on τ_0^c is needed to ensure an interesting Ramsey problem. (Hint: Explore the implications of setting $\tau_t^c = \tau^c$ and $\tau_t^n = -\tau^c$ for all $t \geq 0$, where τ^c is a large positive number.)

Solution

a. We follow the steps described in the paragraph "constructing the Ramsey Plan".

Step 1: Write the household's problem

We first recall the household problem when trading is sequential. The household chooses consumption, labor, capital and bond holdings $\{c_t, n_t, k_{t+1}, b_{t+1}\}_{t=0}^{+\infty}$ so as to maximize

$$\sum_{t=0}^{+\infty} \beta^t u(c_t, 1 - n_t), \tag{141}$$

subject to

$$(1 + \tau_t^c) c_t + k_{t+1} + \frac{b_{t+1}}{R_t} = (1 - \tau_t^n) w_t n_t + (r_t + (1 - \delta))\, k_t + b_t. \tag{142}$$

Attach a Lagrange multiplier $\beta^t \lambda_t$ to time t budget constraint. The first order conditions of the household's problem are:

$$\begin{aligned}
c_t: &\quad \beta^t u_c(t) &= \lambda_t(1+\tau_t^c)\\
n_t: &\quad \beta^t u_n(t) &= \lambda_t(1-\tau_t^n)w_t\\
k_{t+1}: &\quad \lambda_t &= \beta\lambda_{t+1}\left[r_{t+1}+(1-\delta)\right]\\
b_{t+1}: &\quad \tfrac{\lambda_t}{R_t} &= \beta\lambda_{t+1}.
\end{aligned}$$

Observe that the previous equations imply the no-arbitrage condition

$$R_t = r_{t+1} + (1-\delta). \tag{143}$$

To apply the primal approach, we need to reformulate the household problem in the context of time zero trading. We thus define $1/q_t^0 \equiv R_{t-1}R_{t-2}\dots R_0$. We have

$$\beta^t u_c(t) = \lambda_0 q_t^0 (1+\tau_t^c) \tag{144}$$

$$\beta^t u_n(t) = \lambda_0 q_t^0 (1-\tau_t^n) w_t. \tag{145}$$

Observe that (144) and (145) imply in particular that

$$\begin{aligned}
q_t^0(1+\tau_t^c) &= \beta^t \tfrac{u_c(t)}{u_c(0)}(1+\tau_0^c)\\
q_t^0(1-\tau_t^n)w_t &= \beta^t \tfrac{u_l(t)}{u_c(0)}(1+\tau_0^c).
\end{aligned}$$

Furthermore, (144) and (145) also show that the household choice of consumption and labor maximizes

$$\sum_{t=0}^{+\infty}\beta^t u(c_t, 1-n_t), \tag{146}$$

subject to an *intertemporal* budget constraint

$$\sum_{t=0}^{+\infty} q_t^0(1+\tau_t^c)c_t \le \sum_{t=0}^{+\infty} q_t^0 w_t (1-\tau_t^n) n_t + \left(r_0 + (1-\delta)\right)k_0 + b_0. \tag{147}$$

Step 2: Write the intertemporal budget constraint.

The intertemporal budget constraint is

$$\sum_{t=0}^{+\infty}\beta^t(1+\tau_0^c)(u_c(t)c_t - u_l(t)n_t) = u_c(0)\left((r_0+(1-\delta))k_0 + b_0\right).$$

Let $A \equiv u_c(0)\left((r_0+(1-\delta))k_0+b_0\right)$.

Step 3: Form the Lagrangian.

Now define:

$$V(c_t, n_t, \Phi) \equiv u(c_t, 1-n_t) + \Phi(1+\tau_0^c)(u_c(t)c_t - u_l(t)n_t).$$

The Lagrangian associated with the Ramsey plan is:

$$J = \sum_{t=0}^{+\infty} \beta^t \left\{V(c_t, n_t, \Phi) + \theta_t \left(F(k_t, n_t) - (1-\delta)k_t - c_t - g_t - k_{t+1}\right)\right\} - \Phi A.$$

The first order conditions are:

$$\begin{array}{rrcl} c_t: & V_c(t) & = & \theta_t, \quad t \geq 1 \\ n_t: & V_n(t) & = & -\theta_t F_n(t), \quad t \geq 1 \\ k_{t+1}: & \theta_t & = & \beta\theta_{t+1}[F_k(t+1) + (1-\delta)], \quad t \geq 0 \\ c_0: & V_c(0) & = & \theta_0 + \Phi A_c \\ n_0: & V_n(0) & = & -\theta_0 F_n(0) + \Phi A_n. \end{array}$$

The Ramsey Plan is thus solution of the following system of difference equations:

$$\begin{array}{rcl} V_c(t) & = & \beta V_c(t+1)[F_k((t+1) + 1 - \delta], \quad t \geq 1 \\ V_n(t) & = & -V_c(t)F_n(t), \quad t \geq 1 \\ V_n(0) & = & [\Phi A_c - V_c(0)]F_n(0) + \Phi A_n. \end{array}$$

$$\begin{array}{l} c_t + g_t + k_{t+1} = F(k_t, n_t) + (1-\delta)k_t \\ \sum_{t=0}^{+\infty} \beta^t (1+\tau_0^c)(u_c(t)c_t - u_l(t)n_t) - A = 0. \end{array}$$

b. Assume this system converges to a steady state. Remember that the no-arbitrage condition (143) must hold, that is $R_t = \frac{q_t^0}{q_{t+1}^0} = (1-\tau_{t+1}^k)F_k(t+1)+1-\delta$. The steady state version of the first difference equation defining the Ramsey plan is $1 = \beta\big[(1-\tau_{t+1}^k)F_k(t+1) + 1 - \delta\big]$. So that:

$$\frac{q_t^0}{q_{t+1}^0} = \frac{1}{\beta}.$$

On the other hand, the first order conditions of the household problem gives:

$$\frac{q_t^0(1+\tau_t^c)}{q_{t+1}^0(1+\tau_{t+1}^c)} = \frac{u_c(t)}{\beta u_c(t+1)}.$$

In steady state, this becomes:

$$\frac{(1+\tau_t^c)}{(1+\tau_{t+1}^c)} = 1.$$

Which proves that the steady state consumption tax is constant.

c. Consider the household problem under the suggested taxation scheme:

$$\max_{\{c_t\}_{t=0}^{+\infty},\{l_t\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \beta^t u(c_t, 1-n_t)$$
subject to
$$\sum_{t=0}^{+\infty} q_t^0(1+\tau^c)c_t = \sum_{t=0}^{+\infty} q_t^0(1+\tau^c)w_t n_t + \left(r_0 + (1-\delta)\right) k_0 + b_0.$$

The budget set is more conveniently described as:

$$\sum_{t=0}^{+\infty} q_t^0(c_t - w_t n_t) = \frac{(r_0+(1-\delta))k_0 + b_0}{1+\tau^c}.$$

So that it is now apparent that this taxation scheme is equivalent to a lump sum tax on time 0 assets (capital and bonds) and and no other tax afterwards. As we know, this scheme is optimal because it does not induce distortion. Therefore, in order to study what an optimal distortive tax on consumption would be, we need to impose an upper bound on τ_0^c.

Exercise 12.3. *Specific utility function (Chamley, 1986)*

Consider the non stochastic model with capital and labor in this chapter, and assume that the period utility function in equation (12.1) is given by

$$u(c_t, \ell_t) = \frac{c_t^{1-\sigma}}{1-\sigma} + v(\ell_t),$$

where $\sigma > 0$. When σ is equal to one, the term $c_t^{1-\sigma}/(1-\sigma)$ is replaced by $\log(c_t)$.

a. Show that the optimal tax policy in this economy is to set capital taxes equal to zero in period 2 and from thereon, i.e., $\tau_t^k = 0$ for $t \geq 2$. (Hint: Given the preference specification, evaluate and compare equations (12.30) and (12.35a))

b. Suppose there is uncertainty in the economy as in the stochastic model with capital and labor in this chapter. Derive the optimal *ex ante capital tax rate* for $t \geq 2$.

Solution

a. We use the notations developed in the paragraph "constructing the Ramsey plan". We have

$$\begin{aligned} V(c,n,\Phi) &\equiv u(c,1-n) + \Phi(u_c c - u_l n) \\ &= c^{1-\sigma}\left(\tfrac{1}{1-\sigma} + \Phi\right) + v(1-n) - \Phi v_l n \end{aligned}.$$

This shows in particular that:

$$\frac{V_c(t+1)}{V_c(t)} = \frac{u_c(t+1)}{u_c(t)}.$$

Now recall that the first of the difference equations defining the Ramsey plan is:

$$V_c(t) = \beta V_c(t+1)[F_k(t+1) + 1 - \delta] \quad t \geq 1.$$

Also, the Ramsey plan is a competitive equilibrium, so that the Euler equation of the household problem is satisfied:

$$u_c(t) = \beta u_c(t+1)\big[(1-\tau_{t+1}^k)r_{t+1} + 1 - \delta\big] = \beta u_c(t+1)\big[(1-\tau_{t+1}^k)F_k(t+1) + 1 - \delta\big].$$

Combined with the previous equation and with the identity $\frac{V_c(t+1)}{V_c(t)} = \frac{u_c(t+1)}{u_c(t)}$, it implies that, for $t \geq 1$:

$$[F_k(t+1) + 1 - \delta] = \big[(1-\tau_{t+1}^k)F_k(t+1) + 1 - \delta\big].$$

So that $\tau_t^k = 0$ for all $t \geq 2$.

b. The reasoning is very similar. Under uncertainty, the first of the difference equations defining a Ramsey plan is:

$$V_c(s^t) = \beta \sum_{s_{t+1}} \pi(s_{t+1} \mid s^t) V_c(s_{t+1}, s^t)\big[F_k(s_{t+1}, s^t) + 1 - \delta\big] \quad t \geq 1.$$

As before, the Ramsey plan is a competitive equilibrium, so that the Euler equation of the household problem is verified:

$$\begin{aligned} u_c(s^t) &= \beta \textstyle\sum_{s^{t+1}} \pi(s_{t+1} \mid s^t) u_c(s_{t+1}, s^t)\big[\big(1-\tau^k(s_{t+1}, s^t)\big)\, r(s_{t+1}, s^t) + 1 - \delta\big] \\ &= \beta \textstyle\sum_{s^{t+1}} \pi(s_{t+1} \mid s^t) u_c(s_{t+1}, s^t)\big[\big(1-\tau^k(s_{t+1}, s^t)\big)\, F_k(s_{t+1}, s^t) + 1 - \delta\big]. \end{aligned}$$

We use again the fact that:

$$\frac{V_c(s_{t+1}, s^t)}{V_c(s^t)} = \frac{u_c(s_{t+1}, s^t)}{u_c(s^t)}.$$

And we find:

$$\beta \sum_{s_{t+1}} \pi(s_{t+1} \mid s^t) \frac{u_c(s_{t+1}, s^t)}{u_c(s^t)} \tau^k(s_{t+1}, s^t) F_k(s_{t+1}, s^t) = 0 \quad t \geq 1.$$

With $p(s_{t+1} \mid s^t) = \beta\pi(s_{t+1} \mid s^t)\frac{u_c(s_{t+1}, s^t)}{u_c(s^t)}$ and $r(s_{t+1}, s^t) = F_k(s_{t+1}, s^t)$, this can be rewritten:

$$\beta \sum_{s_{t+1}} p(s_{t+1} \mid s^t) \tau^k(s_{t+1}, s^t) = 0 \quad t \geq 1.$$

i.e., the ex-ante capital tax is zero for $t \geq 2$.

Exercise 12.4. *Two labor inputs*

Consider the non stochastic model with capital and labor in this chapter, but assume that there are two labor inputs, n_{1t} and n_{2t}, entering the production function, $F(k_t, n_{1t}, n_{2t})$. The household's period utility function is still given by $u(c_t, \ell_t)$ where leisure is now equal to

$$\ell_t = 1 - n_{1t} - n_{2t}.$$

Let τ_{it}^n be the flat-rate tax at time t on wage earnings from labor n_{it}, for $i = 1, 2$, and τ_t^k denotes the tax on earnings from capital.

a. Solve for the Ramsey plan. What is the relationship between the optimal tax rates τ_{1t}^n and τ_{2t}^n for $t \geq 1$? Explain why your answer is different for period $t = 0$. As an example, assume that k and n_1 are complements while k and n_2 are substitutes.

We now assume that the period utility function is given by $u(c_t, \ell_{1t}, \ell_{2t})$ where

$$\ell_{1t} = 1 - n_{1t}, \qquad \text{and} \qquad \ell_{2t} = 1 - n_{2t}.$$

Further, the government is now constrained to set the same tax rate on both types of labor, i.e., $\tau_{1t}^n = \tau_{2t}^n$ for all $t \geq 0$.

b. Solve for the Ramsey plan. (Hint: Using the household's first-order conditions, we see that the restriction $\tau_{1t}^n = \tau_{2t}^n$ can be incorporated into the Ramsey problem by adding the constraint $u_{\ell_1}(t)F_{n_2}(t) = u_{\ell_2}(t)F_{n_1}(t)$.)

c. Suppose that the solution to the Ramsey problem converges to a steady state where the constraint that the two labor taxes should be equal is binding. Show that the limiting capital tax is not zero unless $F_{n_1}F_{n_2k} = F_{n_2}F_{n_1k}$.

Solution

a. We follow the steps described in the paragraph "constructing the Ramsey Plan".

Step 1: Solve the household problem.

The household problem is:

$$\max_{\{c_t\}_{t=0}^{+\infty}, \{l_t\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \beta^t u(c_t, 1 - n_{1t} - n_{2t})$$
subject to
$$\sum_{t=0}^{+\infty} q_t^0 (1 + \tau_t^c) c_t = \sum_{t=0}^{+\infty} q_t^0 \left((1 - \tau_t^{n_1}) w_{1t} n_{1t} + (1 - \tau_t^{n_2}) w_{2t} n_{2t} \right) + \left(r_0 + (1 - \delta) \right) k_0 + b_0.$$

The first order conditions are

$$\begin{array}{rrcl} c_t: & \beta^t u_c(t) & = & \lambda q_t^0 (1 + \tau_t^c) \\ n_{1t}: & \beta^t u_l(t) & = & \lambda q_t^0 (1 - \tau_t^{n_1}) w_{1t} \\ n_{2t}: & \beta^t u_l(t) & = & \lambda q_t^0 (1 - \tau_t^{n_2}) w_{2t}. \end{array}$$

Taking time zero consumption as numeraire we find:

$$\begin{array}{rcl} q_t^0 & = & \beta^t \frac{u_c(t)}{u_c(0)} \\ q_t^0 (1 - \tau_t^{n_1}) w_{1t} & = & q_t^0 (1 - \tau_t^{n_2}) w_{2t} = \beta^t \frac{u_l(t)}{u_c(0)}. \end{array}$$

Step 2: Write the intertemporal budget constraint.

The intertemporal budget constraint is:

$$\sum_{t=0}^{+\infty} \beta^t \left(u_c(t)c_t - u_l(t)(n_{1t}+n_{2t})\right) = u_c(0)\left((r_0+(1-\delta))k_0+b_0\right).$$

Let $A \equiv u_c(0)\left((r_0+(1-\delta))k_0+b_0\right)$.

Step 3: Form the Lagrangian.

Now define:

$$V(c_t, n_{1t}, n_{2t}, \Phi) \equiv u(c_t, 1-n_{1t}-n_{2t}) + \Phi(1+\tau_0^c)\left(u_c(t)c_t - u_l(t)(n_{1t}+n_{2t})\right).$$

The Lagrangian associated with the Ramsey plan is:

$$J = \sum_{t=0}^{+\infty} \beta^t \left\{V(c_t, n_{1t}, n_{2t}, \Phi) + \theta_t\left(F(k_t, n_{1t}, n_{2t}) - (1-\delta)k_t - c_t - g_t - k_{t+1}\right)\right\} - \Phi A.$$

The first order conditions are:

$$\begin{array}{lcl}
c_t: \quad V_c(t) & = & \theta_t, \quad t \geq 1 \\
n_{1t}: \quad V_{n_1}(t) = V_{n_2}(t) & = & -\theta_t F_{n_1}(t), \quad t \geq 1 \\
n_{2t}: \quad V_{n_2}(t) = V_{n_1}(t) & = & -\theta_t F_{n_2}(t), \quad t \geq 1 \\
k_{t+1}: \quad \theta_t & = & \beta\theta_{t+1}[F_k(t+1)+(1-\delta)], \quad t \geq 0 \\
c_0: \quad V_c(0) & = & \theta_0 + \Phi A_c \\
n_{10}: \quad V_{n_1}(0) & = & -\theta_0 F_{n_1}(0) + \Phi A_{n_1} \\
n_{20}: \quad V_{n_2}(0) & = & -\theta_0 F_{n_2}(0) + \Phi A_{n_2}.
\end{array}$$

The Ramsey Plan is thus solution of the following system of difference equations:

$$\begin{array}{lcl}
V_c(t) & = & \beta V_c(t+1)[F(_k(t+1)+1-\delta], \quad t \geq 1 \\
V_{n_1}(t) = V_{n_2}(t) & = & -V_c(t)F_{n_1}(t), \quad t \geq 1 \\
V_{n_2}(t) = V_{n_1}(t) & = & -V_c(t)F_{n_2}(t), \quad t \geq 1 \\
V_{n_1}(0) & = & [\Phi A_c - V_c(0)]F_{n_1}(0) + \Phi A_{n_1} \\
V_{n_2}(0) & = & [\Phi A_c - V_c(0)]F_{n_2}(0) + \Phi A_{n_2}.
\end{array}$$

$$\begin{array}{c}
c_t + g_t + k_{t+1} = F(k_t, n_{1t}, n_{2t}) + (1-\delta)k_t \\
\sum_{t=0}^{+\infty} \beta^t \left(u_c(t)c_t - u_l(t)(n_{1t}+n_{2t})\right) - A = 0.
\end{array}$$

Now note that, from the first order conditions of the household problem, we have:

$$(1-\tau_t^{n_1})w_{1t} = (1-\tau_t^{n_2})w_{2t}.$$

From the first order condition of the firm's problem we have $w_{1t} = F_{n_1}(t)$ and $w_{2t} = F_{n_2}(t)$. But the difference equations defining the Ramsey plan imply that $F_{n_1}(t) = F_{n_2}(t)$, for all $t \geq 1$. Therefore :

$$(1-\tau_t^{n_1}) = (1-\tau_t^{n_2}).$$

That is, $\tau_t^{n_1} = \tau_t^{n_2}$, for all $t \geq 1$. At $t = 0$ we don't have $F_{n_1}(0) = F_{n_2}(0)$ because time zero capital tax is fixed exogenously. The following applies instead:

$$\theta_0(F_{n_1}(0) - F_{n_2}(0)) = \Phi(A_{n_1} - A_{n_2}) = \Phi u_c(0)(1-\tau_0^k)k_0(F_{kn_1}(0) - F_{kn_2}(0)),$$

where the last equality is found by differentiating A with respect to n_1 and n_2. Note that when $\Phi = 0$, i.e. when we don't put restrictions on τ_0^k, then we have $F_{n_1}(0) = F_{n_2}(0)$.
As before use the first order conditions of the household's problem and of the firm's problem to write that $(1-\tau_0^{n_1})F_{n_1}(0) = (1-\tau_0^{n_2})F_{n_2}(0)$, so that $\frac{F_{n_2}(0)}{F_{n_1}(0)} = \frac{1-\tau_0^{n1}}{1-\tau_0^{n2}}$. Replacing this relationship in the equation above, we find:

$$\theta_0 F_{n_1}(0)\frac{\tau_0^{n_1} - \tau_0^{n_2}}{1-\tau_0^{n_2}} = \Phi u_c(0)(1-\tau_0^k)k_0(F_{kn_1}(0) - F_{kn_2}(0)).$$

Recall that θ_0 and Φ are Lagrange multipliers of inequality constraints and are therefore positive. We have:

$$\text{sign}(\tau_0^{n_1} - \tau_0^{n_2}) = \text{sign}(F_{kn_1}(0) - F_{kn_2}(0)).$$

Now assume as an example that k and n_1 are complement, so that the marginal return with respect to k is increasing in n_1. Also assume that k and n_2 are substitute, so that the marginal return with respect to k is decreasing in n_2. This implies:

$$\tau_0^{n_1} > \tau_0^{n_2}.$$

Some intuition may be gained by remembering that the welfare cost of distortionary taxation is measured by the derivative of the Lagrangian with respect to τ_0^k :

$$\frac{\partial J}{\partial \tau_k^0} = \phi u_c(0)F_k(0)k_0. \tag{148}$$

Note that, if the marginal product of capital is high, then the welfare cost of using distortionary taxation is high – equivalently welfare could be improved a lot by increasing τ_0^k. Therefore, one can reduce the welfare cost of distortionary taxation by lowering the marginal product of capital. Therefore, the planner should choose a relatively high n_2 and a relatively low n_1, which is achieved by taxing n_2 less than n_1.

b. We follow the usual steps.

Step 1: Solve the household problem.

The household problem is:

$$\max_{\{c_t\}_{t=0}^{+\infty},\{l_t\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \beta^t u(c_t, 1-n_{1t}, 1-n_{2t})$$
subject to
$$\sum_{t=0}^{+\infty} q_t^0(1+\tau_t^c)c_t = \sum_{t=0}^{+\infty} q_t^0\left((1-\tau_t^n)w_{1t}n_{1t} + (1-\tau_t^n)w_{2t}n_{2t}\right) + \left(r_0 + (1-\delta)\right)k_0 + b_0.$$

The first order conditions are

$$\begin{array}{rrcl} c_t: & \beta^t u_c(t) & = & \lambda q_t^0(1+\tau_t^c) \\ n_{1t}: & \beta^t u_{l_1}(t) & = & \lambda q_t^0(1-\tau_t^n)w_{1t} \\ n_{2t}: & \beta^t u_{l_2}(t) & = & \lambda q_t^0(1-\tau_t^n)w_{2t}. \end{array}$$

Taking time zero consumption as numeraire we find:

$$q_t^0 = \beta^t \tfrac{u_c(t)}{u_c(0)}$$
$$q_t^0(1-\tau_t^n)w_{1t} = \beta^t \tfrac{u_{l_1}(t)}{u_c(0)}$$
$$q_t^0(1-\tau_t^n)w_{2t} = \beta^t \tfrac{u_{l_2}(t)}{u_c(0)}.$$

Step 2: Write the intertemporal budget constraint.

The intertemporal budget constraint is:

$$\sum_{t=0}^{+\infty} \beta^t \left(u_c(t)c_t - u_{l_1}(t)n_{1t} - u_{l_2}(t)n_{2t}\right) = u_c(0)\left((r_0 + (1-\delta))k_0 + b_0\right).$$

Let $A \equiv u_c(0)\left((r_0 + (1-\delta))k_0 + b_0\right)$.

Step 3: Form the Lagrangian.

We follow the hint and incorporate constraints forcing the planner to raise equal taxes on both labor inputs. Recall that from the first order conditions of the household problem, we have $(1-\tau_t^{n_i})w_1 = u_{li}(t)$, $i = 1, 2$. From the first order conditions of the firm's problem, we have $w_{it} = F_{n_i}(t)$, $i = 1, 2$. In a competitive equilibrium, both labor taxes are equal if and only if:

$$\frac{u_{l_1}(t)}{F_{n_1}(t))} = \frac{u_{l_1}(t)}{F_{n_2}(t)} \Leftrightarrow u_{l_1}(t)F_{n_2}(t) = u_{l_2}(t)F_{n_1}(t).$$

Define as usual:

$$V(c_t, n_{1t}, n_{2t}, \Phi) \equiv u(c_t, 1-n_{1t}, 1-n_{2t}) + \Phi\left(u_c(t)c_t - u_{l_1}(t)n_{1t} - u_{l_2}(t)n_{2t}\right).$$

The Lagrangian associated with the Ramsey plan is, with the new constraint:

$$J = \sum_{t=0}^{+\infty} \beta^t \left\{V(c_t, n_{1t}, n_{2t}, \Phi) + \theta_t\left(F(k_t, n_{1t}, n_{2t}) - (1-\delta)k_t - c_t - g_t - k_{t+1}\right)\right\}$$
$$+ \sum_{t=0}^{+\infty} \beta^t \psi_t \left\{u_{l_1}(t)F_{n_2}(t) - u_{l_2}(t)F_{n_1}(t)\right\} - \Phi A.$$

We only state the associated first order conditions :

$$
\begin{array}{llll}
c_t: & V_c(t) & = & \theta_t - \psi_t[u_{l_1,c}(t)F_{n_2}(t) - u_{l_2,c}(t)F_{n_1}(t)] \\
n_{1t}: & V_{n_1}(t) & = & -\theta_t F_{n_1}(t) \\
 & & & -\psi_t[-u_{l_1,l_1}(t)F_{n_2}(t) + u_{l_1}(t)F_{n_1,n_2}(t) + u_{l_1,l_2}(t)F_{n_1}(t) - u_{l_1}(t)F_{n_1,n_1}(t)] \quad t \geq 1 \\
n_{2t}: & V_{n_2}(t) & = & -\theta_t F_{n_2}(t) \\
 & & & -\psi_t[-u_{l_1,l_2}(t)F_{n_2}(t) + u_{l_1}(t)F_{n_2,n_2}(t) + u_{l_2,l_2}(t)F_{n_1}(t) - u_{l_2}(t)F_{n_1,n_2}(t)] \quad t \geq 1 \\
k_{t+1}: & \theta_t & = & \beta\theta_{t+1}[F_k(t+1) + (1-\delta)] \\
 & & & -\beta\psi_{t+1}[u_{l_1}(t+1) + F_{n_2,k}(t+1) - u_{l_2}(t+1)F_{n_1,k}(t+1)], \quad t \geq 0 \\
c_0: & V_c(0) & = & \theta_0 - \psi_0[u_{l_1,c}(0)F_{n_2}(0) - u_{l_2,c}(0)F_{n_1}(0)] + \Phi A_c \\
n_{10}: & V_{n_1}(0) & = & -\theta_0 F_{n_1}(0) \\
 & & & -\psi_0[-u_{l_1,l_1}(0)F_{n_2}(0) + u_{l_1}(0)F_{n_1,n_2}(0) + u_{l_1,l_2}(0)F_{n_1}(0) - u_{l_1}(0)F_{n_1,n_1}(0)] \\
 & & & +\Phi A_{n_1} \\
n_{20}: & V_{n_2}(0) & = & -\theta_0 F_{n_2}(0) \\
 & & & -\psi_0[-u_{l_1,l_2}(0)F_{n_2}(0) + u_{l_1}(0)F_{n_2,n_2}(0) + u_{l_2,l_2}(0)F_{n_1}(0) - u_{l_2}(0)F_{n_1,n_2}(0)] \\
 & & & +\Phi A_{n_2}.
\end{array}
$$

And:

$$
\begin{array}{l}
c_t + g_t + k_{t+1} = F(k_t, n_{1t}, n_{2t}) + (1-\delta)k_t, \quad t \geq 0 \\
\sum_{t=0}^{+\infty} \beta^t \left(u_c(t)c_t - u_{l_1}(t)n_{1t} - u_{l_2}(t)n_{2t}\right) - A = 0 \\
\psi_t \left\{u_{l_1}(t)F_{n_2}(t) - u_{l_2}(t)F_{n_1}(t)\right\} = 0, \quad t \geq 0.
\end{array}
$$

c. Assume that the solution of this Ramsey converges to a steady state for which the constraint that the two labor taxes should be equal binds. Thus $\psi_t \to \psi \neq 0$ as $t \to +\infty$. The steady state version of the first order condition associated with k_{t+1} can be written:

$$1 = \beta(F_k + 1 - \delta) - \beta\frac{\psi}{\theta}(u_{l_1}F_{n_2,k} - u_{l_2}F_{n_1,k}).$$

On the other hand the no arbitrage condition for capital is, in steady state:

$$\frac{q_t^0}{q_{t+1}^0} = \frac{1}{\beta}\frac{u_c(t)}{u_c(t+1)} = \frac{1}{\beta} = (1-\tau_{t+1}^k)F_k + 1 - \delta.$$

Combining the last two equations gives:

$$\tau_{t+1}^k = -\frac{\psi}{\theta}(u_{l_1}F_{n_2,k} - u_{l_2}F_{n_1,k}),$$

which is different from zero unless $u_{l_1}F_{n_2,k} - u_{l_2}F_{n_1,k}$ is zero. Now recall that

$$u_{l_1}F_{n_2,k} = u_{l_2}F_{n_1,k} \Leftrightarrow (1-\tau^n)w_1F_{n_2,k} = (1-\tau^n)F_{n_1,k} \Leftrightarrow F_{n_1}F_{n_2,k} = F_{n_2}F_{n_1,k}.$$

Thus the steady state tax on capital is not zero unless $F_{n_1}F_{n_2,k} = F_{n_2}F_{n_1,k}$.

CHAPTER 13

Self-insurance

Exercise 13.1.

A single consumer has preferences over sequences of a single consumption good that are ordered by $\sum_{t=0}^{\infty} \beta^t u(c_t)$ where $\beta \in (0,1)$ and $u(\cdot)$ is strictly increasing, twice continuously differentiable, strictly concave, and satisfies the Inada condition $\lim_{c \searrow 0} u'(c) = +\infty$. The one good is not storable. The consumer has an endowment sequence of the one good $y_t = \lambda^t, t \geq 0$, where $|\lambda\beta| < 1$. The consumer can borrow or lend at a constant and exogenous risk-free net interest rate of r that satisfies $(1+r)\beta = 1$. The consumer's budget constraint at time t is

$$b_t + c_t \leq y_t + (1+r)^{-1} b_{t+1},$$

for all $t \geq 0$, where b_t is the *debt* (if positive) or *assets* (if negative) due at t, and the consumer has initial debt $b_0 = 0$.

Part I. In this part, assume that the consumer is subject to the *ad hoc* borrowing constraint $b_t \leq 0 \ \forall t \geq 0$. Thus, the consumer can lend but not borrow.

a. Assume that $\lambda < 1$. Compute the household's optimal plan for $\{c_t, b_{t+1}\}_{t=0}^{\infty}$.

b. Assume that $\lambda > 1$. Compute the household's optimal plan $\{c_t, b_{t+1}\}_{t=0}^{\infty}$.

Part II. In this part, assume that the consumer is subject to the natural borrowing constraint associated with the given endowment sequence.

c. Compute the natural borrowing limits for all $t \geq 0$.

d. Assume that $\lambda < 1$. Compute the household's optimal plan for $\{c_t, b_{t+1}\}_{t=0}^{\infty}$.

e. Assume that $\lambda > 1$. Compute the household's optimal plan $\{c_t, b_{t+1}\}_{t=0}^{\infty}$.

Solution

We let $\{\underline{b}_{t+1}\}_{t=0}^{+\infty} \geq 0$ be a sequence of borrowing limits. The first order necessary conditions of the agent's problem are

$$u'(c_t) \geq u'(c_{t+1}) \quad = \quad \text{if } b_t < \underline{b}_t \tag{149}$$

$$c_t + b_t = y_t + \beta b_{t+1}. \tag{150}$$

Sufficient conditions for optimality are given as follows. A sequence $\{c_t^*, b_{t+1}^*\}_{t=0}^{+\infty}$ is a solution of the agent's problem if it satisfies (186) and (150) together with the "transversality condition"

$$\lim_{T \to +\infty} \beta^{T+1} u'(c_{T+1}^*)(b_{T+1}^* - \underline{b}_{T+1}) = 0 \tag{151}$$

a. We guess and verify that consumption c_t is constant and equal to the annuity value $\bar{c}$ of the agents' endowment

$$\bar{c} = (1-\beta) \sum_{t=0}^{+\infty} \beta^t y_t = \frac{1-\beta}{1-\beta\lambda}. \tag{152}$$

Because $|\beta\lambda| < 1$, we can compute the corresponding sequence of borrowing b_{t+1} by iterating forward on the budget constraint $b_t = y_t - c_t + \beta b_{t+1}$. Namely

$$\begin{aligned} b_t &= \sum_{j=0}^{+\infty} \beta^j \left(\lambda^{t+j} - \frac{1-\beta}{1-\beta\lambda} \right) \\ b_t &= \frac{\lambda^t - 1}{1-\beta\lambda} \end{aligned} \tag{153}$$

The sequence $\{b_t, c_t = \bar{c}\}_{t=0}^{+\infty}$ satisfies the first order conditions (186), (150) together with the transversality condition (151) and the borrowing constraint $b_t \leq 0$.

The intuition is that, if $\lambda < 1$, the endowment sequence is decreasing monotonically. The consumer can achieve a constant consumption stream by saving. More specifically, the consumer will consume the annuity value of his total endowment stream:

$$\sum_{j=0}^{\infty} (\beta\lambda)^j = \frac{1}{1-\beta\lambda}$$

The annuity value, and hence the constant consumption, is:

$$c_t \equiv \bar{c} = \frac{r}{1+r} \frac{1}{1-\beta\lambda} = \frac{1-\beta}{1-\beta\lambda}$$

b. When $\lambda > 1$, we guess that $c_t = y_t = \lambda^t$ and $b_t = 0$. This sequence of consumption and borrowings clearly satisfies (186), (150), and (151).

When $\lambda > 1$, the consumer faces a monotonically increasing endowment sequence. She wants to borrow in the earlier periods, and repay the loan in later periods. As $\beta\lambda < 1$, he would like to consume the annuity value of the discounted sum of her endowment stream, $\frac{1-\beta}{1-\beta\lambda}$, at all periods. However, at time 0, this amount exceeds the time 0 endowment, 1. Hence, the ad hoc borrowing constraint is binding. Similarly, given that the consumer consumed all her endowment at time $t-1$, she wants to consume $\lambda^t \frac{1-\beta}{1-\beta\lambda}$ at time t, which, again, is more than what she has at time t, λ^t. That is, the ad hoc borrowing constraint is binding for all t. In summary, $c_t = y_t \equiv \lambda^t$ and $b_t \equiv 0$ for all $t \geq 0$. Note that the consumption does diverge to infinity, although there is no uncertainty.
c. The sequence of natural borrowing limits is given by

$$\underline{b}_t = \sum_{j=0}^{+\infty} \beta^j y_{t+j} = \frac{\lambda^t}{1-\beta\lambda} \tag{154}$$

Under the non-negativity constraint on consumption, the maximum repayment will be in the form of surrendering all future endowment from time t on.

d. and e. We guess and verify that the consumption is constant and equal to the annuity value $\bar{c}$ of the agent's endowment, given by (152). The corresponding sequence of borrowing is given by (153), clearly satisfies the natural borrowing limit, $b_t \leq \underline{b}_t$, and the transversality condition (151).

When $|\lambda| < 1$, $b_t < 0$, meaning that the agent is saving. The natural borrowing limit is less restrictive than the ad hoc borrowing constraint. In part a, we derived the consumption and borrowing process. The ad hoc borrowing constraint was not binding at any period. Hence, relaxing the borrowing constraint will not produce any different outcome. The same $\{c_t\}_{t=0}^{\infty}$ and $\{b_t\}_{t=1}^{\infty}$ will be chosen.

When $|\lambda| > 1$, $b_t > 0$, meaning that the agent is a perpetual borrower. From part b, we know that the desired consumption sequence is $c_t \equiv \frac{1-\beta}{1-\beta\lambda}$. b_t, the amount of borrowing that the consumer enter time t with, is the cumulative sum of his past dissavings.

$$\begin{aligned} b_t &= \sum_{j=0}^{t-1}(1+r)^{t-j}\left[\frac{1-\beta}{1-\beta\lambda} - \lambda^j\right] \\ &= \beta^{-t}\left[\frac{1-\beta}{1-\beta\lambda}\frac{1-\beta^t}{1-\beta} - \frac{1-(\beta\lambda)^t}{1-\beta\lambda}\right] \\ &= \frac{\lambda^t - 1}{1-\beta\lambda} \end{aligned}$$

Comparing to the borrowing limit in part b, we readily see that the natural borrowing constraint is never binding. Hence, the desired consumption sequence, $c_t \equiv \frac{1-\beta}{1-\beta\lambda}$, for all t is admissible under the natural borrowing constraint. The sequence $\{b_t\}_{t=1}^{\infty}$ follows the formula above.

Exercise 13.2.

The household has preferences over stochastic processes of a single consumption good that are ordered by $E_0 \sum_{t-0}^{\infty} \beta^t \ln(c_t)$ where $\beta \in (0,1)$ and E_0 is the mathematical expectation with respect to the distribution of the consumption sequence of a single nonstorable good, conditional on the value of the time 0 endowment. The consumer's endowment is the following stochastic process: at times $t = 0, 1$, the household's endowment is drawn from the distribution $\text{prob}(y_t = 2) = \pi$, $\text{prob}(y_t = 1) = 1 - \pi$, where $\pi \in (0,1)$. At all times $t \geq 2$, $y_t = y_{t-1}$. At each date $t \geq 0$, the household can lend, but not borrow, at an exogenous and constant risk-free one-period net interest rate of r that satisfies $(1+r)\beta = 1$. The consumer's budget constraint at t is $a_{t+1} = (1+r)(a_t - c_t) + y_{t+1}$, subject to the initial condition $a_0 = y_0$. One-period assets carried $(a_t - c_t)$ over into period $t+1$ from t must be nonnegative, so that the no-borrowing constraint is $a_t \geq c_t$.

a. Draw a tree that portrays the possible paths for the endowment sequence from date 0 onward.

b. Assume that $y_0 = 2$. Compute the consumer's optimal consumption and lending plan.

c. Assume that $y_0 = 1$. Compute the consumer's optimal consumption and lending plan.

d. Under the two assumptions on the initial condition for y_0 in the preceding two questions, compute the asymptotic distribution of the marginal utility of consumption $u'(c_t)$ (which in this case is the distribution of $u'(c_t) = V_t'(a_t)$ for $t \geq 2$), where $V_t(a)$ is the consumer's value function at date t).

e. Discuss whether your results in part d conform to Chamberlain and Wilson's application of the supermartingale convergence theorem.

Solution

a. The tree is displayed in figure 1.

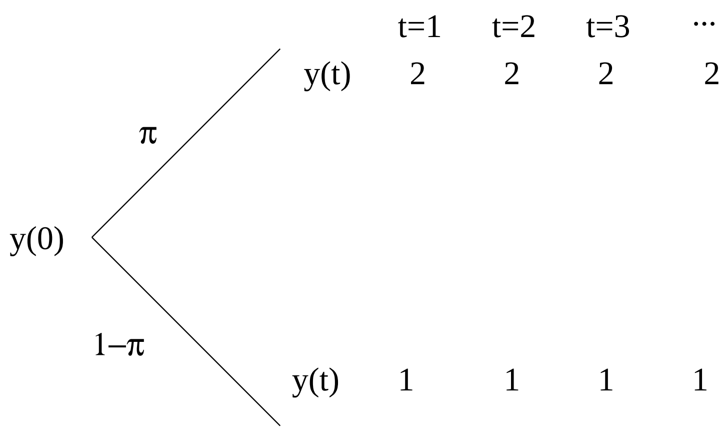


FIGURE 1. Exercise 13.2

The four possibilities are:

$$\{y_0 = 1, y_1 = 1, y_2 = 1, y_3 = 1, \ldots, y_t = 1, \ldots\}$$
$$\{y_0 = 1, y_1 = 2, y_2 = 2, y_3 = 2, \ldots, y_t = 2, \ldots\}$$
$$\{y_0 = 2, y_1 = 1, y_2 = 1, y_3 = 1, \ldots, y_t = 1, \ldots\}$$
$$\{y_0 = 2, y_1 = 2, y_2 = 2, y_3 = 2, \ldots, y_t = 2, \ldots\}$$

b.and c. We solve the consumer problem backward. Given an asset position a_1 at $t = 1$, and given that the endowment stays constant for all $t \geq 1$, we know that the agent's optimal consumption plan is to consume the annuity value c_1 of his asset position a_1 and of his stream of income

$$c_1 = (1 - \beta)a_1 + \beta y_1. \tag{155}$$

where $a_1 = y_1 + \beta^{-1}(y_0 - c_0)$. Therefore, the agent's problem can be reduced to

$$\max_{0 \leq c_0 \leq y_0} u(c_0) + \beta(1-\beta)^{-1} E\left(u\left(y_1 + (1-\beta)\beta^{-1}(y_0 - c_0)\right)\right). \tag{156}$$

The first order condition of this program is

$$u'(c_0) \geq E\left(u'\left(y_1 + (1-\beta)\beta^{-1}(y_0 - c_0)\right)\right), \quad = \quad \text{if } c_0 < y_0. \tag{157}$$

Clearly, because $u'(c) \to +\infty$ when $c \to 0$, we have $c_0 > 0$. Furthermore, the agent is saving ($c_0 < y_0$) if and only if

$$u'(y_0) - E(u'(y_1)) < 0 \tag{158}$$

If $y_0 = 2$, then (158) reduces to $(1-\pi)(u'(2) - u'(1)) < 0$.

Given that all the uncertainty is resolved at time 1, and that the endowment is constant for all $t \geq 1$, we can write the problem.

$$\begin{aligned} \max_{c} \quad & \ln(c) + \frac{(1-\pi)\beta}{1-\beta} \ln((1+r)(a-c)(1-\beta)+1) \\ & + \frac{\pi\beta}{1-\beta} \ln((1+r)(a-c)(1-\beta)+2) \\ & s.t. \quad c \leq a, \end{aligned}$$

where a is the time zero asset wealth and c the time zero consumption level. The first-order condition w.r.t. c can be easily derived.

$$\frac{1}{c} = \frac{1-\pi}{(1+r)(a-c)(1-\beta)+1} + \frac{\pi}{(1+r)(a-c)(1-\beta)+2}$$

Since $a = 2$, we can solve for c to obtain the following.

$$c^* = 2 - \frac{\pi - r - 2 + \sqrt{(\pi - r - 2)^2 + 8(r^2 + r)(1-\pi)}}{2(r^2 + r)}$$

It can be shown that $1 < c_0 = c^* < 2$, as long as $\pi \in (0, 1)$.

On the other hand, if $y_0 = 1$, the (158) reduces to $\pi(u'(1) - u'(2)) > 0$. Intuitively, if $y_0 = 1$, the consumer realizes that her endowment for all $t \geq 1$ will be greater than or equal to her current endowment. We know that she would like to consume more today (equalizing current marginal utility with the expected future marginal utility), but it is impossible due to the ad hoc borrowing constraint. She will consume all her endowment, 1, at time 0. If $y_1 = 2$, she will consume 2 for all $t \geq 1$. Likewise, if $y_1 = 1$, she will consume 1 for all $t \geq 1$. Her lending (saving) will be $s_t = 0$ for all $t \geq 0$.

In other words, if the initial endowment is high ($y_0 = 2$) the agent saves, and if the initial endowment is low ($y_0 = 1$), the agent consume all her endowment and does not save.

d. We let c_0 be the agent's consumption at time 0. When $y_0 = 2$, the distribution of the marginal utility of consumption for $t \geq 2$ is $(1 + \beta^{-1}(2 - c_0))^{-1}$ with probability $1 - \pi$ and $(2 + \beta^{-1}(2 - c_0))^{-1}$ with probability π. When $y_0 = 1$, for all $t \geq 1$, $u'(c_t) = \frac{1}{2}$ with probability π $(y_1 = 2)$, and $u'(c_t) = 1$ with probability $1 - \pi$ $(y_1 = 1)$.

e. The results do not conform to Chamberlain and Wilson application of the supermartingale convergence theorem because the sufficient condition of proposition 2 is not satisfied. Namely, the annuity value of the endowment process is not "sufficiently stochastic".

Exercise 13.3.

Consider the stochastic version of the savings problem under the following *natural borrowing constraints*. At each date $t \geq 0$, the consumer can issue risk-free one-period debt up to an amount that it is feasible for him to repay almost surely, given the nonnegativity constraint on consumption $c_t \geq 0$ for all $t \geq 0$.

a. Verify that the natural debt limit is $(1+r)^{-1} b_{t+1} \leq \frac{\overline{y}_1}{r}$.

b. Show that the natural debt limit can also be expressed as $a_{t+1} - y_{t+1} \geq -\frac{(1+r)\overline{y}_1}{r}$ for all $t \geq 0$.

c. Assume that y_t is an i.i.d. process with nontrivial distribution $\{\Pi_s\}$, in the sense that at least two distinct endowments occur with positive probabilities. Prove that optimal consumption diverges to $+\infty$ under the natural borrowing limits.

d. For identical realizations of the endowment sequence, get as far as you can in comparing what would be the sequences of optimal consumption under the natural and *ad hoc* borrowing constraints.

Solution

a. Assume that, at time t, the agent enters a loan of size βb_{t+1}. Then, she pays to the lender the stream $\{z_{t+j}\}_{j=1}^{+\infty}$. The borrowing position of the agents evolves according to the difference equation

$$\beta b_{t+1+j} = b_{t+j} - z_{t+j}. \tag{159}$$

A payment plan is some adapted sequence $\{z_{t+j}\}_{j=1}^{+\infty}$. Iterating forward on (159), we obtain that the borrowing position of the agent after J payments to the lender is

$$b_{t+J+1} = \beta^{-(J+1)} \left(\beta b_{t+1} - \sum_{j=1}^{J} \beta^j z_{t+j} \right) \tag{160}$$

We adopt the following definition: a loan of size b_{t+1} can be repaid if there exists a payment (state contingent) plan $\{z_{t+j}\}_{j=1}^{+\infty}$ such that

$$P\left(\limsup_{J\to+\infty} b_{t+J+1} < 0\right) = 1. \tag{161}$$

In words,the loan can be repaid almost surely if, for some payment plan, the borrowing position b_{t+J+1} is negative for J large enough, for almost all income realizations $\{y_{t+j}\}_{j=1}^{+\infty}$.

We now verify that $\beta b_{t+1} = \bar{y}_1/r$ is the natural borrowing limit, that is, the largest loan size that can be repaid almost surely. For simplicity we assume that the income process y_t satisfies the assumptions of question c. A loan of size $\beta b_{t+1} \leq \bar{y}_1/r$ can be repaid almost surely by consuming $c_{t+j} = 0$ and paying y_{t+j} to the lender, for all $j \geq 1$. The time t present value of this stream of payments is

$$\sum_{j=1}^{+\infty} \beta^j y_{t+j} \geq \frac{\bar{y}_1}{r} \geq \beta b_{t+1}, \tag{162}$$

because $y_{t+j} \geq \bar{y}_1$, for all $j \geq 1$. Since the event $\{y_{t+j} = \bar{y}_1, j \geq 0\}$ has probability zero, (162) holds with a strict inequality almost surely. Therefore, for J sufficiently large, b_{t+J+1} is negative. In other words, the loan can be repaid almost surely.

We let $\beta b_{t+1} = \bar{y}_1/r + \varepsilon$, for some $\varepsilon > 0$. First, since $\beta < 1$, there is a $J \in \mathbb{N}$ such that,

$$\beta b_{t+1} > \sum_{j=1}^{J} \beta^j \bar{y}_1 + \sum_{j=J+1}^{+\infty} \beta^j \bar{y}_S = \frac{\beta}{1-\beta}(1-\beta^J)\bar{y}_1 + \frac{1}{1-\beta}\beta^{J+1}\bar{y}_S \tag{163}$$

Now consider the set of income streams starting with J "low" income realizations $y_{t+j} = \bar{y}_1$

$$D = \left\{\{y_{t+j}\}_{j=1}^{+\infty} : y_{t+j} = \bar{y}_1, 1 \leq j \leq J\right\}. \tag{164}$$

The inequality (163) ensures that, for all income realizations in D, the time t present value of the agent's income stream is less than the value of the loan. Since y_t is assumed $i.i.d.$, the probability of D is positive and equal to Π_1^K. In other words, with positive probability, the loan cannot be repaid.

b. This follows from the definition $b_{t+1} = a_{t+1} - y_{t+1}$.

c. Because the agent faces a borrowing constraint and $(1+r)^{-1} = \beta$, the first order condition of her program is $u(c_t) \geq E(u(c_{t+1}))$. The proof of $a_t \to +\infty$ then follow from the argument outlined in the text, in the case $\underline{b}_t = 0$.

d. Under either borrowing constraint, the asset and consumption level will eventually diverge to infinity. However, we can characterize the differences in consumption patterns associated with these two borrowing constraints.

When $a = y$ and the endowment realization is low enough, the natural borrowing constraint allows the agent to bring forward some of her future endowment by borrowing, whereas the ad hoc constraint precludes this option. That is, she can consume more under the natural borrowing constraint.

When $a - y \geq 0$ and the endowment realization is high enough, the agent will save some of her current endowment. The optimal consumption requires intertemporal smoothing. Hence, a binding borrowing constraint translates into sub-optimality. Given the same asset level and endowment realization, the agent will save more when she is facing the ad hoc constraint, because this constraint is more likely to be binding in the future. Thus, $c^N(a, y) \geq c^A(a, y)$, for all (a, y) pairs such that $a \geq y$, where the optimal policy functions c^N and c^A correspond to natural borrowing constraint and ad hoc constraint, respectively.

However, we cannot claim that c^N weakly dominates c^A for identical realization of endowment sequence, because the same endowment process will generate different asset level sequences for each borrowing limit specification. For example, if the worst case endowment sequence ($y_t = \bar{y}_1, \ \forall t \leq \tau$) is realized and the natural borrowing constraint is "almost" binding at time τ, we can easily see that $\bar{y}_1 = c^A_\tau > c^N_\tau = \bar{y}_1 - R_\tau$, where R_τ denotes the interest repayment.[1]

The bottom line is that the ability to borrow provides a degree of buffer for the consumer, especially when the endowment realization is low and the asset level is close to zero. As a result, the marginal utility process under the natural borrowing constraint will be less volatile. In addition, the expected utility is higher under the natural borrowing limit.

[1] If we assume $\lim_{c\to 0} u(c) = -\infty$ and/or $\lim_{c\to 0} u'(c) = +\infty$, the natural borrowing constraint will never be binding, although it does affect the allocations.

CHAPTER 14

Incomplete markets models

Exercise 14.1. *Stochastic discount factor (Bewley-Krusell-Smith)*

A household has preferences over consumption of a single good ordered by a value function defined recursively according to $v(\beta_t, a_t, s_t) = u(c_t) + \beta_t E_t v(\beta_{t+1}, a_{t+1}, s_{t+1})$, where $\beta_t \in (0, 1)$ is the time-t value of a discount factor, and a_t is time-t holding of a single asset. Here v is the discounted utility for a consumer with asset holding a_t, discount factor β_t, and employment state s_t. The discount factor evolves according to a three-state Markov chain with transition probabilities $P_{i,j} = \text{Prob}(\beta_{t+1} = \bar{\beta}_j | \beta_t = \bar{\beta}_i)$. The discount factor and employment state at t are both known. The household faces the sequence of budget constraints

$$a_{t+1} + c_t \leq (1 + r)a_t + ws_t$$

where s_t evolves according to an n-state Markov chain with transition matrix P. The household faces the borrowing constraint $a_{t+1} \geq -\phi$ for all t. Formulate Bellman equations for the household's problem. Describe an algorithm for solving the Bellman equations. *Hint:* Form three coupled Bellman equations.

Solution

Let P be the transition matrix for β and Q the transition matrix for s. The household's problem is to solve

$$v(\beta, a, s) = \max_{a', c} \left\{ u(c) + \beta \sum_{\beta'} \sum_{s'} v(\beta', a', s') P(\beta'|\beta) Q(s'|s) \right\},$$

subject to

$$\begin{aligned} a' + c &\leq (1 + r)a + ws \\ a' &\geq -\phi. \end{aligned}$$

By substituting in for consumption and the restriction on asset holdings we obtain the Bellman equation:

$$v(\beta, a, s) = \max_{a' \geq -\phi} \left\{ u\left[(1 + r)a + ws - a'\right] + \beta \sum_{\beta'} \sum_{s'} v(\beta', a', s') P(\beta'|\beta) Q(s'|s) \right\}.$$

To solve the household's problem assume that the household can choose from a finite set of asset holdings (assume M gridpoints for the asset holdings), and write the value function as a $2 + N + M \times 1$ vector. Starting from an initial guess v_0, solve the Bellman equation by policy iteration, in each iteration respecting the borrowing constraint. Iterate until convergence.

Exercise 14.2. *Mobility Costs (Bertola)* $\diamond$

A worker seeks to maximize $E \sum_{t=0}^{\infty} \beta^t u(c_t)$, where $\beta \in (0, 1)$ and $u(c) = \frac{c^{1-\sigma}}{(1-\sigma)}$, and E is the expectation operator. Each period, the worker supplies one unit of labor inelastically (there is no unemployment) and either w^g or w^b, where $w^g > w^b$. A new "job" starts off paying w^g the first period. Thereafter, a job earns a wage governed by the two-state Markov process governing transition between good and bad wages on all jobs; the transition matrix is $\begin{bmatrix} p & (1-p) \\ (1-p) & p \end{bmatrix}$. A new (well-paying) job is always available, but the worker must pay mobility cost $m > 0$ to change jobs. The mobility cost is paid at the beginning of the period that a worker decides to move. The worker's period-t budget constraint is

$$A_{t+1} + c_t + mI_t \leq RA_t + w_t,$$

where R is a gross interest rate on assets, c_t is consumption at t, $m > 0$ is moving costs, I_t is an indicator equaling 1 if the worker moves in period t, zero otherwise, and w_t is the wage. Assume that $A_0 > 0$ is given and that the worker faces the no-borrowing constraint, $A_t \geq 0$ for all t.

a. Formulate the Bellman equation for the worker.

b. Write a Matlab program to solve the worker's Bellman equation. Show the optimal decision rules computed for the following parameter values: $m = .9, p = .8, R = 1.02, \beta = .95, w^g = 1.4, w^b = 1, \sigma = 4$. Use a range of assets levels of $[0, 3]$. Describe how the decision to move depends on wealth.

c. Compute the Markov chain governing the transition of the individual's state (A, w). If it exists, compute the invariant distribution.

d. In the fashion of Bewley, use the invariant distribution computed in part **c** to describe the distribution of wealth across a large number of workers all facing this same optimum problem.

Solution

a. A worker enters the period with asset holdings A and a job that payed $w^k \in \{w^g, w^b\}$ *last* period. To fix notation, if $w^k = w^g$ then $w^{-k} = w^b$ and vice versa. At the beginning of the period the worker decides to move (M) or stay (S). After that decision she chooses asset holdings optimally. Consumption is such that the budget constraint is satisfied.

$$\begin{aligned} v(A, w^k) &= \max_{M,S} \left\{ q_M, p q_S(w^k) + (1-p) q_S(w^{-k}) \right\} \\ q_M &= \max_{A'} \left\{ u(RA + w^g - A' - m) + \beta p v(A', w^g) + \beta(1-p) v(A', w^b) \right\} \\ q_S(w^g) &= \max_{A'} \left\{ u(RA + w^g - A') + \beta p v(A', w^g) + \beta(1-p) v(A', w^b) \right\} \\ q_S(w^b) &= \max_{A'} \left\{ u(RA + w^b - A') + \beta p v(A', w^b) + \beta(1-p) v(A', w^g) \right\}, \end{aligned}$$

A_0 is given and there are no future binding borrowing constraints.

b. Correction: moving cost $m = 0.4$. The value conditional on a good wage in the last period is higher than the value conditional on a low wage (see figure 14. Conditional on the bad wage in the previous period, it is always optimal to move (see figure 14). Conditional on the good wage, it is always optimal to not move (see figure 14). The matlab program is in `ex1402b.m` .

c. The matlab program is in `ex1402b.m`.

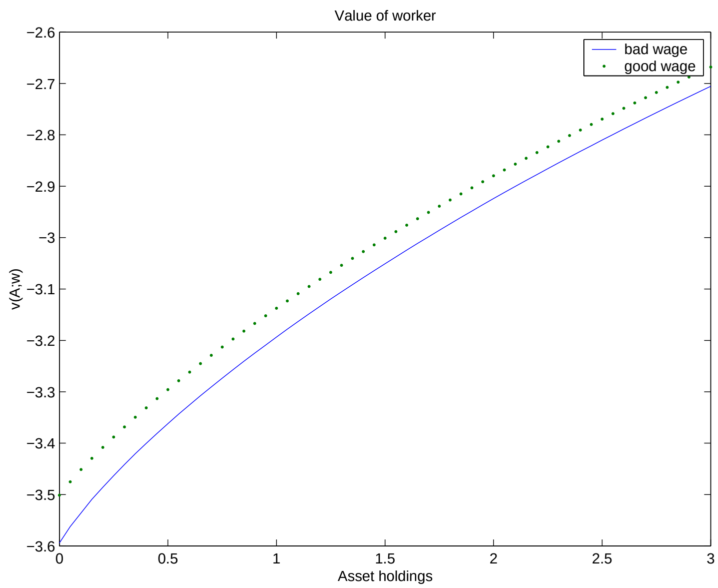


FIGURE 1. Exercise 14.2 a

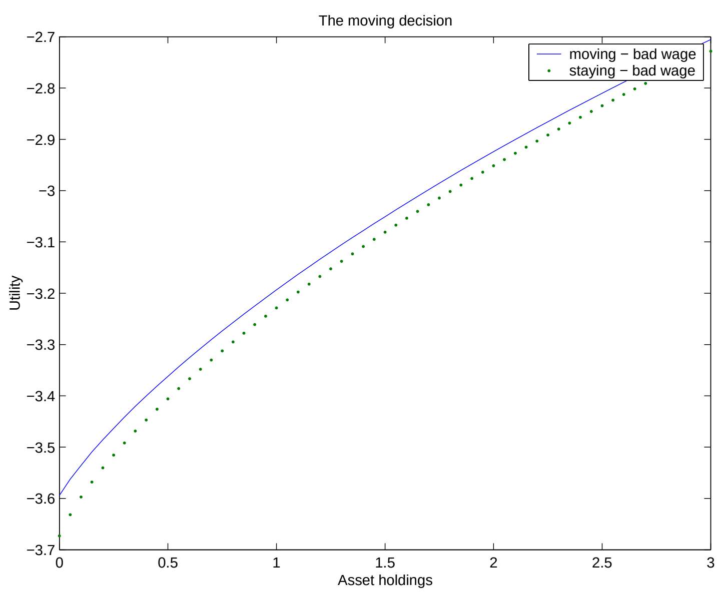


FIGURE 2. Exercise 14.2 b

d. The matlab program is in `ex1402b.m`.

Exercise 14.3. *Unemployment*

There is a continuum of workers with identical probabilities λ of being fired each period when they are employed. With probability $\mu \in (0, 1)$, each unemployed worker receives one offer to work at wage w drawn from the cumulative distribution function $F(w)$. If he accepts the offer, the worker receives the offered

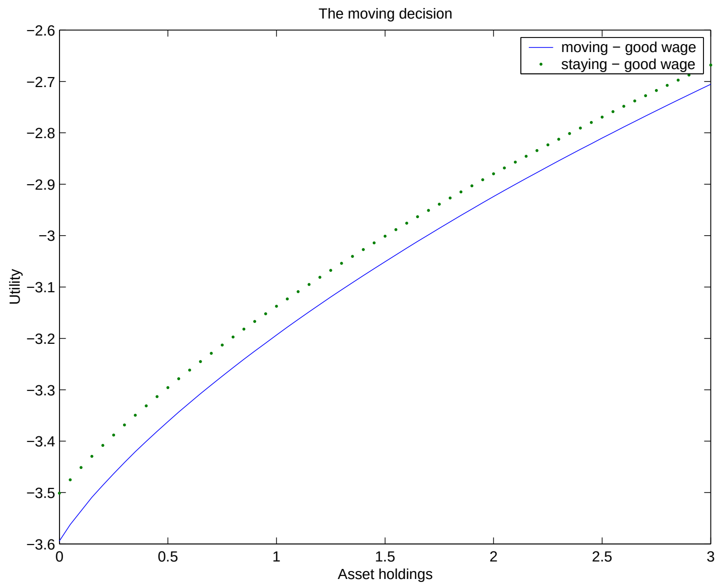


FIGURE 3. Exercise 14.2 c

wage each period until he is fired. With probability $1-\mu$, an unemployed worker receives no offer this period. The probability μ is determined by the function $\mu = f(U)$, where U is the unemployment rate, and $f'(U) < 0, f(0) = 1, f(1) = 0$. A worker's utility is given by $E\sum_{t=0}^{\infty} \beta^t y_t$, where $\beta \in (0,1)$ and y_t is income in period t, which equals the wage if employed and zero otherwise. There is no unemployment compensation. Each worker regards U as fixed and constant over time in making his decisions.

a. For fixed U, write the Bellman equation for the worker. Argue that his optimal policy has the reservation wage property.

b. Given the typical worker's policy (i.e., his reservation wage), display a difference equation for the unemployment rate. Show that a stationary unemployment rate must satisfy

$$\lambda(1-U) = f(U)\big[1 - F(\bar{w})\big]U,$$

where $\bar{w}$ is the reservation wage.

c. Define a *stationary equilibrium.*

d. Describe how to compute a stationary equilibrium. You don't actually have to compute it.

Solution

a. Assume the support of F is $[0,B]$ and $F(B)=1, F(0)=0$. The Bellman equation for an unemployed and the employed worker is

$$\begin{aligned} V^u(U) &= f(U)\int_0^B \max_{A,R}\left\{V^e(w'), \beta\int_0^1 V^u(U')df(U')\right\}dF(w') + (1-f(U))\beta\int_0^1 V^u(U')df(U') \\ V^e(w) &= w + \lambda\beta\int_0^1 V^u(U')df(U') + (1-\lambda)\beta V^e(w). \end{aligned}$$

Rewrite the value of being employed

$$V^e(w) = \left[\frac{1}{1-\beta+\lambda\beta}\right]w + \left[\frac{\lambda\beta}{1-\beta+\lambda\beta}\right]\int_0^1 V^u(U')df(U').$$

Substituting this into the value function of being unemployed we obtain the Bellman equation

(165)

$$V^u(U) = f(U)\int_0^B \max_{A,R}\left\{Aw + \lambda\beta A E\left(V^u\right), \beta E\left(V^u\right)\right\}dF(w') + (1-f(U))\beta E\left(V^u\right),$$

where

$$\begin{aligned} E\left(w\right) &= \int_0^B w'dF(w') \\ E\left(V^u\right) &= \int_0^1 V^u(U')df(U') \\ A &= \left[\frac{1}{1-\beta+\lambda\beta}\right]. \end{aligned}$$

For a fixed U the optimal policy will be a reservation policy. Conditional on receiving an offer, the unemployed worker accepts the draw w iff $w \geq \bar{w}$, where

$$\begin{aligned} \bar{w} &= \beta\frac{1-\lambda A}{A}E\left(V^u\right) \\ &= \beta(1-\beta-\lambda+\lambda\beta)E\left(V^u\right). \end{aligned}$$

The reservation wage depends on the expected value of unemployment, because that will determine the probability of getting an offer when fired in the future.

b. The unemployment rate evolves as follows: a fraction λ of the employed $(1-U)$ get fired and a fraction $f(U)(1-F(\bar{w}))$ of the unemployed receive an offer and accept it. The rest of the unemployed stay in unemployment.

$$U' = \lambda(1-U) + U(1-[f(U)(1-F(\bar{w}))]).$$

In a stationary equilibrium $U' = U$. The above equation simplifies to

$$\lambda(1-U) = Uf(U)(1-F(\bar{w})). \tag{166}$$

c. A stationary equilibrium is an unemployement rate U, a policy funtion $\bar{w}$, such that the policy function solves the worker's problem in equation (165) and the equilibrium unemployment rate satisfies equation (166).

d. Fix the unemployment rate and solves the household problem by iterating to convergence on the Bellman equation (165). Calculate the cutoff rule for the household and the resulting unemployment rate in the next period U'. If $U' > U$, lower U and solve the household problem again. Continue this algorithm until the the resulting $U' = U$.

Exercise 14.4. *Asset insurance*

Consider the following setup. There is a continuum of households who maximize

$$E \sum_{t=0}^{\infty} \beta^t u(c_t),$$

subject to

$$c_t + k_{t+1} + \tau \leq y + \max(x_t, g) k_t^{\alpha}, \quad c_t \geq 0,\ k_{t+1} \geq 0,\ t \geq 0,$$

where $y > 0$ is a constant level of income not derived from capital, $\alpha \in (0,1)$, τ is a fixed lump sum tax, k_t is the capital held at the beginning of t, $g \leq 1$ is an "investment insurance" parameter set by the government, and x_t is a stochastic household-specific gross rate of return on capital. We assume that x_t is governed by a two-state Markov process with stochastic matrix P, which takes on the two values $\bar{x}_1 > 1$ and $\bar{x}_2 < 1$. When the bad investment return occurs, $(x_t = \bar{x}_2)$, the government supplements the household's return by $\max(0, g - \bar{x}_2)$.
The household-specific randomness is distributed identically and independently across households. Except for paying taxes and possibly receiving insurance payments from the government, households have no interactions with one another; there are no markets.
Given the government policy parameters τ, g, the household's Bellman equation is

$$v(k,x) = \max_{k'} \{ u\big[y + \max(x,g)k^{\alpha} - k' - \tau\big] + \beta \sum_{x'} v(k',x') P(x,x') \}.$$

The solution of this problem is attained by a decision rule

$$k' = G(k,x),$$

that induces a stationary distribution $\lambda(k,x)$ of agents across states (k,x).
The average (or per capita) physical output of the economy is

$$Y = \sum_k \sum_x (x \times k^{\alpha}) \lambda(k,x).$$

The average return on capital to households, *including* the investment insurance, is

$$\nu = \sum_k \bar{x}_1 k^{\alpha} \lambda(k, x_1) + \max(g, \bar{x}_2) \sum_k k^{\alpha} \lambda(k, x_2),$$

which states that the government pays out insurance to all households for which $g > \bar{x}_2$.
Define a stationary equilibrium.

Solution

A stationary equilibrium is a policy function $G(k,x)$, a probability distribution $\lambda(k,x)$, and positive real numbers (Y,ν) such that

(i) The policy function solves the household problem in (**??**)

(ii) The stationary distribution $\lambda(k,x)$ is induced by $P(x',x)$ and $G(k,x)$:

$$\lambda(k',x') = \sum_{\{k:k'=G(k,x)\}} \sum_{x'} \lambda(k,x)P(x',x).$$

(iii) The average value of output is implied by the households' decision rules

$$Y = \sum_k \sum_x \left[xG(k,x)^\alpha\right] \lambda(k,x),$$

and the average return on capital is implied by the households' decision rules

$$\nu = \sum_k \left[\bar{x}_1 G(k,\bar{x}_1)^\alpha\right] \lambda(k,\bar{x}_1) + \max(g,\bar{x}_2) \sum_k \left[\bar{x}_2 G(k,\bar{x}_2)^\alpha\right] \lambda(k,\bar{x}_2).$$

Exercise 14.5.

One of a continuum of *ex ante* identical households, each with initial assets A_0, wants to maximize

$$E_0 - .5\sum_{t=0}^{\infty} \beta^t (c_t - b)^2, \quad \beta \in (0,1)$$

subject to the sequence of budget constraints

$$A_{t+1} = R(A_t + y_t - c_t), \quad A_0 \text{ given} \tag{1}$$

where y_t is an i.i.d. endowment sequence with mean $\overline{y}$, $R = \beta^{-1}$, c_t is consumption at t. Here b is a bliss level of consumption satisfying $b >> \overline{y}$. The household is also subject to the following constraint on its assets:

$$\lim_{t\to\infty} \beta^t A_t^2 < +\infty.$$

This condition rules out 'Ponzi schemes'.

a. Show that the household's optimal decision rule can be expressed as

$$c_t = (1 - R^{-1})(y_t + A_t).$$

Please interpret this condition. Show under the optimal consumption plan that c_t is a martingale. Prove that under the optimal consumption plan $c_{t+1} = c_t + (1 - R^{-1})(y_{t+1} - \bar{y})$.

b. Show under the optimal consumption plan

$$A_{t+1} = A_t + R^{-1} y_t.$$

Show that assets are a martingale.

c. Does the martingale convergence theorem tell you anything about the convergence of c_t or A_t?

d. Suppose that y_t is an i.i.d. Gaussian process with mean $\overline{y}$ and variance σ_y^2. Consider a cohort of *ex ante* identical agents each of whom starts with the same A_0. Compute the means and variances of the cross section distributions of consumption for $t \geq 0$. Compute the means and variances of the cross section distributions of assets for $t \geq 0$. Plot these for $t = 0, \ldots, 200$. Comment.

e. *Extra credit* Solve the typical household's problem by formulating it as an optimal linear regulator using `olrp.m` in Matlab. Does the solution from Matlab agree with what you found by hand? Why or why not? *Hint:* Does your optimal linear regulator impose condition (1)?

Solution

a. Solve the difference equation

$$\frac{1}{R}A_{t+1} = A_t + (y_t - c_t)\,,$$

forward to obtain

$$\sum_{j=0}^{\infty}\left(\frac{1}{R}\right)^j E_t c_{t+j} = A_t + \sum_{j=0}^{\infty}\left(\frac{1}{R}\right)^j E_t y_{t+j},$$

maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t u(y_t + A_t - R^{-1}A_{t+1}).$$

The Euler equations are :

$$u_c(c_t) = E_t R\beta u_c(c_{t+1}).$$

This and R$\beta = 1$ implies that consumption is a martingale

$$c_t = E_t c_{t+1}. \tag{167}$$

Substituting this condition into the budget constraint gives

$$\begin{aligned} c_t \sum_{j=0}^{\infty}\left(\frac{1}{R}\right)^j &= A_t + \sum_{j=0}^{\infty}\left(\frac{1}{R}\right)^j E_t y_{t+j} \\ c_t &= (1 - R^{-1})\left(A_t + \sum_{j=0}^{\infty}\left(\frac{1}{R}\right)^j E_t y_{t+j}\right) \\ c_t &= (1 - R^{-1})\,(A_t + y_t) + R^{-1}\bar{y}. \qquad (168) \end{aligned}$$

Equations (167) and (168) characterize the optimal consumption plan. Writing the consumption rule for period t and $t+1$ and using the martingale property of consumption we obtain:

$$\begin{aligned} c_{t+1} &= (R-1)A_t + (R-1)\,(y_t - c_t) + (1 - R^{-1})y_{t+1} + R^{-1}\bar{y} \\ E_t c_{t+1} &= (R-1)A_t + (R-1)\,(y_t - c_t) + \bar{y} \\ c_{t+1} - c_t &= (1 - R^{-1})(y_{t+1} - \bar{y}). \end{aligned}$$

Thus, $c_{t+1} = c_t + (1 - R^{-1})(y_{t+1} - \bar{y})$.

b. The optimal asset holdings satisfy:

$$\begin{aligned} c_t &= \left(1 - R^{-1}\right)\left(A_t + y_t\right) + R^{-1}\bar{y} \\ c_{t+1} &= \left(1 - R^{-1}\right)\left(A_{t+1} + y_{t+1}\right) + R^{-1}\bar{y} \\ (1 - R^{-1})(y_{t+1} - \bar{y}) &= (1 - R^{-1})(A_{t+1} + y_{t+1} - A_t - y_t). \end{aligned}$$

Taking expectations on each side, we find that assets follow a random walk: $E_t A_{t+1} = A_t$.Using (168) into the law of motion of assets we get:

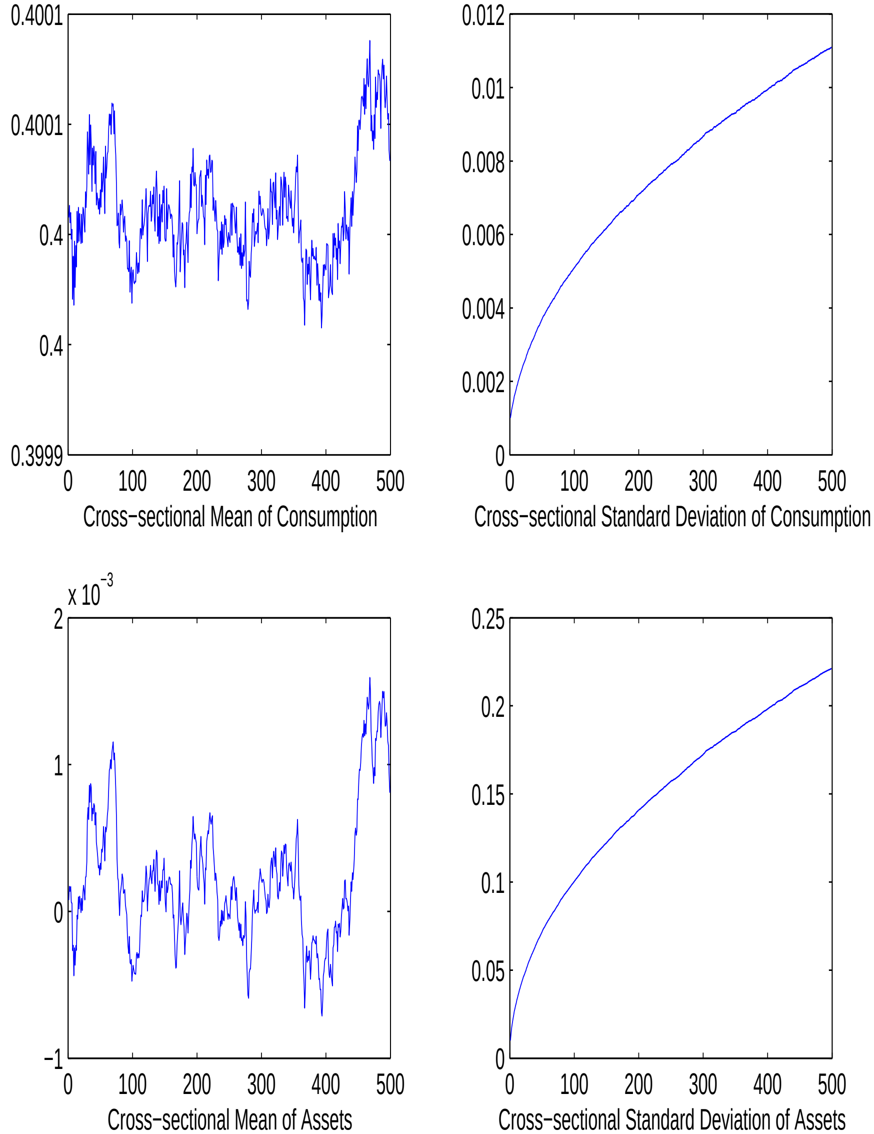
$$\begin{aligned} R^{-1} A_{t+1} &= R^{-1}\left(A_t + y_t\right) - R^{-1}\bar{y} \\ A_{t+1} &= A_t + y_t - \bar{y}. \end{aligned}$$

c. The martingale convergence theorem says that bounded martingales converge. Boundedness requires the existence of a finite M such that $E[|A_t|] < M$. The asset holdings and consumption are a martingale. But they are not bounded. The theorem does not apply. Assets and consumption scaled by the standard deviation of the income process converge to a Brownian motion.

d. The cross-sectional mean of consumption and assets stays constant but their cross-sectional dispersion increase. The matlab code is in `ex1405.m` . This confirms the lack of convergence found in the previous part.

e. Upon solving this problem with the olrp we find that the optimal solution is to consumpe the bliss level of consumption b. This solution does not impose the no-Ponzi scheme condition.

FIGURE 4. Exercise 14.5 : Cross-sectional Mean and Dispersion of Consumption and Assets



CHAPTER 15

Optimal social insurance

Exercise 15.1. *Lagrangian method with two-sided no commitment*

Consider the model of Kocherlakota with two-sided lack of commitment. There are two consumers, each having preferences $E_0 \sum_{t=0}^{\infty} \beta^t u[c_i(t)]$, where u is increasing, twice differentiable, and strictly concave, and where $c_i(t)$ is the consumption of consumer i. The good is not storable, and the consumption allocation must satisfy $c_1(t)+c_2(t) \leq 1$. In period t, consumer 1 receives an endowment of $y_t \in [0,1]$, and consumer 2 receives an endowment of $1-y_t$. Assume that y_t is i.i.d. over time and is distributed according to the discrete distribution $\text{Prob}(y_t = y_s) = \Pi_s$. At the start of each period, after the realization of y_s but before consumption has occurred, each consumer is free to walk away from the loan contract.

a. Find expressions for the expected value of autarky, before the state y_s is revealed, for consumers of each type. (*Note:* These need not be equal.)

b. Using the Lagrangian method, formulate the contract design problem of finding an optimal allocation that for each history respects feasibility and the participation constraints of the two types of consumers.

c. Use the Lagrangian method to characterize the optimal contract as completely as you can.

Solution

a. The value of autarchy consists of the present discounted value of future labor income. For agent 1 it is

$$v_1^{aut} = E_{-1} \sum_{t=0}^{\infty} \beta^t u(y_t),$$

and for agent 2, it is

$$v_2^{aut} = E_{-1} \sum_{t=0}^{\infty} \beta^t u(1-y_t).$$

b. The Lagrangean associated with the optimal contract, for given initial utility promise to agent 1, v, is:

$$\begin{aligned} J &= E_{-1} \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{c} u(1-c_t) + \alpha_t^1 \left[E_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j}) - \left[u(y_t) + \beta v_1^{aut}\right] \right] \\ +\alpha_t^2 \left[E_t \sum_{j=0}^{\infty} \beta^j u(1-c_{t+j}) - \left[u(1-y_t) + \beta v_2^{aut}\right] \right] \end{array} \right\} \\ &\quad + \phi \left[E_{-1} \sum_{t=0}^{\infty} \beta^t u(c_t) - v \right]. \end{aligned}$$

c. Using Abel's partial summation formula and the law of iterated expectations, we can rewrite J as

$$J = E_{-1} \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{c} (1+\mu_t^2) u(1-c_t) + (\phi + \mu_t^1) u(c_t) - (\mu_t^1 - \mu_{t-1}^1) \left[u(y_t) + \beta v_1^{aut}\right] \\ -(\mu_t^2 - \mu_{t-1}^2) \left[u(1-y_t) + \beta v_2^{aut}\right] \end{array} \right\} - \phi v,$$

where

$$\begin{aligned} \mu_t^1 &= \mu_{t-1}^1 + \alpha_t^1 \qquad \mu_{-1}^1 = 0 \\ \mu_t^2 &= \mu_{t-1}^2 + \alpha_t^2 \qquad \mu_{-1}^2 = 0. \end{aligned}$$

In addition to the complementary slackness conditions, the first order condition with respect to c_t is

$$\frac{1+\mu_t^2}{\phi+\mu_t^1} = \frac{u_c(c_t)}{u_c(1-c_t)}.$$

Assume that both agent's have the same history of binding constraints up to time $t-1$ ($\mu_{t-1}^2 = \mu_{t-1}^1$). When agent 1's participation constraint binds whereas agent 2's constraint is not binding then agent 1's multiplier is increased whereas agent 2's multiplier stays constant ($\alpha_t^1 > 0, \alpha_t^2 = 0$). Then the optimal contract prescribes increasing agent 1's consumption c_t at the expense of agent 2's consumption $(1-c_t)$. When nobody's participation constraint binds the allocation of consumption is identical to the one of the previous period and a function of all past binding constraints.

Exercise 15.2. *Optimal unemployment compensation*

a. Write a program to compute the autarky solution, and use it to reproduce Hopenhayn and Nicolini's calibration of r, as described in text.

b. Use your calibration from part a. Write a program to compute the optimum value function $C(V)$ for the insurance design problem with incomplete information. Use the program to form versions of Hopenhayn and Nicolini's table 1, column 4 for three different initial values of V, chosen by you to belong to the set (V_{aut}, V^e).

Solution

a. See first part of program `hopnic.m` . The value under autarky is 16759.
b. See `hopnic.m`, `hnval.m` and `hnval2.m` . The replacement rate graph is plotted for initial promised utilities $[16800, 16942, 17050]$. The graph is in figure **??**

Exercise 15.3. *Taxation after employment*

Show how the functional equation (15.61) and (15.62) would be modified if the planner were permitted to tax workers after they became employed.

Solution

If the planner were permitted to tax workers after they became employed the equations for the optimal unemployment contract would become

$$C(V) = \min_{c,a,V^u} \left\{ c + \beta p(a) C(V^e) + \beta(1-p(a))C(V^u) \right\},$$

subject to the promise keeping constraint

$$u(c) - a + \beta \left[p(a)V^e + (1-p(a))V^u \right] \geq V,$$

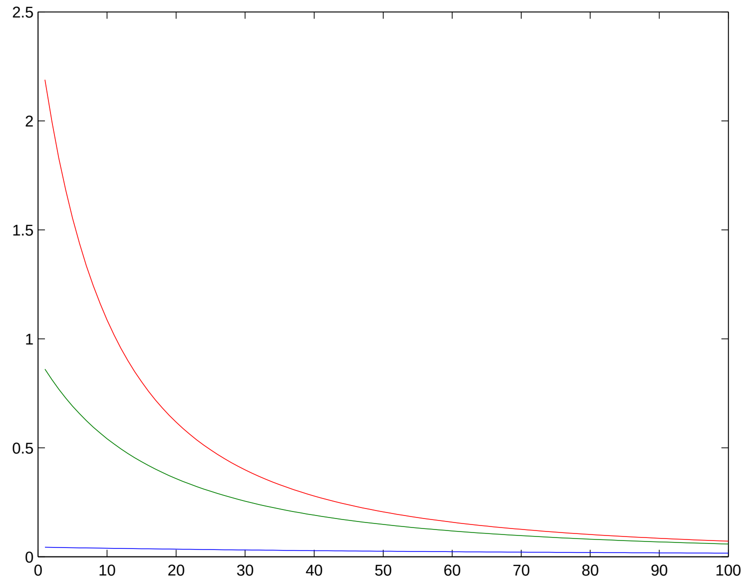


FIGURE 1. Exercise 15.2

and the incentive compatibility constraint

$$\beta p'(a)\,[V^e - V^u] \leq 1 \qquad = 1\ if\ a > 0,$$

but were the definition for the value of employment is adjusted for a permanent tax on labor income τ:

$$V^e = \frac{u(w-\tau)}{1-\beta}.$$

Exercise 15.4. *A model of Dixit, Grossman, and Gul*

For each date $t \geq 0$, two political parties divide a "pie" of fixed size 1. Party 1 receives a sequence of shares $y = \{y_t\}_{t\geq 0}$ and has utility function $E\sum_{t=0}^{\infty}\beta^t U(y_t)$, where $\beta \in (0,1)$, E is the mathematical expectation operator, and $U(\cdot)$ is an increasing, strictly concave, twice differentiable period utility function. Party 2 receives share $1-y_t$ and has utility function $E\sum_{t=0}^{\infty}\beta^t U(1-y_t)$. A state variable X_t is governed by a Markov process; X resides in one of K states. There is a partition S_1, S_2 of the state space. If $X_t \in S_1$, party 1 chooses the division $y_t, 1-y_t$, where y_t is the share of party 1. If $X_t \in S_2$, party 2 chooses the division. At each point in time, each party has the option of choosing "autarky," in which case its share is 1 when it is in power and zero when it is not in power. Formulate the optimal history-dependent sharing rule as a recursive contract. Formulate the Bellman equation. *Hint:* Let $V[u_0(x), x]$ be the optimal value for party 1 in state x when party 2 is promised value $u_0(x)$.

Solution

The Bellman equation for $V(u_0(x), x)$ is

$$V(u_0(x), x) = \max_{(w(.), y)} \{U(y) + \beta E[V(w(x'), x')|x]\}$$

where the maximization is subject to the following several constraints:

Promise Keeping:

$$u_0(x) = U(1 - y) + \beta E[w(x')|x], \qquad \forall x.$$

Incentive constraints (choice of sharing rule):

$$U(1 - y) + \beta E[w(x')|x] \geq U(1) + \beta E[U^2_{aut}(x')|x] \ \forall x \in S_2.$$
$$U(y) + \beta E[V(w(x'), x')|x] \geq U(1) + \beta E[U^1_{aut}(x')|x] \ \forall x \in S_1.$$

Incentive Constraint (No Quitting):

$$w(x') \geq U^2_{aut}(x') \ \forall x' \in S_2$$
$$V(w(x'), x') \geq U^1_{aut}(x') \ \forall x' \in S_1.$$

Exercise 15.5. *Two-state numerical example of social insurance*

Consider an endowment economy populated by a large number of individuals with identical preferences,

$$E \sum_{t=0}^{\infty} \beta^t u(c_t) = E \sum_{t=0}^{\infty} \beta^t \left(4c_t - \frac{c_t^2}{2}\right), \qquad \text{with} \ \ \beta = 0.8.$$

With respect to endowments, the individuals are divided into two types of equal size. All individuals of a particular type receive 0 goods with probability 0.5 and 2 goods with probability 0.5 in any given period. The endowments of the two types of individuals are perfectly negatively correlated so that the per capita endowment is always 1 good in every period. The social planner attaches the same welfare weight to all individuals. Without access to outside funds or borrowing and lending opportunities, the social planner seeks to provide insurance by simply reallocating goods between the two types of individuals. The design of the social insurance contract is constrained by a lack of commitment on behalf of the individuals. The individuals are free to walk away from any social arrangement, but they must then live in autarky evermore.

a. Compute the optimal insurance contract when the social planner lacks memory; that is, transfers in any given period can be a function only of the current endowment realization.

b. Can the insurance contract in part a be improved if we allow for history-dependent transfers?

c. Explain how the optimal contract changes when the parameter β goes to one. Explain how the optimal contract changes when the parameter β goes to zero.

Solution

a. It is useful to compute a benchmark utility level v_{pool} that is attained in the unconstrained symmetric allocation,

$$v_{\text{pool}} = E\sum_{t=0}^{\infty} \beta^t u(1) = \frac{3.5}{1-\beta} = 17.5,$$

and the utility level associated with autarky,

$$v_{\text{aut}} = E\sum_{t=0}^{\infty} \beta^t [0.5u(0) + 0.5u(2)] = \frac{3}{1-\beta} = 15.$$

In the memory-less insurance contract, let c and $2-c$ be the consumption level of an agent with a current endowment of 2 and 0 units, respectively. Evidently, if any participation constraint is binding in the design of the optimal contract, it is the participation constraint of the former agent,

$$u(c) + \beta v \geq u(2) + \beta v_{\text{aut}},$$

where v is an agent's continuation value of remaining within the social insurance arrangement. The continuation value satisfies

$$v = 0.5[u(c) + \beta v] + 0.5[u(2-c) + \beta v],$$

or

$$v = \frac{0.5u(c) + 0.5u(2-c)}{1-\beta}.$$

After substituting equation 15 into equation 15 at equality, and invoking the specific parametrization in the question, we arrive at the following quadratic equation

$$0.625c^2 - 2c + 1.5 = 0.$$

The equation has two roots, as given by the quadratic formula,

$$c = \frac{2 \pm \sqrt{4 - 3.75}}{1.25} = \begin{array}{c} 2, \\ 1.2. \end{array}$$

Thus, the optimal contract has $c = 1.2$ and it attains the expected utility level

$$v = \frac{0.5u(1.2) + 0.5u(0.8)}{1-\beta} = 17.4.$$

b. The solution in part a cannot be improved upon. The reason is that the shocks are i.i.d. The solution described in the previous part smooths consumption as much as possible when agents can revert to autarky.

c. For a sufficiently high β, the first-best outcome (1,1) with utility v_{pool} is attainable since constraint 15 will no longer be binding. We can solve for the range of β where this is true,

$$v_{\text{pool}} \geq u(2) + \beta v_{\text{aut}},$$

or

$$\frac{3.5}{1-\beta} > 6 + \beta\frac{3}{1-\beta} \qquad \Longrightarrow \qquad \beta \geq \frac{5}{6}.$$

In contrast, when β goes to zero, less and less risk sharing can be achieved. For $\beta < .66$ the autarkic solution (2,0) results.

Exercise 15.6. *Optimal unemployment compensation with unobservable wage offers*

Consider an unemployed person with preferences given by

$$E\sum_{t=0}^{\infty}\beta^t u(c_t)\,,$$

where $\beta \in (0,1)$ is a subjective discount factor, $c_t \geq 0$ is consumption at time t, and the utility function $u(c)$ is strictly increasing, twice differentiable, and strictly concave. Each period the worker draws one offer w from a uniform wage distribution on the domain $[w_L, w_H]$ with $0 \leq w_L < w_H < \infty$. Let the cumulative density function be denoted $F(x) = \text{prob}\{w \leq x\}$, and denote its density by f, which is constant on the domain $[w_L, w_H]$. After the worker has accepted a wage offer w, he receives the wage w per period forever. He is then beyond the grasp of the unemployment insurance agency. During the unemployment spell, any consumption smoothing has to be done through the unemployment insurance agency because the worker holds no assets and cannot borrow or lend.
a. Characterize the worker's optimal reservation wage when he is entitled to a time-invariant unemployment compensation b of indefinite duration.

b. Characterize the optimal unemployment compensation scheme under full information. That is, we assume that the insurance agency can observe and control the unemployed worker's consumption and reservation wage.

c. Characterize the optimal unemployment compensation scheme under asymmetric information where the insurance agency cannot observe wage offers, though it can observe and control the unemployed worker's consumption. Discuss the optimal time profile of the unemployed worker's consumption level.

Solution

a. When the worker is entitled to a time-invariant unemployment compensation b his problem is

$$V^u = \max_{\bar{w}\geq 0}\left\{u(b) + \beta F(\bar{w})V^u + \beta(1-F(\bar{w}))V^e\right\}, \tag{169}$$

where the value of being employed is

$$V^e = \frac{1}{1-F(\bar{w})}\int_{\bar{w}}^{w_H}\frac{u(w')}{1-\beta}dF(w'). \tag{170}$$

The first order condition w.r.t. $\bar{w}$ is

$$V^u \leq \frac{u(\bar{w})}{1-\beta}, \tag{171}$$

with equality if $\bar{w} > 0$.

The worker searches until she finds a job, which leaves her at least as well off as the value of unemployment. The optimal policy is to accept a job when $w \geq \bar{w}$ and to reject otherwise.

b. Under full information the optimal compensation scheme is

$$C(V) = \min_{c,\bar{w},V^u} \left\{ c + \beta F(\bar{w}) C(V^u) \right\},$$

subject to the promise keeping constraint

$$V \leq u(c) + \beta F(\bar{w}) V^u + \beta(1 - F(\bar{w})) V^e, \tag{172}$$

where the value of employment is as in equation (170). Denote by θ the Lagrange multiplier on the promise keeping constraint. The first order conditions are

$$\begin{aligned} u'(c) &= \frac{1}{\theta} \\ C(V^u) &= \theta \left[V^u - \frac{u(\bar{w})}{1-\beta} \right]. \\ C'(V^u) &= \theta. \end{aligned}$$

The Benveniste-Scheinkman condition is

$$C'(V) = \theta.$$

Therefore the value of unemployment is constant

$$C'(V^u) = \theta = C'(V) \Longrightarrow V^u = V,$$

given that C is continuous and convex. Likewise, the optimal consumption pattern c_{FI} is constant. This consumption level c_{FI} is higher than the unemployment benefit b of part a, because the insurance scheme is costly: $(C(V^u) \geq 0)$. The latter implies that $V^u_{FI} > \frac{u(\bar{w}_{FI})}{1-\beta}$, whereas this relationship holds with equality in the autarchy case of part a.

c. Under asymmetric info the optimal contract is

$$C(V) = \min_{c,\bar{w},V^u} \left\{ c + \beta F(\bar{w}) C(V^u) \right\},$$

subject to the promise keeping constraint

$$V \leq u(c) + \beta F(\bar{w}) V^u + \beta(1 - F(\bar{w})) V^e, \tag{173}$$

the incentive compatibility constraint

$$V^u \leq \frac{u(\bar{w})}{1-\beta},$$

where the value of employment is as in equation (170). Denote by θ the Lagrange multiplier on the promise keeping constraint and by η the Lagrange multiplier on the incentive compatibility constraint. The first order conditions w.r.t. c, $\bar{w}$, and V^u are

$$\begin{aligned}
u'(c) &= \frac{1}{\theta}, \\
\beta f(\bar{w}) C(V^u) &= \theta \left[\beta f(\bar{w}) V^u - \beta f(\bar{w}) \frac{u(\bar{w})}{1-\beta} \right] + \eta \left[\frac{u'(\bar{w})}{1-\beta} \right], \\
\beta F(\bar{w}) C'(V^u) &= \theta \beta F(\bar{w}) - \eta.
\end{aligned}$$

For $\bar{w} > 0$, the second first order condition simplifies to

$$C(V^u) = \eta \frac{1}{\beta f(\bar{w})} \frac{u'(\bar{w})}{1-\beta},$$

by virtue of the first order condition from the autarky problem holding with equality. The third first order condition simplifies to

$$C'(V^u) = \theta - \frac{1}{\beta F(\bar{w})} \eta.$$

The Benveniste-Scheinkman condition is

$$C'(V) = \theta.$$

Therefore, because $\eta > 0,\ F(\bar{w}) > 0$, the value of unemployment is decreasing as the unemployment spell continues:

$$C'(V^u) < C'(V) \Longrightarrow V^u < V.$$

by virtue of the convexity of C. The optimal consumption pattern c is decreasing as well in order to provide the proper incentives to set a low enough $\bar{w}$. That is, $\bar{w}$ decreases over time.

Exercise 15.7. *Convergence in Kocherlakota model*

We return to Kocherlakota's study of the conditions under which there obtains convergence to a unique nontrivial invariant distribution of continuation values. Suppose that there exists a first-best sustainable allocation. Among such allocations let v^{FB} be the highest initial utility that can be to the first agent and let v_{FB} be the lowest possible utility that can be assigned to the first agent. Then prove that for any $v < v_{FB}$, $\lim_{t\to\infty} v_t = v_{FB}$, and that and for any initial utility satisfying $v > v^{FB}$, $\lim_{t\to+\infty} v_t = v^{FB}$.

Solution

In the first best allocation neither agent's participation constraint binds with strictly positive probability. Let v denote the promised utility for agent 1. Assume its initial level is $v < v_{FB}$. Denote by λ_s the Lagrange multiplier on agent 1's participation constraint in state s and by θ_s the multiplier on agent 2's participation constraint in state s. Let μ be the multiplier on the promise keeping constraint. Since agent 1's utility promise is less than v_{FB}, her participation constraint binds and agent 2's constraint is not binding $(\lambda_s > 0, \theta_s = 0)$. From the first order conditions for the closed economy model, we know

$$\begin{aligned} \pi_s u_c(c_s) &= \mu \pi_s u_c(c_s) + \lambda_s u_c(c_s) \\ \pi_s \beta P_w(w_s) + \pi_s \mu \beta + \lambda_s &= 0, \end{aligned}$$

using these conditions and the Benveniste-Scheinkman condition $P_w(v) = -\mu$ we obtain $P_w(w_s) < P_w(v)$ which implies (by the concavity of P) that $w_s > v$. Agent 1's utility promise is a nondecreasing sequence in this region. The set of sample paths along which every income state is realized has measure one. Along each of these sample paths the utility promise for agent 1 is increasing. Agent 2's participation constraint does not bind in this region. Note that $w_s > v$ implies that $E(w_s) > v$. Therefore w is a sub-martingale, bounded by v_{FB}. By the martingale convergence theorem it converges to v_{FB}. Agent 1's v will not increase above v_{FB} in a Pareto optimal solution. Suppose it did, that is for a time $\tau : v_{\tau-1} \leq v_{FB}$ and $v_\tau > v_{FB}$. If $v_{\tau-1} = v_{FB}$ then by definition of the first best allocation $v_\tau = v_{FB}$, a contradiction. If $v_{\tau-1} < v_{FB}$ then setting $v_\tau^* = v_{FB}$ would satisfy agent 1 and 2's participation constraint and make agent 2 strictly better off $P(v_\tau^*) > P(v_\tau)$,a contradiction to a Pareto optimal allocation. We have shown that starting at $v < v_{FB}$, w is a monotone increasing sequence which converges to v_{FB}. Likewise, for $v > v^{FB} : \lim_{t\to\infty} w_t = v^{FB}$

Exercise 15.8. *Full unemployment insurance*

An unemployed worker orders stochastic processes of consumption, search effort $\{c_t, a_t\}_{t=0}^{\infty}$ according to

$$E \sum_{t=0}^{\infty} \beta^t \left[u(c_t) - a_t \right],$$

where $\beta \in (0,1)$ and $u(c)$ is strictly increasing, twice differentiable, and strictly concave. It is required that $c_t \geq 0$ and $a_t \geq 0$.

All jobs are alike and pay wage $w > 0$ units of the consumption good each period forever. After a worker has found a job, the unemployment insurance agency can tax the employed worker at a rate τ consumption goods per period. The unemployment agency can make τ depend on the worker's unemployment history. The probability of finding a job is $p(a)$ where p is an increasing and strictly concave and twice differentiable function of a, satisfying $p(a) \in [0,1]$ for $a \geq 0$, $p(0) = 0$. The consumption good is non-storable. The unemployed person cannot borrow or lend and holds no assets. If the unemployed worker is to do any consumption smoothing, it has to be through the unemployment insurance agency. The insurance agency can observe the worker's search effort and can control his consumption. An employed worker's consumption is $w - \tau$ per period.

a. Let V_{aut} be the value of an unemployed worker's expected discounted utility when he has no access to unemployment insurance. An unemployment insurance agency wants to insure unemployed workers and to deliver expected discounted

discounted utility $V > V_{\text{aut}}$ at minimum expected discounted cost $C(V)$. The insurance agency also uses the discount factor β. The insurance agency controls c, a, τ, where c is consumption of an unemployed worker. The worker pays the tax τ only after he becomes employed. Formulate the Bellman equation for $C(V)$.

b. Prove that the optimal policy of the insurance agency is a policy that satisfies $c = w - \tau$.

Solution

a. The Bellman equation is:

$$C(V) = \min_{c,a,V^u,V^e} \left\{ c + \beta[1-p(a)]C(V^u) - \beta p(a)\left[\frac{\tau}{1-\beta}\right]\right\},$$

where the minimization is subject to the promise keeping constraints

$$\begin{aligned} V \leq u(c) - a + \beta\left[p(a)V^e + (1-p(1))V^u\right] \\ V^e = \frac{u(w-\tau)}{1-\beta}. \end{aligned}$$

b. The optimal policy is to set a constant consumption level $c = w - \tau$. Note that this makes the worker equally well off in employment as in unemployment. Denote by λ the Lagrange multiplier on the promise keeping constraint. The first order conditions of the problem with respect to c, a, V^u respectively are:

$$1 = \lambda u'(c) \tag{174}$$

$$\beta p'(a)\left[C(V^u) + \frac{\tau}{1-\beta}\right] = \lambda\left[1 - \beta p'(a)(V^e - V^u)\right] \tag{175}$$

$$\beta(1-p(a))C'(V^u) = \beta(1-p(a))\lambda. \tag{176}$$

The Benveniste-Scheinkman condition says

$$C'(V) = \lambda. \tag{177}$$

Combining equations (177) and (176), we see that $V^u = V$. It is optimal to keep the utility promises constant across time. From this and from equations (174) and (175) it follows that it is also optimal to keep consumption and search effort constant during the unemployment spell. Consumption is fully smoothed in unemployment. Constant consumption c and search effort a during unemployment require a constant tax rate τ. To find the optimal constant tax rate, take the first order condition with respect to τ.

$$\frac{1}{\lambda} = u'(w-\tau).$$

This and equation (174) proof the claim. Consumption is only smoothed across states of employment and unemployment when $V^e = V^u$. This is only the case when $c = w - \tau$. Risk aversion of the agent makes it optimal for the social insurance agency to equate marginal utility across states of employment and unemployment by setting a tax/transfer τ such that $c = w - \tau$.

Exercise 15.9. *Kocherlakota meets Markov*

A household orders sequences $\{c_t\}_{t=0}^{\infty}$ of a single nondurable good by

$$E\sum_{t=0}^{\infty}\beta^t u(c_t), \beta \in (0,1)$$

where u is strictly increasing, twice continuously differentiable, and strictly concave with $u'(0) = +\infty$. The household receives an endowment of the consumption good of y_t that obeys a discrete state Markov chain with $P_{ij} = \text{Prob}(y_{t+1} = \overline{y}_j | y_t = \overline{y}_i)$, where the endowment y_t can take one of the I values $[\overline{y}_1, \ldots, \overline{y}_I]$.

a. Conditional on having observed the time t value of the household's endowment, a social insurer wants to deliver expected discounted utility v to the household in the least cost way. The insurer observes y_t at the beginning of every period, and contingent on the observed history of those endowments, can make a transfer τ_t to the household. The transfer can be positive or negative and can be enforced without cost. Let $C(v, i)$ be the minimum expected discounted cost to the insurance agency of delivering promised discounted utility v when the household has just received endowment $\overline{y}_i$. (Let the insurer discount with factor β.) Write a Bellman equation for $C(v, i)$.

b. Characterize the consumption plan and the transfer plan that attains $C(v, i)$; find an associated law of motion for promised discounted value.

c. Now assume that the household is isolated and has no access to insurance. Let $v^a(i)$ be the expected discounted value of utility for a household in autarky, conditional on current income being $\overline{y}_i$. Formulate Bellman equations for $v^a(i), i = 1, \ldots, I$.

d. Now return to the problem of the insurer mentioned in part b, but assume that the insurer cannot enforce transfers because each period the consumer is free to walk away from the insurer and live in autarky thereafter. The insurer must structure a history-dependent transfer scheme that prevents the household from every exercising the option to revert to autarky. Again, let $C(v, i)$ be the minimum cost for an insurer that wants to deliver promised value discounted utility v to a household with current endowment i. Formulate Bellman equations for $C(v, i), i = 1, \ldots, I$. Briefly discuss the form of the law of motion for v associated with the minimum cost insurance scheme.

Solution

a. Let v be the promised value. The Bellman equation for the planner is

$$C(v, i) = \min_{\tau, \{w_j\}_{j=1}^{I}} \left\{ \tau + \beta \sum_j P_{ij} C(w_j, j) \right\},$$

subject to the promise keeping constraint

$$v \leq u(y_i + \tau) + \beta \sum_j P_{ij} w_j$$

,

or equivalently

$$C(v, i) = -y_i + \min_{c, \{w_j\}_{j=1}^I} \left\{ c + \beta \sum_j P_{ij} C(w_j, j) \right\},$$

subject to the promise keeping constraint

$$v \leq u(c) + \beta \sum_j P_{ij} w_j.$$

b. The conjecture is that, without enforcement problem, the planner can achieve full insurance. To prove this conjecture, we first prove that the cost function is separable in v and i: $C(v, i) = \phi(v) + \psi(i)$. Separability requires that (i) the set of functions of this form is closed and (ii) that the Bellman operator maps separable functions of this form into separable functions.

(i) Let $C_n(v, i) \rightarrow C(v, i)$ and $C_n(v, i) = \phi_n(v) + \psi_n(i)$ $\forall n$. Then $C_n(v, i) - C_n(v, 1) = \psi_n(i) - \psi_n(1) = f_n(i) \rightarrow C(v, i) - C(v, 1) = f(i)$. The latter is independent of v since it is the limit of a sequence that is independent of v. Thus $C(v, i) = C(v, 1) + f(i)$ is separable.

(ii) We have to check that $-y_i + \min_{c, \{w_j\}_{j=1}^I} \left\{ c + \beta \sum_j P_{ij} C(w_j, j) \right\}$ is of the same form as $C(v, i) = \phi(v) + \psi(i)$. Let λ be the Lagrange multiplier on the promise keeping constraint, then the first order condition w.r.t. w_j yields

$$\beta P_{ij} \phi'(v'_j) = \lambda \beta P_{ij} \ \ \forall j \in \{1, ..., I\}$$

This implies that w_j does not depend on j. Therefore we can rewrite the program as

$$-y_i + \min_{c,w} \{c + \beta \phi(w)\} + \sum_j P_{ij} \psi(j),$$

subject to the promise keeping constraint

$$v \leq u(c) + \beta w.$$

The solution to this program is separable in v and i.

The separability of the cost function, the first order condition and the Benveniste-Scheinkman condition $C'(v, i) = \lambda$ imply that $w_j = v$ $\forall j \in \{1, ..., I\}$.

So, for any given realization of today's income y_i, the planner offers a constant utility promise in every state of nature tomorrow. From the first order condition w.r.t. consumption $1 = u'(c)\phi'(w)$, we find that the optimal consumption (and hence transfer τ) is also constant.

c. The Bellman equation in autarky is:

$$v_i^a = u(y_i) + \beta \sum_j P_{ij} v_j^a.$$

This is a linear system of n equations in n unknowns.

d. The problem with limited commitment imposes an additional participation constraint on the allocations. The Bellman equation for the planner is

$$C(v,i) = \min_{c, \{w_j\}_{j=1}^I} \left\{ c - y_i + \beta \sum_j P_{ij} C(w,j) \right\},$$

subject to the promise keeping constraint

$$v \leq u(c) + \beta \sum_j P_{ij} w_j,$$

and subject to the enforcement constraint

$$u(c) + \beta \sum_j P_{ij} w_j \geq u(y_i) + \beta \sum_j P_{ij} v_j^a = v^a(i).$$

The first order condition w.r.t. w_j now becomes: $\beta P_{ij} C'(w_j, j) = (\lambda + \eta)\beta P_{ij}$, where η is the Lagrange multiplier associated with the participation constraint. The Benveniste-Scheinkman condition still is $C'(v,i) = \lambda$.

Therefore, $C'(w_j, j) \geq C'(v,i)$ in states of the world i where the enforcement constraint binds. By the strict convexity of C, this implies that $w_j > v_i$. From the first order condition w.r.t. the transfer τ, it follows that the transfer should be strictly greater when $\eta > 0$. The current period transfer and the promised utility rise at the same time for the agent whose constraint is binding. The planner keeps the agent who threatens to walk away in the contract by increasing today's transfers and increasing tomorrow's promises. When the constraint is not binding, a constant utility promise and transfer level are optimal.

Alternatively, the separability of the cost function shown in part (b) breaks down because states with a binding promise keeping constraint introduce dependence on i into the cost function. Full insurance is no longer optimal.

Exercise 15.10. *Wealth dynamics in money lender model*

Consider the model in the text of the village with a money lender. The village consists of a large number (e.g., a continuum) of households each of whom has an i.i.d. endowment process that is distributed as

$$\text{Prob}(y_t = \overline{y}_s) = \frac{1-\lambda}{1-\lambda^S} \lambda^{s-1}$$

where $\lambda \in (0,1)$ and $\overline{y}_s = s + 5$ is the s^{th} possible endowment value, $s = 1, \ldots, S$. Let $\beta \in (0,1)$ be the discount factor and β^{-1} the gross rate of return at which the money lender can borrow or lend. The typical household's one-period utility function is $u(c) = (1-\gamma)^{-1}c^{1-\gamma}$ where γ is the household's coefficient of relative risk aversion. Assume the parameter values $(\beta, S, \gamma, \lambda) = (.95, 20, 2, .95)$.

Hint: The formulas given in the section 'Recursive computation of the optimal contract' will be helpful in answering the following questions.

a. Using matlab, compute the optimal contract that the money lender offers a villager, assuming that the contract leaves the villager indifferent between refusing and accepting the contract.

b. Compute the expected profits that the money lender earns by offering this contract for an initial discounted utility that equals the one that the household would receive in autarky.

c. Let the cross section distribution of consumption at time $t \geq 0$ be given by the c.d.f. $\mathrm{Prob}(\mathrm{c_t} \leq \overline{\mathrm{C}}) = F_t(\overline{C})$. Compute F_t. Plot it for $t = 0$, $t = 5$, $t = 10$, $t = 500$.

d. Compute the money lender's savings for $t \geq 0$ and plot it for $t = 0, \ldots, 100$.

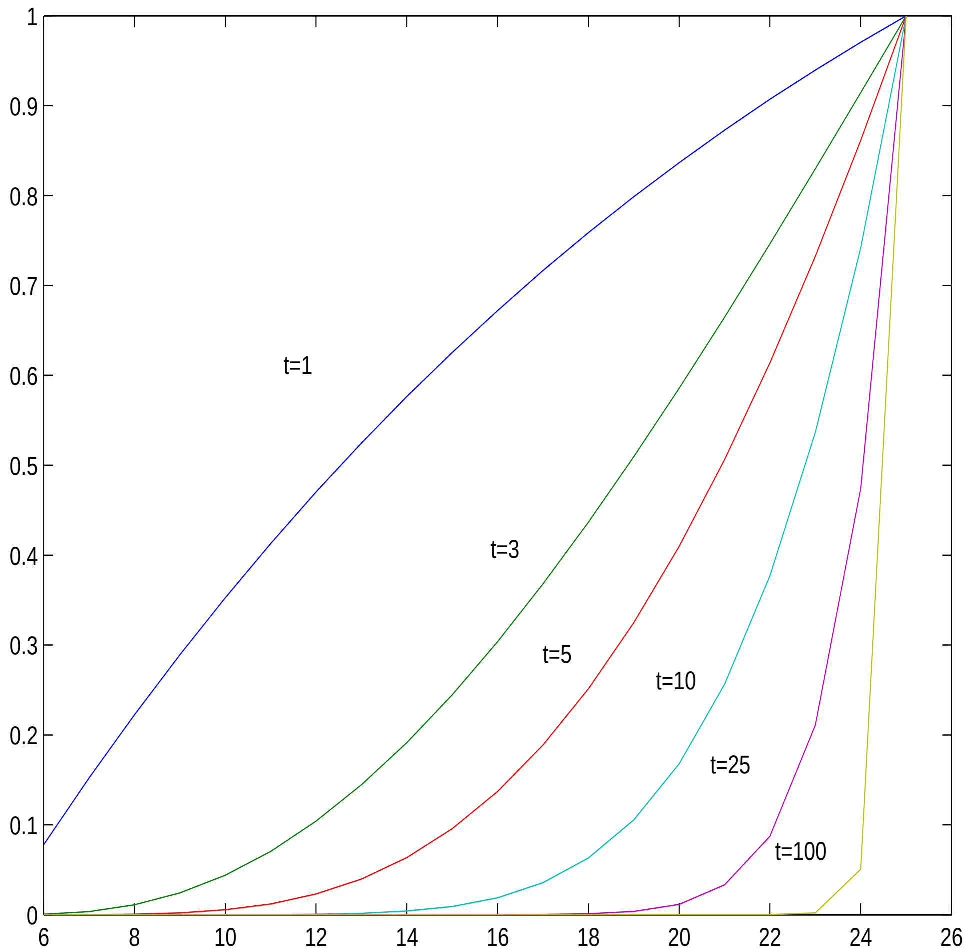
e. Now adapt your program to find the initial level of promised utility $v > v_{\rm aut}$ that would set $P(v) = 0$. *Hint:* Think of an iterative algorithm to solve $P(v) = 0$.

Solution

a. The matlab code is in `ex1510.m` , which uses `Kochrecur.m` . The optimal contract is to give $c_0 = 6$ and $v_{aut} = -1.73156$ as long as the highest endowment realization is lower than $y_{j\,\min} = y_2 = 7$. For higher income realizations the contract offers higher consumption and promised utility. For the highest income state $s = 20$, $c_S = 11.87$ and $w_S = -1.685$.
b. Following the algorithm in the text, the money lender makes an expected profit of 44.06 when holding the households at their reservation value.
c. The cumulative consumption distribution shifts to the right as agents largest income realization thus far increases. In the limit it collapses to a spike at $s = S$. See figure 2.
d. The money lender's bank balance increases exponentially. His first period saving equal his first period profits which are positive. He earns an interest rate β^{-1} on this balance. In addition he obtains increasing profits as agents' income realizations become higher. See figure 3.
e. The initial value which makes the moneylender's profits equal to zero will be such that the moneylender pays out all expected earnings. For the parameters that are given, $E[y_s] = 13.82$. The value associated with paying a constant consumption $c = 13.82$ forever is $\frac{u(c)}{1-\beta} = -1.4468$. Note that this level is substantially higher than $w_S = -1.685$ in the contract derived in part a, which maximizes the lender's profits. Figure 4 plots the optimal contract, expected profits and bank balance for the money lender for this initial value $v = -1.4468$.

Exercise 15.11. *Folk theorem*

FIGURE 2. Exercise 15.10 a : Consumption Distribution



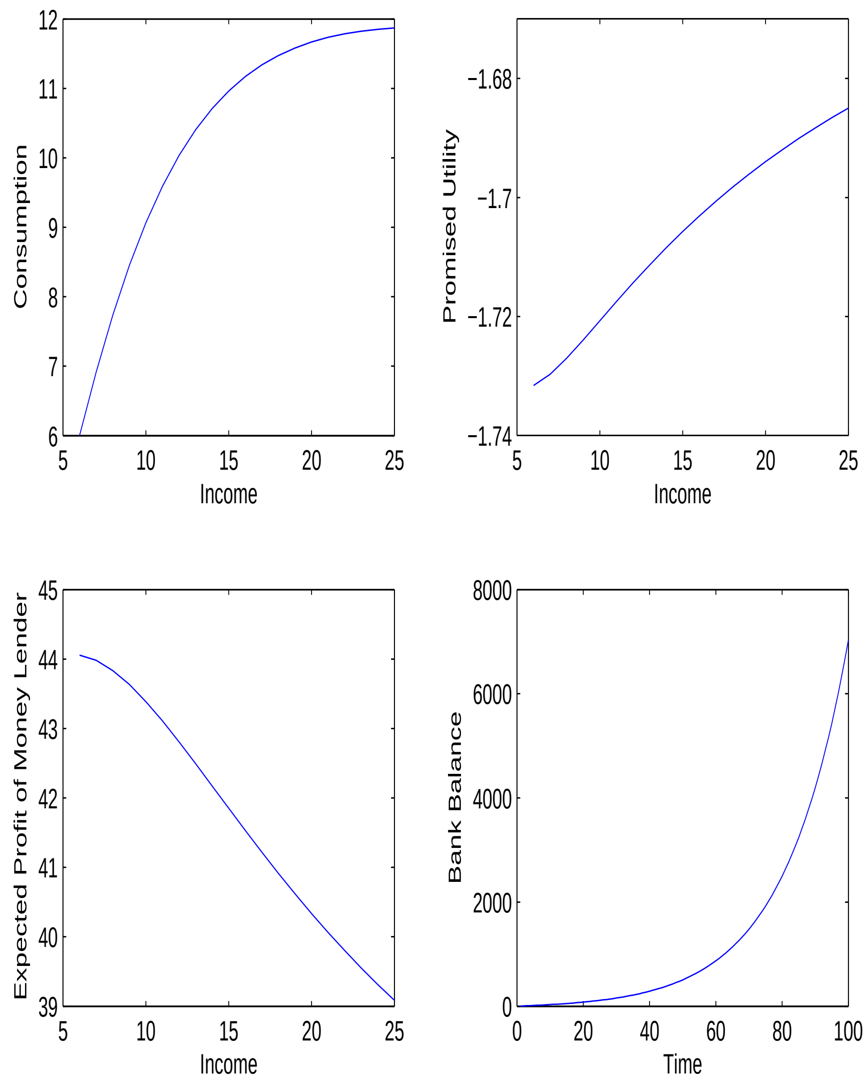
Consider the following version of Kocherlakota's model. The one-period utility function is $u(c) = (1-\gamma)^{-1}(c+b)^{1-\gamma}$, where $b = 5, \gamma = 2$. The endowment of agent 1 is y_t and the endowment of agent 2 is $1 - y_t$, where y_t is i.i.d. and $\text{Prob}(y_t = \overline{y}_s) = \Pi_s = S^{-1}$. Assume that $\overline{y}_s = s/S, s = 1, \ldots, S$. To begin assume that $\beta = .95$ and $\sigma = 4$.

a. Compute v_{aut}.

b. Consider the case of full risk-sharing. Let $v = \frac{u(\overline{c})}{1-\beta}$ and $P(v) = \frac{u(1-\overline{c})}{1-\beta}$; the locus $(v, P(v))$ traces out a Pareto frontier as $\overline{c}$ ranges from 0 to 1. Write a Matlab program to compute that (unconstrained) Pareto frontier and plot it.

c. Compute $v_{\text{aut}} = (1-\beta)^{-1}\sum_{s=1}^{S}\Pi_s u(\overline{y}_s)$. Plot $(v_{\text{aut}}, v_{\text{aut}})$ on the figure from part b.

FIGURE 3. Exercise 15.10 b : Consumption, Promised Utility, Profits and Bank Balance in Contract that Maximizes the Money Lender's Profits



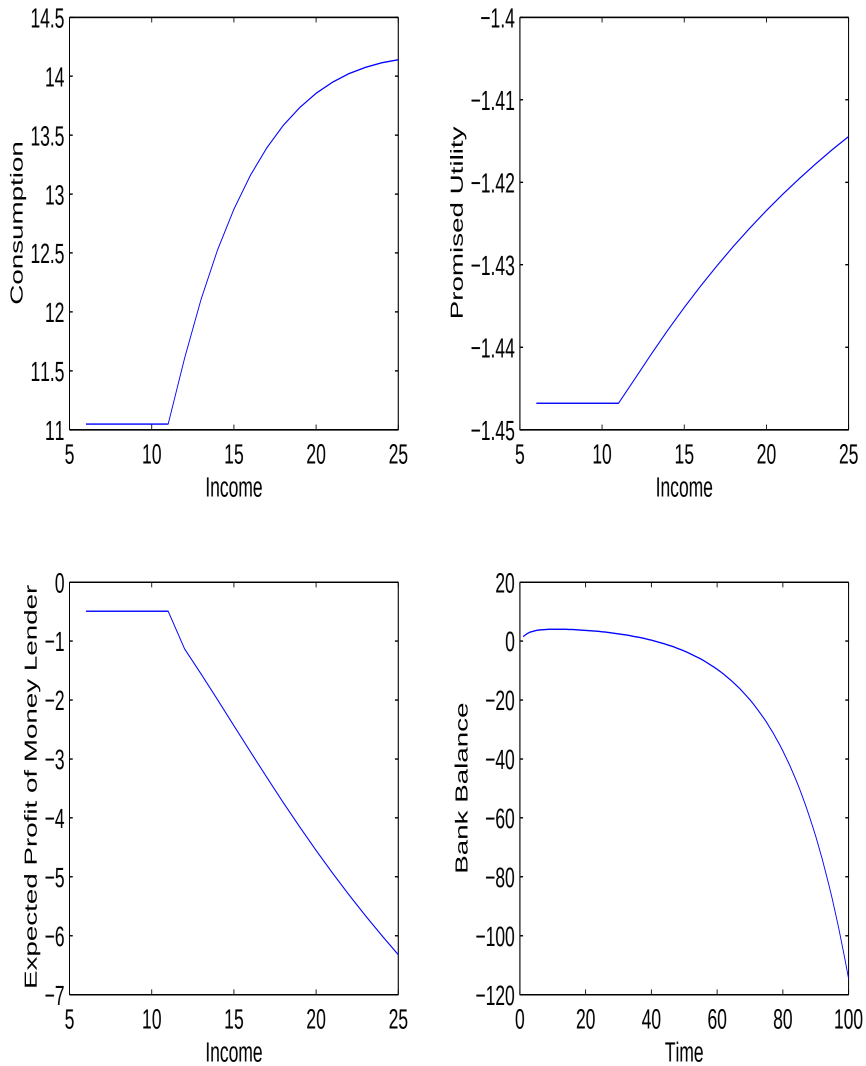
d. Consider the two participation constraints

$$\begin{array}{lcl} u(c_t) + \beta E_t\left(\sum_{j=0}^{+\infty} \beta^j u(c_{t+j})\right) & \geq & u(y_t) + v_{\text{aut}} \\ u(1-c_t) + \beta E_t\left(\sum_{j=0}^{+\infty} \beta^j u(1-c_{t+j})\right) & \geq & u(1-y_t) + v_{\text{aut}}. \end{array}$$

Find the values $\check{v}$ and $\hat{v}$ that solve

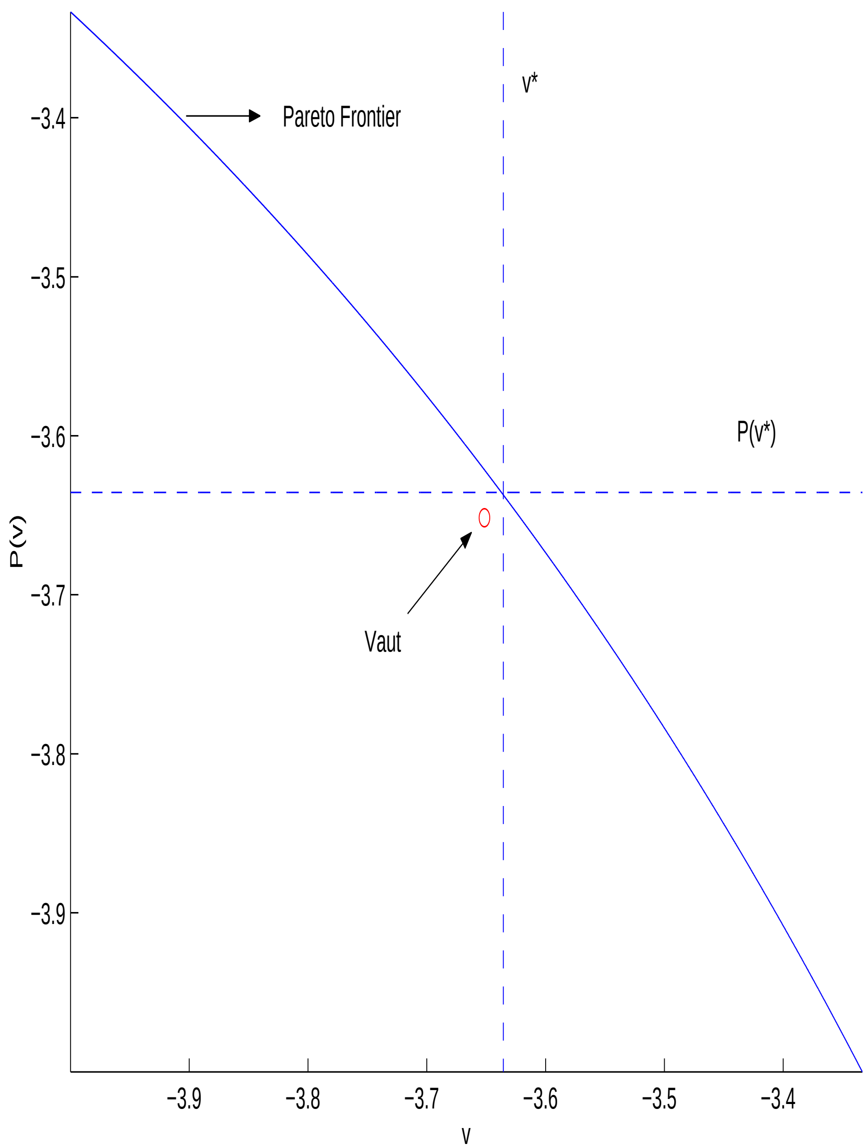
$$\begin{array}{ll} \check{v} & = u(y_S) + \beta v_{\text{aut}} \\ P(\hat{v}) & = u(1-y_1) + \beta v_{\text{aut}}. \end{array}$$

FIGURE 4. Exercise 15.10 c : Consumption, Promised Utility, Profits and Bank Balance in Contract that Gives Zero Profits to Money Lender



Plot a vertical line at $\check{v}$ and a horizontal line at $P(\hat{v})$ on the figure from part b. Please interpret $\check{v}$ as the minimum value of v such that the participation constraint for agent 1 will never bind; and interpret $P(\hat{v})$ as the minimum value of $P(v)$ such that the continuation value of agent two will never bind. Check whether there is a piece of the $(v, P(v))$ frontier each point of which satifies $v \geq \check{v}$ and $P(v) \geq P(\hat{v})$.

FIGURE 5. Exercise 15.11 a : Pareto Frontier, $\beta = 0.95$



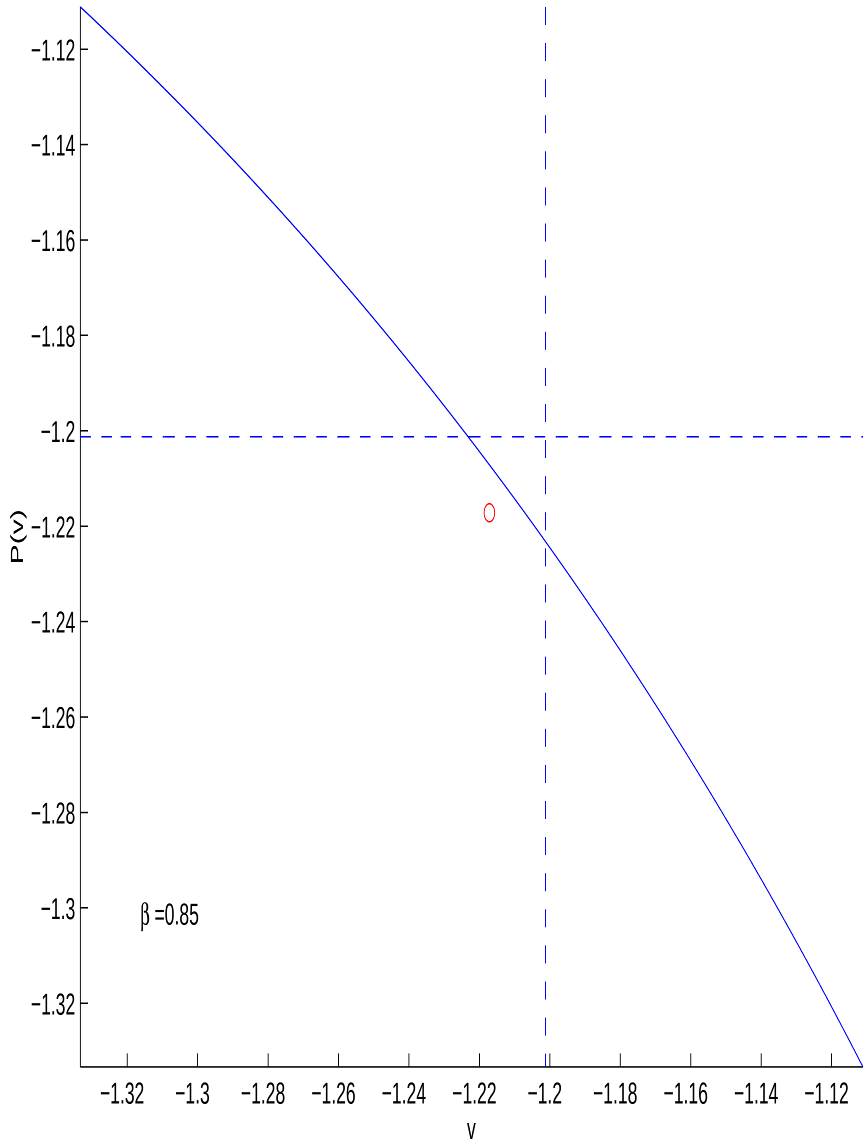
e. Show that there is a set of assignments of initial values $(v, P(v))$ that (i) lie on the Pareto frontier traced out in part b, such that (ii) the participation of neither agent will ever bind. Argue that complete risk sharing then occurs forevermore.

f. Lower β to such a value that the region defined in part c no longer exists. Argue that in this case, perpetually incomplete risk sharing must occur.

Solution

a.-d. Figure 5 summarizes the first 4 parts for $\beta = 0.95$. The matlab code is in `ex1511.m` . We see that there is a region where both agents are unconstrained.

FIGURE 6. Exercise 15.11 b : Pareto Frontier, $\beta = 0.85$

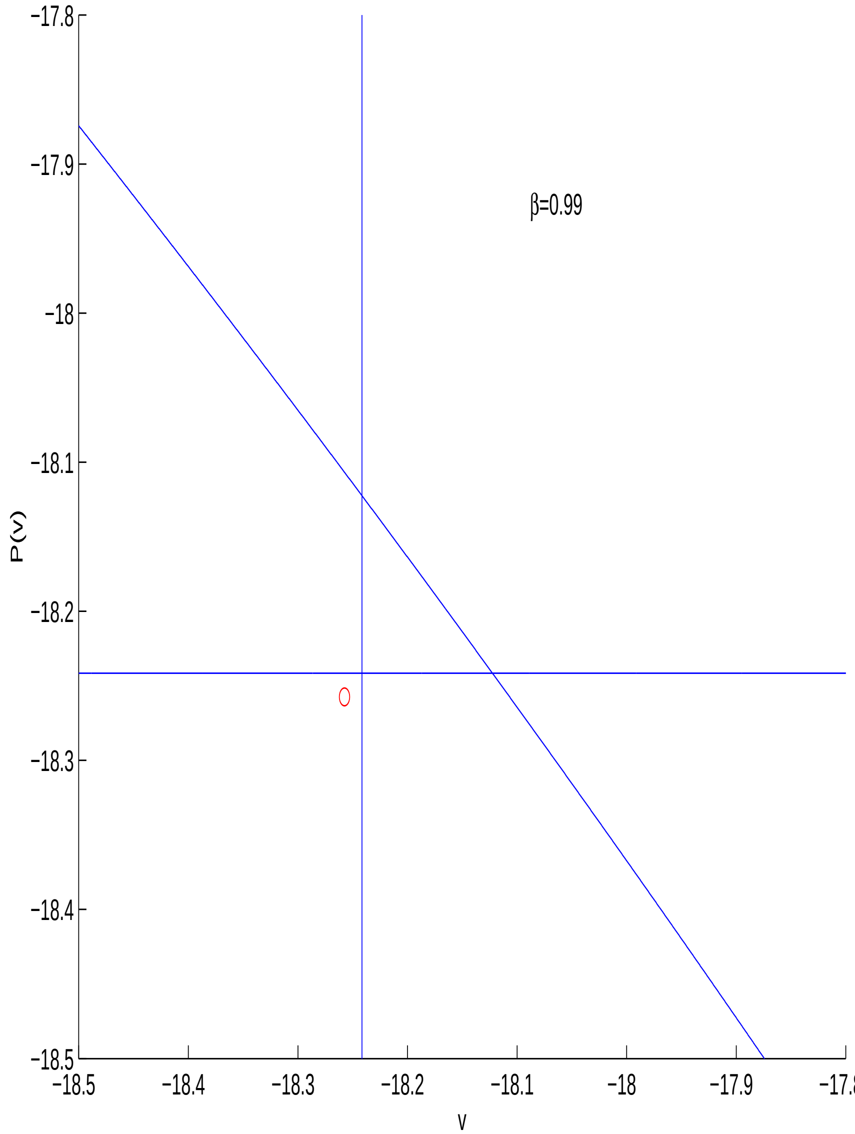


We obtain perfect risk sharing. Starting from outside this region, we won't enter in it as shown in exercise 15.9. That is, the participation constraints won't bind and hence the promised utilities won't need to be changed.

e. Figures 6 and 7 plot the same graph for $\beta < 0.9473$ and , there is no region with complete insurance. Whenever one agent's constraint binds, the planner increases the stochastic Negishi weight of that agent. One agent's constraint will always bind.

Exercise 15.12. *Thomas and Worrall (1988)*

FIGURE 7. Exercise 15.11 c : Pareto Frontier, $\beta = 0.99$



There is a competitive spot market for labor always available to each of a continuum of workers. Each worker is endowed with one unit of labor each period that he supplies inelastically to work either permanently for "the company" or each period in a new one-period job in the spot labor market. The worker's productivity in either the spot labor market or with the company is an i.i.d. endowment process that is distributed as

$$\mathrm{Prob}(w_t = \overline{w}_s) = \frac{1-\lambda}{1-\lambda^S}\lambda^{s-1},$$

where $\lambda \in (0,1)$ and $\overline{w}_s = s+5$ is the sth possible marginal product realization, $s = 1, \ldots, S$. In the spot market, the worker is paid w_t. In the the company, the worker is offered a history-dependent payment $\omega_t = f_t(h_t)$ where $h_t = w_t, \ldots, w_0$. Let $\beta \in (0,1)$ be the discount factor and β^{-1} the gross rate of return at which the company can borrow or lend. The worker cannot borrow or lend. The worker's one-period utility function is $u(\omega) = (1-\gamma)^{-1} w^{1-\gamma}$ where ω is the period wage from the company, which equals consumption, and γ is the worker's coefficient of relative risk aversion. Assume the parameter values $(\beta, S, \gamma, \lambda) = (.95, 20, 2, .95)$.

The company's discounted expected profits are

$$E \sum_{t=0}^{\infty} \beta^t \left(w_t - \omega_t \right). \tag{178}$$

The worker is free to walk away from the company at the start of any period, but must then stay in the spot labor market forever. In the spot labor market, the worker receives continution value

$$v_{\text{spot}} = \frac{Eu(w)}{1-\beta}.$$

The company designs a history-dependent compensation contract that must be sustainable (i.e., self-enforcing) in the face of the worker's freedom to enter the spot labor market at the beginning of period t *after* he has observed w_t but before he receives the t period wage.

Hint: Do these questions ring a bell? See exercise 5.10.

a. Using Matlab, compute the optimal contract that the company offers the worker, assuming that the contract leaves the worker indifferent between refusing and accepting the contract. **b.** Compute the expected profits that the firm earns

by offering this contract for an initial discounted utility that equals the one that the worker would receive by remaining forever in the spot market.

c. Let the distribution of wages that the firm offers to its workers at time $t \geq 0$ be given by the c.d.f. $\mathrm{Prob}(\omega_\mathrm{t} \leq \overline{\mathrm{w}}) = F_t(\overline{w})$. Compute F_t. Plot it for $t = 0$, $t = 5$, $t = 10$, $t = 500$.

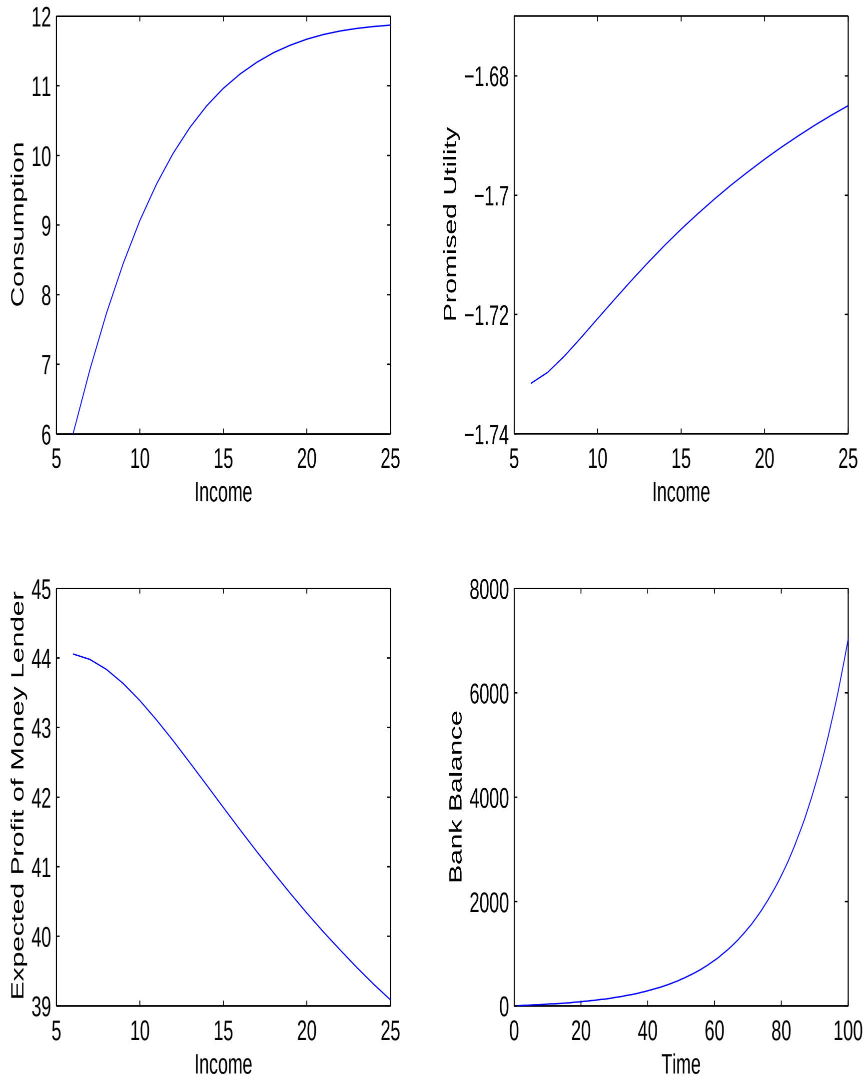
d. Plot an expected wage-tenure profile for a new worker.

e. Now assume that there is competition among companies and free entry. New companies enter by competing for workers by raising initial promised utility with the company. Adapt your program to find the initial level of promised utility $v > v_{\text{spot}}$ that would set expected profits from the averager worker $P(v) = 0$.

Solution

The solution to this problem is identical to the solution to problem 15.10. Figures 15,15 and 15 plot the optimal contract that promises $v_s pot$, the consumption CDF and its evolution as time goes on, and the wagee-tenure profile. The initial promised utility that makes expected profits zero is -1.4468. The matlab program is `ex1512.m` .

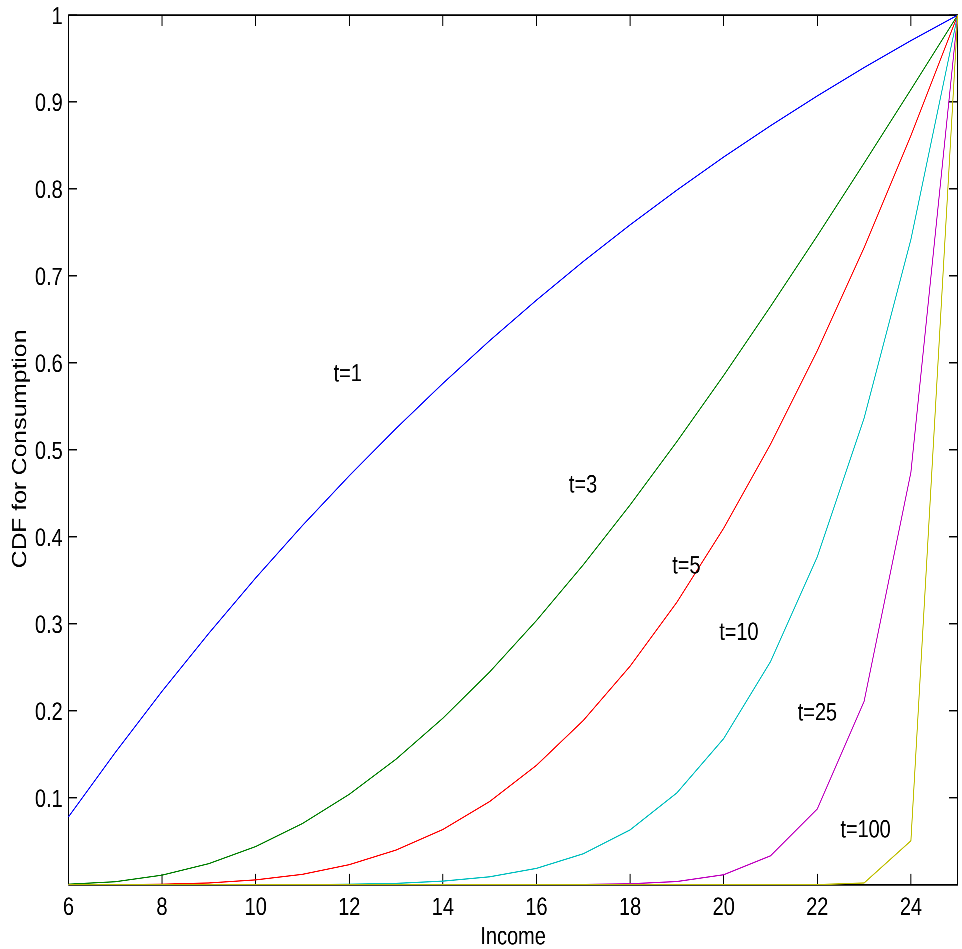
FIGURE 8. Exercise 15.12 a : Consumption, Promised Utility, Profits and Bank Balance in Contract that Maximizes the Money Lender's Profits



Exercise 15.13. *Cole and Kocherlakota (2001)*

Consider a *c*losed version of our two period model (T=2) based upon Cole and Kocherlakota's (2001) framework, where the planner has no access to outside borrowing. In this economy, suppose that an incomplete-markets equilibrium would give rise to an interest rate on bonds equal to $1 + r > \beta^{-1}$. Show that this decentralized outcome is inefficient. That is, show that there exists an incentive-feasible allocation that yields a higher ex-ante utility than the decentralized outcome.

FIGURE 9. Exercise 15.12 b : Consumption Distribution



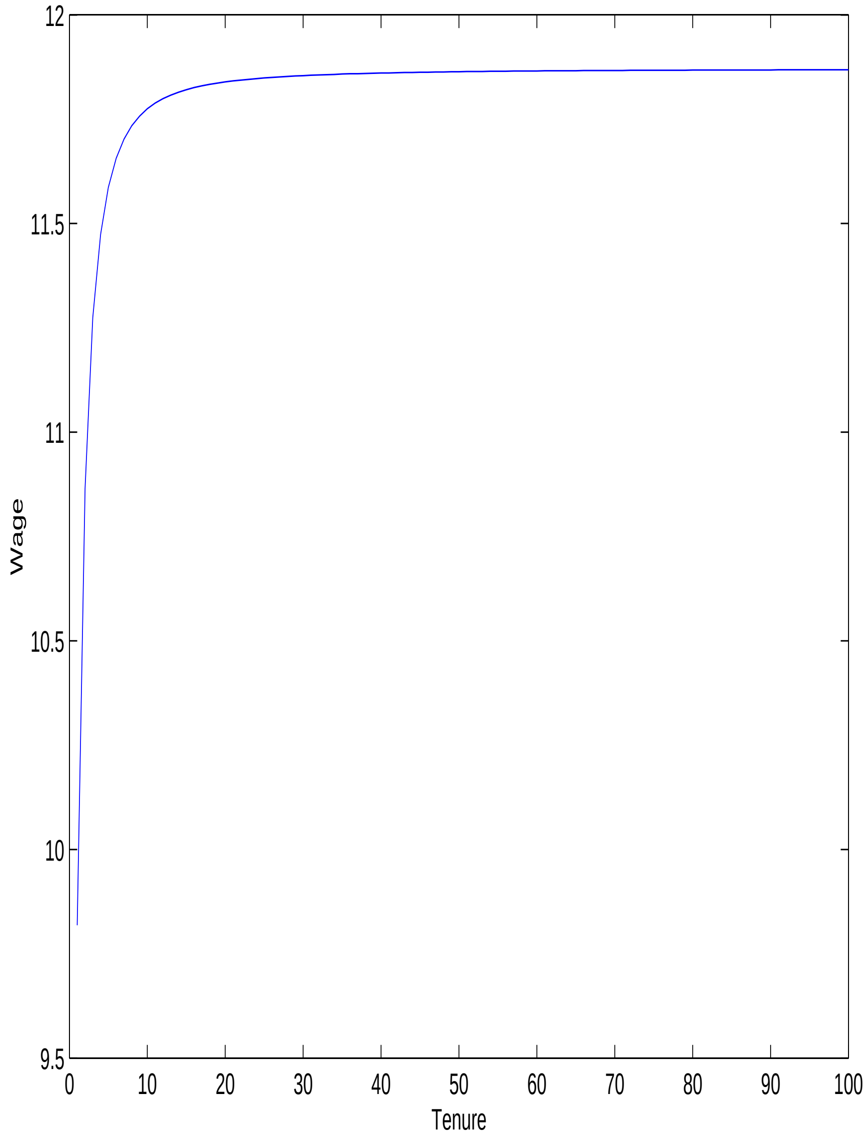
Solution

Let (c, A, r) be the incomplete markets equilibrium. Let $\beta^{-1} = R$. Recall from the FOC that

$$\begin{aligned} u'(c_1(\bar{y}_s)) &= \beta(1+r)\sum_{j=1}^{S}\Pi_j u'(c_2(\bar{y}_s, \bar{y}_j)) \\ &< \beta R \sum_{j=1}^{S}\Pi_j u'(c_2(\bar{y}_s, \bar{y}_j)). \end{aligned}$$

The last equation says that individuals have no incentive to store at rate R. Let $b_t(h_t) = y_t(h_t) - c_t(h_t)$ and $K_t = \sum_{h_t}\pi(h_t)A_t(h_t) = 0$. The incentive-feasible allocation $(c, 0, b, 0)$ is not efficient because there exists another incentive-feasible allocation that raises ex-ante utility by redistributing from high income agents to low income agents. Let the candidate allocation satisfy: $\tilde{c}_1(\bar{y}_j) = c_1(\bar{y}_j) - \epsilon$, where

FIGURE 10. Exercise 15.12 c : Wage-Tenure Profile



$\epsilon < \bar{y}_j - \bar{y}_{j-1}$ and $\tilde{c}_1(\bar{y}_{j-1}) = c_1(\bar{y}_{j-1}) + \delta(\epsilon)$ so that ex-ante utility is unchanged. A sufficiently small ϵ can be found such that (1) the agent with income state j has no incentive to lie down to state $j-1$ (2) the agents still have no incentive to store under the candidate allocation. Because of the concavity of the utility function $\epsilon > \delta(\epsilon)$. This relaxes the resource constraint. With the extra resources available, the candidate allocation can strictly improve upon the allocation implied by the incomplete markets equilibrium.

Exercise 15.14. *Thomas and Worrall meet Phelan-Townsend*

Consider the Thomas Worrall environment and denote $\Pi(y)$ the density of the i.i.d. endowment process, where y belongs to the discrete set of endowment levels $Y = [\overline{y}_1, \ldots, \overline{y}_S]$. The one-period utility function is $u(c) = (1-\gamma)^{-1}(c-a)^{1-\gamma}$ where $\gamma > 1$ and $\overline{y}_S > a > 0$.

Discretize the set of transfers B and the set of continuation values W. We assume that the discrete set $B \subset (a - \overline{y}_S, \overline{b}]$. Notice that with the one period utility function above, the planner could never extract more than $a - \overline{y}_S$ from the agent. Denote $\Pi^v(b, w|y)$ the joint density over (b, w) that the planner offers the agent who reports y and to whom he has offered beginning of period promised value v. For each $y \in Y$ and each $v \in W$, the planner chooses a set of conditional probabililites $\Pi^v(b, w|y)$ to satisfy the Bellman equation

$$P(v) = \max_{\Pi^v(b,w,y)} \sum_{B \times W \times Y} [-b + \beta P(w)]\, \Pi^v(b, w, y), \tag{179}$$

subject to the following constraints:

$$\begin{aligned}
v &= \sum_{B\times W\times Y} [u(y+b) + \beta w]\, \Pi^v(b,w,y) \\
\sum_{B\times W} [u(y+b) + \beta w]\, \Pi^v(b,w|y) &\geq \sum_{B\times W} [u(y+b) + \beta w]\, \Pi^v(b,w|\tilde{y}) \\
& \quad \forall (y, \tilde{y}) \in Y \times Y \\
\Pi^v(b,w,y) &= \Pi(y)\Pi^v(b,w|y) \ \ \forall (b,w,y) \in B \times W \times Y \\
\sum_{B\times W\times Y} \Pi^v(b,w,y) &= 1.
\end{aligned}$$

Here (180) is the promise keeping constraint, (180) are the truth-telling constraints, and (180), (180) are restrictions imposed by the laws of probability.

a. Verify that that given $P(w)$, one step on the Bellman equation is a linear programming problem.

b. Set $\beta = .94, a = 5, \gamma = 3$. Let S, N_B, N_W be the number of points in the grids for Y, B, W, respectively. Set $S = 10$, $N_B = N_W = 25$. Set $Y = \begin{bmatrix} 6 & 7 & \ldots 15 \end{bmatrix}$, $\text{Prob}(y_t = \overline{y}_s) = S^{-1}$. Set $W = [w_{\min}, \ldots, w_{\max}]$ and $B = [b_{\min}, \ldots, b_{\max}]$, where the intermediate points in W and B, respectively, are equally spaced. Please set $w_{\min} = \frac{1}{1-\beta}\frac{1}{1-\gamma}(y_{\min} - a)^{1-\gamma}$ and $w_{\max} = w_{\min}/20$ (these are negative numbers, so $w_{\min} < w_{\max}$). Also set $b_{\min} = (1 - y_{\max} + .33)$ and $b_{\max} = y_{\max} - y_{\min}$. For these

parameter values, compute the optimal contract by formulating a linear program for one step on the Bellman equation, then iterating to convergence on it.

c. Notice the following probability laws:

$$\begin{aligned}
\text{Prob}(b_t, w_{t+1}, y_t | w_t) &\equiv \Pi^{w_t}(b_t, w_{t+1}, y_t) \\
\text{Prob}(w_{t+1}|w_t) &= \textstyle\sum_{b \in B, y \in Y} \Pi^{w_t}(b, w_{t+1}, y) \\
\text{Prob}(b_t, y_t|w_t) &= \textstyle\sum_{w_{t+1} \in W} \Pi^{w_t}(b_t, w_{t+1}, y_t).
\end{aligned} \tag{181}$$

Please use these and other probability laws to compute $\text{Prob}(w_{t+1}|w_t)$. Show how to compute $\text{Prob}(c_t)$, assuming a given initial promised value w_0. **d.** Assume that $w_0 \approx -2$. Compute and plot $F_t(c) = \text{Prob}(c_t \leq c)$ for $t = 1, 5, 10, 100$. Qualitatively, how do these distributions compare with those for the simple village and money lender model with no information problem and one-sided lack of commitment?

Solution

a. The objective function and all constraints are linear in the probabilities. This makes this problem into a linear programming problem.
b. The program `linprogTWexp.m` solves the problem.
c. The probability laws for w and c are also computed in `linprogTWexp.m`. The unconditional distribution over consumption at time t is obtained by first computing the t-period transition probabilities for the promised utilities w and multiplying those with the probability distribution over consumption states conditional on w, $P(c_t|w_t)$.
d. The successive consumption cdf's are moving to the left. Ever more agents are receiving a low consumption allocation. The reason is that promised utilities are headed south in the Thomas-Worral model. In contrast, in the village lender economy, the consumption distribution converges eventually becomes degenerate when all agents have obtained the highest income distribution. Promised utilities are non-decreasing processes in that model.

FIGURE 11. Exercise 15.14 a : Profits of Money Lender in Thomas-Worral Model

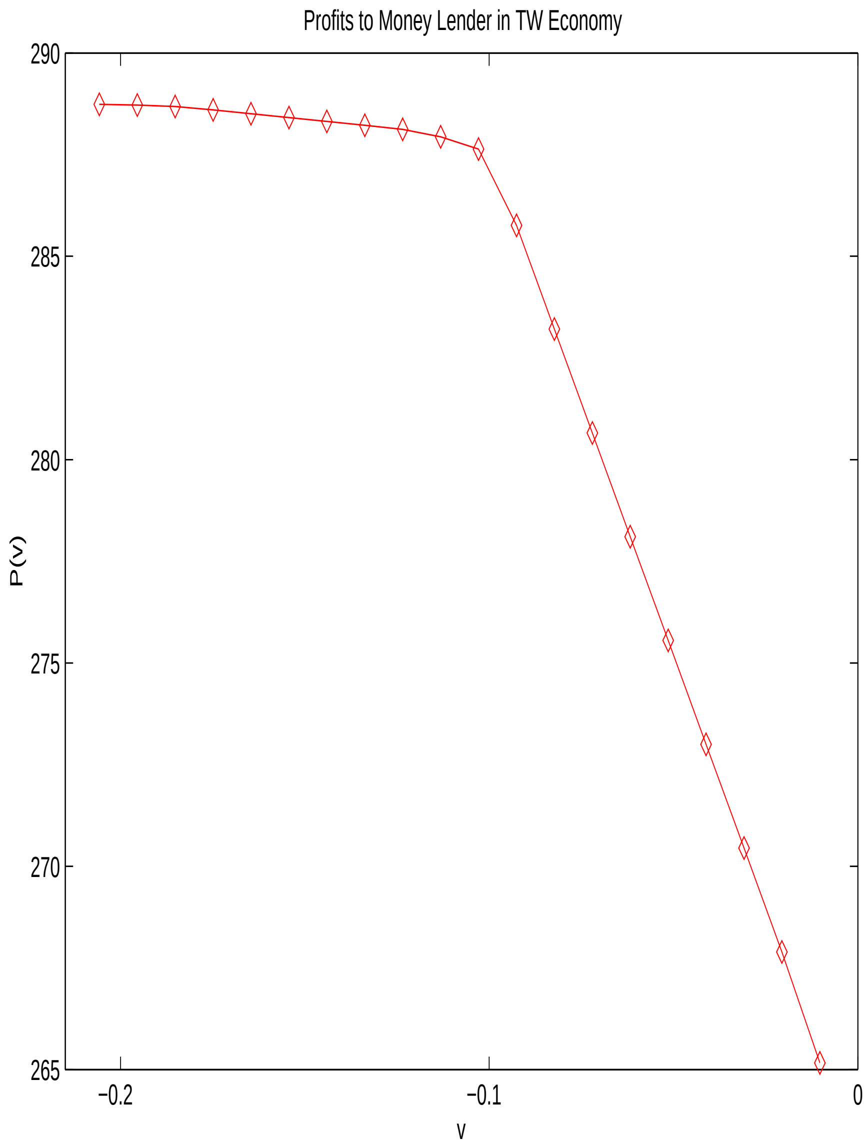
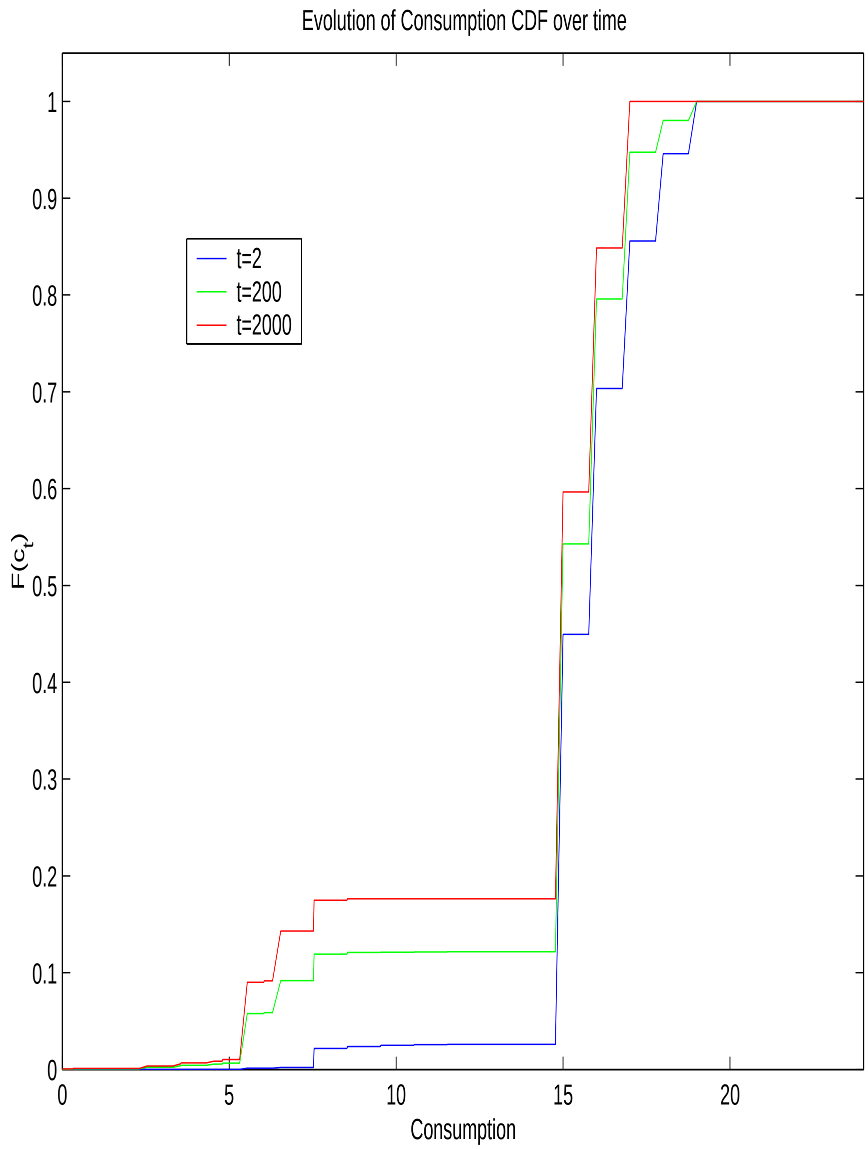


FIGURE 12. Exercise 15.14 b Evolution of Consumption Distribution over Time



Exercise 15.15. *Kehoe-Levine-Kocherlakota without risk*

Consider an economy in which each of two types of households has preferences over streams of a single good that are ordered by $v = \sum_{t=0}^{\infty} \beta^t u(c_t)$ where $u(c) = (1-\gamma)^{-1}(c+b)^{1-\gamma}$ for $\gamma \geq 1$ and $\beta \in (0,1)$, and $b > 0$. For $\epsilon > 0$ and $t \geq 0$, households of type 1 are endowed with an endowment stream $y_{1,t} = 1+\epsilon$ in even numbered periods and $y_{1,t} = 1-\epsilon$ in odd numbered periods. Households of type 2 own an endowment stream of $y_{2,t}$ that equals $1-\epsilon$ in even periods and $1+\epsilon$ in odd periods. There are equal numbers of the two types of household. For convenience, you can assume that there is one of each type of household.

Assume that $\beta = .8$, $b = 5$, $\gamma = 2$, and $\epsilon = .5$.

a. Compute autarky levels of discounted utility v for the two types of households. Call them $v_{\text{aut},h}$ and $v_{\text{aut},\ell}$.

b. Compute the competitive equilibrium allocation and prices. Here assume that there are no enforcement problems.

c.Compute the discounted utility to each household for the competitive equilibrium allocation. Denote them v_i^{CE} for $i = 1, 2$.

d. Verify that the competitive equilibrium allocation is not self-enforcing in the sense that at each $t > 0$, some households would prefer autarky to the competitive equilibrium allocation.

e. Now assume that there are enforcement problems because at the beginning of each period, each household can renege on contracts and other social arrangments with the consequence that it receives the autarkic allocation from that period on. Let v_i be the discounted utility at time 0 of consumer i. Formulate the consumption smoothing problem of a planner who wants to maximize v_1 subject to $v_2 \geq \tilde{v}_2$, and constraints that the express that the allocation must be self-enforcing.

f. Find an efficient self-enforcing allocation of the periodic form $c_{1,t} = \check{c}, 2-\check{c}, \check{c}, \ldots$ and $c_{2,t} = 2-\check{c}, \check{c}, 2-\check{c}, \ldots$, where continuation utilities of the two agents oscillate between two values v_h and v_ℓ. Compute $\check{c}$. Compute discounted utilities v_h for the agent who receives $1+\epsilon$ in the period and v_ℓ for the agent who receives $1-\epsilon$ in the period.

Plot consumption paths for the two agents for (i) autarky, (ii) complete markets without enforcement problems, (iii) complete markets with the enforcement constraint. Plot continuation utilities for the two agents for the same three allocations. Comment on them.

g. Compute one-period gross interest rates in the complete market economies with and without enforcement constraints. Plot them over time. In which economy is the interest rate higher? Explain.

h. Keep all parameters the same, but gradually increase the discount factor. As you raise β toward one, compute interest rates as in part (g). At what value of β

do interest rates in the two economies become equal. At that value of β, is either participation constraint ever binding?

Solution

a. The value of autarky for agent h is

$$\begin{aligned} v^h_{aut} &= u(1+\epsilon) + \beta u(1-\epsilon) + \beta^2 u(1+\epsilon) + \beta^3 u(1-\epsilon) + ... \\ &= u(1+\epsilon)\left[1+\beta^2+\beta^4+...\right] + \beta u(1-\epsilon)\left[1+\beta^2+\beta^4+...\right] \\ &= \frac{u(1+\epsilon)+\beta u(1-\epsilon)}{1-\beta^2} \\ &= -0.8314. \end{aligned}$$

Analogously, for agent $l : v^l_{aut} = \frac{\beta u(1+\epsilon)+u(1-\epsilon)}{1-\beta^2} = -0.8469$

b. A competitive equilibrium is an allocation $(c^1, c^2) = \{c^1_t, c^2_t\}_{t=0}^{\infty}$ and a sequence of Arrow-Debreu prices $P = \{P_t\}_{t=0}^{\infty}$ such that
(i) given prices $\{P_t\}$ each agent solves

$$\max_c \sum_t \beta^t \frac{(b+c_t)^{1-\gamma}}{1-\gamma},$$

subject to the AD budget constraint

$$\sum_t P_t c_t \leq \sum_t P_t y_t.$$

(ii) markets clear: $\forall t$

$$\sum_i c^i_t = \sum_i y^i_t.$$

Guess that the solution satisfies $c^i_t = c^i_{CE}, i = 1, 2$. The first order conditions for agent $i \in \{l, h\}$ are:

$$\beta^t \left(c^i_t + b\right)^{-\gamma} = \lambda^i P_t.$$

Using our guess in the first order condition and the normalization $P_0 = 1$ gives us the expression for the competitive equilibrium prices

$$P_t = \beta^t \frac{(c^i_{CE} + b)^{-\gamma}}{\lambda^i} = \beta^t P_0 = \beta^t.$$

The budget constraint for agent i becomes

$$\sum_{t=0}^{\infty} \beta^t c^i_{CE} = \sum_{t=0}^{\infty} \beta^t y_t.$$

For agent 1 this gives

$$\begin{aligned} \frac{c^1_{CE}}{1-\beta} &= \frac{(1+\epsilon)+\beta(1-\epsilon)}{(1-\beta)(1+\beta)} \\ c^1_{CE} &= 1 + \epsilon\left(\frac{1-\beta}{1+\beta}\right) = 1.0556. \end{aligned}$$

Using market clearing consumption of agent 2 is

$$c_{CE}^2 = 1 - \epsilon\left(\frac{1-\beta}{1+\beta}\right) = 0.9444.$$

It is easy to prove that c_t^1 and c_t^2 are constant. From the first order condition $\beta^t u'(c_t^1) = \lambda p_t$ and $\beta^t u'(c_t^2) = \mu p_t$ it follows that the ratio of marginal utilities is constant. Together with the resource constraint $c_t^1 + c_t^2 = 2$ this implies that the consumption shares are constant.

c. The value from this allocation is $v_{CE}^1 = \frac{\left(b+c_{CE}^1\right)^{1-\gamma}}{(1-\gamma)(1-\beta)} = -0.8257$ and $v_{CE}^l = \frac{\left(b+c_{CE}^2\right)^{1-\gamma}}{(1-\gamma)(1-\beta)} = -0.8411$.

d. Combining the results of part a and part c, in odd periods, the household with the high endowment (type 2) would like to walk away ($v^{aut} = -.8314 > v^{CE} = -.844$). In even periods, nobody wants to default.

e. Formulate the planner problem for the case with limited commitment

$$\max_{c^1,c^2} v_0^1 = \sum_{t=0}^{\infty} \beta^t \frac{(b+c_t^1)^{1-\gamma}}{1-\gamma},$$

subject to the participation constraints for agent 2 :

$$v_t^2 = \sum_{t=0}^{\infty} \beta^t u\left(c_t^2\right) \geq \bar{v}^2,$$

and enforcement constraints for each agent, $\forall t$:

$$\begin{aligned} v_{CE,t}^2 &\geq v_{aut,t}^2 \\ v_{CE,t}^1 &\geq v_{aut,t}^1. \end{aligned}$$

f. The contract can be formulated recursively as $\{c^1, c^2, v^l, v^h\}$ where $c_t^1 = \check{c}, 2-\check{c}, \check{c}, 2-\check{c}, ...$ and $c_t^2 = 2 - c_t^1$. The discounted utility v^h for the agent who receives $(1+\epsilon)$ in that period and v^l for the agent who receives $(1-\epsilon)$ are given by:

$$\begin{aligned} v^h &= \frac{u(\check{c}) + \beta u(2-\check{c})}{1-\beta^2} \\ v^l &= \frac{\beta u(\check{c}) + u(2-\check{c})}{1-\beta^2}. \end{aligned}$$

In line with part d, we guess that tonly the enforcement constraint of the high endowment household is binding:

$$v^h = \frac{u(\check{c}) + \beta u(2-\check{c})}{1-\beta^2} = v_{aut}^h.$$

From that equation we compute $\check{c} = 1.1645$ and $v^h = -0.8314$. It follows that $v^l = -0.8365$. The planner thus achieves more risk sharing (consumption smoothing) than in autarky but less than in the perfect enforcement case.

g. In complete markets and with perfect enforcement we know that agents equate their intertemporal marginal rates of consumption. Because consumption allocations are constant, the IMRS equals one for both agents in every period. It follows that the stochastic discount factor is simply β. The interest rate is the inverse of the stochastic discount factor. Therefore, the interest rate equals $\beta^{-1} = 1.25$. In the case of limited commitment the stochastic discount factor is the maximal IMRS among the agents. This is the IMRS of the unconstrained agent. We know that the agent with the low income realization is unconstrained. Therefore the SDF is

$$m_{t,t1} = \beta \frac{u'(2-\check{c})}{u'(\check{c})} = \beta \left(\frac{b+2-\check{c}}{b+\check{c}} \right)^{-\gamma}.$$

The risk-free interest rate is $m_{t,t1}^{-1} = 1.1202$. It is constant. The interest rate is lower than the complete markets interest rate, because at any higher rate than the prevailing one, the unconstrained household wants to save more. One has to lower the interest rate from the complete markets benchmark interest rate until the unconsatrined household will find the current allocation optimal. A Pareto-improvement implies more saving buy the unconstrained household, who is the household that is pricing the assets in the economy (the highest IMRS). This necessitates a higher interest rate or a lower IMRS.

h. For $\beta = 0.843265$ the interest rates are the same with or without perfect enforcement. The interest rate is 1.185866. For $\beta = 0.92$ the constraints stop binding. For β high enough, the CE allocation is sustainable. The interest rate of the unconstrained economy is $\frac{1}{\beta} = \frac{13}{11}$ and is decreasing in β. The interest rate of the constrained economy is $\frac{1}{\beta}\left(\frac{\check{c}}{2-\check{c}}\right)^{-\gamma}$ which is smaller and increasing in β, until β becomes $\frac{11}{13}$. At that point the economy collapses to the frictionless economy where the participation constraint is never binding.

CHAPTER 16

Credible government policies

Exercise 16.1.

Consider the following one-period economy. Let (ξ, x, y) be the choice variables available to a representative agent, the market as a whole, and a benevolent government, respectively. In a rational expectations equilibrium or competitive equilibrium, $\xi = x = h(y)$, where $h(\cdot)$ is the "equilibrium response" correspondence that gives competitive equilibrium values of x as a function of y; that is, $[h(y), y]$ is a competitive equilibrium. Let C be the set of competitive equilibria.

Let $X = \{x_M, x_H\}, Y = \{y_M, y_H\}$. For the one-period economy, when $\xi_i = x_i$, the payoffs to the government and household are given by the values of $u(x_i, x_i, y_j)$ entered in the following table:

One-period payoffs to the government–household
[values of $u(x_i, x_i, y_j)$]

	x_M	x_H
y_M	10*	20
y_H	4	15*

*Denotes $(x, y) \in C$.

The values of $u(\xi_k, x_i, y_j)$ not reported in the table are such that the competitive equilibria are the outcome pairs denoted by an asterisk (*).

a. Find the *Nash equilibrium* (in pure strategies) and *Ramsey outcome* for the one-period economy.

b. Suppose that this economy is repeated twice. Is it possible to support the Ramsey outcome in the first period by reverting to the Nash outcome in the second period in case of a deviation?

c. Suppose that this economy is repeated three times. Is it possible to support the Ramsey outcome in the first period? In the second period? Consider the following expanded version of the preceding economy. $Y = \{y_L, y_M, y_H\}$, $X = \{x_L, x_M, x_H\}$. When $\xi_i = x_i$, the payoffs are given by $u(x_i, x_i, y_j)$ entered here:

One-period payoffs to the government–household
[values of $u(x_i, x_i, y_j)$]

	x_L	x_M	x_H
y_L	3*	7	9
y_M	1	10*	20
y_H	0	4	15*

*Denotes $(x, y) \in C$.

d. What are Nash equilibria in this one-period economy?

e. Suppose that this economy is repeated twice. Find a subgame perfect equilibrium that supports the Ramsey outcome in the first period. For what values of δ will this equilibrium work? **f.** Suppose that this economy is repeated three times. Find a subgame perfect equilibrium that supports the Ramsey outcome in the first two periods (assume $\delta = 0.8$). Is it unique?

Solution

a. The Nash equilibrium is $(x_M, y_M) \in C$ because $10 = u(x_M, x_M, y_M) > u(x_M, x_M, y_H) = 4$. The Ramsey outcome is $(x_H, y_H) \in C$ because $15 = u(x_H, x_H, y_H) > u(x_M, x_M, y_M) = 10$.
b. It is not possible to support the Ramsey by Nash reversion for the 2-period economy. In the second period agent and government play the Nash equilibrium. The condition to support the Ramsey equilibrium is violated:

$$(1-\delta)v^R + \delta v^N \geq (1-\delta)\tilde{v} + \delta v^N \Longrightarrow 15 \geq 20$$

c. There is a unique NE in the stage game. In a finitely repeated game, repetition of the NE of the stage game is the only subgame perfect equilibrium. Therefore, it is not possible to support the Ramsey in the first nor in the second period by Nash reversion for the 3-period economy.
d. The Nash equilibrium are $\{(x_L, y_L), (x_M, y_M)\} \in C$. The Ramsey outcome is $(x_H, y_H) \in C$.
e. There is a unique SPE for the 2-period economy if we punish with the worst NE $(x_L, y_L) = N_2$ in case of a deviation and prescribe the best NE $(x_M, y_M) = N_1$ in case of adherence. We can support Ramsey in the first period when the discount factor is at least 0.4166

$$\begin{aligned}
(1-\delta)v^R + \delta v^{N_1} &\geq (1-\delta)\tilde{v} + \delta v^{N_2} \Longrightarrow \\
(1-\delta)15 + \delta 10 &\geq (1-\delta)20 + \delta 3 \Longrightarrow \\
\delta &\geq \frac{5}{12}
\end{aligned}$$

f. Because $0.8 > \frac{5}{12}$, in the second period we can sustain playing Ramsey in the second period if we play N_1 in case of adherence and N_2 in case of a deviation. This follows immediately from part e. Working backwards, in period 1 we can sustain Ramsey if we play Ramsey in period 2 and N_1 in period 3. In case of deviation we play N_2 forever after. This works because the gain from deviating is less than the loss from deviating: $5 \leq \delta 12 + \delta^2 7 = 14.08$. A second equilibrium that works is to punish with N_2 in period 2 and with N_1 in period 3, because $5 \leq \delta 12 + \delta^2 0 = 9.6$. The third and last equilibrium that works is to punish with N_1 in period 2 and with N_2 in period 3, because $5 \leq \delta 5 + \delta^2 7 = 8.48$. Therefore, there is no unique SPE.

Exercise 16.2.

Consider a version of the setting studied by Stokey (1989). Let (ξ, x, y) be the choice variables available to a representative agent, the market as a whole, and a benevolent government, respectively. In a rational expectations or competitive equilibrium, $\xi = x = h(y)$, where $h(\cdot)$ is the "equilibrium response" correspondence that gives competitive equilibrium values of x as a function of y; that is, $[h(y), y]$ is a competitive equilibrium. Let C be the set of competitive equilibria.

Consider the following special case. Let $X = \{x_L, x_H\}$ and $Y = \{y_L, y_H\}$. For the one-period economy, when $\xi_i = x_i$, the payoffs to the government are given by the values of $u(x_i, x_i, y_j)$ entered in the following table:

One-period payoffs to the government–household
[values of $u(x_i, x_i, y_j)$]

	x_L	x_H
y_L	0*	20
y_H	1	10*

* Denotes $(x, y) \in C$.

The values of $u(\xi_k, x_i, y_j)$ not reported in the table are such that the competitive equilibria are the outcome pairs denoted by an asterisk (*).

a. Define a *Ramsey plan* and a *Ramsey outcome* for the one-period economy. Find the Ramsey outcome.

b. Define a *Nash equilibrium* (in pure strategies) for the one-period economy.

c. Show that there exists no Nash equilibrium (in pure strategies) for the one-period economy.

d. Consider the infinitely repeated version of this economy, starting with $t = 1$ and continuing forever. Define a *subgame perfect equilibrium*.

e. Find the value to the government associated with the *worst* subgame perfect equilibrium.

f. Assume that the discount factor is $\delta = .8913 = (1/10)^{1/20} = .1^{.05}$. Determine whether infinite repetition of the Ramsey outcome is sustainable as a subgame perfect equilibrium. If it is, display the associated subgame perfect equilibrium.

g. Find the value to the government associated with the *best* subgame perfect equilibrium.

h. Find the outcome path associated with the *worst* subgame perfect equilibrium.

i. Find the one-period continuation value v_1 and the outcome path associated with the one-period continuation strategy σ^1 that induces adherence to the worst subgame perfect equilibrium.

j. Find the one-period continuation value v_2 and the outcome path associated with the one-period continuation strategy σ^2 that induces adherence to the first-period outcome of the σ^1 that you found in part i.

k. Proceeding recursively, define v_j and σ^j, respectively, as the one-period continuation value and the continuation strategy that induces adherence to the first-period outcome of σ^{j-1}, where (v_1, σ^1) were defined in part i. Find v_j for $j = 1, 2, \ldots$, and find the associated outcome paths.

l. Find the lowest value for the discount factor for which repetition of the Ramsey outcome is a subgame perfect equilibrium.

Solution

a. A Ramsey plan: the government chooses y and walks away. Then the public plays a competitive equilibrium as response:

$$\max_{(x,y)\in C} u(x,x,y) \qquad or \qquad \max_{y} u(h(y),h(y),y)$$

The Ramsey plan in the example is y_H, the public's response is x_H and the Ramsey outcome is $v^R = 10$.

b. A nash equilibrium satisfies (1) $(x^N, y^N) \in C$ and (2) $u(x^N, x^N, y^N) = \max_{\eta\in Y} u(x^N, x^N, \eta)$

c. For each of the two competitive equilibria the condition for second consition for a NE is violated:

$$\begin{aligned} 10 &= u(x_H, x_H, y_H) < u(x_H, x_H, y_L) = 20 \\ 0 &= u(x_L, x_L, y_L) < u(x_L, x_L, y_H) = 1 \end{aligned}$$

d. A strategy profile $\sigma = (\sigma^h, \sigma^g)$ is a SPE if $\forall t, \forall (x^{t-1}, y^{t-1})$: (1) $(x_t, y_t) \in C$, $x_t = \sigma^h(x^{t-1}, y^{t-1}), y_t = \sigma^g(x^{t-1}, y^{t-1})$ and (2) $\forall \eta \in Y$:

$$(1-\delta)r(x_t, y_t) + \delta V\left(\sigma|_{(x^t,y^t)}\right) \geq (1-\delta)r(x_t, \eta) + \delta V\left(\sigma|_{(x^t,y^{t-1},\eta)}\right)$$

A SPE prescribes to play a NE in every period and specifies that no deviations can be optimal.

e. The worst SPE is self-enforcing and has associated value $\underline{v}$ obtained from

$$\underline{v} = \min_{v_1\in V, y\in Y} \{(1-\delta)r(h(y), y) + \delta v_1\}$$

subject to

$$(1-\delta)r(h(y), y) + \delta v_1 \geq (1-\delta)r(h(y), H(h(y))) + \delta\underline{v}$$

where the worst SPE is used as the continuation value in the event opf a deviation. The minimum is attained when the constraint is binding:

$$\underline{v} = \min_{y\in Y} (1-\delta)r(h(y), H(h(y))) + \delta\underline{v}$$

For $y = y_L$, $h(y_L) = x_L$, $H(x_L) = y_H$ and $r(x_L, y_H) = 1$ whereas for $y = y_H$, $h(y_H) = x_H$, $H(x_H) = y_L$ and $r(x_H, y_L) = 20$. The minimum is attained for $y = y_L$ and the value associated with the worst SPE is $\underline{v}= 1$. We find $v_1 = \frac{1}{\delta}\left[\underline{v} - (1-\delta)r(h(y_L), H(h(y_L)))\right] = \frac{1}{\delta}$.

f. We can sustain infinite repitition of the Ramsey outcome by reverting to the worst SPE calculated in the previous part in case of deviation.

$$\frac{10}{1-0.8913} \geq 20 + \frac{0.8913}{1-0.8913}1$$

For this value of δ, $v_1 = 1.122$

g. The best SPE is self-rewarding

$$\bar{v} = \max_{y\in Y} r(h(y), y)$$

subject to

$$r(h(y), y) \geq (1-\delta)r(h(y), H(h(y))) + \delta \underline{v}$$

When $y = y_L$, the constraint is violated, whereas for $y = y_H$ the constraint is satisfied for $\delta = 0.8913$:

$$10 \geq (1 - 0.8913)20 + 0.8913$$

Therefore $\bar{v} = 10$

h. the outcome path associated with the worst SPE is (x_L, y_L) in the first $(p-1)$ periods, (x_H, y_H) in the p^{th} period, (x_L, y_L) in the next $(p-1)$ periods, (x_H, y_H) in the $2p^{th}$ period, etc. This output path implies that $p = \lfloor 20\frac{\log(11)}{\log(10)} \rfloor = \lfloor 20.83 \rfloor = 21.$

i. We know that $v_1 = \frac{1}{\delta}$.The output path is (x_L, y_L) in the first $(p-1)$ periods, (x_H, y_H) in the p^{th} period, (x_L, y_L) in the next $(p-1)$ periods, (x_H, y_H) in the $2p^{th}$ period, etc. The resulting p is

$$\frac{1}{0.1^{0.05}} = \frac{10\delta^p}{1-\delta^p}$$

$$p = 20\frac{\log(1 + 10(0.1^{0.05}))}{\log(10)} = 19.92$$

j. $v_2 = \frac{1}{\delta^2}$. The output path is (x_L, y_L) in the first $(p-1)$ periods, (x_H, y_H) in the p^{th} period, (x_L, y_L) in the next $(p-1)$ periods, (x_H, y_H) in the $2p^{th}$ period, etc. Where p is determined from

$$\frac{1}{0.1^{0.10}} = \frac{10\delta^p}{1-\delta^p}$$

$$p = 20\frac{\log(1 + 10(0.1^{0.10}))}{\log(10)} = 19.03$$

k. $v_3 = \frac{1}{\delta^3}$.The output path is (x_L, y_L) in the first $(p-1)$ periods, (x_H, y_H) in the p^{th} period, (x_L, y_L) in the next $(p-1)$ periods, (x_H, y_H) in the $2p^{th}$ period, etc. Where p is determined from

$$p = 20\frac{\log(1 + 10(0.1^{\frac{3}{20}}))}{\log(10)} = 18.15$$

For 4 periods: $p = 17.28$, for 5 periods $p = 16.42$, etc.

l. From part f we know that we can support repetition of the Ramsey outcome by reverting to the worst SPE for δ such that.

$$\begin{aligned} \frac{10}{1-\delta} &\geq 20 + \frac{\delta}{1-\delta} \\ \delta &\geq \frac{10}{19} = 0.526 \end{aligned}$$

Exercise 16.3.

Consider the following model of Kydland and Prescott (1977). A government chooses the inflation rate y from a closed interval $[0, 10]$. There is a family of Phillips curves indexed by the public's expectation of inflation x:

$$U = U^* - \theta(y - x) \tag{1}$$

where U is the unemployment rate, y is the inflation rate set by the government, and $U^* > 0$ is the natural rate of unemployment and $\theta > 0$ is the slope of the Phillips curve, and where x is the average of private agents' setting of a forecast of y, called ξ. Private agents' only decision in this model is to forecast inflation. They choose their forecast ξ to maximize

$$-.5(y - \xi)^2. \tag{2}$$

Thus, if they know y, private agents set $\xi = y$. All agents choose the same ξ so that $x = \xi$ in a rational expectations equilibrium. The government has one-period return function

$$r(x, y) = -.5(U^2 + y^2) = -.5[(U^* - (y - x))^2 + y^2]. \tag{3}$$

Define a *competitive equilibrium* as a 3-tuple U, x, y such that given y, private agents solve their forecasting problem and (1) is satisfied.

a. Verify that in a competitive equilibrium, $x = y$ and $U = U^*$.

b. Define the government best response function in the one-period economy. Compute it.

c. Define a Nash equilibrium (in the spirit of Stokey (1989) or chapter 16). Compute it.

d. Define the Ramsey problem for the one-period economy. Define the Ramsey outcome. Compute it.

e. Verify that the the Ramsey outcome is better than the Nash outcome.

Now consider the repeated economy where the government cares about

$$(1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} r(x_t, y_t), \tag{4}$$

where $\delta \in (0, 1)$.

f. Define a *subgame perfect equilibrium*.

g. Define a *recursive* subgame perfect equilibrium.

h. Find a recursive subgame perfect equilibrium that sustains infinite repetition of the one-period Nash equilibrium outcome.

i. For $\delta = .95$, $U^* = 5, \theta = 1$, find the value of (4) associated with the worst subgame perfect equilibrium. Carefully and completely show your method for computing the worst subgame perfect equilibrium value. Also, compute the values associated with the repeated Ramsey outcome, the Nash equilibrium, and Abreu's simple stick-and-carrot strategy.

j. Compute a recursive subgame perfect equilibrium that attains the worst subgame perfect equilibrium value (4) for the parameter values in part i.

k. For $U^* = 5, \theta = 1$, find the cutoff value δ_c of the discount factor δ below which the Ramsey value v^R cannot be sustained by reverting to repetition of v^N as a consequence of deviation from the Ramsey y.

l. For the same parameter values as in part k, find another cut off value $\tilde{\delta}_c$ for δ below which Ramsey cannot be sustained by reverting after a deviation to an equilibrium attaining the worst subgame perfect equilibrium value. Compute the worst subgame perfect equilibrium value for $\tilde{\delta}_c$.

m. For $\delta = .08$, compute values associated with the best and worst subgame perfect equilibrium strategies. *Hint:* Read the section leading up to formulas equations 16.29–16.32.

Solution

a. In a competitive equilibrium or rational expectations equilibrium the private sectors forecasts materialize: $x = y$. plugging this in into the Phillips curve gives $U = U^*$. denote the set of competitive equilibria by $\mathcal{C}$

b. From the government's first order condition w.r.t. y we find its best response is $y = H(x) = \frac{\theta^2 x + \theta U^*}{1+\theta^2} = 0.5(U^* + x)$.

c. A Nash equilibrium (x^N, y^N) is a competitive equilibrium $(x^N = y^N)$ that satisfies $r(x^N, y^N) > r(x^N, \eta), \forall \in [0, 10]$. The Nash equilibrium is $y^N = x^N = \theta U^* = U^*$ as long as $\theta U^* < 10$. This gives a value of $V^N = -\left(U^*\right)^2$.

d. The Ramsey problem for the government is $\max_{(x,y)\in\mathcal{C}} r(x, y)$. The policy y^R that attains the maximum of the Ramsey problem is the Ramsey outcome. The Ramsey outcome is $x^R = y^R = 0$, which gives a value $V^R = -0.5\left(U^*\right)^2$.

e. The Nash outcome is twice as low as the Ramsey outcome.

f. A strategy profile (σ^h, σ^g) is a subgame perfect equilibrium of the infintely repeated economy if for each t and each history $(x^{t-1}, y^{t-1}) \in X^{t-1}xY^{t-1}$: (1) the outcome $x_t = \sigma_t^h(x^{t-1}, y^{t-1})$ is consistent with a competitve equilibrium when $y_t = \sigma_t^g(x^{t-1}, y^{t-1})$ and (2) $(1-\delta)r(x_t, y_t) + \delta V_g\left(\sigma|_{(x^t, y^t)}\right) \geq (1-\delta)r(x_t, \eta) + \delta V_g\left(\sigma|_{(x^t, y^{t-1}, \eta)}\right) \qquad \forall \eta \in Y$.

g. A recursive strategy (ϕ, v) is a subgame perfect equilibrium if (1) the first period outcome pair is a competitive equilibrium: $x = z^h(v)$ given $y = z^g(v)$, (2) $\mathcal{V}(v, z^h(v), \eta)$ is a value for a subgame perfect equilibrium $\qquad \forall \eta \in Y$ and (3) $v = (1-\delta)r(z^h(v), z^g(v)) + \delta\mathcal{V}(v, z^h(v), z^g(v)) \geq (1-\delta)r(z^h(v), \eta) + \delta\mathcal{V}(v, z^h(v), \eta)$.

h. Infinite repitition of Nash can be sustained by deviating to the Nash outcome. The Nash equilibrium is a competitive equilibrium and the continuation value V^N associated with deviating is itself the outcome of a SPE. In particular $z^h(v) = x^N, z^g(v) = y^N$ and $v = V^N$.

i. The worst SPE is self-enforcing. As shown in the text, to find its value we solve $v^{worst} = \min_{y \in Y} r\{h(y), H(h(y))\}$. Using the expression for the best response H, we minimize

$$\begin{aligned} & \min_{y \in Y} -\frac{1}{2}\left[[U^* - (0.5U^* + 0.5h(y) - h(y))]^2 + h(y)^2\right] \\ = \quad & \min_{y \in Y} -\frac{1}{2}\left[[0.5U^* + 0.5h(y))]^2 + h(y)^2\right] \end{aligned}$$

This function is concave and reaches a minimum at $y = 10$. If follows that $H(h(y)) = 7.5$. The value associated with the worst SPE is then $v^{worst} = -56.25$. For the same parameter values $V^N = -25, V^R = -12.5$. For the stick and carrot stategy, the stick is plaing the worst competitve equilibrium, which value is -62.5. The value of the stick and carrot is is given by $v^{SC} = (1-\delta)(-62.5) + \delta(-12.5) = -15$.

j. We need to compute the best SPE equilibrium. This is self-rewarding. It is easily found to be $v^{best} = V^R = -12.5$. We use the following recursive procedure to find a SPE that achieves the worst. Set the first period promised value $v_0 = v^{worst} = -56.25$. Use v^{worst} in the event of a deviation. In the event of adherence in the first period specify a continuation value $v_1 = \delta^{-1}v^{worst} + \delta^{-1}(1-\delta)(V^R) = -55.9211$. In the event of adherence in the next period, the value is $v_2 = \delta^{-1}v_1 + \delta^{-1}(1-\delta)(V^R) = -55.5748$. We can continue the work recursively forward and find the increasing set of subgame perfect equilibria that are played upon adherence. The government decision $\tilde{y}$ that satisfies $r(\tilde{y}, \tilde{y}) = v_1$ is 9.319. Summarizing: $v^{worst} = -56.25$, $z^h(v) = z^g(v) = 10$ if $v = v^{worst}$ and $z^h(v) = z^g(v) = 9.319$ otherwise. $\mathcal{V}(v_t, x_t, y_t) = v_1$ if $(x_t, y_t) = [z^h(v_t), z^h(v_t)]$ and v^{worst} otherwise.

k. Ramsey cannot be sustained by Nash reversion when

$$\begin{aligned} (1-\delta)r(x^R, y^R) + \delta V^R \quad &> \quad (1-\delta)r(x^R, H(x^R)) + \delta V^N \\ -12.5 \quad &> \quad (1-\delta)(-6.25) + \delta(-25) \end{aligned}$$

we find that $r(x^R, y^R) = -12.5$ and $r(x^R, H(x^R)) = -6.25$. The value $\delta_c = 0.333$.

l. Proceeding likewise

$$(1-\delta)r(x^R, y^R) + \delta V^R > (1-\delta)r(x^R, H(x^R)) + \delta v^{worst}$$

We find $\tilde{\delta}_c = 0.125$.

m. Assume $\delta = 0.08$, then $15.625 = v_1 > v^{best} = -12.5$.so we need to find a new set $[v^{worst}, v^{best}]$. The formula's (16.29)-(16.31) in the text can be solved to get $v^{worst} = -43.56$, $v^{best} = -14.12$, $y^{best} = 1.8$ and $y^{worst} = 8.2$. We can verify that now $v_1 = v^{best} = -14.12$.

CHAPTER 17

Fiscal-monetary theories of inflation

Exercise 17.1. *Why deficits in Italy and Brazil were once extraordinary proportions of GDP*

The government's budget constraint can be written as

$$g_t - \tau_t + \frac{b_t}{R_{t-1}}(R_{t-1} - 1) = \frac{b_{t+1}}{R_t} - \frac{b_t}{R_{t-1}} + \frac{M_{t+1}}{p_t} - \frac{M_t}{p_t}. \tag{1}$$

The left side is the real gross-of-interest government deficit; the right side is change in the real value of government liabilities between $t-1$ and t. Government budgets often report the *nominal* gross-of-interest government deficit, defined as

$$p_t(g_t - \tau_t) + p_t b_t \left(1 - \frac{1}{R_{t-1}p_t/p_{t-1}}\right),$$

and their ratio to nominal GNP, $p_t y_t$, namely,

$$\left[(g_t - \tau_t) + b_t\Big(1 - \frac{1}{R_{t-1}p_t/p_{t-1}}\Big)\right] /y_t.$$

For countries with a large b_t (e.g., Italy) this number can be very big even with a moderate rate of inflation. For countries with a rapid inflation rate, like Brazil in 1993, this number sometimes comes in at 30 percent of GDP. Fortunately, this number overstates the magnitude of the government's "deficit problem," and there is a simple adjustment to the interest component of the deficit that renders a more accurate picture of the problem. In particular, notice that the real values of the interest component of the real and nominal deficits are related by

$$b_t\left(1 - \frac{1}{R_{t-1}}\right) = \alpha_t b_t \left(1 - \frac{1}{R_{t-1}p_t/p_{t-1}}\right),$$

where

$$\alpha_t = \frac{R_{t-1} - 1}{R_{t-1} - p_{t-1}/p_t}.$$

Thus, we should multiply the real value of nominal interest payments $b_t(1-\frac{p_{t-1}}{R_{t-1}p_t})$ by α_t to get the real interest component of the debt that appears on the left side of equation (1).

a. Compute α_t for a country that has a b_t/y ratio of .5, a gross real interest rate of 1.02, and a zero net inflation rate.

b. Compute α for a country that has a b_t/y ratio of .5, a gross real interest rate of 1.02, and a 100 percent per year net inflation rate.

Solution

a. Zero net inflation rate means that $\frac{p_{t-1}}{p_t} = 1$, so that $\alpha_t = 1$ as well.

b. 100% inflation rate means that $\frac{p_{t-1}}{p_t} = 0.5$. This time:

$$\alpha_t = \frac{1.02 - 1}{1.02 - 0.5} = \frac{1}{26}.$$

Exercise 17.2. *A strange example of Brock (1974)*

Consider an economy consisting of a government and a representative household. There is one consumption good, which is not produced and not storable. The exogenous supply of the good at time $t \geq 0$ is $y_t = y > 0$. The household owns the good. At time t the representative household's preferences are ordered by

$$\sum_{t=0}^{\infty} \beta^t \{\ln c_t + \gamma \ln(m_{t+1}/p_t)\}$$

where c_t is the household's consumption at t, p_t is the price level at t, and m_{t+1}/p_t is the real balances that the household carries over from time t to $t+1$. Assume that $\beta \in (0,1)$ and $\gamma > 0$. The household maximizes the above utility function over choices of $\{c_t, m_{t+1}\}$ subject to the sequence of budget constraints

$$c_t + m_{t+1}/p_t = y_t - \tau_t + m_t/p_t, \quad t \geq 0$$

where τ_t is a lump-sum tax due at t. The household faces the price sequence $\{p_t\}$ as a price taker and has given initial value of nominal balances m_0.
At time t the government faces the budget constraint

$$g_t = \tau_t + (M_{t+1} - M_t)/p_t, \quad t \geq 0$$

where M_t is the amount of currency that the government has outstanding at the beginning of time t and g_t is government expenditures at time t. In equilibrium, we require that $M_t = m_t$ for all $t \geq 0$. The government chooses sequences of $\{g_t, \tau_t, M_{t+1}\}_{t=0}^{\infty}$ subject to the government budget constraints being satisfied for all $t \geq 0$ and subject to the given initial value $M_0 = m_0$.
a. Define a *competitive equilibrium.*

For the remainder of this problem assume that $g_t = g < y$ for all $t \geq 0$, and that $\tau_t = \tau$ for all $t \geq 0$. Define a *stationary equilibrium* as an equilibrium in which the rate of return on currency is constant for all $t \geq 0$.

b. Find conditions under which there exists a stationary equilibrium for which $p_t > 0$ for all $t \geq 0$. Derive formulas for real balances and the rate of return on currency in that equilibrium, given that it exists. Is the stationary equilibrium unique?

c. Find a first-order difference equation in the equilibrium level of real balances $h_t = M_{t+1}/p_t$ whose satisfaction assures equilibrium (possibly nonstationary).

d. Show that there is a fixed point of this difference equation with positive real balances, provided that the condition that you derived in part b is satisfied. Show that this fixed point agrees with the level of real balances that you computed in part b.

f. Within which of the equilibria that you found in parts b and e is the following "old-time religion" true: "Larger sustained government deficits imply permanently larger inflation rates"?

Solution

a. We state the following definition:

DEFINITION 18. *An equilibrium is a sequence of price* $\{p_t\}_{t=0}^{+\infty}$*, sequences of household consumption and money holding* $\{c_t, m_{t+1}\}$*, sequences of government policy* $\{g_t, \tau_t, M_{t+1}\}$ *such that the following conditions are satisfied:*

(i) *Optimality: given* $\{p_t\}_{t=0}^{+\infty}$*,* $\{c_t, m_{t+1}\}_{t=0}^{+\infty}$ *solves the household problem.*

(ii) *Feasibility: the market for good and the market for money clear for all* $t \geq 0$*:*

$$\begin{aligned} c_t + g_t &= y \\ m_{t+1} &= M_{t+1}. \end{aligned}$$

(iii) *The government's budget constraint is satisfied for all* $t \geq 0$*:*

$$g_t = \tau_t + \frac{M_{t+1} - M_t}{p_t}.$$

b, c, and d. We form the Lagrangian of the household problem:

$$\sum_{t=0}^{+\infty} \beta^t \left\{ \ln(c_t) + \gamma \ln\left(\frac{m_{t+1}}{p_t}\right) - \lambda_t \left(c_t + \frac{m_{t+1}}{p_t} - y - \tau - \frac{m_t}{p_t}\right)\right\}.$$

The first order conditions are :

$$\begin{aligned} \tfrac{1}{c_t} &= \lambda_t \\ \tfrac{\gamma}{m_{t+1}} &= \tfrac{\lambda_t}{p_t} - \beta \tfrac{\lambda_{t+1}}{p_{t+1}}, \end{aligned} \tag{182}$$

and an appropriate sufficient transversality condition is

$$\lim_{T \to +\infty} \lambda_{T+1} \frac{m_{T+1}}{p_{T+1}} = 0. \tag{183}$$

Market clearing imposes that $c_t = y - g$ and $m_{t+1} = M_{t+1}$. This implies in turn that $\lambda_t = \frac{1}{y-g}$. Replacing these expressions into the second first order condition gives:

$$\frac{\gamma}{M_{t+1}} = \frac{1}{y-g}\left(\frac{1}{p_t} - \frac{\beta}{p_{t+1}}\right).$$

Note also that market clearing and the household budget constraint imply the government budget constraint:

$$y - \tau - c_t = g - \tau = \frac{M_{t+1} - M_t}{p_t}.$$

Therefore, equilibrium price sequences and money holding sequences are characterized by the following system of difference equation:

$$\begin{aligned} t \geq 0 \quad & \tfrac{\gamma}{M_{t+1}} = \tfrac{1}{y-g}\left(\tfrac{1}{p_t} - \tfrac{\beta}{p_{t+1}}\right) \\ t \geq 0 \quad & g - \tau = \tfrac{M_{t+1} - M_t}{p_t}. \end{aligned}$$

The first equation is the first order conditions evaluated at the equilibrium allocation and the second is the government budget constraint.
Anticipating question (c.) and (d.) , we rewrite this system of difference equations using auxiliary variables, the level of real balance $h_t \equiv \frac{m_{t+1}}{p_t}$ and the return on currency $R_t = \frac{p_t}{p_{t+1}}$. We find, after some manipulations:

$$\begin{array}{ll} t \geq 0 & \gamma(y-g) = h_t(1-\beta R_t) \\ t = 0 & g - \tau = h_0 - \frac{M_0}{p_0} \\ t \geq 1 & g - \tau = h_t - R_{t-1}h_{t-1} \\ R_t = \frac{p_t}{p_{t+1}} & \text{and} \quad h_t = \frac{M_{t+1}}{p_t}. \end{array}$$

The first equation allows to express $h_{t-1}R_{t-1}$ as a function of h_{t-1}. Replacing this expression in the third equation yields to:

$$\begin{array}{ll} t \geq 0 & R_t = \frac{1}{\beta}\left(1 - \frac{\gamma(y-g)}{h_t}\right) \\ t = 0 & g - \tau = h_0 - \frac{M_0}{p_0} \\ t \geq 1 & h_t = g - \tau - \frac{\gamma}{\beta}(y-g) + \frac{1}{\beta}h_{t-1} \\ R_t = \frac{p_t}{p_{t+1}} & \text{and} \quad h_t = \frac{M_{t+1}}{p_t}. \end{array}$$

The linear difference equation for h_t is easily solved. First we solve for its unique fixed point $h^* = g - \tau - \frac{\gamma}{\beta}(y-g) + \frac{1}{\beta}h^*$. Then we substract the equation defining h^* to the difference equation to find $h_t - h^* = \frac{1}{\beta}(h_{t-1} - h^*)$. We iterate on it and obtain $h_t = h^* + \left(\frac{1}{\beta}\right)^t (h_0 - h^*)$.
Equilibria are then constructed the following way:

(1) Choose $h_0 > 0$
(2) Solve for p_0 using $g - \tau = h_0 - \frac{M_0}{p_0}$
(3) Solve for h_t using $h_t = h^* + \left(\frac{1}{\beta}\right)^t (h_0 - h^*)$.
(4) Solve for R_t using $R_t = \frac{1}{\beta}\left(1 - \frac{\gamma(y-g)}{h_t}\right)$
(5) Solve for p_t and M_{t+1} using $R_t = \frac{p_t}{p_{t+1}}$ and $h_t = \frac{M_{t+1}}{p_t}$.
(6) Accept this equilibrium candidate only if p_0, h_t and R_t are positive, and satisfy the transversality condition (183).

Stationary equilibria are such that the rate of return on currency R_t is constant. From the above equations it implies that h_t is constant as well. But the difference equation for h_t has a unique fixed point $h^* = \frac{1}{1-\beta}\left(\gamma(y-g) - \beta(g-\tau)\right)$. Thus there is a unique candidate stationary equilibrium:

$$\begin{aligned} h^* &= \frac{1}{1-\beta}\left(\gamma(y-g) - \beta(g-\tau)\right) \\ R^* &= \frac{1}{\beta}\frac{\gamma(y-g)-(y-\tau)}{\gamma(y-g)-\beta(y-\tau)} \\ \frac{M_0}{p_0^*} &= \frac{1}{1-\beta}\left(\gamma(y-g) - (y-\tau)\right). \end{aligned}$$

The positivity restrictions on h_t, R_t and p_0 imposes the following necessary and sufficient condition for existence of a stationary equilibrium:

$$\gamma(y-g) > (y-\tau).$$

Exercise 17.3. *Optimal inflation tax in a cash-in-advance model*

Consider the version of Ireland's (1997) model described in the text but assume perfect competition (i.e., $\alpha = 0$) with flexible market-clearing wages. Suppose now that the government must finance a constant amount of purchases g in each period by levying flat-rate labor taxes and raising seigniorage. Solve the optimal taxation problem under commitment.

Solution

We follow the usual steps to solve this Ramsey problem (see chapter 12).

Step 1: Define a competitive equilibrium.

The household problem is:

$$\begin{array}{ll} & \max_{\{c_t,n_t,m_{t+1},b_{t+1}\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \beta^t \left(\frac{c_t^\gamma}{\gamma} - n_t\right) \\ \text{subject to} & 0 \le n_t \le 1 \\ & m_{t+1} \ge 0 \\ & 0 \le c_t \le \frac{m_t}{p_t} + b_t - \frac{b_{t+1}}{R_t} \\ & c_t + \frac{b_{t+1}}{R_t} + \frac{m_{t+1}}{p_t} \le \frac{m_t}{p_t} + b_t + (1-\tau_t)n_t. \end{array}$$

Definition 19. *An equilibrium is a price and an interest rate sequence* $\{p_t, R_t\}_{t=0}^{+\infty}$, *a sequence of household decisions,* $\{c_t, n_t, m_{t+1}, b_{t+1}\}_{t=0}^{+\infty}$ *and a sequence of government policy* $\{\tau_t, M_{t+1}\}_{t=0}^{+\infty}$ *such that the following conditions are satisfied:*

(i) *Optimality: given prices and interest rates, the household decisions solves the household problem.*

(ii) *Feasibility: the market for good, the market for money and the market for bond clear for all* $t \ge 0$*:*

$$\begin{aligned} c_t + g &= n_t \\ m_{t+1} &= M_{t+1} \\ b_{t+1} &= 0. \end{aligned}$$

(iii) *The government budget constraint is satisfied:*

$$g = \tau_t n_t + \frac{M_{t+1} - M_t}{p_t}.$$

The Ramsey problem is to choose a competitive equilibrium that maximizes the household welfare.

Step 2: Characterize a competitive equilibrium.

We form the Lagrangian of the household problem, ignoring the positivity constraints on consumption, labor and money holding:

$$
\begin{aligned}
\sum_{t=0}^{+\infty} \beta^t \left(\frac{c_t^\gamma}{\gamma} - n_t\right) \quad & -\beta^t \mu_t \left(c_t - \frac{m_t}{p_t} - b_t + \frac{b_{t+1}}{R_t}\right) \\
& -\beta^t \lambda_t \left(c_t + \frac{b_{t+1}}{R_t} + \frac{M_{t+1}}{p_t} - b_t - \frac{m_t}{p_t} - (1-\tau_t) n_t\right) \\
& -\beta^t \nu_t (n_t - 1).
\end{aligned}
$$

The first order conditions are:

$$
\begin{array}{lccl}
c_t: & c_t^{\gamma-1} & = & \mu_t + \lambda_t \\
n_t: & 1 & = & \lambda_t(1-\tau_t) + \nu_t \\
m_{t+1}: & \frac{\lambda_t}{p_t} & = & \beta \frac{\mu_{t+1}+\lambda_{t+1}}{p_{t+1}} \\
b_{t+1}: & \frac{\lambda_t + \mu_t}{p_t} & = & \beta(\mu_{t+1} + \lambda_{t+1}).
\end{array}
$$

The market clearing conditions are, assuming as in the text that the CIA binds.

$$
\begin{array}{lcl}
c_t & = & n_t - g \\
b_t & = & 0 \\
\frac{M_t}{p_t} & = & c_t \\
\frac{M_{t+1}}{p_t} & = & (1-\tau_t) n_t.
\end{array}
$$

Note that the household is taxed "twice": once by the tax on labor and once by the inflation tax through the cash in advance constraint. To see this, use the market clearing conditions to write

$$
c_t = \frac{M_t}{p_t} = \frac{M_t}{M_{t+1}} \frac{M_{t+1}}{p_t} = \frac{1-\tau_t}{x_t} n_t.
$$

This implies that the effective tax rate on labor income is given by $\alpha_t \equiv 1 - \frac{1-\tau_t}{x_t}$. This also tells that, in this set-up, the labor tax and the inflation tax are indeterminate: the same effective tax rate α_t can be achieved with different combinations of x_t and τ_t.

Step 3: Solve for the Ramsey plan.

The Ramsey problem is to choose a competitive equilibrium that maximizes the household welfare.
Given g, the household utility at a feasible allocation is given by:

$$
\sum_{t=0}^{+\infty} \beta^t \left(\frac{(n_t - g)^\gamma}{\gamma} - n_t\right).
$$

What is the first best ? Assume that the planner is free to impose a choice of n_t to the agent. It is easily shown that the function $\frac{(n_t-g)^\gamma}{\gamma} - n_t$ is increasing for $n_t \in [0,1]$. Thus, the first best is to choose $n_t = 1$ for all $t \geq 0$, so that $c_t = 1 - g$ for all $t \geq 0$.

We now show that the first best allocation is implementable as a competitive equilibrium. Specifically, we construct (positive) Lagrange multipliers μ_t, λ_t, ν_t

such that the first order conditions of the household problem are satisfied at the first best allocation.

Note first that, at the first best allocation, $\alpha_t n_t = \alpha_t 1 - \frac{1-\tau_t}{x_t} = g$. Second, market clearing implies that $\frac{p_t}{p_{t+1}} = \frac{M_t}{M_{t+1}} \frac{c_{t+1}}{c_t} = \frac{1}{x_t}$. Third, replacing the first best allocation in the first order conditions gives, after some manipulations:

$$\begin{aligned}
&\tfrac{1-\tau_t}{x_t} = 1 - g \\
&R_t = \tfrac{1}{\beta} \\
&\lambda_t = \beta \tfrac{(1-g)^\gamma}{1-\tau_t} \\
&\mu_t = (1-g)^{\gamma-1} - \beta \tfrac{(1-g)^\gamma}{1-\tau_t} \\
&\nu_t = 1 - \beta(1-g)^\gamma .
\end{aligned}$$

Note that the positivity of μ_t imposes a restriction on the τ_t:

$$1 - \tau_t \geq \beta(1-g) \Leftrightarrow x_t \geq \beta.$$

This shows that, apart from the above restriction, the optimal labor tax rate and optimal money growth are indeterminate.

Exercise 17.4.

Solution

a. The household problem is to maximize

$$\sum_{t=0}^{+\infty} \beta^t \left(c_t^{\alpha - \eta(1-\alpha)} \left(\frac{m_{t+1}}{p_t} \right)^{1-\alpha} \right), \tag{184}$$

with respect to $\{c_t, m_{t+1}, b_{t+1}\}_{t=0}^{+\infty}$ and subject to the budget constraint

$$c_t + \frac{m_{t+1}}{p_t} + \frac{b_{t+1}}{R_t} = y - \tau_t + \frac{m_t}{p_t} + b_t. \tag{185}$$

Letting $\lambda_t > 0$ be the Lagrange multiplier on the time t budget constraint, the first order conditions are

$$\frac{u(c_t, m_{t+1}/p_t)}{c_t}(\alpha - \eta(1-\alpha)) = \lambda_t \tag{186}$$

$$\frac{u(c_t, m_{t+1}/p_t)}{m_{t+1}/p_t}(1-\alpha) = \lambda_t - \beta \frac{p_t}{p_{t+1}} \lambda_{t+1} \tag{187}$$

$$\lambda_t = \beta R_t \lambda_{t+1}, \tag{188}$$

and the transversality condition is

$$\lim_{T \to +\infty} \beta^T \lambda_T \left(\frac{m_T}{p_T} + b_T \right) = 0. \tag{189}$$

In order to derive the demand for money, we substitute equation (188) in (187). We obtain the equation

$$\frac{u(c_t, m_{t+1}/p_t)}{m_{t+1}/p_t}(1-\alpha) = \lambda_t \left(1 - 1/(\pi_t R_t)\right), \tag{190}$$

which, combined with (186), gives the demand for money

$$\begin{aligned} \frac{m_{t+1}}{p_t} &= c_t \frac{1-\alpha}{\alpha - \eta(1-\alpha)}\left(1 - 1/(\pi_t R_t)\right)^{-1} \\ &= c_t \frac{1-\alpha}{\alpha - \eta(1-\alpha)}(1 + 1/i_t), \end{aligned} \tag{191}$$

which is a decreasing function of the nominal interest rate i_t.

b. and c. We characterize the unique equilibrium with $\beta = 1/R$. The market clearing conditions are $y = c_t + g$, $b_t = B$, and

$$h_t \equiv m_{t+1}/p_t = (y-g)\frac{1-\alpha}{\alpha - \eta(1-\alpha)}(1 - \beta/\pi_t))^{-1} \equiv f(\pi_t) \tag{192}$$

This equation is of the form $h_t = k(1 - \beta/\pi_t)^{-1}$, with $k = (y-g)(1-\alpha)/(\alpha - \eta(1-\alpha))$. It implies in particular that

$$\pi_t = \beta h_t/(k - h_t). \tag{193}$$

We use the government budget constraint to compute the path of real money balances. Specifically, we have

$$t = 0 \qquad d = h_t - \frac{M_0}{p_0} \tag{194}$$

$$t = 1, 2, \dots \quad d = h_t - \frac{h_{t-1}}{\pi_{t-1}}, \tag{195}$$

where $d \equiv g + B(1-\beta) - \tau$ is the deficit. Substituting expression (193) into (195), we obtain that the path of real money balances follows the linear difference equation

$$h_t = d + \frac{1}{\beta}(h_{t-1} - k) = h^* + \frac{1}{\beta}(h_{t-1} - h^*) = h^* + \left(\frac{1}{\beta}\right)^t (h_0 - h^*), \tag{196}$$

where $h^* = (k - \beta d)/(1-\beta)$ is the stationary point of (196). Clearly, if $h_0 < h^*$, then, for t large enough, $h_t < 0$, which cannot be the basis of an equilibrium. Also, if $h_0 > h^*$, then h_t grows at rate $1/\beta$, violating the transversality condition (189), and therefore cannot be the basis of an equilibrium. The only candidate is $h_t = h^*$, for all $t \geq 0$. If $h^* \leq k$, this candidate cannot be the basis of an equilibrium, because (193) implies that the inflation rate is negative. Otherwise,

$h_t = h^* > k$, together with $c_t = y - g$ and $b_t = B$, satisfies the first order conditions (186) as well as the transversality condition (189). The maximum level of deficit is the that can be financed in this economy is therefore the maximum d consistent with the condition $h^* > k$, that is

$$d < \bar{d} \equiv k. \tag{197}$$

d. Since the government announces its intention to cut back the deficit to zero at $t \geq T+1$, the inflation rate is $\pi_t = 1$, for $t \geq T$. The price level at time T is found by writing the government budget constraint

$$g + B = \tau + B - \beta \Delta B + f(1) - \frac{M}{p_T}, \tag{198}$$

where ΔB is the quantity of bonds purchased by the government and $g + B = \tau + \beta B$ by assumption. The quantity ΔM of money used to purchase these bonds satisfies $\Delta M / p_T = \beta \Delta B$. Substituting this last expression into (198), we obtain

$$\frac{(1+\mu)M}{p_T} = f(1). \tag{199}$$

Since, on the other hand, the inflation rate is 1 for all $t \leq T-1$, we must have $f(1) = M/p_{T-1}$, which implies, together with (199), that $p_T = (1+\mu)p_{T-1}$.

e. As in the previous question, the inflation rate is $\pi_t = 1$, for all $t \geq T$. The government budget constraints are

$$t = T \qquad \frac{1}{\pi_{T-1}} f(\pi_{T-1}) = f(1) - \beta \Delta B < f(1) \tag{200}$$

$$t = 1, \ldots, T-1 \quad \frac{1}{\pi_{t-1}} f(\pi_{t-1}) = f(\pi_t) \tag{201}$$

$$t = 0 \qquad f(\pi_0) = \frac{M}{p_0}. \tag{202}$$

The time T constraint (200) implies that $\pi_{T-1} > 1$. Using the constraints for $t = 1, \ldots, T-1$, we obtain by induction that $1/\pi_{t-1} f(\pi_{t-1}) = f(\pi_t) < f(1)$, and thus that $\pi_{t-1}` > 1$. We also observe that $\pi_t > 1$ implies that $f(\pi_{t-1}) = \pi_{t-1} f(\pi_t) > f(\pi_t)$ and therefore that $\pi_t > \pi_{t-1}$. Thus, the sequence of inflation rate is increasing. Lastly, $\pi_0 > 1$ implies that

$$f(\pi_0) = \frac{M}{p_0} < f(1) = \frac{(1+\mu)M}{p_T}. \tag{203}$$

Therefore, the inflation rate between $t = 0$ and $t = T$ is equal to $1 + \mu$. However, because the increase of the money supply is announced at $t = 0$, the inflation rate is greater than 1 and increasing for $t = 0, \ldots, T - 1$. Therefore, we have

$$1 < \pi_{T-1} < 1 + \mu. \tag{204}$$

Exercise 17.5.

We interpret the exercise as follows. We let $\{M_{t+1}, B_{t+1}, \tau_t, g_t\}_{t=0}^{+\infty}$ be a feasible government policy, and the associated path of prices and inflation rate $\{p_t, \pi_t\}_{t=0}^{+\infty}$. We let the demand for money be

$$\frac{m_{t+1}}{p_t} = f(\pi_t) \tag{205}$$

The inflation elasticity of the demand for money is

$$\varepsilon(\pi) = -\frac{\pi f'(\pi)}{f(\pi)}. \tag{206}$$

Since the nominal interest rate is $1 + i_t = R_t \pi_t$, the interest elasiticity is equal to the inflation elasiticity. We assume that the government sells a small quantity ΔB_1 of bonds at $t = 0$, and *repurchases* these bonds with money at $t = 1$. We let Δp_t and $\Delta \pi_t$ be changes in prices and inflations rates induced by this policy experiment. Because bonds are repurchased at $t = 1$ and all variables except p_t and π_t are unchanged by assumption, we know that $\Delta \pi_t = 0$, for all $t = 1, 2, \ldots$. We differentiate the government's budget constraint at $t = 0$ and $t = 1$ to obtain

$$0 = \frac{\Delta B_1}{R_0} + f'(\pi_0)\Delta\pi_0 + \frac{M_0}{p_0^2}\Delta p_0 \tag{207}$$

$$\Delta B_1 = \Delta\pi_0 \left(-\frac{1}{\pi_0} f'(\pi_0) + \frac{1}{\pi_0^2} f(\pi_0) \right). \tag{208}$$

We use equation (208) to express $\Delta \pi_0$ as a function of ΔB_1, and we substitute the resulting expression in (207). We obtain

$$\pi_0 \Delta B_1 \left(\frac{1}{\pi_0 R_0} - \frac{\varepsilon(\pi_0)}{1 + \varepsilon(\pi_0)} \right) = -\frac{M_0}{p_0^2}\Delta p_0 \tag{209}$$

This shows that $\Delta p_0 < 0$ if and only if

$$\frac{\varepsilon(\pi_0)}{1 + \varepsilon(\pi_0)} < \frac{1}{\pi_0 R_0} \tag{210}$$

In other words, if the interest elasticity of the money demand is sufficiently small, the open market operation results in a decrease of the price at $t = 0$. However, since money needs to be printed at $t = 1$ to repurchase the bonds, equation (208) shows that the inflation rate between $t = 0$ and $t = 1$ increases.

Exercise 17.6.

Solution

a. The "law of one price" implies that the price level in country A after the dollarization is equal to the US price level p^*, and the inflation rate is equal to the US inflation rate π^*. Given the money demand equation, the country A demand the US government a quantity $M^* = p^* F(c, \pi^*)$ US dollars. The quantity M of peso is exchanged for M^* US dollars, at an exchange rate $e = M/M^*$.

b. Since dollarization implies that the government stops printing money, it stops raising the inflation tax of $\tau = (1 - R_m)F(y - g, \beta R_m)$ per period. However, the US government is now raising an inflation tax on country A, $\tau^* \equiv F(y - g, \beta R_m^*)(1 - R_m^*)$ per period. Thus, if the government of country A is a good negociator, it could claim this inflation tax to the US government. In that even, the required increase in tax would be $\tau - \tau^*$ (which is positive, by the assumption that country A is on the good part of the Laffer curve.)

Exercise 17.7.

Solution

a. Since the foreign inflation is zero, $R_{mt} = 1$ for all $t \geq 0$. We write the government budget constraints, using the fact that, for all $t \geq 0$, $M_{t+1}/p_t = F(y - g, R_{mt}/R)$ and $M_{t+1} = eB^*_{t+1}$. The time zero constraint is

$$g_0 + B(1 - 1/R) - \tau_0 = (1 - 1/R)F(y_0 - g, 1/R) + \frac{B_0^* e}{p_0} - \frac{M_0}{p_0}. \tag{211}$$

and, for $t \geq 1$, the budget constraint is

$$g_t + B(1 - 1/R) - \tau_t = (1 - 1/R)F(y_t - g, 1/(R\pi^*)). \tag{212}$$

The left-hand side of the government budget constraint can be viewed as the current deficit. The right-hand side is the income that the government raises by printing money. We observe in particular that, in spite of the currency board contract, the government is still raising a positive income by printing money. This is because the money supply M_{t+1} of time $t+1$ is backed by dollar denominated bond bought at time t. In other words, to back the issue of one additional unit of real money balance, the government is only "paying" $1/R < 1$ unit of consumption good. The difference $1 - 1/R$ is government income. Therefore, a permanent increase in y raises permanently government income and allow to lower permanently the level of taxes.

b. We derive the government budget constraint, taking into account that inflation is π^* and that the price of a one-period dollar-denominated bond is $1/(R\pi^*)$. At time zero, the government budget constraint is

$$g_0 + B(1 - 1/R) - \tau_0 = (1 - 1/R\pi^*)F(y_0 - g, 1/(R/\pi^*)) + \frac{B_0^* e}{p_0} - \frac{M_0}{p_0}. \tag{213}$$

and, for $t \geq 1$, the budget constraint is

$$g_t + B(1 - 1/R) - \tau_t = (1 - 1/R\pi^*)F(y_t - g, 1/(R\pi^*)). \tag{214}$$

Thus, the government is facing a modified Laffer curve. In particular, a raise in foreign inflation raise the government income if $1/(R\pi^*)$ is on the "good part" of the curve.

c. For $t \geq 1$, the "currency board contract" takes the form

$$M_t(e + \Delta e) = B_t^*. \tag{215}$$

Therefore, the government budget constraint for $t \geq 1$ (214) is unchanged. At time zero, however, (213) becomes

$$g_0 + B(1 - 1/R) - \tau_0 = (1 - 1/R\pi^*)F(y_0 - g, 1/(R/pi^*)) + \frac{B_0^*(e + \Delta e)}{p_0} - \frac{M_0}{p_0}.$$

The currency board contract holds at $t = 0$ with the exchange rate e that was anticipated at $t = -1$, that is $M_0 = eB_0^*$. Thus, the time zero budget constraint can be rewritten

$$g_0 + B(1 - 1/R) - \tau_0 = (1 - 1/R\pi^*)F(y_0 - g, 1/(R/pi^*)) + \frac{B_0^* \Delta e}{p_0}. \tag{216}$$

After the devaluation, the dollar value of the current money supply is

$$M_0/(e + \Delta_e) = B_0^* e/(e + \Delta e) < B_0^*. \tag{217}$$

In words, the dollar value of the government liability has decreased. Since, on the other hand, the government holds a quantity B_0^* of dollars, the dollar value of its income is unchanged. This results in a (real) surplus of $(B_0^* \Delta e)/p_0 > 0$ that it can use to lower temporarily the level of per capita taxes.

Exercise 17.8.

Solution

a. and b. For "transversality," we assume that $\beta\gamma^{\phi(1-\sigma)} < 1$. We solve for a "balanced growth path" of the economy in which y_t, g_t, c_t and m_{t+1}/p_t grow at the same rate $\gamma > 1$, and l_t and s_t as well as R_t and R_{mt} are constant. The model is a special case of the shopping time economy studied in chapter 17, with $H(c, m/p) = \psi c(m/p)^{-1}$. The first order necessary conditions are the same as in the text, and the transversality condition is

$$\lim_{T \to +\infty} \beta^T u_c(T) \left(\frac{m_T}{p_T} + b_T \right) = 0. \tag{218}$$

Equilibrium on the good market imposes that $c_t = y_t - g_t = \gamma^t(y - g)$. We now solve for the real rate and the real money balances. Using the expression (17.14) derived in the text we obtain that, on a balanced growth path, the real rate on government bond is constant and equal to

$$R = \gamma(\beta\gamma^{\phi(1-\sigma)})^{-1} > 1. \tag{219}$$

Equation (17.16) derived in the text, defining the money demand is

$$\begin{aligned}
& (1 - R_{mt}/R) \left(\frac{u_c(t)}{u_l(t)} - H_c(t) \right) + H_{m/p}(t) = 0 \\
\Leftrightarrow \quad & (1 - R_{mt}/R) \left(\frac{\phi}{1-\phi} \frac{l_t}{c_t} - \psi \left(\frac{m_{t+1}}{p_t} \right)^{-1} \right) - \psi c_t \left(\frac{m_{t+1}}{p_t} \right)^{-2} = 0 \\
\Leftrightarrow \quad & (1 - R_{mt}/R)(\phi - s_t) - s_t^2 = 0. \qquad (220)
\end{aligned}$$

Equation (220) describes the demand for real money balance per unit of consumption good $s_t^{-1} = 1/c_t m_{t+1}/p_t$ as a function $f(\cdot)$, with $s_t^{-1} = f(R_{mt})$. To determine the sequence $\{p_t, R_{mt}\}_{t=0}^{\infty}$, we substitute $f(R_{mt})$ into the government budget constraint. We find, assuming that $B_t = \tau_t = 0$ and dividing through by c_t

$$\begin{aligned}
\frac{g}{y-g} &= f(R_{m1}) - \frac{M_0}{p_0} && (221) \\
\frac{g}{y-g} &= f(R_{mt}) - \frac{1}{\gamma} R_{mt-1} f(R_{mt-1}), \quad t \geq 1. && (222)
\end{aligned}$$

An equilibrium constant inflation rate $R_{mt} = R_m$ solves

$$\frac{g}{y-g} = f(R_m) - \frac{1}{\gamma} R_m f(R_m). \tag{223}$$

Then, the initial price level p_0 is implied by (221) and, for $t \geq 1$, the price level is $p_t = R_m^{-t} p_0$. The ratio of real money balance to consumption is constant and equal to $f(R_m)$, implying that real money balance grows at rate γ. The transversality condition (218) is satisfied because of the assumption $\beta\gamma^{\phi(1-\sigma)} < 1$.

c. In an equilibrium for which the inflation rate is equal to 1, demand for real money balance is growing at rate γ. More consumption requires more money to keep the shopping time s_t constant. The government raises a postive amount of revenue

$$\frac{M_t}{p}(\gamma - 1) > 0. \tag{224}$$

d. The previous discussion suggests that a monetary policy that promote growth may also promote the demand for real money balance. Thus, it may allow the government increase the money supply and raise revenue without generating inflation.

CHAPTER 18

Credit and currency

Exercise 18.1. *Arrow-Debreu*

Consider an environment with equal numbers N of two types of agents, odd and even, who have endowment sequences

$$\begin{aligned}\{y_t^o\}_{t=0}^{\infty} &= \{1,1,0,1,1,0,\dots\}\\ \{y_t^e\}_{t=0}^{\infty} &= \{0,0,1,0,0,1,\dots\}.\end{aligned}$$

Households of each type h order consumption sequences by $\sum_{t=0}^{\infty}\beta^t u(c_t^h)$. Compute the Arrow-Debreu equilibrium for this economy.

Solution

Let $\beta^t q_t^0$ be the time zero price of a unit of consumption at time t. We form the Lagrangian of the household problem:

$$\sum_{t=0}^{+\infty}\beta^t u(c_t^h) - \lambda^h\left(\sum_{t=0}^{+\infty}\beta^t q_t^0(c_t^h - y_t^h)\right).$$

The first order condition is :

$$u'(c_t^h) = \lambda^h q_t^0 \Rightarrow c_t^h = (u')^{-1}(\lambda^h q_t^0).$$

On the other hand market clearing imposes:

$$\sum_{h=o,e} N c_t^h = \sum_{h=o,e} N(u')^{-1}(\lambda^h q_t^0) = N.$$

The left hand side is a decreasing function of q_t^0 since $(u')^{-1}$ is decreasing. This implies that the solution of the above equation is unique, that q_t^0 is a constant, and, from the first order condition of the household problem, that c_t^h is constant as well.
Now normalize $q_0^0 = 1$ to have $q_t^0 = 1$ for all t. We use the even agent budget constraint to find c^e:

$$\begin{aligned} & \textstyle\sum_{t=0}^{+\infty}\beta^t(c^e - y_t^e) = 0 \Leftrightarrow \frac{c^e}{1-\beta} = \beta^3 + \beta^6 + \dots\\ \Leftrightarrow\quad & c^e = (1-\beta)\beta^3(\textstyle\sum_{k=0}^{+\infty}\beta^{3k}) \Leftrightarrow c^e = (1-\beta)\frac{\beta^3}{1-\beta^3}.\end{aligned}$$

Use the identity $(1-\beta^3) = (1-\beta)(1+\beta+\beta^2)$ and the market clearing condition to obtain:

$$\begin{aligned} c^e &= \tfrac{\beta^3}{1+\beta+\beta^2}\\ c^o &= 1 - c^e.\end{aligned}$$

Exercise 18.2. *One-period consumption loans*

Consider an environment with equal numbers N of two types of agents, odd and even, who have endowment sequences

$$\begin{aligned} \{y_t^o\}_{t=0}^{\infty} &= \{1,0,1,0,\dots\} \\ \{y_t^e\}_{t=0}^{\infty} &= \{0,1,0,1,\dots\}. \end{aligned}$$

Households of each type h order consumption sequences by $\sum_{t=0}^{\infty}\beta^t u(c_t^h)$. The only market that exists is for one-period loans. The budget constraints of household h are

$$c_t^h + b_t^h \leq y_t^h + R_{t-1}b_{t-1}^h, \quad t \geq 0,$$

where $b_{-1}^h = 0, h = o, e$. Here b_t^h is agent h's lending (if positive) or borrowing (if negative) from t to $t+1$, and R_{t-1} is the gross real rate of interest on consumption loans from $t-1$ to t.

a. Define a competitive equilibrium with one-period consumption loans.

b. Compute a competitive equilibrium with one-period consumption loans.

c. Is the equilibrium allocation Pareto optimal? Compare the equilibrium allocation with that for the corresponding Arrow-Debreu equilibrium for an economy with identical endowment and preference structure.

Solution

a. The household problem is to maximize

$$\sum_{t=0}^{+\infty}\beta^t u(c_t^h),$$

with respect to $\{c_t^h, b_t^h\}_{t=0}^{+\infty}$, and subject to

$$\begin{aligned} &c_t^h + b_t^h \leq y_t^h + R_{t-1}b_{t-1}^h \\ &b_t^h \geq -B \\ &b_{-1} \text{ given,} \end{aligned}$$

where $B \geq 0$ is some borrowing constraint, chosen large enough so that it never binds.

Definition 20. *A competitive equilibrium is an interest rate sequence* $\{R_t\}_{t=0}^{+\infty}$*, sequences of household decisions* $\{c_t^h, b_t^h\}_{t=0}^{+\infty}$*,* $h = o, e$*, such that the following conditions are satisfied:*

(i) *Optimality: given interest rates, the household decisions solve the household problem.*
(ii) *Feasibility: the market for bond and the market for good clear for all* t*:*

$$\begin{aligned} \textstyle\sum_{h=o,e} Nc_t^h &= N \\ \textstyle\sum_{h=o,e} Nb_t^h &= 0. \end{aligned}$$

b. We form the Lagrangian of the household problem:

$$L^h = \sum_{t=0}^{+\infty} \beta^t \left\{ u(c_t^h) - \lambda_t^h \left(c_t^h + b_t^h - y_t^h - R_{t-1} b_{t-1}^h \right) \right\}.$$

The first order conditions are:

$$\begin{array}{lll} c_t^h : & u'(c_t^h) & = & \lambda_t^h \\ b_t^h : & \lambda_t^h & = & \beta \lambda_{t+1}^h, \end{array}$$

and an appropriate sufficient transversality condition is

$$\lim_{T \to +\infty} \beta^T \lambda_T \left(B + b_T^h \right) = 0.$$

We guess and verify that there is an equilibrium in which $R_t = \beta^{-1}$ for all t. The second equation implies that, in such an equilibrium, λ_t^h is constant. Then, the first equation implies that consumption is constant as well. Now consider the budget constraint of agent h, it implies that:

$$\begin{aligned} b_t^h &= y_t^h - c^h + \tfrac{1}{\beta} b_{t-1}^h \\ &= y_t^h - c^h + \tfrac{1}{\beta}(y_{t-1}^h - c^h) + \tfrac{1}{\beta^2} b_{t-2}^h. \end{aligned}$$

Remember that the endowment process is of period 2. This means that $y_t^h - c^h + \frac{1}{\beta}(y_{t-1}^h - c^h)$ is constant. Therefore b_t^h follows a time invariant linear difference equation of the form $b_t^h = \text{constant} + \frac{1}{\beta^2} b_{t-2}^h$. This difference equation has one stationary solution and an infinity of non-stationary solutions growing at rate $\frac{1}{\beta}$. But the transversality condition imposes that bond holdings cannot grow at a rate greater or equal than β. This implies that

$$b_t^h = b_{t-2}^h \quad \forall t \geq 1$$

.

Since $b_{-1}^h = 0$, the above restriction implies that, for $t = 2k+1$, $b_{2k+1}^h = 0$. What's left to compute is b_t^h for $t = 2k$, c^o and c^h. To do so write the odd agent budget constraints:

$$\begin{array}{ll} t = 2k : & c^o + b_{2k}^o = 1 \\ t = 2k+1 : & c^o = \frac{1}{\beta} b_{2k}^o, \end{array}$$

which implies that $b_{2k}^o = -b_{2k}^e = \frac{\beta}{1+\beta}$. The corresponding equilibrium consumptions are $c^o = \frac{1}{1+\beta}$ and $c^e = \frac{\beta}{1+\beta}$.

c. The equilibrium allocation is Pareto optimal since

$$\frac{u'(c_t^o)}{u'(c_t^e)} = \text{constant}.$$

Calculations identical to those of exercise 18.1 show that the equilibrium allocation is the one corresponding to the Arrow-Debreu equilibrium of this economy.

Exercise 18.3. *Stock Market*

Consider a "stock market" version of an economy with endowment and preference structure identical to the one in the previous economy. Now odd and even agents begin life owning one of two types of "trees." Odd agents own the "odd" tree, which is a perpetual claim to a dividend sequence

$$\{y_t^o\}_{t=0}^{\infty} = \{1, 0, 1, 0, \ldots\},$$

while even agents initially own the "even" tree, which entitles them to a perpetual claim on dividend sequence

$$\{y_t^e\}_{t=0}^{\infty} = \{0, 1, 0, 1, \ldots\}.$$

Each period, there is a stock market in which people can trade the two types of trees. These are the only two markets open each period. The time-t price of type j trees is $a_t^j, j = o, e$. The time-t budget constraint of agent h is

$$c_t^h + a_t^o s_t^{ho} + a_t^e s_t^{he} \leq (a_t^o + y_t^o) s_{t-1}^{ho} + (a_t^e + y_t^e) s_{t-1}^{he},$$

where s_t^{hj} is the number of shares of stock in tree j held by agent h from t to $t+1$. We assume that $s_{-1}^{oo} = 1, s_{-1}^{ee} = 1, s_{-1}^{jk} = 0$ for $j \neq k$.

a. Define an equilibrium of the stock market economy.

b. Compute an equilibrium of the stock market economy.

c. Compare the allocation of the stock market economy with that of the corresponding Arrow-Debreu economy.

Solution

a. The household problem is to maximize

$$\sum_{t=0}^{+\infty} \beta^t u(c_t^h),$$

with respect to $\{c_t^h, s_t^{ho}, s_t^{he}\}_{t=0}^{+\infty}$, and subject to

$$\begin{aligned}
&c_t^h + a_t^o s_t^{ho} + a_t^e s_t^{he} \leq (a_t^o + y_t^o) s_{t-1}^{ho} + (a_t^e + y_t^e) s_{t-1}^{he} \\
&s_t^{ho} a_t^o + s_t^{he} a_t^e \geq -A,
\end{aligned}$$

where $A \geq 0$ is some borrowing constraint which is chosen large enough so that it never binds.

Definition 21. *A competitive equilibrium is a two price sequences $\{a_t^o, a_t^e\}_{t=0}^{+\infty}$, sequences of household decisions $\{c_t^h, s_t^{ho}, s_t^{he}\}_{t=0}^{+\infty}$, $h = o, e$, such that the following conditions are satisfied:*

(i) *Optimality: given tree prices, the household decisions solve the household problem.*

(ii) *Feasibility: the tree market and the good market clear for all $t \geq 0$:*

$$\begin{array}{c} \sum_{h=o,e} N s_t^{ho} = 1 \\ \sum_{h=o,e} N s_t^{he} = 1 \\ \sum_{h=o,e} N c_t^h = 0. \end{array}$$

b. We form the Lagrangian of the household problem:

$$L^h = \sum_{t=0}^{+\infty} \beta^t \left\{ u(c_t^h) - \lambda_t^h \left(c_t^h + a_t^o s_t^{ho} + a_t^e s_t^{he} - (a_t^o + y_t^o) s_{t-1}^{ho} - (a_t^e + y_t^e) s_{t-1}^{he} \right) \right\}.$$

The first order conditions are:

$$\begin{array}{ll} c_t^h : & u'(c_t^h) = \lambda_t^h \\ s_t^{ho} : & \lambda_t^h a_t^o = \beta \lambda_{t+1}^h (a_{t+1}^o + y_{t+1}^o) \\ s_t^{he} : & \lambda_t^h a_t^e = \beta \lambda_{t+1}^h (a_{t+1}^e + y_{t+1}^e). \end{array}$$

An appropriate sufficient transversality condition is

$$\lim_{T \to +\infty} \beta^T \lambda_T \left(s_t^{ho} a_t^o + s_t^{he} a_t^e - A \right) = 0.$$

We guess and verify that there is a competitive equilibrium with $c_t^h = c^h =$ constant. Then, the first equation implies that λ_t^h is also constant and, with the transversality condition, the second and third equation can thus be written:

$$\begin{array}{c} a_t^o = \beta(a_{t+1}^o + y_{t+1}^o) = \sum_{\tau=1}^{+\infty} \beta^t y_{t+\tau}^o \\ a_t^e = \beta(a_{t+1}^e + y_{t+1}^e) = \sum_{\tau=1}^{+\infty} \beta^t y_{t+\tau}^e. \end{array}$$

In odd period (resp. even), the odd tree (resp. even) has value $\beta^2 + \beta^4 + \ldots = \frac{\beta^2}{1-\beta^2}$ and the even tree (resp. odd) has value $\beta + \beta^3 + \ldots = \frac{\beta}{1-\beta^2}$.
A constant consumption c^h can be achieved by the constant portfolio $(s^{ho}, s^{he}) = (c^h, c^h)$. Note that the price of this portfolio is itself constant, equal to $\frac{\beta}{1-\beta} c^h$.
Our candidate equilibrium has the following features: at time $t = 0$, the household trades its initial portfolio (its periodic endowment) for a portfolio of the type (c^h, c^h), which synthesizes the constant consumption stream $c_t^h = c^h$. Then, from $t = 1$ on, the household holds the same portfolio: there is no trade on the tree market. The c^h, $h = o, e$, are given by the budget constraint at time 0:

$$\begin{array}{ll} c^o : & c^o + \dfrac{\beta}{1-\beta} c^o = 1 + \dfrac{\beta^2}{1-\beta^2} \\ c^e : & c^e + \dfrac{\beta}{1-\beta} c^e = \dfrac{\beta}{1-\beta^2}, \end{array}$$

which imply that $c^o = 1/(1+\beta)$ and $c^e = \beta/(1+\beta)$.

c. As in question (18.3) this allocation is the one corresponding to the Arrow-Debreu economy.

Exercise 18.4. *Inflation*

Consider a Townsend turnpike model in which there are N odd agents and N even agents who have endowment sequences, respectively, of

$$\begin{aligned} \{y_t^o\}_{t=0}^{\infty} &= \{1, 0, 1, 0, \ldots\} \\ \{y_t^e\}_{t=0}^{\infty} &= \{0, 1, 0, 1, \ldots\}. \end{aligned}$$

Households of each type order consumption sequences by $\sum_{t=0}^{\infty} \beta^t u(c_t)$. The government makes the stock of currency move according to

$$M_t = z M_{t-1}, \quad t \geq 0.$$

At the beginning of period t, the government hands out $(z-1)m_{t-1}^h$ to each type-h agent who held m_{t-1}^h units of currency from $t-1$ to t. Households of type $h = o, e$ have time-t budget constraint of

$$p_t c_t^h + m_t^h \leq p_t y_t^h + m_{t-1}^h + (z-1) m_{t-1}^h.$$

a. Guess that an equilibrium endowment sequence of the periodic form (18.9) exists. Make a guess at an equilibrium price sequence $\{p_t\}$ and compute the equilibrium values of $(c_0, \{p_t\})$. *Hint:* Make a "quantity theory" guess for the price level.

b. How does the allocation vary with the rate of inflation? Is inflation "good" or "bad"? Describe odd and even agents' attitudes toward living in economies with different values of z.

Solution

a. The first order condition of the household problem has the same form as in the text:

$$\beta \frac{p_t}{p_{t+1}} \frac{u'(c_{t+1}^h)}{u'(c_t^h)} \leq 1 \qquad = 1 \quad \text{if} \quad m_t^h > 0.$$

We guess a periodic equilibrium of the kind described in the text. The consumption stream and the money holding sequence of the agents has the following form:

$$\begin{aligned} c^o &= \{c^o, 1 - c^o, c^o, \ldots\} \\ c^e &= \{1 - c^o, c^o, 1 - c^o, \ldots\} \\ m^o &= \{M_1, 0, M_3, 0, \ldots\} \\ m^e &= \{0, M_2, 0, M_4, \ldots\}. \end{aligned}$$

We make the expected "quantity theory" guess for the price level:

$$p_t = z^t p_0.$$

This allocation of consumption and money is feasible. We just need to set c^o and p^o such that it satisfies both the first order condition and the budget constraints. c^o must be chosen such that:

$$\frac{\beta}{z}\frac{u'(1-c^o)}{u'(c^o)} = 1 \Leftrightarrow \frac{u'(1-c^o)}{u'(c^o)} = \frac{z}{\beta} > 1.$$

The left hand size is an increasing function of c^o since $u'(x)$ is a positive and decreasing function. Furthermore, from the Inada conditions, the left hand side goes to 0 as $c^o \to 0$ and to $+\infty$ as $c^o \to 1$. Therefore, the above equation has a unique solution $c^o(z)$. Since the left hand side is increasing and is 1 at $c^o = \frac{1}{2}$, we know that $c^o(z) > \frac{1}{2}$. Also, since the left hand side is increasing in c^o and the right hand side is increasing in z, $c^o(z)$ is an increasing function of z. In other words, more inflation increases consumption fluctuations. Lastly, since both the left and the right hand side are differentiable, $c^o(z)$ is differentiable – this is needed because we'll take derivative in the next question.
Note that the first order condition is also met in period of zero money holding:

$$\frac{\beta}{z}\frac{u'(c^o)}{u'(1-c^o)} = \left(\frac{\beta}{z}\right)^2 < 1.$$

Now the price level p_0 is found by using the agent budget constraint when money holding is positive:

$$p_t c^o + \frac{M_t}{N} = p_t \Leftrightarrow p_t c^o + z^t \frac{M_0}{N} = p_t,$$

which implies that p_t has the expected form:

$$p_t = z^t \frac{M_0}{N(1-c^o)} \Rightarrow p_0 = \frac{M_0}{N(1-c^o)}.$$

Note that the budget constraint when money holding is zero is also verified :

$$p_t c^o + \frac{M_t}{N} = p_t \Leftrightarrow p_t(1-c^o) = \frac{M_{t-1}}{N} + (z-1)\frac{M_{t-1}}{N}.$$

b. The intertemporal utilities of the agents are:

$$\begin{aligned} U^o(z) &= \tfrac{1}{1-\beta^2}\left(u\left(c^o(z)\right) + \beta u\left(1-c^o(z)\right)\right) \\ U^e(z) &= \tfrac{1}{1-\beta^2}\left(u\left(1-c^o(z)\right) + \beta u\left(c^o(z)\right)\right). \end{aligned}$$

Taking derivative with respect to z and using the fact that $\frac{\beta}{z}\frac{u'(1-c^o)}{u'(c^o)} = 1$, we find:

$$\begin{aligned} \tfrac{dU^o}{dz} &= \tfrac{1}{1-\beta^2}\tfrac{dc^o}{dz}u'(c^o)(1-z) < 0 \\ \tfrac{dU^e}{dz} &= \tfrac{1}{1-\beta^2}\tfrac{dc^o}{dz}u'(c^o)\left(\beta - \tfrac{z}{\beta}\right) < 0. \end{aligned}$$

The sign of these derivatives is found by noting that $u' > 0$, $\frac{dc^o}{dz} > 0$ and $z > 1$. This means that both agents are worse off when inflation increase. This is because inflation increases consumption fluctuations and therefore pushes the equilibrium allocation further away from the Pareto frontier where consumption stream are constant. Note also that

$$\frac{dU^e}{dz} < \frac{dU^o}{dz} < 0.$$

Which means that the even agent suffers higher welfare loss than the odd agent.

Exercise 18.5. *A Friedman-like scheme*

Consider Friedman's scheme to improve welfare by generating a deflation. Suppose that the government tries to boost the rate of return on currency above β^{-1} by setting $\beta > (1+\tau)$. Show that there exists no equilibrium with an allocation of the class (18.9) and a price level path satisfying $p_t = (1+\tau)p_{t-1}$, with odd agents holding $m_0^o > 0$. [That is, the piece of the "restricted Pareto optimality frontier" does not extend above the allocation (.5,.5) in Figure 18.3.]

Solution

From the text we know that a candidate equilibrium must satisfy the two following first order condition:

$$\begin{array}{lll} \text{When} \quad m_t > 0: & \frac{1}{1+\tau} = \frac{u'(c^o)}{\beta u'(1-c^o)} \\ \text{When} \quad m_t = 0: & \frac{1}{1+\tau} \leq \frac{u'(1-c^o)}{\beta u'(c^o)}. \end{array}$$

Replacing the first equation in the second one yields the following necessary condition for equilibrium existence:

$$\left(\frac{\beta}{1+\tau}\right)^2 \leq 1,$$

which shows the claim: there exists no equilibrium of the required class such that $\beta > 1+\tau$.

Exercise 18.6. *Distribution of currency*

Consider an economy consisting of large and equal numbers of two types of infinitely lived agents. There is one kind of consumption good, which is nonstorable. "Odd" agents have period-2 endowment pattern $\{y_t^o\}_{t=0}^{\infty}$, while "even" agents have period-2 endowment pattern $\{y_t^e\}_{t=0}^{\infty}$. Agents of both types have preferences that are ordered by the utility functional

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t^i), \quad i = o, e, \quad 0 < \beta < 1,$$

where c_t^i is the time-t consumption of the single good by an agent of type i.

Assume the following endowment pattern:

$$y_t^o = \{1, 0, 1, 0, 1, 0, \ldots\}$$
$$y_t^e = \{0, 1, 0, 1, 0, 1, \ldots\}.$$

Now assume that all borrowing and lending is prohibited, either ex cathedra through legal restrictions or by virtue of traveling and locational restrictions of the kind introduced by Robert Townsend. At time $t = 0$, all odd agents are endowed with αH units of an unbacked, inconvertible currency, and all even units are endowed with $(1-\alpha)H$ units of currency, where $\alpha \in [0,1]$. The currency is denominated in dollars and is perfectly durable. Currency is the only object that agents are permitted to carry over from one period to the next. Let p_t be the price level at time t, denominated in units of dollars per time-t consumption good.

a. Define an *equilibrium with valued fiat currency.*

b. Let an "eventually stationary" equilibrium with valued fiat currency be one in which there exists a $\bar{t}$ such that for $t \geq \bar{t}$, the equilibrium allocation to each type of agent is of period 2 (i.e., for each type of agent, the allocation is a periodic sequence that oscillates between two values). Show that for each value of $\alpha \in [0,1]$, there exists such an equilibrium. Compute this equilibrium.

Solution

a. Let us first write the household problem:

$$\begin{array}{ll} & \max_{\{c_t^i, m_t^i\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \beta^t \ln(c_t^i) \\ \text{subject to} & p_t c_t^i + m_t^i \leq p_t y_t^i + m_{t-1}^i \\ & m_t^i \geq 0. \end{array}$$

Definition 22. *A competitive equilibrium with valued fiat currency is a positive price process $\{p_t\}_{t=0}^{+\infty}$ with $p_t < +\infty$ and a sequence of household decisions $\{c_t^i, m_t^i\}_{t=0}^{+\infty}$, $i = o, e$, such that the two following conditions are satisfied:*

(i) *Optimality Given prices, the household decisions solve the household problem.*

(ii) *Feasibility The market for consumption and the market for money clear for all $t \geq 0$:*

$$\sum_{i=o,e} N c_t^i = N$$
$$\sum_{i=o,e} N m_t^i = H.$$

b. We guess and verify that there exists an eventually stationary equilibrium such that $\bar{t} = 1$. In other words, we look for an equilibrium of the form:

$$\begin{aligned} c^o &= \{c_\alpha^o, 1 - c^o, c^o, \ldots\} \\ c^e &= \{1 - c_\alpha^o, c^o, 1 - c^o, \ldots\} \\ m^o &= \{H, 0, H, 0, \ldots\} \\ m^e &= \{0, H, 0, H, \ldots\} \\ p &= \{p_0, p_1, p_1, p_1 \ldots\}. \end{aligned}$$

The first order conditions are the same as in the text. Specifically, we have:

$$\frac{1}{c_t^i} \geq \beta \frac{1}{c_{t+1}^i} \frac{p_t}{p_{t+1}} \qquad = \text{if} \quad m_t^i > 0.$$

First, when money holdings are positive and $t \geq 1$, we have:

$$\frac{1 - c^o}{c^o} = \beta \Rightarrow c^o = \frac{1}{1+\beta}.$$

When money holdings are zero, we check that:

$$\frac{c^o}{1-c^o} = \frac{1}{\beta} > \beta.$$

The price level for $t \geq 1$ is found using the odd agent budget constraint:

$$p_1 c^o + \frac{H}{N} = p_1 \Rightarrow p_1 = \frac{H}{N(1-c^o)}.$$

Similarly, at $t = 0$, the odd agent first order condition implies:

$$\frac{1-c^o}{c^o_\alpha} = \beta \frac{p_0}{p_1}.$$

Also, the odd agent budget constraint gives:

$$p_0 c^o_\alpha + \frac{H}{N} = p_0 + \alpha \frac{H}{N} \Rightarrow p_0 = \frac{(1-\alpha)H}{N(1-c^o_\alpha)}.$$

Replacing the expression for p_1 and p_0 into the odd agent first order condition gives:

$$c^o_\alpha = \frac{1}{1+(1-\alpha)\beta}.$$

Note that when $\alpha = 0$ we find, as expected, the equilibrium consumption described in the text.

We now need to verify that the first order condition of the even agent is satisfied. We have, using the expression for p_0 and p_1:

$$\begin{array}{rccc} & \frac{c^o}{1-c^o_\alpha} \geq \beta \frac{p_0}{p_1} & \Leftrightarrow & \frac{c^o}{1-c^o_\alpha} \geq \beta \frac{1-c^o}{1-c^o_\alpha} \\ \Leftrightarrow & c^o \geq \beta(1-c^o) & \Leftrightarrow & \frac{1}{1+\beta} \geq \beta^2 \frac{1}{1+\beta}, \end{array}$$

which is true since $\beta < 1$.

Exercise 18.7. *Capital overaccumulation*

Consider an environment with equal numbers N of two types of agents, odd and even, who have endowment sequences

$$\begin{aligned} \{y^o_t\}_{t=0}^\infty &= \{1-\varepsilon, \varepsilon, 1-\varepsilon, \varepsilon, \ldots\} \\ \{y^e_t\}_{t=0}^\infty &= \{\varepsilon, 1-\varepsilon, \varepsilon, 1-\varepsilon, \ldots\}. \end{aligned}$$

Here ε is a small positive number that is very close to zero. Households of each type h order consumption sequences by $\sum_{t=0}^\infty \beta^t \ln(c^h_t)$ where $\beta \in (0,1)$. The one good in the model is storable. If a nonnegative amount k_t of the good is stored at time t, the outcome is that δk_t of the good is carried into period $t+1$, where $\delta \in (0,1)$. Households are free to store nonnegative amounts of the good.

a. Assume that there are no markets. Households are on their own. Find the autarkic consumption allocations and storage sequences for the two types of agents. What is the total per-period storage in this economy?

b. Now assume that there exists a fiat currency, available in fixed supply of M, all of which is initially equally distributed among the even agents. Define an equilibrium with valued fiat currency. Compute a stationary equilibrium with valued fiat currency. Show that the associated allocation Pareto dominates the one you computed in part a.

c. Suppose that in the storage technology $\delta = 1$ (no depreciation) and that there is a fixed supply of fiat currency, initially distributed as in part b. Define an "eventually stationary" equilibrium. Show that there is a continuum of eventually stationary equilibrium price levels and allocations.

Solution

a. In autarky the household problem is:

$$\begin{array}{rl} & \max_{\{c_t,k_t\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \beta^t \ln(c_t) \\ \text{subject to} & k_t = y_t + \delta k_{t-1} - c_t \\ & k_t \geq 0, \quad k_{-1} = 0. \end{array}$$

The first order necessary condition takes the usual form:

$$u'(c_t) \geq \beta\delta u'(c_{t+1}) \quad = \quad \text{if} \quad k_t > 0.$$

It is natural to guess that there is a periodic solution, that the household stores in high endowment periods and eat all storage (and store nothing) in low endowment periods.

More precisely, we guess that the odd agent chooses:

$$\begin{aligned} c &= (c^o, c^e, c^o, c^e, \ldots) \\ k &= (k^o, 0, k^o, 0, \ldots), \end{aligned}$$

and that the even agent chooses:

$$\begin{aligned} c &= (\varepsilon, c^o, c^e, c^o, \ldots) \\ k &= (0, k^o, 0, k^o, \ldots). \end{aligned}$$

Notice that $\varepsilon > 0$ is needed to make sure that the even agent has something to eat in period 0.

Let's verify this guess. First, use the budget constraint to write all expressions in term of k^o. We obtain $c^o = 1 - \varepsilon - k^o$ and $c^e = \varepsilon + \delta k^o$. We use the first order condition in high endowment period to find k^o:

$$\begin{aligned} & \tfrac{1}{c^o} &&= \tfrac{\beta\delta}{c^e} \\ \Leftrightarrow\ & \tfrac{1}{1-\varepsilon-k^o} &&= \tfrac{\beta\delta}{\varepsilon+\delta k^o} \\ \Leftrightarrow\ & k^o &&= \tfrac{\beta}{1+\beta}(1-\varepsilon) - \varepsilon\tfrac{1}{\delta(1+\beta)}. \end{aligned}$$

Note that k^o is positive provided that ε is small enough. The first order condition holds as well in low endowment period:

$$\frac{1}{c^e} > \frac{(\beta\delta)^2}{c^e} = \frac{\beta\delta}{c^o}.$$

b. In the monetary economy the household problem is:

$$\begin{array}{rl} & \max_{\{c_t, m_t, k_t\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \beta^t \ln(c_t) \\ \text{subject to} & c_t + \frac{m_t}{p_t} + k_t = y_t + \frac{m_{t-1}}{p_t} + \delta k_{t-1} \\ & k_t \geq 0, \quad k_{-1} = 0 \\ & m_t \geq 0, m_{-1}, \end{array}$$

and an equilibrium is defined as follows:

DEFINITION 23. *A competitive equilibrium with valued fiat currency is a positive price process* $\{p_t\}_{t=0}^{+\infty}$ *with* $p_t < +\infty$ *and a sequence of household decisions* $\{c_t^i, m_t^i, k_t^i\}_{t=0}^{+\infty}$, $i = o, e$, *such that the two following conditions are satisfied:*

(i) *Optimality Given prices, the household decisions solve the household problem.*
(ii) *Feasibility The market for consumption and the market for money clear for all* $t \geq 0$*:*

$$\begin{array}{l} \sum_{i=o,e} N c_t^i + \sum_{i=o,e} N k_t^i = N \\ \sum_{i=o,e} N m_t^i = M. \end{array}$$

We guess and verify that there exists a stationary equilibrium of the usual form. Price p_t = constant = p so that the gross return on money is 1, allocations are periodic and capital is not stored since it is dominated in return by money. Precisely, the odd agent choice is:

$$\begin{array}{rcl} c & = & (c^o, 1-c^o, c^o, 1-c^o \ldots) \\ m & = & (M, 0, M, 0 \ldots) \\ k & = & (0, 0, 0, 0 \ldots), \end{array}$$

and the even agent choice is

$$\begin{array}{rcl} c & = & (1-c^o, c^o, 1-c^o, c^o \ldots) \\ m & = & (0, M, 0, M \ldots) \\ k & = & (0, 0, 0, 0 \ldots). \end{array}$$

This candidate allocation is feasible. We just need to check optimality, i.e. the first order conditions and the budget constraints. The first order condition in high endowment period is:

$$\frac{1}{c^o} = \frac{\beta}{1-c^o} \Leftrightarrow c^o = \frac{1}{1+\beta}.$$

The first order condition in low endowment period is also satisfied since

$$\frac{1}{1-c^o} > \frac{\beta^2}{1-c^o} = \frac{\beta}{c^o}.$$

The price level is set to satisfy the budget constraint in high endowment period.

$$c^o + \frac{M}{p} = 1 - \varepsilon \Leftrightarrow p = \frac{M}{1 - \varepsilon - c^o}.$$

Note that we need ε small enough to ensure that $p > 0$.

The utility of the odd agent in one of the previous periodic allocation is of the form:

$$U^o = \frac{1}{1-\beta^2} \left(\ln(c^o) + \beta \ln(c^e) \right).$$

The utility of the even agent is of the form:

$$U^e = \ln(c_0^e) + \beta U^o.$$

Note that, if ε is small enough, the time zero consumption of the even agent is lower in autarky (question a.) than in the monetary economy (question b.). Thus, in order to prove that the agents are better off in the monetary economy it is enough to show that:

$$U^o(\text{question} \quad \text{a.}) > U^o(\text{question} \quad \text{b.}).$$

It is convenient to write the utility difference as:

$$\begin{cases} \frac{1}{1+\beta} \left(U^o(\text{question} \quad \text{b}) - U^o(\text{question} \quad \text{a}) \right) = \frac{1}{1+\beta} \ln\left(x^o(\delta)\right) + \frac{\beta}{1+\beta} \ln\left(x^e(\delta)\right) \\ x^o(\delta) = 1 - \varepsilon + \frac{\varepsilon}{\delta} \\ x^e(\delta) = \delta(1-\varepsilon) + \varepsilon. \end{cases}$$

Note that the first equation has the form of an expected utility. The associated lottery has prize $(x^o(\delta), x^e(\delta))$, and probabilities are $\frac{1}{1+\beta}$ and $\frac{\beta}{1+\beta}$. The expectation of this lottery is, when $\varepsilon = 0$:

$$\frac{1+\beta\delta}{1+\beta} < 1.$$

So that it is still less than 1 provided ε is small enough. Using Jensen's inequality we obtain:

$$\frac{1}{1+\beta} \left(U^o(\text{question} \quad \text{b.}) - U^o(\text{question} \quad \text{a.}) \right) \leq \frac{1}{1+\beta} \ln(1) + \frac{\beta}{1+\beta} \ln(1) = 0.$$

Therefore, agents are better off in the monetary equilibrium than in autarky.

c. An eventually "stationary equilibrium" is an equilibrium which becomes stationary after some $T \geq 0$. For simplicity we focus on eventually stationary equilibria with $T = 1$. Precisely, the price sequence is of the form

$$p = (p, p, p, p, \ldots),$$

where p is the price level

The choice of the odd agent is

$$\begin{cases} c(\text{odd}) & = & (c^o, 1-c^o, c^o, 1-c^o, \ldots) \\ m(\text{odd}) & = & (M, 0, M, 0, \ldots) \\ k(\text{odd}) & = & (k^o, 0, k^o, 0, k^o \ldots) \quad k_{-1} = 0, \end{cases}$$

and the choice of the even agent is:

$$\begin{cases} c(\text{even}) & = & (1-c^o-k^o, c^o, 1-c^o, c^o, \ldots) \\ m(\text{even}) & = & (0, M, 0, M, \ldots) \\ k(\text{even}) & = & (0, k^o, 0, k^o, 0 \ldots) \quad k_{-1} = 0, \end{cases}$$

where p, c^o, k^o are constant to be determined.

This candidate equilibrium is feasible by construction. We only need to check optimality, i.e. the first order conditions and the budget constraints.

First, since capital and money earn the same return, they can be both held in equilibrium.

In high endowment periods, the first order condition is:

$$\frac{1}{c^o} = \beta \frac{1}{1-c^o} \Leftrightarrow c^o = \frac{1}{1+\beta}.$$

In low endowment periods, the first order condition is also verified:

$$\frac{1}{1-c^o} > \frac{\beta}{c^o} = \frac{\beta^2}{1-c^o}.$$

Lastly, at time $t = 0$, the first order condition for the even agent is also verified:

$$\frac{1}{1-c^o-k^o} > \frac{1}{1-c^o} > \frac{\beta}{c^o} = \frac{\beta^2}{1-c^o}.$$

p is set so that the budget constraint holds in high endowment period:

$$c^o + k^o + \frac{M}{p} = 1 - \varepsilon.$$

Note that any $0 < k^o < 1 - \varepsilon - c^o$ defines an eventually stationary equilibrium. Thus, there is a continuum of eventually stationary equilibrium.

Exercise 18.8. *Altered endowments*

Consider a Bewley model identical to the one in the text, except that now the odd and even agents are endowed with the sequences

$$\begin{aligned} y_t^0 &= \{1-F, F, 1-F, F, \ldots\} \\ y_t^e &= \{F, 1-F, F, 1-F, \ldots\}, \end{aligned}$$

where $0 < F < (1 - c^o)$, where c^o is the solution of equation (18.10).

Compute the equilibrium allocation and price level. How do these objects vary across economies with different levels of F? For what values of F does a stationary equilibrium with valued fiat currency exist?

Solution

The first order condition of the household's problem is the same as in the text:

$$\beta \frac{u'(c_{t+1})}{p_{t+1}} \leq \frac{u'(c_t)}{p_t} \quad = \quad \text{if} \quad m_t > 0.$$

As in the text we guess and verify a stationary solution of the following form. The price level is constant equal to p. The odd agent's choice is:

$$\begin{cases} c &= (c^o, 1-c^o, c^o, 1-c^o, \ldots) \\ m &= (M, 0, M, 0, \ldots), \end{cases}$$

and the even agent's choice is:

$$\begin{cases} c &= (1-c^o, c^o, 1-c^o, c^o, \ldots) \\ m &= (0, M, 0, M, \ldots). \end{cases}$$

This candidate equilibrium is feasible. We only need to check optimality, i.e. the first order conditions and the budget constraints. The first order conditions are the same as in the text. In high endowment periods, we have:

$$\beta u'(1-c^o) = u'(c^o).$$

So that c^o solves the same equation as in the text. Note also that the previous equation implies the first order condition in low endowment periods:

$$\beta u'(c^o) = \beta^2 u'(1-c^o) < u'(1-c^o).$$

p is set so that the budget constraint holds in high endowment period:

$$c^o + \frac{M}{p} = 1 - F.$$

So that an equilibrium with valued fiat currency exists if and only if $F < 1 - c^o$. Furthermore, the price level is increasing in F, or, equivalently, the value of money is decreasing in F. This reflects the fact that, as agents' endowment get closer to a full insurance point, the smoothing service provided by money becomes less valuable.

Exercise 18.9. *Inside money* $\diamond$

Consider an environment with equal numbers N of two types of households, odd and even, who have endowment sequences

$$\begin{aligned} \{y_t^o\}_{t=0}^{\infty} &= \{1, 0, 1, 0, \ldots\} \\ \{y_t^e\}_{t=0}^{\infty} &= \{0, 1, 0, 1, \ldots\}. \end{aligned}$$

Households of type h order consumption sequences by $\sum_{t=0}^{\infty}\beta^t u(c_t^h)$. At the beginning of time 0, each even agent is endowed with M units of an unbacked fiat currency and owes F units of consumption goods; each odd agent is owed F units of consumption goods and owns 0 units of currency. At time $t \geq 0$, a household of type h chooses to carry over $m_t^h \geq 0$ of currency from time t to $t+1$. (We start households out with these debts or assets at time 0 to support a stationary equilibrium.) Each period $t \geq 0$, households can issue indexed one-period debt in amount b_t, promising to pay off $b_t R_t$ at $t+1$, subject to the constraint that $b_t \geq -F/R_t$, where $F > 0$ is a parameter characterizing the borrowing constraint and R_t is the rate of return on these loans between time t and $t+1$. (When $F = 0$, we get the Bewley-Townsend model.) A household's period-t budget constraint is

$$c_t + m_t/p_t + b_t = y_t + m_{t-1}/p_t + b_{t-1}R_{t-1},$$

where R_{t-1} is the gross real rate of return on indexed debt between time $t-1$ and t. If $b_t < 0$, the household is borrowing at t, and if $b_t > 0$, the household is lending at t.

a. Define a competitive equilibrium in which valued fiat currency and private loans coexist.

b. Argue that, in the equilibrium defined in part a, the real rates of return on currency and indexed debt must be equal.

c. Assume that $0 < F < (1-c^o)/2$, where c^o is the solution of equation (18.10). Show that there exists a stationary equilibrium with a constant price level and that the allocation equals that associated with the stationary equilibrium of the $F = 0$ version of the model. How does F affect the price level? Explain.

d. Suppose that $F = (1-c^o)/2$. Show that there is a stationary equilibrium with private loans but that fiat currency is valueless in that equilibrium.

e. Suppose that $F = \frac{1}{2(1+\beta)}$. For a stationary equilibrium, find an equilibrium allocation and interest rate.

f. Suppose that $F \in [(1-c^o)/2, \frac{1}{2(1+\beta)}]$. Argue that there is a stationary equilibrium (without valued currency) in which the real rate of return on debt is $R \in (1, \beta^{-1})$.

Solution

a. The household problem is:

$$\begin{array}{ll} & \max_{\{c_t,m_t,b_t\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty}\beta^t u(c_t) \\ \text{subject to} & c_t + \frac{m_t}{p_t} + b_t = y_t + \frac{m_{t-1}}{p_t} + b_{t-1}R_{t-1} \\ & m_t \geq 0, \quad m_{-1} \\ & b_t \geq -\frac{F}{R_t}, \quad b_{-1}, \end{array}$$

and an equilibrium is defined as follows:

DEFINITION 24. *A competitive equilibrium with valued fiat currency is a positive price process* $\{p_t\}_{t=0}^{+\infty}$ *with* $p_t < +\infty$ *and a sequence of household decisions* $\{c_t^i, m_t^i, b_t^i\}_{t=0}^{+\infty}$, $i = o, e$, *such that the two following conditions are satisfied:*

(i) *Optimality: Given prices, the household decisions solve the household problem.*
(ii) *Feasibility: The market for consumption good, the market for money and the market for private loans clear for all* $t \geq 0$*:*

$$\sum_{i=o,e} Nc_t^i = N$$
$$\sum_{i=o,e} Nm_t^i = M$$
$$\sum_{i=o,e} Nb_t^i = 0.$$

b. We need to prove that, in an equilibrium in which private loans and money coexist, $R_t = \frac{p_t}{p_{t+1}}$.

If $R_t > \frac{p_t}{p_{t+1}}$, private loans earn a better return than money. Thus, no agent is willing to hold money unless it has no value ($p = +\infty$). Therefore, in an equilibrium with valued fiat currency, it must be that $R_t \leq \frac{p_t}{p_{t+1}}$. If, on the other hand, $R_t < \frac{p_t}{p_{t+1}}$, it is optimal for the agents to borrow as much as possible at the rate R_t (i.e. choose $b_t = \frac{-F}{R_t}$) in order to buy money. This cannot be an equilibrium on the market for private loans. Therefore, in an equilibrium with valued fiat currency, it must be that $R_t \geq \frac{p_t}{p_{t+1}}$. The last two inequalities imply that $R_t = p_t/p_{t+1}$, meaning that money and private loans earn the same return in equilibrium.

c. We guess and verify a stationary solution of the following form. The price level is constant equal to p. The return on private loan is constant equal to 1. The odd agent's choice is:

$$\begin{cases} c &= (c^o, 1-c^o, c^o, 1-c^o, \ldots) \\ m &= (M, 0, M, 0, \ldots) \\ b &= (F, -F, F, -F, \ldots). \end{cases}$$

And the even agent's choice is:

$$\begin{cases} c &= (1-c^o, c^o, 1-c^o, c^o, \ldots) \\ m &= (0, M, 0, M, \ldots) \\ b &= (-F, F, -F, F, \ldots). \end{cases}$$

This candidate equilibrium is feasible and satisfies the borrowing constraints. We only need to check optimality, i.e. the first order conditions and the budget constraints.

When money and private loans coexist and earn the same return, the first order condition is a slightly modified version of (18.8) :

$$\frac{u'(c_t)}{p_t} \geq \frac{\beta u'(c_{t+1})}{p_{t+1}} \quad = \text{if } b_t + m_t > \frac{-F}{R_t}.$$

In high endowment periods, the usual (18.10) holds, namely:

$$u'(c^o) = \beta u'(1 - c^o),$$

which also implies the first order condition in low endowment periods. The price level p is set so that the budget constraints hold. Specifically, in high endowment period, we have:

$$c^o + \frac{M}{p} + F = 1 - F.$$

Note that it implies the budget constraint in low endowment period. It is clear that the above has a solution $0 < p < +\infty$ if and only if:

$$F < \frac{1 - c^o}{2}.$$

d. If $F = \frac{(1-c^o)}{2}$, we can repeat the analysis with $p_t = +\infty$. Precisely, we guess and verify that there exists an equilibrium of the same form as in the previous question, but in which agents hold no money. The return on private loans is $R = 1$. The first order condition becomes:

$$u'(c_t) \geq \beta R_t u'(c_{t+1}) \quad = \text{if } b_t > \frac{-F}{R_t}.$$

The same analysis go through.

e. If $F = \frac{1}{2(1+\beta)}$, we show that we can support a Pareto optimal allocation. As the aggregate endowment is constant, it gives constant consumption streams to the agents. It is natural to guess that both agents have the same consumption stream, with initial bond holdings b_{-1} that are to be chosen to ensure this symmetry.

$$c = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots).$$

We guess that the gross interest rate is $R = \beta^{-1}$. The loan stream of an odd agent is $b = (\beta F, -\beta F, \beta F, -\beta F, \ldots)$, while the one of an even agent is $b = (-\beta F, \beta F, -\beta F, \beta F, \ldots)$.

As usual we have chosen a feasible candidate which satisfies the borrowing constraint $b_t \geq -\frac{F}{R_t}$. We only need to check the first order conditions and the budget constraints. The first order condition clearly holds:

$$\beta R u'\left(\frac{1}{2}\right) = u'\left(\frac{1}{2}\right),$$

while the budget constraint is, in high endowment periods:

$$\frac{1}{2}+\frac{\beta}{2(1+\beta)}=1-\frac{1}{2(1+\beta)}.$$

Note that it implies the budget constraint in low endowment periods.

f. For $\frac{1-c^o}{2}<F<\frac{1}{2(1+\beta)}$, we guess and verify an equilibrium of the following form. Money is valueless so that we can assume that no agent hold money. Private loans earn a gross return $R_t=R>1$. The odd agent choice is:

$$\begin{cases} c &= (c^o(F),1-c^o(F),c^o(F),1-c^o(F),\ldots) \\ b &= \left(\frac{F}{R},-\frac{F}{R},\frac{F}{R},-\frac{F}{R},\ldots\right), \end{cases}$$

and the even agent's choice is:

$$\begin{cases} c &= (1-c^o(F),c^o(F),1-c^o(F),c^o(F),\ldots) \\ b &= \left(-\frac{F}{R},\frac{F}{R},-\frac{F}{R},\frac{F}{R},\ldots\right). \end{cases}$$

This candidate equilibrium is feasible and satisfies the borrowing constraints. We only need to check optimality, i.e. the first order condition and the budget constraint. The first order condition and the budget constraint in high endowment period are:

$$\begin{cases} u'(c^o) &= \beta R u'(1-c^o) \\ c^o+\frac{F}{R} &= 1-F. \end{cases}$$

To complete the construction of an equilibrium, it is enough to solve the above system in (c^o,R). Note that the budget constraint in high endowment period implies the budget constraint in low endowment period. Also the first order condition in high endowment period implies the first order condition in low endowment periods provided $R\beta<1$. Therefore, we look for a solution $R<\frac{1}{\beta}$.

The second equation gives $1-c^o=F\left(\frac{1+R}{R}\right)$. Replacing this expression in the first equation, we obtain:

$$\beta R\frac{u'\left(F\left(\frac{1+R}{R}\right)\right)}{u'\left(1-F\left(\frac{1+R}{R}\right)\right)}=1. \tag{225}$$

The left hand side of (225)is an increasing function of R. When evaluated at $R=1$ it is:

$$\beta\frac{u'(2F)}{u'(1-2F)}, \tag{226}$$

a decreasing function of F which is 1 when $F=\frac{1-c^o}{2}$. Since $F>\frac{1-c^o}{2}$, (226) is less than 1. At $R=\frac{1}{\beta}>1$, the left hand side of (225) is:

$$\frac{u'\left(F(1+\beta)\right)}{u'\left(1-F(1+\beta)\right)}, \tag{227}$$

a decreasing function of F which is 1 when $F = \frac{1}{2(1+\beta)}$. Since $F < \frac{1}{2(1+\beta)}$, (227) is greater than 1. This shows that (225) has a unique solution $1 < R < \frac{1}{\beta}$.

Exercise 18.10. *Initial conditions and inside money* $\diamond$

Consider a version of the preceding model in which each odd person is initially endowed with no currency and no IOUs, and each even person is initially endowed with M/N units of currency, but no IOUs. At every time $t \geq 0$, each agent can issue one-period IOUs promising to pay off F/R_t units of consumption in period $t+1$, where R_t is the gross real rate of return on currency or IOUs between periods t and $t+1$. The parameter F obeys the same restrictions imposed in exercise 18.9.

a. Find an equilibrium with valued fiat currency in which the "tail" of the allocation for $t \geq 1$ and the tail of the price level sequence, respectively, are identical with that found in exercise 18.9.

b. Find the price level, the allocation, and the rate of return on currency and consumption loans at period 0.

Solution

Since we look for an equilibrium with valued fiat currency we impose $0 < F < \frac{1-c^o}{2}$. Our candidate equilibrium is of the following form. The sequence of price is $(p_0, p, p, \ldots)$, where p is the price in exercise 18.9. The sequence of interest rates is $(R_0, 1, 1, \ldots)$. The choice of an odd agent is:

$$\begin{cases} c &= (c_0^o, 1-c^o, c^o, 1-c^o \ldots) \\ m &= (M, 0, M, 0, \ldots) \\ b &= (\frac{F}{R_0}, -F, F, -F, \ldots), \end{cases}$$

and the choice of the even agent is:

$$\begin{cases} c &= (1-c_0^o, c^o, 1-c^o, c^o \ldots) \\ m &= (0, M, 0, M, \ldots) \\ b &= (\frac{-F}{R_0}, F, -F, F, \ldots). \end{cases}$$

As in exercise 18.9, private loans and money must earn the same return. Therefore $R_0 = \frac{p_0}{p}$.

The candidate allocation is feasible and satisfies the borrowing constraints. We only need to check optimality, i.e. the first order condition and the budget constraint. From exercise 18.9, we already know that they are satisfied for $t \geq 1$. We only need to check them at $t = 0$. Let's focus first on the odd agent. The first order condition and the budget constraint are:

$$\begin{cases} \frac{\beta R_0 u'(1-c^o)}{u'(c_0^o)} &= 1 \\ c_0^o + \frac{M}{p_0} + \frac{F}{R_0} &= 1. \end{cases}$$

Remember $\frac{p_0}{p} = R_0$. Now we use the budget constraint to express c_0^o in term of R_0 and replace it in the first order condition. We find:

$$\frac{\beta R_0 u'(1-c^o)}{u'\left(1 - \frac{1}{R_0}\left(\frac{M}{p} + F\right)\right)} = 1.$$

The left hand side is an increasing function of R_0. At $R_0 = 1$, it is

$$\frac{\beta u'(1-c^o)}{u'\left(1 - \frac{M}{p} - F\right)} > \frac{\beta u'(1-c^o)}{u'\left(1 - \frac{M}{p}\right)} = \frac{\beta u'(1-c^o)}{u'(c^o)} = 1,$$

and, when $R_0 \to \frac{1}{\frac{M}{p}+F}$, it goes to 0 because of the Inada conditions. Thus, the above system of equation has a unique solution $\frac{1}{\frac{M}{p}+F} < R_0 < 1$. Note that the lower bound on R_0 implies that $c_0^o > 0$, as expected.

Let's now check optimality for the even agent. Her budget constraint is implied by the odd agent's one. We only need to show that $\frac{\beta u'(c^o)}{u'(1-c_0^o)} < 1$. To do so, use first the odd agent's first order condition to notice that:

$$1 = \frac{\beta R_0 u'(1-c^o)}{u'(c_0^o)} < \beta \frac{u'(1-c^o)}{u'(c_0^o)}.$$

The right hand side is an increasing function of c_0^o which takes value 1 at $c_0^o = c^o$. Therefore $c_0^o > c^o$. Now consider the even agent's first order condition:

$$\frac{\beta R_0 u'(c^o)}{u'(1-c_0^o)} < \frac{\beta R_0 u'(c^o)}{u'(1-c^o)} < 1.$$

The first inequality holds because $c_0^o > c^o$. The second one because $R_0 < 1$ and $\frac{\beta u'(c^o)}{u'(1-c^o)} < 1$.

Exercise 18.11.

Solution

a. and b. At $t = 0$, the government issues $\bar{M} - M$ units of money and purchases Δ units of IOU issued by the private sector. Then, the government uses the interest payments to decrease the stock of money. Since the initial condition is an equilibrium in which fiat money and private IOU coexists (an equilibrium of the type derived question c of exercise 18.9), we impose $F < (1-c^o)/2$, where c^o solves equation (18.10).

We guess and verify that there exists an equilibrium with zero inflation, that is $p_t = p$ for all $t \geq 0$. The consumption c^o is the solution of equation (18.10). The gross return on private IOU must be the same as the return on money, that is 1. The equilibrium is the one described in question c of exercise 18.9, with M there replaced here by $\bar{M}$.

Since the government does not earn any interest on private IOU, it cannot decrease the stock of money. The real bill experiment amounts to replace intrisically worthless and unbacked pieces of paper issued by the private sector by some issued by the government. The amount of consumption smoothing that can be achieved is dictated by the condition

$$\beta \frac{u'(1-c^o)}{u'(c^o)} = 1. \tag{228}$$

and does not depend on the quantity of money $\bar{M}$. The price level is determined by writing the budget constraint of an agent in an high endowment period

$$c^o + F + \frac{\bar{M}}{p} = 1 - F. \tag{229}$$

which implies

$$\Delta + \frac{M}{p} + 2F = 1. \tag{230}$$

Therefore, the price level increases with Δ.

c. The "quantity theory of money" do hold in the sense that

$$\frac{M}{p} = 1 - c^o - 2F. \tag{231}$$

An increase of the stock of money of $M + \Delta M = M(2+\mu)$ results in an increase of the price of $p + \Delta p = (1+\mu)p$.

CHAPTER 19

Equilibrium search and matching

Exercise 19.1. *An island economy (Lucas and Prescott, 1974)*

Let the island economy in this chapter have a productivity shock that takes on two possible values, $\{\theta_L, \theta_H\}$ with $0 < \theta_L < \theta_H$. An island's productivity remains constant from one period to another with probability $\pi \in (.5, 1)$, and its productivity changes to the other possible value with probability $1 - \pi$. These symmetric transition probabilities imply a stationary distribution where half of the islands experience a given θ at any point in time. Let $\hat{x}$ be the economy's labor supply (as an average per market).

a. If there exists a stationary equilibrium with labor movements, argue that an island's labor force has two possible values, $\{x_1, x_2\}$ with $0 < x_1 < x_2$.

b. In a stationary equilibrium with labor movements, construct a matrix Γ with the transition probabilities between states (θ, x), and explain what the employment level is in different states.

c. In a stationary equilibrium with labor movements, we observe only four values of the value function $v(\theta, x)$ where $\theta \in \{\theta_L, \theta_H\}$ and $x \in \{x_1, x_2\}$. Argue that the value function takes on the same value for two of these four states.

d. Show that the condition for the existence of a stationary equilibrium with labor movements is

$$\beta(2\pi - 1)\theta_H > \theta_L , \tag{232}$$

and, if this condition is satisfied, an implicit expression for the equilibrium value of x_2 is

$$[\theta_L + \beta(1-\pi)\theta_H]\, f'(2\hat{x} - x_2) = \beta\pi\theta_H f'(x_2) . \tag{233}$$

e. Verify that the allocation of labor in part d coincides with a social planner's solution when maximizing the present value of the economy's aggregate production. Starting from an initial equal distribution of workers across islands, condition (232) indicates when it is optimal for the social planner to increase the number of workers on high-productivity islands. The first-order condition for the social planner's choice of x_2 is then given by equation (233).
$\{$*Hint*: Consider an employment plan (x_1, x_2) such that the next period's labor force is x_1 (x_2) for an island currently experiencing productivity shock θ_L (θ_H). If $x_1 \le x_2$, the present value of the economy's production (as an island average) becomes

$$0.5 \sum_{t=0}^{\infty} \beta^t \left[\theta_L f(2\hat{x} - x_2) + (1-\pi)\theta_H f(2\hat{x} - x_2) + \pi\theta_H f(x_2)\right] .$$

Examine the effect of a once-and-for-all increase in the number of workers allocated to high-productivity islands.$\}$

Solution

a. We know from the text that labor movements are characterized by two increasing functions $X^-(\theta) \le X^+(\theta)$. Assume that the current shock is θ. If, at the beginning of period, the island labor force is $x < X^-(\theta)$, then outside workers move in the island so that *next period* labor force is $X^-(\theta)$. If, at the beginning of a period, the island labor force is $x > X^+(\theta)$, then workers move out of the island so that *next period* labor force is $X^+(\theta)$. Agents who move out cannot work this period. Otherwise, that is if $X^-(\theta) \leq x \leq X^+(\theta)$, all workers stay and no outside workers move in. We discuss two cases.

Case 1, no movements : $X^-(\theta_H) \leq X^+(\theta_L)$

This implies that, since X^- and X^+ are increasing in θ:

$$X^-(\theta_L) \leq X^-(\theta_H) \leq X^+(\theta_L) \leq X^+(\theta_H).$$

If the island labor force is $x \in [X^-(\theta_H), X^+(\theta_L)]$, then for all $s = L, H$, it is true that $X^-(\theta_s) \leq x \leq X^+(\theta_s)$. Thus, x is within the "moving boundaries" for all possible θ. It implies that the island labor force never change in equilibrium.

If the initial island labor force is $x < X^-(\theta_H)$, then, at the first $\theta = \theta_H$ the island labor force becomes $X^-(\theta_H)$ and stay constant afterward. Similarly, if the initial labor force is $x > X^+(\theta_L)$, then, at the first $\theta = \theta_L$, the island labor force becomes $x = X^+(\theta_L)$ and stay constant afterward.

Case 2, movements: $X^+(\theta_L) < X^-(\theta_H)$

Since $X^- \leq X^+$ are increasing, this implies :

$$X^-(\theta_L) \leq X^+(\theta_L) < X^-(\theta_H) \leq X^+(\theta_H).$$

First observe that if the initial island labor force is $x \leq X^+(\theta_L)$, then at the first $\theta = \theta_H$, the island labor force is $X^-(\theta_H)$. Similarly, if the initial island labor force is $x \geq X^-(\theta_L)$, then, at the first $\theta = \theta_L$, the island labor force is $X^+(\theta_L)$. Lastly, assume that the initial island labor force is $X^+(\theta_L) \leq x \leq X^-(\theta_H)$. If $\theta = \theta_L$, workers move out and next period labor force is $X^+(\theta_L)$. If $\theta = \theta_H$, workers move in and next period labor force is $X^-(\theta_H)$.

The above discussion shows that, for any initial labor force, the island labor force lies eventually in the set $\{X^+(\theta_L), X^-(\theta_H)\}$. Furthermore, once in the set, the labor force switch back and forth between $X^+(\theta_L)$ and $X^-(\theta_H)$. From $X^+(\theta_L)$ to $X^-(\theta_H)$ after a "positive shock" θ_H, and back to $X^-(\theta_L)$ after a "negative shock" θ_L. Therefore, in a *stationary* equilibrium with movement, an island labor force take only two possible values :

$$x_1 \equiv X^+(\theta_L) < X^-(\theta_H) \equiv x_2.$$

b. In a stationary equilibrium, an island is in one of the following states. Remember that workers who are moving do not work.

(i) (θ_L, x_1). No movement. Employment is $n = x_1$.
(ii) (θ_H, x_1). Outside worker move in, next period labor force is x_2. Employment is $n = x_1$.
(iii) (θ_L, x_2). Workers move out, next period labor force is x_1. Since worker move this period, employment is $n = x_1$.
(iv) (θ_H, x_2). No movement. Employment is $n = x_2$.

The above description implies that the transition matrix is :

$$\Gamma = \begin{pmatrix} \pi & 1-\pi & 0 & 0 \\ 0 & 0 & 1-\pi & \pi \\ \pi & 1-\pi & 0 & 0 \\ 0 & 0 & 1-\pi & \pi \end{pmatrix}.$$

We solve for the stationary distribution of island across states. It is a vector $q \in {\mathbb{R}_+}^4$ such that :

$$\begin{array}{rcl} q'\Gamma & = & q \\ \sum_{i=1}^4 q_i & = & 1. \end{array}$$

Simple algebra shows that :

$$\begin{array}{rcccl} q_1 & = & q_4 & = & \pi/2 \\ q_2 & = & q_3 & = & (1-\pi)/2. \end{array}$$

Observe that the proportion of islands with current shock θ_1 is $1/2$ as expected.

c. We know write the system of Bellman equations. We can drop the max operator since know the employment level and next period labor force in each state.

$$v(x_1, \theta_L) = \theta_L f'(x_1) + \beta E\left[v(x_1, \theta')|\theta_L\right] \tag{234}$$
$$v(x_1, \theta_H) = \theta_H f'(x_1) + \beta E\left[v(x_2, \theta')|\theta_H\right] \tag{235}$$
$$v(x_2, \theta_L) = \theta_L f'(x_1) + \beta E\left[v(x_1, \theta')|\theta_L\right] \tag{236}$$
$$v(x_2, \theta_H) = \theta_H f'(x_2) + \beta E\left[v(x_2, \theta')|\theta_H\right]. \tag{237}$$

Clearly $v(x_1, \theta_L) = v(x_2, \theta_L)$. This reflect the fact that, if the initial labor force is x_2 and the current shock is θ_L, agents are moving out of the island and thus are not working.

d. To simplify notations we number the states $1, 2, 3, 4$ as in question b. We note $f_1 = f'(x_1)$ and $f_2 = f'(x_2)$. The system of Bellman equations is:

$$w_2 = \theta_H f_1 + \beta(\pi w_4 + (1-\pi)w_3) \tag{238}$$
$$w_3 = \theta_L f_1 + \beta(\pi w_3 + (1-\pi)w_2) \tag{239}$$
$$w_4 = \theta_H f_2 + \beta(\pi w_4 + (1-\pi)w_3). \tag{240}$$

Those are only three equations and we have 5 unknowns : w_2, w_3, w_4, x_1, x_2. We need two more equations. The first expresses that in state 3 (x_2, θ_L), agents are indifferent between staying and moving :

$$w_3 = \beta(\pi w_4 + (1-\pi)w_3). \tag{241}$$

The left hand side is the value of staying and the right hand side is the value of moving (to an island who is currently in θ_H). The second additional equation expresses that the steady state labor force is $\hat{x}$:

$$\hat{x} = 1/2(x_1 + x_2). \tag{242}$$

We solve the system as follows. First substitute equation (241) in equation (238) to obtain :

$$w_2 = \theta_H f_1 + w_3, \tag{243}$$

substitute (243) in (239) :

$$w_3(1-\beta) = \theta_L f_1 + \beta(1-\pi)\theta_H f_1, \tag{244}$$

now eliminate w_4 in equations (240) and (241). This gives :

$$w_3(1-\beta) = \beta\pi\theta_H f_2. \tag{245}$$

Equate (244) and (245) :

$$(\theta_L + \theta_H\beta(1-\pi))\, f'(x_1) = \beta\pi\theta_H f'(x_2). \tag{246}$$

Using $x_1 = 2\hat{x} - x_2$ (that is equation (242)) gives the expression of the text. This last equation characterize uniquely a stationary equilibrium with labor movement. Remember that it is only a "necessary condition". We need to verify that the candidate equilibrium we found is indeed an equilibrium. In particular, we need to check that:

$$x_1 < x_2.$$

Since f' is a strictly decreasing function, it is equivalent to check that $f'(x_1) > f'(x_2)$. From (246) this is equivalent to $\theta_L + \theta_H\beta(1-\pi) < \beta\pi\theta_H$. Rearranging gives:

$$\theta_L < \beta\theta_H(2\pi - 1). \tag{247}$$

e. In this question we show the following: (247) is a necessary condition for a steady state to be an optimum in a planning problem. The planner is constraint by the same "moving technology" as the agents. Namely, if the planner moves some worker from island A to island B, then the workers cannot work this period and are available for work next period in island B.

Consider a steady state that has the same form as in the competitive equilibrium. The steady state is characterized by two numbers $x_1 < x_2$. x_1 is the labor force in an island who experienced shock θ_L last period, and x_2 is the labor force in an island who experienced θ_H last period. The steady state distribution and the employment levels that we derived in question b imply that, the aggregate production in this steady state is, per period :

$$1/2\theta_L f(x_1) + 1/2(1-\pi)\theta_H f(x_1) + 1/2\pi\theta_H f(x_2). \tag{248}$$

We now conduct two variational experiments. The first one goes as follows. Suppose that the planner wants to move the economy to the steady state $(x_1 - \varepsilon, x_2 + \varepsilon)$, for some $\varepsilon > 0$ small. In order to do so she moves *today* ε of agents from islands experiencing shock θ_L. Then, from tomorrow on, the economy is in the steady state $(x_1 - \varepsilon, x_2 + \varepsilon)$.

The payoff associated with this variational experiment is :

$$-1/2\varepsilon\theta_L f'(x_1) + 1/2\varepsilon\frac{\beta}{1-\beta}\left(-\theta_L f'(x_1) - (1-\pi)\theta_H f'(x_1) + \pi\theta_H f'(x_2)\right). \tag{249}$$

The first term reflect the cost of moving ε agents (who cannot work) from θ_L islands. The second term is the gain of moving to steady state $(x_1 - \varepsilon, x_2 + \varepsilon)$, from tomorrow on.

If (x_1, x_2) is an optimum then (249) must be negative. Rearranging gives :

$$\theta_L f'(x_1) + \beta(1-\pi)\theta_H f'(x_1) \geq \beta\pi\theta_H f'(x_2). \tag{250}$$

To obtain the reverse inequality, we conduct the following variational experiment. We start from the steady state (x_1, x_2). In a given period there is a fraction $(1-\pi)/2$ of islands in state (x_2, θ_L) and a fraction $(1-\pi)/2$ of islands in state (x_1, θ_H). Suppose that the planner moves $x_2 - x_1 - \varepsilon$ from (x_2, θ_L) to (x_1, θ_H) where, as before, ε is a small positive number. Then, from tomorrow on, a fraction $1-\pi$ of the islands lives in steady state $(x_1 + \varepsilon, x_2 - \varepsilon)$ and a fraction π still lives in steady state (x_1, x_2). In other word, this experiment engineers a change of steady state for a fraction $1-\pi$ of the population. The payoff associated with this experiment is:

$$\varepsilon(1-\pi)/2\left[\theta_L f'(x_1) + \frac{\beta}{1-\beta}\left(\theta_L f'(x_1) + (1-\pi)\theta_H f'(x_1) - \pi\theta_H f'(x_2)\right)\right]. \tag{251}$$

The first term is positive because the planner moves *less* agents today than it would have in steady state (x_1, x_2). Rearranging produce the inequality :

$$\theta_L f'(x_1) + \beta(1-\pi)\theta_H f'(x_1) \leq \beta\pi\theta_H f'(x_2). \tag{252}$$

Thus, if a steady state solves the planning problem, it is uniquely characterized by :

$$\theta_L f'(x_1) + \beta(1-\pi)\theta_H f'(x_1) = \beta\pi\theta_H f'(x_2).$$

Exercise 19.2. *Business cycles and search (Gomes, Greenwood and Rebelo, 1997)*

Part 1 *The worker's problem*

Think about an economy in which workers all confront the following common environment: Time is discrete. Let $t = 0, 1, 2, \ldots$ index time. At the beginning of each period, a previously employed worker can choose to work at her last period's wage or to draw a new wage. If she draws a new wage, the old wage is lost and she will be unemployed in the current period. She can start work at the new wage in the next period. New wages are independent and identically distributed from the cumulative distribution function F, where $F(0) = 0$, and $F(M) = 1$ for $M < \infty$. Unemployed workers face a similar problem. At the beginning of each period, a previously unemployed worker can choose to work at last period's wage offer or to draw a new wage from F. If she draws a new wage, the old wage offer is lost and she can start working at the new wage in the following period. Someone offered a wage is free to work at that wage for as long as she chooses (she cannot be fired). The income of an unemployed worker is b, which includes unemployment insurance and the value of home production. Each worker seeks to maximize $E_0 \sum_{t=0}^{\infty} (1-\mu)^t \beta^t I_t$, where μ is the probability that a worker dies at the end of a period, β is the subjective discount factor, and I_t is the worker's income in period t; that is, I_t is equal to the wage w_t when employed and the income b when unemployed. Here E_0 is the mathematical expectation operator, conditioned on information known at time 0. Assume that $\beta \in (0, 1)$ and $\mu \in (0, 1)$.

a. Describe the worker's optimal decision rule. In particular, what should an employed worker do? What should an unemployed worker do?

b. How would an unemployed worker's behavior be affected by an increase in μ?

Part 2 *Equilibrium unemployment rate*

The economy is populated with a continuum of the workers just described. There is an exogenous rate of new workers entering the labor market equal to μ, which equals the death rate. New entrants are unemployed and must draw a new wage.

c. Find an expression for the economy's unemployment rate in terms of exogenous parameters and the endogenous reservation wage. Discuss the determinants of the unemployment rate.

We now change the technology so that the economy fluctuates between booms (B) and recessions (R). In a boom, all employed workers are paid an extra $z > 0$.

That is, the income of a worker with wage w is $I_t = w + z$ in a boom and $I_t = w$ in a recession. Let whether the economy is in a boom or a recession define the *state* of the economy. Assume that the state of the economy is i.i.d. and that booms and recessions have the same probabilities of 0.5. The state of the economy is publicly known at the beginning of a period before any decisions are made.

d. Describe the optimal behavior of employed and unemployed workers. When, if ever, might workers choose to quit?

e. Let w_B and w_R be the reservation wages in booms and recessions, respectively. Assume that $w_B < w_R$. Let G_t be the fraction of workers employed at wages $w \in [w_B, w_R]$ in period t. Let U_t be the fraction of workers unemployed in period t. Derive difference equations for G_t and U_t in terms of the parameters of the model and the reservation wages, $\{F, \mu, w_B, w_R\}$.

f. The following time series is a simulation from the solution of the model with booms and recessions. Interpret the time series in terms of the model.

Solution

Part 1 a. Since an employed worker who quits receives unemployment compensation in the first period of unemployment, we can describe the optimal decisions using only one value function. Let $V(w)$ be the value of an employed (unemployed) worker with wage w (wage offer w) in hand at the beginning of the period and who behaves optimally. The employed worker decides whether to stay or quit and the unemployed worker whether to accept or reject. The Bellman equation is:

$$V(w) = \max_{\text{accept,reject}} \left\{ w + \beta(1-\mu)V(w), b + \beta(1-\mu)\int_0^M V(w')dF(w') \right\}.$$

As we know from chapter 5, this Bellman equation implies that the optimal policy of a worker is described by a reservation wage $\bar{w}$. An unemployed worker accepts the offer and stay forever if $w \geq \bar{w}$ and reject the offer otherwise. An implicit equation for $\bar{w}$ can be derived as in McCall's model described in chapter 5. One finds :

$$\bar{w} - b = \frac{\beta(1-\mu)}{1-\beta(1-\mu)} \int_{\bar{w}}^M (1 - F(w'))dw'. \tag{253}$$

Part 1 b. This question is answered by writing equation (253) as:

$$\bar{w} - b - \frac{\beta(1-\mu)}{1-\beta(1-\mu)} \int_{\bar{w}}^M (1 - F(w))dw = 0.$$

Observe that the left hand side is increasing in $\bar{w}$ and in μ. Therefore, an increase in μ results in a decrease in $\bar{w}$. Similarly, the left hand side is decreasing in b.

Thus, an increase in b results in an increase in $\bar{w}$.

Part 2 c. The dynamic of the unemployment rate U_t can be described as follows. In any given period, a mass of μ of newborn agents enters the labor market, and they start unemployed. Also, out of the $(1-\mu)$ unemployed workers who do not die, a fraction $F(\bar{w})$ rejects the offer they draw. This is summarized by:

$$U_{t+1} = \mu + (1-\mu)F(\bar{w})U_t.$$

The steady state level of unemployment is therefore:

$$U^* = \frac{\mu}{1-(1-\mu)F(\bar{w})}.$$

An increase in b increases $\bar{w}$ which in turn increases the unemployment rate.
An increase in μ has two opposite effects. First, more newborn agents enter the labor market every period, which increases the unemployment rate. Second, it increases the incentive to accept a wage offer, which reduces $\bar{w}$ and in turn decreases the unemployment rate.

Part 2 d. As in part 1, both employed and unemployed workers share the same value function. Let $V(w,s)$ $(s=R,B)$ be the value of a worker with wage w in hand at the beginning of a period where the state of the economy is $s \in \{\text{Recession}, \text{Boom}\}$ and who behaves optimally. The Bellman equations are:

$$\begin{array}{rcl} V(w,R) & = & \max\left\{w + \sum_s \frac{\beta}{2}(1-\mu)V(w,i), b + \sum_s \frac{\beta}{2}(1-\mu)\int V(w',s)dF(w')\right\} \\ V(w,B) & = & \max\left\{w + z + \sum_s \frac{\beta}{2}(1-\mu)V(w,s), b + \sum_s \frac{\beta}{2}(1-\mu)\int V(w',s)dF(w')\right\}, \end{array}$$

where it is understood that sums are over $s=R,B$. The value functions are weakly increasing in w. The left hand side of the Bellman equations is thus increasing and the right hand side is a constant Q. This implies that the optimal policy is characterized by a pair of reservation wage w_R and w_B. The worker accepts a job in recession (boom) if the wage offer is greater than w_R (w_B) and rejects otherwise. The reservation wages solve :

$$\begin{array}{rcl} w_R + \sum_s \frac{\beta}{2}(1-\mu)V(w_R,s) & = & Q \\ w_B + \sum_s \frac{\beta}{2}(1-\mu)V(w_B,s) & = & Q - z. \end{array}$$

Since the left hand side of those two equations is weakly increasing, this implies that $w_B < w_R$. Thus a worker may accept a job in a boom (when the job is more productive) and quit it in a recession.
One can characterize further those two quantities as follows. We consider the three regions $w \geq w_R$, $w_B \leq w \leq w_R$ and $w \leq w_B$. In each of those three regions we know which term is greater in each Bellman equations. We can drop the max and solve for the value functions. We find, for $w \geq w_R$:

$$\begin{array}{rcl} V(w,R) & = & \frac{w}{1-(1-\mu)\beta} + \frac{(1-\mu)\beta z}{2(1-(1-\mu)\beta)} \\ V(w,B) & = & V(w,R)+z. \end{array}$$

Similarly for $w_B \leq w \leq w_R$:

$$\begin{array}{rcl} V(w,R) & = & Q \\ V(w,B) & = & \frac{w+z}{1-1/2(1-\mu)\beta} + \frac{1/2(1-\mu)\beta}{1-1/2(1-\mu)\beta}Q. \end{array}$$

For all other $w \leq w_B$:

$$V(w,R) = V(w,B) = Q.$$

Now use the above expressions to write the indifference conditions $V(w_B,B) = Q$ and and $V(w_R,R) = Q$. It gives

$$\begin{array}{rcl} \frac{w_B+z}{1-\beta(1-\mu)} & = & Q \\ \frac{w_R+\beta z/2}{1-\beta(1-\mu)} & = & Q, \end{array}$$

which shows that $w_R = w_B + (1-\beta/2)z$. Now we can write an implicit equation for w_B:

$$\begin{array}{rcl} \frac{w_B+z}{1-\beta} & = & Q \\ & = & b \quad +\beta(1-\mu)\left[\int_0^{w_B} \frac{w_B+z}{1-\beta(1-\mu)} dF(w') \right. \\ & & \quad +1/2\int_{w_B}^{w_R} \frac{w'+z}{1-\beta/2(1-\mu)} + \frac{w_B+z}{(1-\beta(1-\mu))(1-\beta/2(1-\mu))} dF(w') \\ & & \quad \left. +\int_{w_R}^{M} \frac{w'+z/2}{1-\beta(1-\mu)} dF(w') \right]. \end{array}$$

Part 2 e. Let $B = 1$ if the economy is in a boom and zero otherwise. The difference equations for U_t, G_t is given by:

$$\begin{array}{rcl} U_{t+1} & = & \mu + (1-\mu)(F(w_B)B + F(w_R)(1-B))U_t + (1-\mu)G_t(1-B) \\ G_{t+1} & = & B(1-\mu)\left(G_t + (F(w_R)-F(w_B))U_t\right). \end{array}$$

Part 2 f. The graph presented feature an asymmetry: increases in unemployment rate are sharper than decreases. This reflect the fact that all workers in G_t quit their jobs when the economy experiences a recession.

Exercise 19.3. *Business cycles and search again*

The economy is either in a boom (B) or recession (R) with probability .5. The state of the economy (R or B) is i.i.d. through time. At the beginning of each period, workers know the state of the economy for that period. At the beginning of each period, a previously employed worker can choose to work at her last period's wage or draw a new wage. If she draws a new wage, the old wage is lost, b is received this period, and she can start working at the new wage in the following period. During recessions, new wages (for jobs to start next period) are i.i.d. draws from the c.d.f. F, where $F(0) = 0$ and $F(M) = 1$ for $M < \infty$. During booms, the worker can choose to quit and take *two* i.i.d. draws of a possible new wage (with the option of working at the higher wage, again for a job to start the next period) from the *same* c.d.f. F that prevails during recessions. (This ability to choose is what "Jobs are more plentiful during booms" means to workers.) Workers who are unemployed at the beginning of a period receive b this period and draw either one (in recessions) or two (in booms) wages offers from the c.d.f. F to start work next period. A worker seeks to maximize $E_0 \sum_{t=0}^{\infty}(1-\mu)^t \beta^t I_t$, where μ is the probability that a worker dies at the end of a period, β is the subjective discount factor, and I_t is the worker's income in period t; that is, I_t is equal to the wage w_t when employed and the income b when unemployed.

a. Write the Bellman equation(s) for a previously employed worker.

b. Characterize the worker's quitting policy. If possible, compare reservation wages in booms and recessions. Will employed workers ever quit? If so, who will quit and when?

Solution

a. The value function for a previously employed worker v with current wage w in hand in the recession state $s = R$ is given by

(254)
$$V(w,R) = \max_{\text{stay,quit}} \left\{ w + \tilde{\beta}/2 \sum_s V(w,s), b + \tilde{\beta}/2 \sum_s \int_0^M V(w',s)dF(w') \right\}.$$

where $\tilde{\beta} = \beta(1-\mu)$ and sums are over $s = R, B$. In a boom, quitters are allowed to draw twice from F:

(255)
$$V(w,B) = \max_{\text{stay,quit}} \left\{ w + \tilde{\beta}/2 \sum_s V(w,s), b + \tilde{\beta}/2 \sum_s \int_0^M V(w',s)d(F^2)(w') \right\}.$$

b. This problem is analogous to problem 5.2. The left hand side of the Bellman equations is weakly increasing and the right hand side is a constant. It implies that the optimal policy is described by two reservation wages w_R and w_B. In a recession (in a boom), the worker quits if her wage is lower than w_R (lower than w_B) and stays otherwise. The reservation wages in $s = R, B$ solve:

$$w_s + \tilde{\beta}/2 \sum_{s'} V(w_s, s') = b + \tilde{\beta}/2 \sum_{s'} \int_0^M V(w',s')dG_s(w'), \tag{256}$$

where $G_R(w) = F(w)$ and $G_B(w) = F^2(w)$. Observe that $F^2(w) \leq F(w)$ which implies that F^2 first order stochastically dominates F. Also note that $V(w,s)$ is a weakly increasing function of w. This implies:

$$\int_0^M V(w',s)dF(w') \leq \int_0^M V(w',s)d(F^2)(w').$$

Using this inequality in (256) gives :

$$w_R + \tilde{\beta}/2 \sum_{s'} V(w_R, s') \leq w_B + \tilde{\beta}/2 \sum_{s'} V(w_B, s').$$

Since $V(w,s)$ is weakly increasing in w this implies that:

$$w_B > w_R.$$

This shows that, in contrast with the conclusions of exercise 19.2, workers may accept a job in a recession and quit it in a boom. In exercise 19.2, the value of accepting a job increases in a boom because of higher productivity. In this exercise, the value of quitting increases in a boom because job offers are better.

European unemployment

The following three exercises are based on work by Ljungqvist and Sargent (1998), Marimon and Zilibotti (1999), and Mortensen and Pissarides (1999b), who calibrate versions of search and matching models to explain high European unemployment. Even though the specific mechanisms differ, they all attribute the rise in unemployment to generous benefits in times of more dispersed labor market outcomes for job seekers.

Exercise 19.4. *Skill-biased technological change, (Mortensen and Pissarides, 1999b)*

Consider a matching model in discrete time with infinitely lived and risk-neutral workers who are endowed with different skill levels. A worker of skill type i produces h_i goods in each period that she is matched to a firm, where $i \in \{1, 2, \ldots, N\}$ and $h_{i+1} > h_i$. Each skill type has its own but identical matching function $M(u_i, v_i) = Au_i^{\alpha} v_i^{1-\alpha}$, where u_i and v_i are the measures of unemployed workers and vacancies in skill market i. Firms incur a vacancy cost ch_i in every period that a vacancy is posted in skill market i; that is, the vacancy cost is proportional to the worker's productivity. All matches are exogenously destroyed with probability $s \in (0,1)$ at the beginning of a period. An unemployed worker receives unemployment compensation b. Wages are determined in Nash bargaining between matched firms and workers. Let $\phi \in [0,1)$ denote the worker's bargaining weight in the Nash product, and we adopt the standard assumption that $\phi = \alpha$.

a. Show analytically how the unemployment rate in a skill market varies with the skill level h_i.

b. Assume an even distribution of workers across skill levels. For different benefit levels b, study numerically how the aggregate steady-state unemployment rate is affected by mean-preserving spreads in the distribution of skill levels.

c. Explain how the results would change if unemployment benefits are proportional to a worker's productivity.

Solution

a. Since there is no interaction between labor markets for different skill levels, the analysis of the first section on matching model applies. Specifically, equations (19.6) and (19.18) can be used to obtain:

$$\begin{aligned} u_i &= \tfrac{s}{s+\theta_i q(\theta_i)} \\ h_i - b &= \tfrac{r+s+\alpha\theta_i q(\theta_i)}{(1-\alpha)q(\theta_i)}. \end{aligned}$$

We divide both sides of the second equation by h_i. Also, the Cobb-Douglas form of the matching function implies that $q(\theta_i) = A\theta_i^{-\alpha}$. This manipulations give:

$$\begin{aligned} u_i &= \tfrac{s}{s+A\theta_i^{1-\alpha}} \\ 1 - \tfrac{b}{h_i} &= \tfrac{c}{1-\alpha}\left(\tfrac{(r+s)}{A}\theta_i^{\alpha} + \alpha\theta_i\right). \end{aligned}$$

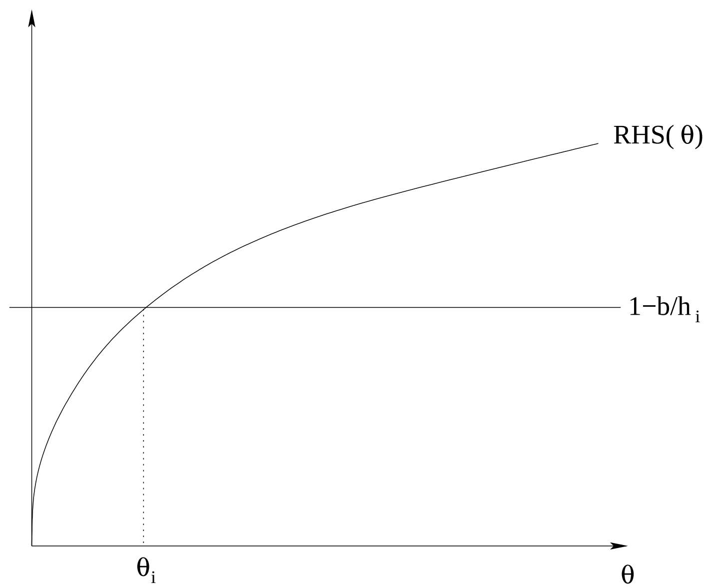


FIGURE 1. Exercise 19.4 a: implicit equation for θ_i

The second equation is illustrated in figure **??**. It is clear from it that an increase in skill is associated with a rise of the horizontal line $1 - b/h_i$, and thus with an increase of θ_i. From the first equation, this is associated in turn with an decrease in the equilibrium unemployment level u_i.

Observe that the assumption of constant unemployment benefit across skill levels is crucial to obtain this result. High skill workers are given a smaller replacement rate so that they have more incentive to accept a job offer.

b. We wrote the following matlab programs. `unemp1.m` computes the equilibrium level of unemployment in market i by solving the implicit equation:

$$h_i - b - \frac{r + s + \alpha\theta_i q(\theta_i)}{(1-\alpha)q(\theta_i)}.$$

`ex1904.m` computes the aggregate unemployment level for various benefit levels and distributions of skills. Specifically, we contrasted the three following distributions. The first is a dirac, for which all the probability mass is concentrated at the mean. The second is a tent function. The third is the uniform distribution. All three distributions have the same mean. The tent has an larger spread than the Dirac, and the uniform has a larger spread than the tent. The results are illustrated in figure 2 and 3.

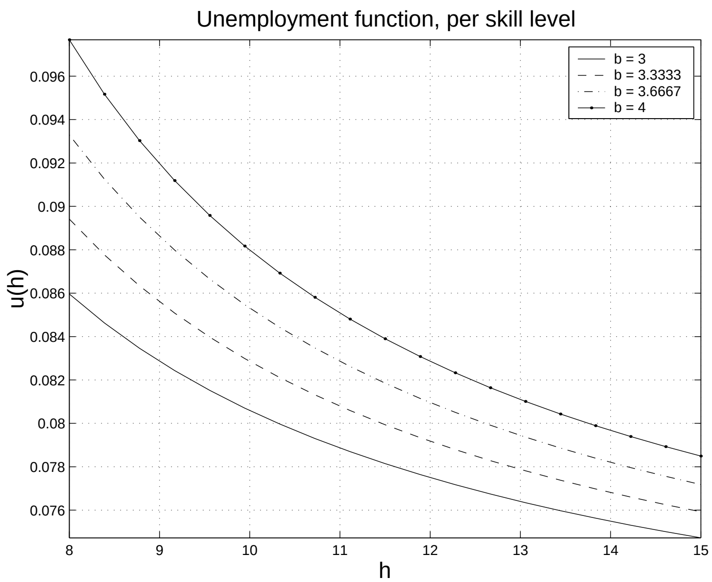


FIGURE 2. Exercise 19.4 b : Solving for unemployment level in each skill market

First, figure 2 shows that the equilibrium unemployment level is a convex and decreasing function of h. Thus, one should expect mean preserving increase in spread of the skill distribution to increase the aggregate unemployment rate. This point is illustrated in figure 3.

Exercise 19.5. *Dispersion of match values (Marimon and Zilibotti, 1999)*

We retain the matching framework of exercise 19.4 but assume that all workers have the same innate ability $h = \bar{h}$ and any earnings differentials are purely match specific. In particular, we assume that the meeting of a firm and a worker is associated with a random draw of a match-specific productivity p from an exogenous distribution $G(p)$. If the worker and firm agree upon staying together, the output of the match is then $p \cdot h$ in every period as long as the match is not

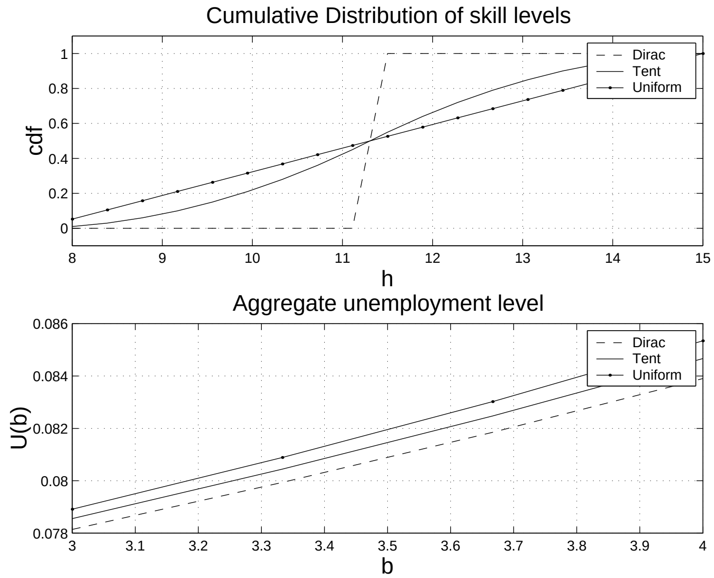


FIGURE 3. Exercise 19.4 b : Solving for the aggregate unemployment level

exogenously destroyed as in exercise 19.4. We also keep the assumptions of a constant unemployment compensation b and Nash bargaining over wages.

a. Characterize the equilibrium of the model.

b. For different benefit levels b, study numerically how the steady-state unemployment rate is affected by mean-preserving spreads in the exogenous distribution $G(p)$.

Solution

a. The analysis of this model parallels the one done in the text, with some adjustments to account for the match specific productivity. We first define Ω, the set of match specific productivities such that the firm and the worker agree to stay together. We will later show that this set is of the form $\{p \geq p_{\min}\}$. The probability that a match results in employment is $G(\Omega)$. The equilibrium unemployment level is thus given by:

$$u = \frac{s}{s + \theta q(\theta) G(\Omega)}. \tag{257}$$

We now write Bellman equations. Let w_p be the wage in a match of productivity p. The firm's value of a filled job with productivity p is J_p. The value of a vacancy is V. The worker's value of accepting a match of productivity p is E_p, and the value of being unemployed is U. Those values solve:

$$\begin{aligned}
(258) \qquad J_p &= ph - w(p) + \beta(sV + (1-s)J_p) \\
(259) \qquad V &= -c + \beta\left(q(\theta)G(\Omega)E\left(J_p \mid p \in \Omega\right) + (1 - q(\theta)G(\Omega))\, V\right) \\
(260) \qquad E_p &= w(p) + \beta(sU + (1-s)E_p) \\
(261) \qquad U &= z + \beta\left(\theta q(\theta)G(\Omega)E\left(E_p \mid p \in \Omega\right) + (1 - q(\theta)G(\Omega))\, U\right).
\end{aligned}$$

We define an equilibrium as follows:

DEFINITION 25. *An equilibrium is a collection of value functions J_p, V, E_p, U, a tightness parameter θ and a set Ω such that:*

(i) *J_p, V, E_p, U solve the Belman equations (258),(259),(260),(261)*
(ii) *Free entry: $V = 0$*
(iii) *Nash Bargaining: $E_p - U = \alpha(E_p - U + J)$*
(iv) *Optimality: $\Omega = \{p : E_p - U + J \geq 0\}$*

Observe that, if we take conditional expectations in equations (258) and (260), we obtain the same system of value functions as in the text, with $q(\theta)$ being replaced by $G(\Omega)q(\theta)$. This implies in particular that equation (19.16) holds, namely:

$$\frac{r}{1+r}U = z + \frac{\alpha}{1-\alpha}c\theta. \tag{262}$$

Note also that from equation (258) and the free entry condition we have $J_p = \frac{ph - w_p}{1-\beta(1-s)}$. Similarly, equation (260) gives $E_p = \frac{w_p}{1-\beta(1-s)} + \frac{\beta s}{1-\beta(1-s)}U$. With the help of equation (262), we can write the surplus of match p :

$$E_p + J_p - U = \frac{ph}{1-\beta(1-s)} - \frac{1}{1-\beta(1-s)}\left(z + \frac{\alpha}{1-\alpha}c\theta\right). \tag{263}$$

Only positive surplus match are formed. The last equation thus implies that there exists p_θ such that a match is accepted if and only if $p > p_\theta$. Solving form p_θ gives:

$$p_\theta = \frac{z}{h} + \frac{\alpha}{1-\alpha}\frac{c}{h}\theta.$$

Therefore $\Omega = \{p \geq p_\theta\}$. Observe that equation (19.18) holds in conditional expectation (when replacing $q(\theta)$ by $G(\Omega)q(\theta)$). Specifically:

$$hE(p \mid \Omega) - z = \frac{r + s + \alpha\theta q(\theta)G(\Omega)}{(1-\alpha)q(\theta)G(\Omega)}c.$$

Rearranging this last equation shows that the equilibrium tightness parameters is a solution of:

$$h\int_{p_\theta}^{1} p dG(p) - z\left(1 - G\left(p_\theta\right)\right) = \frac{c}{1-\alpha}\left(\frac{r+s}{A}\theta^\alpha + \phi\theta\left(1 - G\left(p_\theta\right)\right)\right). \tag{264}$$

b. We wrote the following matlab programs. `impl2.m` is the implicit equation (264) solved by the equilibrium θ. `unemp2.m` computes the equilibrium unemployment level by solving (264) and then using (257). `ex1905.m` plot the results.

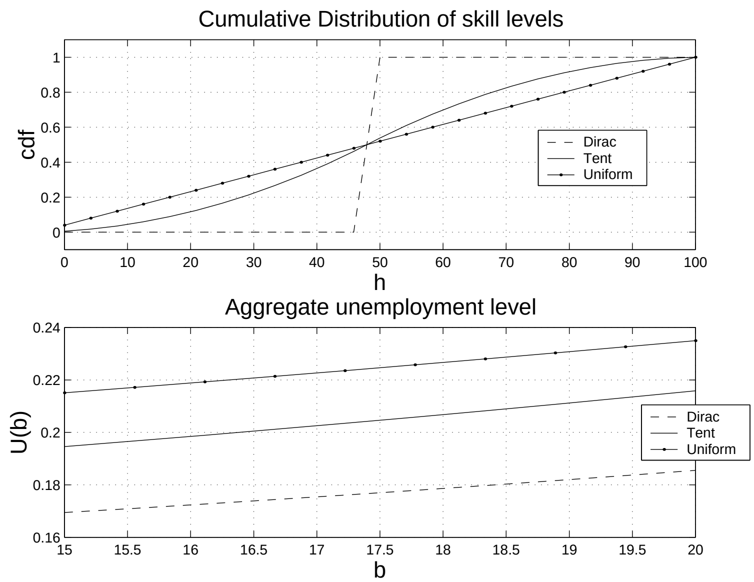


FIGURE 4. Exercise 19.5 : Solving for equilibrium unemployment

The results are illustrated in figure 4. We use three distributions with mean 1/2: Dirac, tent and uniform. The unemployment level is the highest for the uniform distribution, i.e. for the highest spread. When the spread increases, low productivity offers are more likely but a worker has can reject them enjoy unemployment benefit. He is insured against this risk. At the same time, high productivity offers are more likely. Therefore, an unemployed worker has an incentive to stay longer in unemployment in order to obtain a high productivity offer. Also, the unemployment rate increases with b. Higher b corresponds to a more valuable outside option when choosing to accept or reject an offer, and leads therefore to more rejections.

Exercise 19.6. *Idiosyncratic shocks to human capital (Ljungqvist and Sargent, 1998)*

We retain the assumption of exercise 19.5 that a worker's output is the product of his human capital h and a job-specific component which we now denote w, but we replace the matching framework with a search model. In each period of unemployment, a worker draws a value w from an exogenous wage offer distribution $G(w)$ and, if the worker accepts the wage w, he starts working in the following period. The wage w remains constant throughout the employment spell that ends either because the worker quits or the job is exogenously destroyed with probability s at the beginning of each period. Thus, in a given job with wage w, a worker's earnings wh can only vary over time because of changes in human capital h. For simplicity, we assume that there are only two levels of human capital, h_1 and h_2 where $0 < h_1 < h_2 < \infty$. At the beginning of each period of employment, a worker's human capital is unchanged from last period with probability π_e and is equal to h_2 with probability $1 - \pi_e$. Losses of human capital are only triggered by exogenous job destruction. In the period of an exogenous job loss, the laid off worker's human capital is unchanged from last period with probability π_u and is equal to h_1 with probability $1 - \pi_u$. All unemployed workers receive unemployment compensation, and the benefits are equal to a replacement ratio $\gamma \in [0, 1)$ times a worker's last job earnings.

a. Characterize the equilibrium of the model.

b. For different replacement ratios γ, study numerically how the steady-state unemployment rate is affected by changes in h_1.

Solution

a. In this exercise, we assume that there are finitely many wages. The set of wages is denoted by $\mathcal{W} \equiv \{w_1, \ldots, w_N\}$, and an unemployed worker draws wage w_n with probability $G(w_n)$. The corresponding set of unemployment benefit is $\mathcal{B} \equiv \{\gamma w_1 h_1, \ldots \gamma w_N h_1, \gamma w_1 h_2, \ldots, \gamma w_N h_2\}$. Lastly, the set of human capital levels is denoted by $\mathcal{H} \equiv \{h_1, h_2\}$. We let $U(b, h)$ be the value of an unemployed worker, with benefit $b \in \mathcal{B}$ and human capital $h \in \mathcal{H}$, and we let $V(w, h)$ be the value of an employed worker with wage $w \in \mathcal{W}$ and human capital $h \in \mathcal{H}$. The Bellman equation for the an unemployed worker is

$$U(b, h) = b + \beta \sum_{n=1}^{N} G(w_n) \max\{V(w_n, h), U(b, h)\}. \tag{265}$$

The timing of (265) is the following: an unemployed worker receives compensation b and draws an offer w that she either accepts or rejects. Since she starts working in the following period, the value of accepting or rejecting the offer is discounted by β. The Bellman equation for an employed worker with human capital h_1 is

$$\begin{aligned} V(w, h_1) &= wh_1 + \beta s U(\gamma w h_1, h_1) \\ &\quad + \beta(1-s)\pi_e \max\big\{V(w, h_1), U(\gamma w h_1, h_1)\big\} \\ &\quad + \beta(1-s)(1-\pi_e) \max\big\{V(w, h_2), U(w\gamma h_1, h_2)\big\}. \end{aligned} \tag{266}$$

The worker receives the wage wh_1, and next period she can be laid off, or can experience an increase of her human capital. If she is not laid off, she is given the option of quitting her job. Similarly, the Bellman equation for an employed worker with human h_2 is

$$\begin{aligned} V(w,h_2) &= wh_2+\beta s\pi_u U(\gamma wh_2,h_2) \\ &\quad +\beta s(1-\pi_u)U(\gamma wh_2,h_1) \\ &\quad +\beta(1-s)\max\bigl\{V(w,h_2),U(w\gamma h_2,h_2)\bigr\}. \end{aligned} \tag{267}$$

An high-skilled worker who looses her job might experience a loss of human capital. As before, a worker is given the option of quitting her job after any period of employment. The system of Bellman equations (265), (266) and (267) is solved numerically by the program `ex196f.m` using a value function iteration algorithm.

We now characterize the equilibrium unemployment level. We let $\mu(b,h)$ be the fraction of unemployed worker with benefit $b\in\mathcal{B}$ and human capital $h\in\mathcal{H}$. Similarly, $\lambda(w,h)$ is the fraction of employed worker with wage $w\in\mathcal{W}$ and human capital $h\in\mathcal{H}$. These must satisfy the accounting equation

$$\sum_{(b,h)\in\mathcal{B}\times\mathcal{H}}\mu(b,h)+\sum_{(w,h)\in\mathcal{W}\times\mathcal{H}}\lambda(w,h)=1. \tag{268}$$

We define the set of wage accepted by an unemployed worker,

$$A(b,h)=\{w\in\mathcal{W}:\ V(w,h)\geq U(b,h)\}, \tag{269}$$

for all $(b,h)\in\mathcal{B}\times\mathcal{H}$. Similarly, we define a "quit" indicator-function $q:\mathcal{W}\times\mathcal{H}\to\{0,1\}$ as follows: $q(w,h)=1$ if and only if $U(\gamma wh,h)>V(w,h)$. Lastly, we define another "quit" indicator-function $q_e:\mathcal{W}\to\{0,1\}$, which keeps track of quits when a worker human capital increases. Namely, $q_e(w)=1$ if and only if $U(\gamma wh_1,h_2)>V(w,h_2)$. Equipped with these notations, we write the steady-state equations for the distribution of worker types. We start with the unemployed types $(\gamma wh_1,h_1)$:

$$\lambda(w,h_1)\bigl(s+(1-s)\pi_e q(w,h_1)\bigr)=\mu(\gamma wh_1,h_1)G\bigl(A(\gamma wh_1,h_1)\bigr), \tag{270}$$

for $w\in\mathcal{W}$. The left-hand (right-hand) side is the inflow (outflow) of workers in this type. Now, for the unemployed types $(\gamma wh_2,h_1)$,

$$\lambda(w,h_2)s(1-\pi_u)=\mu(\gamma wh_2,h_1)G\bigl(A(\gamma wh_2,h_1)\bigr), \tag{271}$$

for $w\in\mathcal{W}$. For the unemployed types $(\gamma wh_2,h_2)$,

$$\lambda(w,h_2)\bigl(s\pi_u+(1-s)q(w,h_2)\bigr)=\mu(\gamma wh_2,h_2)G\bigl(A(\gamma wh_2,h_2)\bigr), \tag{272}$$

for $w\in\mathcal{W}$. Lastly, for the unemployed type $(\gamma wh_1,h_2)$,

$$\lambda(w,h_1)(1-s)(1-\pi_e)q_e(w)=\mu(\gamma wh_1,h_2)G\bigl(A(\gamma wh_1,h_2)\bigr). \tag{273}$$

Similarly, the steady-state equation for the employed types (w,h_1) is

$$\begin{aligned}(274) \qquad & \sum_{b\in\mathcal{B}} \mu(b,h_1)G(w)\mathbb{I}_{\{w\in A(b,h_1)\}} \\ = \;& \lambda(w,h_1)\big(s+(1-s)\pi_e q(w,h_1)+(1-s)(1-\pi_e)\big),\end{aligned}$$

for $w \in \mathcal{W}$. And, for the employed type (w,h_2),

$$\begin{aligned}(275) \qquad & \sum_{b\in\mathcal{B}} \mu(b,h_2)G(w)\mathbb{I}_{\{w\in A(b,h_2)\}} + \lambda(w,h_1)(1-s)(1-\pi_e)\big(1-q_e(w)\big) \\ = \;& \lambda(w,h_2)\big(s+(1-s)\pi_e q(w,h_2)\big),\end{aligned}$$

for $w \in \mathcal{W}$. Equations (270)-(275) sum to zero, reflecting the fact that an outflow from some type is an inflow in some other type. The Matlab program `ex196f.m` organizes these equations in a large matrix and solve for the steady-state distribution of types. Figure 5 plots the unemployment level as a function of the replacement ratio γ, for two choices of $\mathcal{H}$. The first choice has a low spread $h_2 - h_1$ and the second a large spread. The equilibrium unemployment level increases with the replacement ration, reflecting the fact that unemployed workers with larger benefit have more incentive to reject job offers. The equilibrium unemployment level is also sensitive to the distribution of human capital level. Namely, when the spread is larger, previously high-skilled unemployed workers, with type $(\gamma w h_2, h_1)$, have a larger benefit relative to the job offers they receive. This give them more incentive to reject job offers.

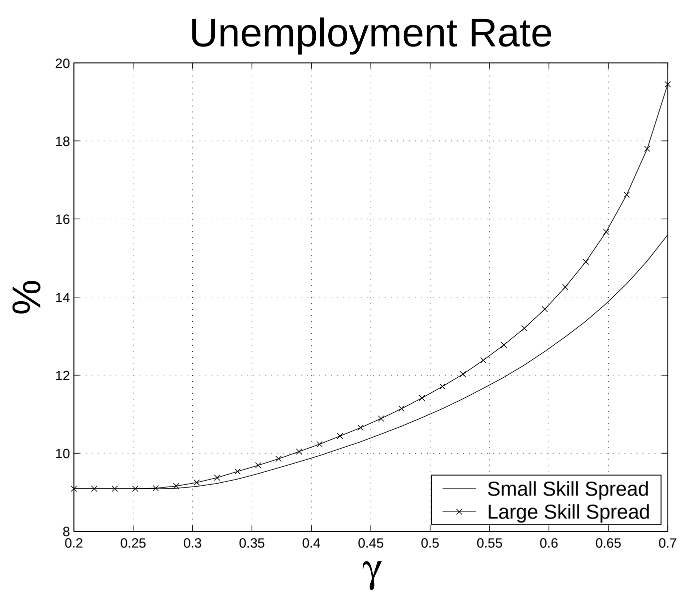


Figure 5. Execise 19.6 : Solving for equilibrium unemployment

Comparison of models

c. Explain how the different models in exercises 19.4–19.6 address the observations that European welfare states have experienced less of an increase in earnings differentials as compared to the United States, but suffer more from long-term unemployment where the probability of gaining employment drops off sharply with the length of the unemployment spell.

Solution

In the three exercises, the increase in earning differentials can be captured by an increase in the spread of the distribution of h. Moreover, a welfare state can be represented by a larger unemployment benefit. The three exercises share the feature that, in a welfare state, an increase in the earning differential results in an higher unemployment level. The mechanisms are different.

In first exercise, the unemployment benefit is constant for all wages. As a result, low-skilled workers have stronger incentives to reject an offer than high-skilled workers. An increase in the skill spread increases the fraction of low-skill workers with stronger incentive to reject, resulting in an increase in unemployment. It also increase the fraction of high-skill worker, with stronger incentive to accept, resulting in an decrease in unemployment. The total effect is to increase unemployment.

In the second exercise, the increase in the spread of match values increases the option value of staying unemployed. The unemployment benefit also increase this option value. The combination of a larger benefit and a larger spread yield to a larger unemployment level.

Lastly, in the last exercise, with an increase in the skill spread, previously high-skilled workers have stronger incentive to reject, resulting in an increase of the unemployment level.

d. Explain why the assumption of infinitely lived agents is innocuous for the models in exercises 19.4 and 19.5, but the alternative assumption of finitely lived agents can make a large difference for the model in exercise 19.6.

Solution

In the three exercises, assuming finite lives would mechanically increase the turnover of workers in jobs. As a result, search-and-matching delays are likely to increase the "frictional" unemployment.

Assuming finitely lives might also change the incentive to accept a job offer. In the first two exercises, the value of being employed does not increase with tenure. As a result, finitely-lived and infinitely-lived workers have similar incentive to accept a job offer. In the third exercise, however, because a worker's skill increases with tenure, a finite life can reduce dramatically the value of accepting a low-skilled job offers. It might result in a further increase in the equilibrium unemployment level.

Exercise 19.7. *Temporary jobs and layoff costs*

Consider a search model with temporary jobs. At the beginning of each period, a previously employed worker loses her job with probability μ, and she can keep her job and wage rate from last period with probability $1 - \mu$. If she loses her job (or chooses to quit), she draws a new wage and can start working at the new wage in the following period with probability one. After a first period on the new job, she will again in each period face probability μ of losing her job. New wages are independent and identically distributed from the cumulative distribution function F, where $F(0) = 0$, and $F(M) = 1$ for $M < \infty$. The situation during unemployment is as follows. At the beginning of each period, a previously unemployed worker can choose to start working at last period's wage offer or to draw a new wage from F. If she draws a new wage, the old wage offer is lost and she can start working at the new wage in the following period. The income of an unemployed worker is b, which includes unemployment insurance and the value of home production. Each worker seeks to maximize $E_0 \sum_{t=0}^{\infty} \beta^t I_t$, where β is the subjective discount factor, and I_t is the worker's income in period t; that is, I_t is equal to the wage w_t when employed and the income b when unemployed. Here E_0 is the mathematical expectation operator, conditioned on information known at time 0. Assume that $\beta \in (0, 1)$ and $\mu \in (0, 1]$.

a. Describe the worker's optimal decision rule.

Suppose that there are two types of temporary jobs: short-lasting jobs with μ_s and long-lasting jobs with μ_l, where $\mu_s > \mu_l$. When the worker draws a new wage from the distribution F, the job is now randomly designated as either short-lasting with probability π_s or long-lasting with probability π_l, where $\pi_s + \pi_l = 1$. The worker observes the characteristics of a job offer, (w, μ).

b. Does the worker's reservation wage depend on whether a job is short-lasting or long-lasting? Provide intuition for your answer.

We now consider the effects of layoff costs. It is assumed that the government imposes a cost $\tau > 0$ on each worker that loses a job (or quits).

c. Conceptually, consider the following two reservation wages, for a given value of μ: (i) a previously unemployed worker sets a reservation wage for accepting last period's wage offer; (ii) a previously employed worker sets a reservation wage for continuing working at last period's wage. For a given value of μ, compare these two reservation wages.

d. Show that an unemployed worker's reservation wage for a short-lasting job exceeds her reservation wage for a long-lasting job.

e. Let $\bar{w}_s$ and $\bar{w}_l$ be an unemployed worker's reservation wages for short-lasting jobs and long-lasting jobs, respectively. In period t, let N_{st} and and N_{lt} be the fractions of workers employed in short-lasting jobs and long-lasting jobs, respectively. Let U_t be the fraction of workers unemployed in period t. Derive difference equations for N_{st}, N_{lt} and U_t in terms of the parameters of the model and the reservation wages, $\{F, \mu_s, \mu_l, \pi_s, \pi_l, \bar{w}_s, \bar{w}_l\}$.

Solution

a. Let $V(w)$ be the value function of a worker (employed or unemployed), with wage w in hand, at the beginning of the period and who behaves optimally. The Bellman equation is :

$$V(w) = \max_{\text{accept,reject}} \{w + \beta (\mu(b + \beta Q) + (1 - \mu)V(w)), b + \beta Q\} \tag{276}$$

$$Q = \int_0^M V(w')dF(w'). \tag{277}$$

The worker's decision is described by a reservation wage $\bar{w}$. If the wage offer is greater than $\bar{w}$ the worker accepts it, otherwise she rejects it.

b. In this situation we can describe the optimal decision by very similar Bellman equations :

$$V(w, \mu) = \max_{\text{accept,reject}} \{w + \beta (\mu(b + \beta Q) + (1 - \mu)V(w, \mu)), b + \beta Q\} \tag{278}$$

$$Q = \pi_s \int_0^M V(w', \mu_s)dF(w') + \pi_l \int_0^M V(w', \mu_l)dF(w'). \tag{279}$$

As before, the optimal policy is described by a reservation wage $\bar{w}(\mu)$. To characterize it further, observe first that when w is greater than $\bar{w}(\mu)$, we have :

$$V(w, \mu) = w + \beta (\mu(b + \beta Q) + (1 - \mu)V(w, \mu)).$$

Solving the above equation for $V(w, \mu)$ gives :

$$V(w, \mu) = \frac{w}{1 - \beta(1 - \mu)} + \frac{\beta\mu}{1 - \beta(1 - \mu)}(b + \beta Q).$$

Now the reservation wage solves

$$V(\bar{w}(\mu), \mu) = b + \beta Q.$$

Solving this equation shows that the reservation wage does not depend on μ :

$$\bar{w}(\mu) = (1 - \beta)(b + \beta Q).$$

An intuition for this result goes as follows. At the reservation wage, agents are indifferent between rejecting, or accepting and then quitting, or accepting and then being fired. It implies that, at the reservation wage, the value of rejecting equals the value of keeping the job *forever*. That is :

$$b + \beta Q = \frac{\bar{w}(\mu)}{1 - \beta}.$$

c. As before, the value functions are:

$$V^u(w,\mu) = \max_{\text{accept,reject}} \left\{ w + \beta\left(\mu(b-\tau+\beta Q) + (1-\mu)V(w,\mu)\right), b+\beta Q \right\} \tag{280}$$

$$V^e(w,\mu) = \max_{\text{accept,reject}} \left\{ w + \beta\left(\mu(b-\tau+\beta Q) + (1-\mu)V(w,\mu)\right), b-\tau+\beta Q \right\} \tag{281}$$

$$Q = \pi_s \int_0^M V(w',\mu_s)dF(w') + \pi_l \int_0^M V(w',\mu_l)dF(w'). \tag{282}$$

The optimal policy is described by reservation wages $\bar{w}^i(\mu)$, $i = u, e$. The left hand side of the Bellman equations is weakly increasing and is the same for both employed and unemployed workers. The right hand side is a constant, lower for an employed worker than for an unemployed worker. Thus the indifference conditions defining the reservation wage imply that

$$\bar{w}^e(\mu) < \bar{w}^u(\mu).$$

The above equation shows in particular that employed worker would never quit a job unless they are fired.

d. We repeat the step of question b. First, we solve for the value function of an employed worker when $w \geq \bar{w}^u(\mu) > \bar{w}^e(\mu)$. We obtain :

$$V^e(w,\mu) = \frac{w}{1-\beta(1-\mu)} + \frac{\beta\mu}{1-\beta(1-\mu)}(b-\tau+\beta Q).$$

Since $\bar{w}^u(\mu) \geq \bar{w}^e(\mu)$, we know that:

$$V^u(\bar{w}^u(\mu),\mu) = V^e(\bar{w}^u(\mu),\mu) = b + \beta Q.$$

Solving this equation shows that :

$$\frac{\bar{w}^u(\mu)}{1-\beta} = (b+\beta Q) + \frac{\beta\mu\tau}{1-\beta}. \tag{283}$$

Now the reservation wage depends on the duration. Short lived jobs (high μ) are associated with higher reservation wage. This follows from the fact that, when there are firing cost, an employed worker is no longer indifferent between rejecting and accepting and then quitting (or being fired). He now needs to be compensated for the firing cost. Thus jobs with higher expected firing cost (short lived) are associated with higher reservation wages.

Equation (283) has the following intuitive interpretation. The left hand side is the value of working at the reservation wage *forever*. This should be equal to the value of accepting the job plus the value of staying forever, that is the value of never paying any firing cost. What is this value ? In each period on the job, a worker need to pay a firing τ cost with probability μ. The expected present value of this stream of cost is :

$$E\sum_{t=1}^{+\infty}\beta^t\tau B_t.$$

Where B_t are i.i.d. binomial random variable which are 1 with probability μ and zero otherwise. The above expression simplifies to :

$$\frac{\beta\mu\tau}{1-\mu}.$$

e. Recall that employed worker never quit jobs unless they are fired. This implies the following difference equations :

$$U_{t+1} = \mu_s N_{st} + \mu_l N_{lt} + (1-\pi_s F(\bar{w}_s) - \pi_l F(\bar{w}_l))U_t \tag{284}$$

$$N_{st+1} = (1-\mu_s)N_{st} + \pi_s(1-F(\bar{w}_s))U_t \tag{285}$$

$$N_{st+1} = (1-\mu_l)N_{lt} + \pi_l(1-F(\bar{w}_l))U_t. \tag{286}$$

Exercise 19.8. *Productivity shocks, job creation, and job destruction, donated by Rodolfo Manuelli*

Consider an economy populated by a large number of identical individuals. The utility function of each individual is

$$\sum_{t=0}^{\infty}\beta^t x_t,$$

where $0<\beta<1$, $\beta=1/(1+r)$, and x_t is income at time t. All individuals are endowed with one unit of labor that is supplied inelastically: if the individual is working in the market, its productivity is y_t, while if he/she works at home productivity is z. Assume that $z<y_t$. Individuals who are producing at home can also — at no cost — search for a market job. Individuals who are searching and jobs that are vacant get randomly matched. Assume that the number of matches per period is given by

$$M(u_t, v_t),$$

where M is concave, increasing in each argument, and homogeneous of degree one. In this setting, u_t is interpreted as the total number of unemployed workers, and v_t is the total number of vacancies. Let $\theta\equiv v/u$, and let $q(\theta)=M(u,v)/v$ be the probability that a vacant job (or firm) will meet a worker. Similarly, let $\theta q(\theta) = M(u,v)/u$ be the probability that an unemployed worker is matched with a vacant job. Jobs are exogenously destroyed with probability s. In order to create a vacancy a firm must pay a cost $c>0$ per period in which the vacancy is "posted" (i.e., unfilled). There is a large number of potential firms (or jobs) and this guarantees that the expected value of a vacant job, V, is zero. Finally, assume that, when a worker and a vacant job meet, they bargain according to the Nash Bargaining solution, with the workers' share equal to φ. Assume that $y_t = y$ for all t. **a.** Show that the zero profit condition implies that,

$$w = y - (r+s)c/q(\theta).$$

b. Show that if workers and firms negotiate wages according to the Nash Bargaining solution (with worker's share equal to φ), wages must also satisfy

$$w = z + \varphi(y - z + \theta c).$$

c. Describe the determination of the equilibrium level of market tightness, θ.

d. Suppose that at $t = 0$, the economy is at its steady state. At this point, there is a once and for all permanent increase in productivity. The new value of y is $y' > y$. Show how the new steady state value of θ, θ', compares with the previous value. Argue that the economy "jumps" to the new value right away. Explain why there are no "transitional dynamics" for the level of market tightness, θ.

e. Let u_t be the unemployment rate at time t. Assume that at time 0 the economy is at the steady state unemployment rate corresponding to θ — the "old" market tightness — and display this rate. Denote this rate as u_0. Let $\theta_0 = \theta'$. Note that that change in unemployment rate is equal to the difference between Job Destruction at t, JD_t and Job Creation at t, JC_t. It follows that

$$\begin{aligned} JD_t &= (1-u_t)s, \\ JC_t &= \theta_t q(\theta_t) u_t, \\ u_{t+1} - u_t &= JD_t - JC_t. \end{aligned}$$

Go as far as you can characterizing job creation and job destruction at $t = 0$ (after the shock). In addition, go as far as you can describing the behavior of both JC_t and JD_t during the transition to the new steady state (the one corresponding to θ').

Solution

a. , b. and c. See pp 575-578, chapter 19, in *Recursive Macroeconomic Theory.*

d. Let's first settle some timing issues. Let u_t be the unemployment rate in period t. In period t, firms post vacancies v_t, which result in $M(u_t, v_t)$ matches. Matched workers start working in period $t+1$. Let $\theta_t \equiv v_t/u_t$. The dynamic of the unemployment rate is :

$$u_{t+1} = u_t + s(1-u_t) - \theta_t q(\theta_t), \quad u_0\text{given.} \tag{287}$$

The rest of the equations defining an equilibrium is:

$$J_t = y' - w_t + \beta\left(sV_{t+1} + (1-s)J_{t+1}\right) \tag{288}$$
$$V_t = -c + \beta\left(q(\theta_t)J_{t+1} + (1-q(\theta_t))V_t\right) \tag{289}$$
$$E_t = w_t + \beta\left(sU_{t+1} + (1-s)E_{t+1}\right) \tag{290}$$
$$U_t = z + \beta\left(\theta_t q(\theta_t)E_{t+1} + (1-\theta_t q(\theta_t))U_{t+1}\right) \tag{291}$$
$$E_t - U_t = \phi(J_t - V_t + E_t - U_t) \tag{292}$$
$$V_t = 0. \tag{293}$$

Equations (288) and (289) are the Bellman equations of the firm. Equations (290) and (291) are the Bellman equations of the worker. Equation (292) expresses Nash bargaining and equation (293) is the free entry condition. An equilibrium is $\{u_t, \theta_t, w_t, J_t, V_t, E_t, U_t\}_{t=0}^{+\infty}$ that satisfies equations (287) to (293) and the transversality conditions

$$\lim_{t\to+\infty} \beta^t W_t = 0. \tag{294}$$

For $W_t = J_t, V_t, E_t, U_t$. Observe that the system of equilibrium equation separates in two blocks. We can first solve for $\{\theta_t, w_t, J_t, V_t, E_t, U_t\}_{t=0}^{+\infty}$ using (288)-(293). Observe that the solution does not depend on u_0. Then, we solve for $\{u_{t+1}\}_{t=0}^{+\infty}$ using (287). Now note that one solution of (288)-(293) is a constant vector $(\theta', w', J', V', E', U')$. The values correspond to the steady state equilibrium in a basic matching model in which productivity is y'. Going back to equation (287) we can analyze the dynamics of the unemployment rate.

This discussion shows that there is an equilibrium in which all equilibrium quantities except the unemployment rate "jump" to the new steady state right away. It does not rule out *a priori* existence of other equilibria.

e. The dynamic of the unemployment rate is given by:

$$u_{t+1} = u_t + (1-u_t)s - q(\theta')\theta' u_t. \tag{295}$$

Let $u_0 = u$ be the old steady state unemployment and u' be the new one. u' solves :

$$u' = u' + (1-u')s - q(\theta')\theta' u'. \tag{296}$$

Subtract (296) to (295) to obtain :

$$u_{t+1} - u' = (1 - s - \theta' q(\theta'))(u_t - u') = \left[1 - s - \theta' q(\theta')\right]^{t+1}(u - u'). \tag{297}$$

Since $0 < s + \theta' q(\theta') < 2$, the above equation implies that $u_t \to u'$ as $t \to +\infty$ (as it should). For simplicity assume that $0 < 1 - s - \theta' q(\theta') < 1$. Since $y' > y$, we know from comparative statics of the basic matching model that $\theta' > \theta$ and thus that $u' < u$. Therefore $u_t = u' + \left[1 - s - \theta' q(\theta')\right]^t (u - u')$ decreases geometrically towards the new steady state at rate $1 - s - \theta' q(\theta')$. This implies in turn that the number of job creations is decreasing over time and the number of job destruction is increasing. Specifically, at the time of the productivity shocks, JC_t jumps to $\theta' q(\theta') u_t$ while JD_t is equal to $(1 - u_t) * s$. Thus, there are more jobs created than destructed. Over time, JC_t decreases while JD_t increases. Asymptotically, job creation is equal to job destruction.

Exercise 19.9. *Workweek restrictions, unemployment, and welfare, donated by Rodolfo Manuelli*

Recently, France has moved to a shorter workweek of about 35 hours per week. In this exercise you are asked to evaluate the consequences of such a move. To this end, consider an economy populated by risk-neutral, income-maximizing workers with preferences given by

$$U = E_t \sum_{j=0}^{\infty} \beta^j y_{t+j}, \quad 0 < \beta < 1, \quad 1 + r = \beta^{-1}.$$

Assume that workers produce z at home if they are unemployed, and that they are endowed with one unit of labor. If a worker is employed, he/she can spend x units of time at the job, and $(1 - x)$ at home, with $0 \leq x \leq 1$. Productivity on the job is yx, and x is perfectly observed by both workers and firms. Assume that if a worker works x hours, his/her wage is wx. Assume that all jobs have productivity $y > z$, and that to create a vacancy firms have to pay a cost of $c > 0$ units of output per period. Jobs are destroyed with probability s. Let the number of matches per period be given by

$$M(u, v),$$

where M is concave, increasing in each argument, and homogeneous of degree one. In this setting, u is interpreted as the total number of unemployed workers, and v is the total number of vacancies. Let $\theta \equiv v/u$, and let $q(\theta) = M(u, v)/v$. Assume that workers and firms bargain over wages, and that the outcome is described by a Nash Bargaining outcome with the workers' bargaining power equal to φ.

a. Go as far as you can describing the unconstrained (no restrictions on x other than it be a number between zero and one) market equilibrium.

b. Assume that $q(\theta) = A\theta^{-\alpha}$, for some $0 < \alpha < 1$. Does the solution of the planner's problem coincide with the market equilibrium?

c. Assume now that the workweek is restricted to be less than or equal to $x^* < 1$. Describe the equilibrium.

d. For the economy in part c go as far as you can (if necessary make additional assumptions) describing the impact of this workweek restriction on wages, unemployment rates, and the total number of jobs. Is the equilibrium optimal?

Solution

a. We first describe the equilibrium equations, following the notation of chapter 19.

$$
\begin{aligned}
s(1-u) &= \theta q(\theta) u && (298)\\
J &= x(y-w) + \beta\left(sV + (1-s)J\right) && (299)\\
V &= -c + \beta\left(q(\theta)J + (1-q(\theta))V\right) && (300)\\
E &= \max_{0\le x\le 1}\left\{wx + z(1-x) + \beta\left(sU + (1-s)E\right)\right\} && (301)\\
U &= z + \beta\left(\theta q(\theta)E + (1-\theta q(\theta))U\right) && (302)\\
E - U &= \phi(J - V + E - U) && (303)\\
V &= 0. && (304)
\end{aligned}
$$

Equation (298) is the steady state condition for the unemployment rate. Equations (299) and (300) are the Bellman equations for the firm. Equations (301) and (302) are the Bellman equations for the worker. Equation (303) reflect Nash bargaining. Lastly, equation (304) is the free entry condition. An equilibrium is $(u, \theta, w, J, V, E, U)$ that satisfies the above system of equations.

We now show that, in equilibrium, $w \ge z$. Suppose not. Then the optimal choice of the worker is $x = 0$. From equations (299) and (304), it follows that $J = 0$. But then equation (300) implies that $V < 0$. A contradiction.
In equilibrium, $w \ge z$. Let's assume that if $w = z$ then the worker chooses $x = 1$. We can then drop the max in equation (301) and replace x by 1 in all other equations. The equilibrium is then the one of the basic matching model described in chapter 19, pp $575 - 578$.

b. First observe that, since $y > z$, the social planner necessarily chooses that all worker spend $x = 1$ on the job. Thus we can replace x by 1 in the social planner objective. Then the question is answered using the argument of section *Analysis of welfare*, page 578 in *Recursive Macroeconomic Theory*. The steady state market equilibrium coincides with steady state of the social optimum if and only if $\alpha = \phi$.

c. and **d.** Under this new assumption the equilibrium equations have the same form as before except that the max is subject to $0 \le x \le x^*$. The same argument as in part **a** shows that, in equilibrium, $w \ge z$. If we resolve indifference by setting $x = x^*$, we can drop the max from equation (301) and replace x by x^*. Now make the following change of variables:

$$\tilde{w} \equiv x^*w \tag{305}$$
$$\tilde{y} \equiv x^*y \tag{306}$$
$$\tilde{z} \equiv x^*z \tag{307}$$
$$\tilde{E} \equiv E - \frac{z(1-x^*)}{1-\beta} \tag{308}$$
$$\tilde{U} \equiv U - \frac{z(1-x^*)}{1-\beta}. \tag{309}$$

It is then easy to verify that the system of equilibrium equation is exactly (298)-(304) when all variables have been replaced by their $\tilde{}$ counterparts. Thus, we can apply formula of chapter 19. We know that the market tightness of the constrained equilibrium, $\tilde{\theta}$), solves:

$$\tilde{y} - \tilde{z} = x^*(y-z) = \frac{r + s + \phi\tilde{\theta}q(\tilde{\theta})}{(1-\phi)q(\tilde{\theta})}c. \tag{310}$$

The right hand side is increasing in $\tilde{\theta}$. The left hand side is increasing in x^*. Thus, lower x^* are associated with lower $\tilde{\theta}$ and higher unemployment. The period wage $\tilde{w}$ solves :

$$wx^* = \tilde{w} = (1-\phi)\tilde{z} + \phi\tilde{y} + \phi\tilde{\theta}c. \tag{311}$$

All the terms on the right hand side increase with x^*. Therefore, lower x^* are associated with lower wages. Note that the wage per hour $\tilde{w}/x^*$ may increase of decrease.

The equilibrium is not optimal. Since $y > z$ a central planner would necessarily impose that employed worker spend $x = 1$ on the job. If the planner is constrained to choose $x \leq x^*$, we find that the equilibrium is optimal if and only if $\phi = \alpha$.

Exercise 19.10. *Costs of creating a vacancy and optimality, donated by Rodolfo Manuelli*

Consider an economy populated by risk-neutral, income-maximizing workers with preferences given by

$$U = E_t \sum_{j=0}^{\infty} \beta^j y_{t+j}, \quad 0 < \beta < 1, \quad 1 + r = \beta^{-1}.$$

Assume that workers produce z at home if they are unemployed. Assume that all jobs have productivity $y > z$, and that to create a vacancy firms have to pay p_A, with $p_A = C'(v)$, per period when they have an open vacancy, with v being the total number of vacancies. Assume that the function $C(v)$ is strictly convex, twice differentiable and increasing. Jobs are destroyed with probability s. Let the number of matches per period be given by

$$M(u, v),$$

where M is concave, increasing in each argument, and homogeneous of degree one. In this setting, u is interpreted as the total number of unemployed workers, and v is the total number of vacancies. Let $\theta \equiv v/u$, and let $q(\theta) = M(u, v)/v$. Assume that workers and firms bargain over wages, and that the outcome is described by a Nash Bargaining outcome with the workers' bargaining power equal to φ.

a. Go as far as you can describing the market equilibrium. In particular, discuss how changes in the exogenous variables, z, y and the function $C(v)$, affect the equilibrium outcomes.

b. Assume that $q(\theta) = A\theta^{-\alpha}$, for some $0 < \alpha < 1$. Does the solution of the planner's problem coincide with the market equilibrium? Describe instances, if any, in which this is the case.

Solution

a. Since $C'(v)$ is taken as given by the entrepreneur when deciding to post a vacancy, the equilibrium equations have the same form as in the basic matching model of chapter 19, replacing c by $C'(v)$. Following the algebra outlined in the book, we find :

$$y - z = \frac{r + s + \phi\theta q(\theta)}{(1-\phi)q(\theta)} C'(v). \tag{312}$$

This equation has two unknowns, $\theta \equiv \frac{v}{u}$ and v. We obtain a second equation in (θ, v) using the steady state condition $u(1-s) = u\theta q(\theta)$:

$$\frac{v}{\theta} = u = \frac{s}{s + \theta q(\theta)}. \tag{313}$$

which implies that $v(\theta) = s\theta/(s + \theta q(\theta))$. Observe that this is an increasing function of θ because $q(\theta)$ is decreasing in θ. Replacing this expression in (312), we obtain :

$$y - z = \frac{r + s + \phi\theta q(\theta)}{(1-\phi)q(\theta)} C'\left[\frac{s\theta}{s + \theta q(\theta)}\right]. \tag{314}$$

$C'(v)$ strictly increasing is because C is strictly convex. $v(\theta)$ is increasing. This implies that the right hand side of the above equation is increasing in θ. Thus, if there is an equilibrium, it is unique. Existence requires that the right hand side evaluated at zero is less than $y - z$.

The usual comparative statics are still true. An increase in y or a decrease in z increases the market tightness θ and reduce employment. An additive shift of the cost function ($C(v) + \Delta c$) do not affect the equilibrium, entry decision depends on the marginal cost. However, a multiplicative shift of the cost function ($(1+k)C(v)$) has an impact on the equilibrium outcome. $k > 0$ shifts the marginal cost C' upward, implying that the equilibrium market tightness is lower. Thus, in equilibrium, less vacancies are created and the unemployment rate is higher.

b. The planner's problem is :

$$\max_{\{v_t,n_{t+1}\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \left[n_t y + (1-n_t)z - C(v_t)\right].$$

Subject to $n_{t+1} \leq (1-s)n_t + M(1-n_t, v_t)$. We attach the multiplier $\beta^t\lambda_t$ to the constraint. With this normalization λ_t is measured in term of time t consumption good. When taking derivative, we observe that $\frac{\partial M}{\partial u} = \alpha M(u,v)/u = \alpha\theta q(\theta)$ and $\frac{\partial M}{\partial v} = (1-\alpha)M(u,v)/v = (1-\alpha)q(\theta)$. The first order conditions are :

$$y - z = \lambda_{t+1}(1-s) + \lambda_t/\beta - \alpha\theta_{t+1}q(\theta_{t+1}) \tag{315}$$
$$0 = -C'(v_{t+1}) + \lambda_{t+1}(1-\alpha)q(\theta_{t+1}). \tag{316}$$

In a steady state θ_t, v_t and λ_t are constant. Rearranging gives the familiar :

$$y - z = \frac{r + s + \alpha\theta q(\theta)}{(1-\alpha)q(\theta)}. \tag{317}$$

The usual result follows. The steady state market equilibrium is the steady state of the social planning problem if and only if $\alpha = \phi$.

Exercise 19.11. *Financial wealth, heterogeneity,and unemployment, donated by Rodolfo Manuelli*

Consider the behavior of a risk-neutral worker that seeks to maximize the expected present discounted value of wage income. Assume that the discount factor is fixed and equal to β, with $0 < \beta < 1$. The interest rate is also constant and satisfies $1 + r = \beta^{-1}$. In this economy, jobs last forever. Once the worker has accepted a job, he/she never quits and the job is never destroyed. Even though preferences are linear, a worker needs to consume a minimum of a units of consumption per period. Wages are drawn from a distribution with support on $[a, b]$. Thus, any employed individual can have a feasible consumption level. There is no unemployment compensation. Individuals of type i are born with wealth a^i, $i = 0, 1, 2$, where $a^0 = 0$, $a^1 = a$, $a^2 = a(1+\beta)$. Moreover, in the period that they are born, all individuals are unemployed. Population, N_t, grows at the constant rate $1 + n$. Thus, $N_{t+1} = (1 + n)N_t$. It follows that, at the beginning of period t, at least nN_{t-1} individuals — those born in that period — will be unemployed. Of the nN_{t-1} individuals born at time t, φ^0 are of type 0, φ^1 of type 1, and the rest, $1-\varphi^0-\varphi^1$, are of type 2. Assume that the mean of the offer distribution (the mean offered, not necessarily accepted, wage) is greater than a/β.

a. Consider the situation of an unemployed worker who has $a^0 = 0$. Argue that this worker will have a reservation wage $w^*(0) = a$. Explain.

b. Let $w^*(i)$ be the reservation wage of an individual with wealth i. Argue that $w^*(2) > w^*(1) > w^*(0)$. What does this say about the cross sectional relationship between financial wealth and employment probability? Discuss the economic reasons underlying this result.

c. Let the unemployment rate be the number of unemployed individuals at t, U_t, relative to the population at t, N_t. Thus, $u_t = U_t/N_t$. Argue that in this economy the unemployment rate is constant.

d. Consider a policy that redistributes wealth in the form of changes in the fraction of the population that is born with wealth a^i. Describe as completely as you can the effect upon the unemployment rate of changes in φ^i. Explain your results.

Extra Credit: Go as far as you can describing the distribution of the random variable "number of periods unemployed" for an individual of type 2.

Solution

a. All worker need to consume at least a. A worker with no financial wealth can consume a only if she works. Thus, workers with no financial wealth always accept the first offer they receive. In other words, their reservation wage is $w^*(i) = a$.

Consider a worker with financial wealth a. If she accepts the offer to work forever at wage w, her utility is :

$$V_{\text{accept}} = \max_{\{c_t\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \beta^t c_t. \tag{318}$$

Subject to the intertemporal budget constraint:

$$\sum_{t=0}^{+\infty} \frac{c_t}{(1+r)^t} \leq a + \sum_{t=0}^{+\infty} \frac{w}{(1+r)^t}. \tag{319}$$

Since the utility function is increasing, the budget constraint is biding. When assuming furthermore that $\beta = 1/(1+r)$, we find that the right hand side of the budget constraint is equal to the objective, i.e. :

$$V_{\text{accept}} = a + \frac{w}{1-\beta}.$$

Observe also that, if she rejects, the unemployed worker need to consume a. She will enter next period with no financial wealth. In other word, if she rejects this period, she will accept any wage offer next period. Thus, the value function equation of an unemployed worker of type 1 with wage w in hand solves the following Bellman equation :

$$V^1(w) = \max_{\text{accept,reject}} \left\{ \frac{w}{1-\beta} + a, a + \beta \frac{E(w)}{1-\beta} \right\}.$$

The above equation implies that the optimal policy is characterized by a reservation wage $w^*(1)$ that solves :

$$\frac{w^*(1)}{1-\beta} = \frac{\beta E(w)}{1-\beta} > \frac{a}{1-\beta}.$$

Since $\beta E(w) > a$ by assumption. Consider now an unemployed worker with financial wealth $(1+\beta)a$. If she accepts, the same reasoning as above shows that her utility is :

$$\frac{w}{1-\beta} + (1+\beta)a.$$

If she rejects, she has to consume a this period, she saves βa so that next period she has financial wealth $\beta(1+r)a = a$. Thus, her value function solves the Bellman equation :

$$V^2(w) = \max_{\text{accept,reject}} \left\{ \frac{w}{1-\beta} + (1+\beta)a, a + \beta E\left[V^1(w')\right] \right\}. \tag{320}$$

The above equation shows that the optimal policy is characterized by a reservation wage $w^*(2)$ that solves :

$$\frac{w^*(2)}{1-\beta} + (1+\beta)a = a + \beta E\left[V^1(w')\right]. \tag{321}$$

To compare $w^*(1)$ and $w^*(2)$ we make the following observation. From the Bellman equation for V^1, it is clear that:

$$V^1(w) \geq \frac{w}{1-\beta} + a.$$

Furthermore, for all $w < w^*(1)$, the inequality is strict. Taking expectations on both sides implies that :

$$E\left[V^1(w')\right] > a + \frac{E(w)}{1-\beta}.$$

Now replace this inequality into (321) to obtain $w^*(2) > w^*(1)$. This model thus implies that, in a cross section, employment probability is negatively correlated with financial wealth. Interestingly, this is in spite of the fact that leisure do not enter in the utility function. There is negative correlation because a wealthier agent can buy the option to wait for a new offer.

c.
Observe first that unemployment lasts at the most 2 periods. Thus, at time t, there is no unemployed agent from generation $t-2$ and earlier. Unemployment at time t is :

$$U_t = nN_{t-1}(\phi^1 F(w_1) + \phi^2 F(w_2)) + nN_{t-2}\phi^2 F(w_2)F(w_1). \tag{322}$$

The first term on the right hand side is the number of newborn agents who refuse their wage offer. The second term is the measure of agents born in period $t-1$ who rejects twice their offer. Dividing by N_t, using the fact that $N_t/N_{t-1} = 1+n$, we find :

$$u_t = \frac{n}{n+1}(\phi^1 F(w_1) + \phi^2 F(w_2)) + \frac{n}{(n+1)^2}\phi^2 F(w_2)F(w_1), \tag{323}$$

which shows that the unemployment rate is constant in this economy.

d. Let's describe a redistributive policy as follows. It is a pair x_1, x_2 , of fractions of agents with wealth a and wealth $(1+\beta)a$ that satisfies the budget constraint :

$$x_1 a + x_2(1+\beta)a = \phi^1 a + \phi^2(1+\beta)a.$$

Observe that we constraint the government to redistribute within generations only. The above budget constraint can be written :

$$x_1 = \phi^1 + \phi^2(1+\beta) - x_2(1+\beta) = \Phi - x_2(1+\beta). \tag{324}$$

Replacing the above expression into the unemployment rate equation, we find :

$$u = \frac{n}{1+n}\Phi F(w_1) + x_2\left[\frac{n}{1+n}F(w_2) + \frac{n}{(1+n)^2}F(w_1)F(w_2) - \frac{n}{1+n}F(w_1)(1+\beta)\right]. \tag{325}$$

As one could expect, an increase in x_2 has two effects. It first increase the fraction of unemployed agents born with wealth $(1+\beta)a$. Since the increase in x_2 is engineered by decreasing x_1, the fraction of unemployed agents born with wealth a is decreasing. The net effect is ambiguous at this stage.

Extra Credit. An agent born with wealth $(1+\beta)a$ stays unemployed one period with probability $F(w_2)(1-F(w_1))$ and stays unemployed two periods with probability $F(w_2)F(w_1)$.

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