

Investment under Uncertainty and the Value of Real and Financial Flexibility*

Patrick Bolton[†] Neng Wang[‡] Jinqiang Yang[§]

December 29, 2014

Abstract

We develop a model of investment timing under uncertainty for a financially constrained firm. Facing external financing costs, the firm prefers to fund its investment through internal funds, so that the firm's optimal investment policy and value depend on both its earnings fundamentals and liquidity holdings. We show that financial constraints significantly alter the standard real options results, with the financial flexibility conferred by internal funds acting as a complement, and at times as a substitute, to the real flexibility given by the optimal timing of investment. We show that: 1) the investment hurdle is highly nonlinear and non-monotonic in the firm's internal funds; 2) in contrast to predictions implied by standard corporate savings models, a financially constrained firm may behave in a risk seeking sense (and thus firm value may be convex in liquidity) due to the interaction between financial and real (growth/abandonment) flexibility; 3) with multiple rounds of growth options, a value-maximizing financially constrained firm may choose to over-invest via accelerated investment timing in earlier stages in order to mitigate under-investment problems in later stages.

*First Draft (December, 2012) was circulated under the title "Investment, Liquidity, and Financing under Uncertainty." We thank Ilona Babenka, Martin Cherkas, Sudipto Dasgupta, Peter DeMarzo, Dirk Hackbarth, Mark Gertler, Simon Gilchrist, Vicky Henderson, Erica Xuenan Li, and seminar participants at American Finance Association (AFA) 2014 meetings in Philadelphia, Boston University, China International Conference in Finance (CICF), Columbia University, HKUST 2013 Finance Symposium, Rutgers University, University of Cambridge, University of Minnesota Carlson School, and Zhejiang University for helpful comments.

[†]Columbia University, NBER and CEPR. Email: pb2208@columbia.edu. Tel. 212-854-9245.

[‡]Columbia Business School and NBER. Email: neng.wang@columbia.edu. Tel. 212-854-3869.

[§]The School of Finance, Shanghai University of Finance and Economics (SUFE). Email: yang.jinqiang@mail.sufe.edu.cn.

1 Introduction

In their influential textbook Dixit and Pindyck (1994) condense the essence of investment decisions to three key attributes: i) the degree of *irreversibility*; ii) the *risk* over future revenue; and, iii) the flexibility in the *timing* of the investment decision. In this paper we add a fourth attribute: the external financing *cost* of the investment. Much of the real-options theory of investment under uncertainty following McDonald and Siegel (1986) assumes that firms operate in frictionless capital markets. This is for a good reason, as the firm's investment decision can then be formulated as a simple *real option problem* involving the optimal exercise and valuation of an American option. The option pricing tools developed by Black and Scholes (1973) and Merton (1973) can then be deployed to analyze this real options problem.

A major limitation of this approach, however, is that firms in practice do not operate in frictionless capital markets. They face significant external financing costs and as a result mostly rely on internally generated cash flows to finance their investments. The recent financial crisis is an important reminder of how severe external financing costs can be in extreme situations and how much they can affect corporate investment and the macroeconomy. The extensive surveys of chief financial officers (CFOs) by Graham and Harvey (2001, 2002) have also revealed the great importance CFOs attach to maintaining financial flexibility. In practice, the value of *real flexibility* obtained from the optimal timing of irreversible investment and abandonment is intertwined with the value of *financial flexibility* derived from retained earnings and optimal timing to tap external capital markets. Accordingly, the important theoretical question we address in this paper is how corporate decisions and the valuation of assets in place and growth options under uncertainty are affected by external financing costs and the firm's efforts to relax these costs by dynamically managing its *liquidity* holdings. In effect, we are integrating two classical strands of literature, one started by Miller and Orr (1966) on corporate cash balances and the other started by McDonald and Siegel (1986) on real options.

Although the classical tools of option pricing theory can no longer be directly applied (and although the analysis of the investment problem and its financing involves solving a considerably more complex two-dimensional partial differential equation), the results we obtain are intuitive and striking. First, both the growth options in the start-up phase and the abandonment options in the mature phase of the firm's life-cycle are worth less when the firm faces external financing constraints. Second, the firm is more likely to abandon an asset (captured by a higher abandonment *hurdle*) when the firm faces external financ-

ing costs: The firm facing external financing costs is capitalizing expected future external financing costs and setting these against the present value of the asset. However, should the financially constrained firm decide to continue operating the asset when it runs out of cash by raising external funds, the firm will tend to raise *more* external funds the *higher* are external financing costs, as the firm seeks to limit the risk of having to return to capital markets again in the future. Third, in contrast to predictions implied by standard corporate savings models, a financially constrained firm may behave in a risk seeking sense (and thus firm value may be convex in liquidity) due to the interaction between financial and real (growth/abandonment) flexibility.

Fourth, with multiple rounds of growth options, a financially constrained firm may surprisingly lower the hurdle for early investments (a form of over-investment) in order to gain access to higher operating cash-flows and accumulate more internal funds sooner, which in turn mitigates the firm's under-investment distortions in the future. This is in sharp contrast to the unambiguous under-investment prediction of the existing financial-constraints models in the vein of Fazzari, Hubbard, and Petersen (1988), Froot, Scharfstein, and Stein (1993), and Kaplan and Zingales (1997), as well as dynamic models such as Hennessy and Whited (2007), Riddick and Whited (2009), and Bolton, Chen, and Wang (2011).

Fifth, the hurdle for investment in the start-up phase is a non-monotonic function of the firm's stock of liquid assets: When the firm's internal funds are sufficient to entirely cover the costs of the investment then the firm's hurdle for investment is lower the higher the firm's internal funds. In contrast, when the firm's internal funds cannot cover the entire cost of the investment then the hurdle may be sharply increasing with the firm's internal funds. The reason is that when the firm is approaching the point where it may be able to entirely fund its investment with retained earnings it has a strong incentive to delay investment until it has sufficient funds to be able to avoid tapping costly external funds. An important implication of this result is that investment is not necessarily more likely when the firm has more cash. Investment could well be delayed further, as the firm's priority becomes avoiding reliance on external funds.

Sixth, external equity financing is also non-monotonic in the firm's internal funds as the firm's option value from accessing external financing is highest when its real option value is moderate. The intuition is as follows. For a firm whose investment option is sufficiently out of the money, financial flexibility has little value. On the other hand, for a firm whose investment option is deep in the money, the value of financial flexibility is of second order. Therefore, the multiple options of financing (internal versus external as well as the flexible timing decision of external financing) are most valuable for a firm whose real option value

is in the medium range. In sum, for firms facing external financing costs the value of an investment opportunity is not just tied to *timing optionality* but also to *financial flexibility*.

Related literature. Following Brennan and Schwartz (1985) and McDonald and Siegel (1986) the basic formulation of the investment under uncertainty problem has been extended in many different directions. Majd and Pindyck (1987) enrich the analysis with a time-to-build feature. Dixit (1989) uses the real option approach to examine entry and exit from a productive activity. Titman (1985) and Williams (1991) analyze real estate development in a real options framework. Abel and Eberly (1994) analyze a unified framework of investment under uncertainty that integrates the q theory of investment with the real options approach. Grenadier (1996), Lambrecht and Perraudin (2003) and others extend the real options decision problem to a game-theoretic environment. Grenadier and Wang (2005) incorporate informational asymmetries and agency problems into the real options framework. Morellec (2004) develops a continuous-time, real-options model in which managers want to invest and also to retain control. Grenadier and Wang (2007) study the impact of time-inconsistent hyperbolic discounting on real-option exercising strategies. Lambrecht and Myers (2007, 2008) apply the real options framework to analyze takeover, divestment and dividend policies in models with managerial agency. Miao and Wang (2007) develop an incomplete-markets real-options problem and show that the standard positive effect of volatility on option value is balanced by the risk-averse entrepreneur's precautionary savings motive.

Boyle and Guthrie (2003) is the first study of real options in the presence of financial constraints. They model financial constraints by assuming that the firm can only pledge a fraction of its value, which constrains its ability to fund investments. Unfortunately, however, their framework is flawed and contains two inconsistencies, the main one being that the firm is allowed to continue operations even when it accumulates arbitrarily large negative earnings, although it is not able to raise funds to finance capital expenditures.¹ An earlier study by Mauer and Triantis (1994) considers a real options problem for a levered firm, which may face recapitalization costs when its operating performance is poor. However, as in Leland (1994) the firm does not otherwise incur any external financing costs. In their model the levered firm has a lower hurdle as it seeks to bring tax-shield benefits of debt financing forward in time.

¹The other inconsistency is that the firm's investment project value is assumed to follow a geometric Brownian motion, while cumulative operating cash-flows are assumed to follow an arithmetic Brownian motion. Basically, Boyle and Guthrie (2003) assume that by exercising its growth option the firm swaps out an asset in place with iid returns for another asset with value V which follows a geometric Brownian motion. Moreover, while swapping out these assets the firm loses all its accumulated cash X .

Closer to our setup, Décamps and Villeneuve (2007) consider a financially constrained firm with an asset in place generating cash-flows that are subject to i.i.d shocks and a growth option, which raises the drift of the cash-flow process and can only be financed with internal funds. They characterize the firm’s optimal investment and dividend policy. Asvanunt, Broadie and Sundaresan (2007) also consider a real options problem for a levered firm. Unlike Mauer and Triantis (1994) and Leland (1994), they do not introduce external equity financing costs in the form of dilution costs and allow the firm to accumulate internal funds. However, neither of these studies provides a systematic analysis of the firm’s optimal cash management problem and the interplay of real and financial flexibility.

More recently, four independent studies have analyzed a similar problem to ours: First, Copeland and Lyasoff (2013) who consider a somewhat narrower framework to ours, which does not allow for either abandonment or sequential growth options. Second, Boot and Vladimirov (2014) consider a financially constrained entrepreneurial firm with an asset in place that generates random cash flows following a geometric Brownian motion and a new investment opportunity. Their focus is on comparing private firms, with less liquid assets but better aligned objectives between the entrepreneur and other investors, and public firms, with more liquid assets but greater potential disagreements between the manager and other investors. Third, Babenko and Tserlukevich (2013) consider the optimal hedging policy (in a dynamic model with cash accumulation) for a financially constrained firm with a decreasing returns to scale technology and lumpy investment opportunities. Fourth, Hugonnier, Malamud, and Morellec (2014) study the impact of financial constraints on growth options under capital supply uncertainty, assuming, however, that the cash-flow process (both before and after the option is exercised) follows an arithmetic Brownian motion (that is, the cash flow shocks are *i.i.d.*).

Our paper is also related to Décamps, Mariotti, Rochet, and Villeneuve (2011) and Bolton, Chen, and Wang (2011), who consider optimal dynamic corporate policies for firms facing external financing costs. Décamps, Mariotti, Rochet, and Villeneuve (2011) consider a financially constrained firm’s optimal dynamic payout and equity issuance policy and Bolton, Chen, and Wang (2011) extend the q -theory of investment to firms facing external financing costs (Anderson and Carverhill (2012) also study a related liquidity management problem).

The major advance of this paper is that it allows for permanent earnings shocks, irreversibility, fixed investment and operating costs, which together generate a demand for optionality and flexibility. In contrast, neither Décamps, Mariotti, Rochet, and Villeneuve (2011) nor Bolton, Chen, and Wang (2011) allow for permanent shocks, a key simplification which allows them to stay within a one-dimensional dynamic optimization problem

formulation. This paper provides a significant generalization of a financially constrained firm’s dynamic optimization problem to two dimensions—earnings fundamentals and liquidity buffers—in contrast to the earlier contributions with the (scaled) liquidity buffer as the single state variable. The economic analysis in the two-dimensional problem is significantly enriched, as the firm now takes account not only of its current *stock* of internal funds but also of the information about its future cash *flow* prospects contained in current persistent earnings shocks. Despite the significantly more complex formulation of the two-dimensional problem, we are still able to provide an intuitive and remarkably tractable analysis of this problem.

Finally, our work is also related to the literature on dynamic contracting models of corporate financing and investment. Thus, DeMarzo, Fishman, He, and Wang (2012) develop a model with a managerial agency problem in a q -theory framework with capital adjustment costs, by building on Bolton and Scharfstein (1990) and using the dynamic contracting framework of DeMarzo and Sannikov (2006) and DeMarzo and Fishman (2007). Rampini and Viswanathan (2010, 2013) develop dynamic models of investment and risk management with endogenous financing constraints, in which the firm is subject to endogenous collateral constraints induced by limited enforcement.

2 Model

Operating revenues and profits. We consider a firm with an investment opportunity modeled as in McDonald and Siegel (1986). At any point in time $t \geq 0$, the firm can exercise an investment opportunity by paying a fixed investment cost $I > 0$. Upon exercising, the firm then immediately obtains a perpetual stream of profits $Y_t - Z$, where Z is a constant and Y is stochastic. As in the real options literature, we assume that Y follows a geometric Brownian motion (GBM) process:

$$dY_t = \mu Y_t dt + \sigma Y_t dB_t, \tag{1}$$

where μ is the drift parameter, σ the volatility parameter, and B is a standard Brownian motion. Note that the operating profit per unit of time, given by $(Y_t - Z)$, can be negative. It is suboptimal for the firm to abandon its asset when Y falls just short of Z because the firm is forward looking and has option value. When Y_t is sufficiently lower than Z , it is no longer optimal for the firm to continue its operations and will then stop the project. We normalize the scrap value of the project upon the firm’s exercising of its abandonment option to zero. The firm, thus, has an American-style perpetual liquidation option where the timing of the

option exercising decision is endogenously chosen. For simplicity, throughout the paper, we will refer to Y as the operating revenue and Z as the operating cost. Obviously, our model specification allows both the revenue and cost to be stochastic and only requires the operating profit, $Y - Z$, to be an affine function in Y that follows a GBM process.

Before undertaking the investment, the firm does not incur any costs, and while the firm is waiting to invest, the process Y_t continues to evolve according to (1). In sum, the simplest formulation of the life-cycle of the firm in our model allows for three phases: a start-up phase, a mature phase, and a liquidation (a strong form of scale-down) phase.

We assume that investors are risk neutral, so that all cash flows are discounted at the risk-free rate, r . Equivalently, we may interpret that the process (1) has already captured the risk adjustment, i.e., under the risk-neutral measure and hence we may use the risk-free rate to discount the firm's profits. One may pursue the risk-neutral measure interpretation when analyzing a firm's risk-return tradeoff.

External financing costs. We introduce the standard specification for the external financing costs as follows: if the firm needs external funds F net of fees, it incurs an external financing cost $\Phi(F)$ through out its life cycle. Hence, the firm must raise a gross amount $F + \Phi(F)$. For simplicity, we assume that the equity issuance cost function for external financing takes the following form:

$$\Phi(F) = \phi_0 + \phi_1 F, \quad (2)$$

where $\phi_0 \geq 0$ is the fixed cost parameter and $\phi_1 \geq 0$ is the marginal cost of external financing. Intuitively, when the fixed equity issuance cost is sufficiently high, the firm prefers liquidation over equity issuance. While in theory, we may allow for different equity issuance cost functions in the start-up and mature phases to capture different degrees of financing frictions (e.g., agency costs and informational frictions) in the start-up and mature phases, we keep the functional forms to be the same in both phases for simplicity.

Liquidity hoarding. For a financially constrained firm, the key is the cash accumulation dynamics. We next discuss cash accumulation in both phases.

The startup phase. At the beginning of the start-up phase ($t = 0$) the firm is endowed with a stock of cash (or, more generally, a liquidity hoard comprising both cash and marketable securities) of W_0 . Over time cash accumulates at the risk-free rate r , so that $dW_t = rW_t dt$.

Note that since the firm earns the risk-free rate r on its cash it does not need to pay out any cash to its shareholders, and shareholders weakly prefer hoarding cash inside the firm.

If the firm were to earn less than the risk-free rate on its cash, it would also face an optimal payout decision. For simplicity we do not consider this generalization of the model.²

When the firm's liquidity W is insufficient to cover investment costs, i.e. $W < I$, obviously the firm will have to raise external financing or continue to accumulate internal funds in order to finance the cost of exercising the growth option. Because the firm will also need funds to finance potential operating losses after the growth option is exercised, it may choose to raise equity more than needed to cover investment cost in the startup phase.

The mature phase. In the mature phase, after exercising its growth option, the firm accumulates its liquidity W_t as follows:

$$dW_t = (rW_t + Y_t - Z)dt + dC_t, \quad W_t \geq 0. \quad (3)$$

The first term in (3) is the sum of its interest income rW and operating profits $Y_t - Z$. The firm's profits increase liquidity hoard W , when $rW_t + Y_t - Z > 0$, and liquidity holding finances losses when $rW_t + Y_t - Z < 0$. The second term in (3), dC_t , denotes the firm's net external equity financing. As the firm incurs no cost when holding cash, it weakly never pays out while in operation and hence $dC_t \geq 0$. Finally, after the firm has chosen to liquidate/scale-down its investment, without loss of generality, it simply pays out its remaining cash W_t to shareholders and closes down.

The firm's dynamic optimization problem thus involves rich interactions among an investment timing decision, an abandonment timing decision, and multiple external equity financing decisions. Importantly, liquidity plays a critical role in both phases and influences the exercising decisions of these various real (investment/abandonment) and financing options. Before providing the solution for a financially constrained firm's optimization problem, we first summarize the main results for a financially unconstrained firm.

3 The First-Best Benchmark

In the perfect capital markets world, where the Modigliani-Miller irrelevance theorem holds, a firm is financially unconstrained and only faces investment and abandonment timing decisions. We first report the first-best solution for the financially unconstrained firm. Appendix A sketches out the derivations for these standard results reported below.

²For dynamic models with cash-carrying costs, see Bolton, Chen, and Wang (2011) and DéCamps, Martiotti, Rochet, and Villeneuve (2011) for example.

3.1 The Mature Phase

In the Modigliani-Miller (MM) world the firm's value $P^*(W, Y)$ is simply given by the sum of its cash W and the value of its asset in place:

$$P^*(W, Y) = Q^*(Y) + W, \quad (4)$$

where the firm's project value, denoted by $Q^*(Y)$, is given by:

$$Q^*(Y) = \left(\frac{Y}{r - \mu} - \frac{Z}{r} \right) + \left(\frac{Y}{Y_a^*} \right)^\gamma \left(\frac{Z}{r} - \frac{Y_a^*}{r - \mu} \right), \quad \text{for } Y \geq Y_a^*, \quad (5)$$

and γ is given by:

$$\gamma = \frac{1}{\sigma^2} \left[- \left(\mu - \frac{\sigma^2}{2} \right) - \sqrt{\left(\mu - \frac{\sigma^2}{2} \right)^2 + 2r\sigma^2} \right] < 0. \quad (6)$$

The first term in (5) is the present discounted value of its operating profits if the firm were to remain in operation forever (which would be suboptimal for sufficiently low Y). The second term in (5) gives the additional value created if the firm were to optimally exercise its abandonment option. The firm optimally operates its physical asset if and only if $Y \geq Y_a^*$, where Y_a^* is the firm's abandonment threshold and is given by:

$$Y_a^* = \frac{\gamma}{\gamma - 1} \frac{r - \mu}{r} Z. \quad (7)$$

For $Y < Y_a^*$ the asset is abandoned, cash is paid out, and $Q^*(Y) = 0$. As is well known in the literature, the firm's value $P^*(W, Y)$ is convex in Y due to its abandonment option in the mature phase. Next, we turn to the firm's value in the start-up phase.

3.2 The Start-up Phase

As for the firm's value $P^*(W, Y)$ in the mature phase, the first-best value $G^*(W, Y)$ in the start-up phase takes the following simple additive form:

$$G^*(W, Y) = H^*(Y) + W. \quad (8)$$

Here, $H^*(Y)$ is the value of the firm's growth option, which includes the present discounted value of its future operating profits $Y_t - Z$, the value of optimal growth option exercising, and the value of optimal abandonment. The firm optimally exercises its growth option if and only if $Y \geq Y_i^*$ where Y_i^* is the optimal investment threshold that satisfies

$$(\beta - \gamma) \left(\frac{Y_i^*}{Y_a^*} \right)^\gamma \left(\frac{Z}{r} - \frac{Y_a^*}{r - \mu} \right) + (\beta - 1) \frac{Y_i^*}{r - \mu} - \beta \left(\frac{Z + rI}{r} \right) = 0, \quad (9)$$

and β is a constant given by:

$$\beta = \frac{1}{\sigma^2} \left[- \left(\mu - \frac{\sigma^2}{2} \right) + \sqrt{\left(\mu - \frac{\sigma^2}{2} \right)^2 + 2r\sigma^2} \right] > 1. \quad (10)$$

The option value $H^*(Y)$ has the following closed-form solution:

$$H^*(Y) = \left(\frac{Y}{Y_i^*} \right)^\beta (Q^*(Y_i^*) - I), \quad \text{for } Y \leq Y_i^*, \quad (11)$$

and $H^*(Y) = Q^*(Y) - I$, for $Y > Y_i^*$. Again, here the firm's value $G^*(W, Y)$ is convex in Y , a well known result in the classic real options theory. These are the MM benchmark results against which we compare the solution for a financially constrained firm.

4 Abandonment and Financing in the Mature Phase

In the mature phase, the firm manages its asset in place and has a liquidity hoard W . With external financing costs, the liquidity hoard W influences its decision and its valuation. We denote the firm's value in the mature phase by $P(W, Y)$. The firm can be in one of the four regions depending on its liquidity W and earning Y :

1. the “financially unconstrained” region where more W does not influence the firm's decision in any way and the firm is permanently financially unconstrained and hence behaves in the same way as it does in an MM world;
2. the “interior” liquidity hoarding region where the firm does not raise any external financing nor pays anything out but simply hoards and accumulates its liquidity W and continues its operation;
3. the “equity issuance” region where the firm issues equity to replenish liquidity. Importantly, as we will show, the firm has to meet at least two conditions for equity issue in our model: *i*) the firm incurs an operating loss, i.e. $Y - Z < 0$ and *ii*) it exhausts its liquidity capacity, i.e., $W = 0$. These two conditions are necessary as it is weakly optimal for the firm to defer the option of financing, whenever feasible, to save the time value of financing costs.
4. the “liquidation” region where the firm chooses to liquidate its operations when Y is sufficiently below its operating cost Z , which reflects the firm's dynamic tradeoff between *i*) covering operating losses $Y - Z$ via its internal funds and *ii*) the option value of capturing the upside when the earning eventually turns out to be high enough.

4.1 The Financially Unconstrained Region

Unlike in a static setting, a firm is financially unconstrained in a dynamic setting if and only if it faces no financial constraint with probability one at the current and all future times. There are two ways that the firm can be financially unconstrained: (i) it internally generates sufficient liquidity at all times; or (ii) the firm already has a sufficient liquidity hoard: $W \geq \Lambda$, where Λ denotes the lowest level of liquidity needed for a mature firm to be permanently financially unconstrained. We will provide an explicit formula for Λ .

Type-1 financially unconstrained firm: $Y \rightarrow \infty$.

If the firm's internally generated cash flow Y is very high, the firm will be fully liquid even without any liquidity hoard W , as internally generated revenue Y is fully sufficient to cover its operate cost Z in each period without ever having to raise external funds (due to the nature of the permanent earnings shock). In the limit where $Y \rightarrow \infty$, the following boundary condition must hold:

$$\lim_{Y \rightarrow \infty} P(W, Y) = \lim_{Y \rightarrow \infty} P^*(W, Y), \quad (12)$$

where $P^*(W, Y)$ is the value for a financially unconstrained firm and is given by (4).

Type-2 financially unconstrained firm: $W \geq \Lambda$.

The firm is able to implement the first-best abandonment option policy and avoid costly external financing permanently with probability one provided that the firm has sufficiently high liquidity hoarding. The question is how high the firm's liquidity hoard W has to be for a firm to be permanently financially unconstrained. As long as the firm can achieve its first-best policy under all circumstances, it is financially unconstrained. In the mature phase, as long as the corporate saving rate ($rW + Y - Z$) is weakly positive when evaluated at its first-best abandonment threshold Y_a^* , then the firm will never be involuntarily liquidated. The condition $rW + Y_a^* - Z \geq 0$ is equivalent to

$$W \geq \Lambda, \quad (13)$$

where we obtain the following explicit formula for Λ :

$$\Lambda = \frac{Z - Y_a^*}{r} = \frac{r - \gamma\mu}{r^2(1 - \gamma)}Z, \quad (14)$$

by substituting the explicit formula (7) for the abandonment hurdle Y_a^* into (13). In summary, as long as condition (13) holds, the firm is permanently financially unconstrained, and hence the firm's value $P(W, Y)$ equals the first-best firm's value:

$$P(W, Y) = P^*(W, Y), \quad \text{for } W \geq \Lambda. \quad (15)$$

Note that the firm can finance its efficient continuation entirely out of its cash hoard in the first-best continuation region, $Y \geq Y_a^*$. And for $Y < Y_a^*$, the firm voluntarily and efficiently abandons its asset and distributes W to shareholders.

In the macroeconomics savings literature following Aiyagari (1994), a core concept is the *natural borrowing limit*, which refers to the maximal amount of risk-free credit that a consumer can tap with no probability of ever defaulting. Hence, the consumer can borrow at the risk-free rate up to that *natural borrowing limit*, but any additional amount of borrowing will give rise to default risk. Here is the analogy. For a firm, Λ given in (14) is the minimum of the liquidity hoard that it needs in order to implement its first-best abandonment strategy. Any liquidity hoard lower than Λ may induce under-investment via inefficient liquidation in the future with positive probability.

Next, we turn to the “interior” region where the firm hoards liquidity and operates its asset in place. In this region, the firm is financially constrained but does not raise any external financing as the firm strictly prefers deferring equity issuance decision to the future.

4.2 The Interior Liquidity Hoarding Region

Using the standard principle of optimality, we characterize the firm’s value $P(W, Y)$ as the solution to the following Hamilton-Jacobi-Bellman (HJB) equation:

$$rP(W, Y) = (rW + Y - Z)P_W(W, Y) + \mu Y P_Y(W, Y) + \frac{\sigma^2 Y^2 P_{YY}(W, Y)}{2}, \quad (16)$$

subject to various boundary conditions to be discussed later. The first term on the right side of (16), given by the product of the firm’s marginal value of cash $P_W(W, Y)$ and the firm’s saving rate $(rW + Y - Z)$, represents the effect of the firm’s savings on its value. The second term (the P_Y term) represents the marginal effect of expected earnings change μY on firm value, and the last term (the P_{YY} term) encapsulates the effects of the volatility of changes in earnings Y on firm value. Intuitively, the expected change of firm value $P(W, Y)$, given by the right side of (16), equals $rP(W, Y)$, as the firm’s expected return is r .

With financial frictions, liquidity generally is more valuable than its pure monetary value. Typically, the firm’s decision of whether to abandon the project or not is influenced not only by its fundamentals (e.g., earning Y) but also by its financial considerations including the prospect of having to incur external financing costs in the future. All else equal, the costs of external financing ought to be an additional inducement to abandon a project. Therefore, one would expect that the prospect of having to incur external financing costs would lower the firm’s valuation for its asset in place and result in a higher abandonment hurdle.

Now consider the situation where a financially constrained firm is just indifferent between abandoning the firm or not. At the moment of indifference, firm value must be continuous,

$$P(W, \underline{Y}(W)) = W, \quad (17)$$

which states the the firm's value equals to its liquidity hoard W at the moment of abandoning its asset. Equation(17) implicitly defines the abandonment hurdle $\underline{Y}(W)$. Moreover, since the abandonment decision is optimally made, the marginal values along the earning Y dimension shall be matched before and after the abandonment of the asset in place,

$$P_Y(W, \underline{Y}(W)) = 0. \quad (18)$$

The smooth-pasting condition (18) states that at the optimal abandonment hurdle $\underline{Y}(W)$, P_Y equals zero.³ Next, we turn to the equity issuance region.

4.3 The Equity Issuance Region

As Bolton, Chen, and Wang (2011) show, in a world with constant financing opportunities where financing terms do not change over time, the firm has no need to issue equity unless it absolutely has to. By delaying equity issuance, the firm saves the time value of money for the financing costs. In the mature phase, with positive liquidity hoarding W , the firm has sufficient slack to cover any operating losses over a given small time interval. Therefore, the firm never issues equity before it exhausts its cash.

When the firm runs out of its cash ($W = 0$), it finds itself in one of the three regions. First, when $Y > Z$, the firm is solvent even without savings as its internally generated cash flow Y covers its operating cost Z , and hence the firm needs no external financing. When the firm's internally generated revenue cannot fully cover its operating cost ($Y < Z$) and with no savings ($W = 0$), the firm has an option to either issue equity or to simply liquidate its asset in place altogether, whichever is in the interest of its shareholders. Intuitively, whether the firm issues equity or liquidates itself depends on how valuable the firm's going concern value is, i.e. how high the revenue Y is.

With a fixed equity issuance cost $\phi_1 > 0$, the firm optimally chooses the amount of financing F and the endogenous hurdle $\underline{Y}(0)$ to satisfy the following value-matching boundary conditions at $W = 0$:

$$P(0, Y) = \begin{cases} P(F, Y) - F - \Phi(F), & \underline{Y}(0) < Y < Z, \\ 0, & Y \leq \underline{Y}(0). \end{cases} \quad (19)$$

³As a result, the firm's marginal value of liquidity $P_W(W, \underline{Y}(W))$ evaluated at the optimal abandonment hurdle $\underline{Y}(W)$ equals one.

The endogenous hurdle $\underline{Y}(0)$ given in condition (19) provides the boundary between the equity-issuance region and the liquidation region, $Y \leq \underline{Y}(0)$. The first case in (19) corresponds to the equity issuance region, $\underline{Y}(0) < Y < Z$. Let $F_a(Y)$ denote the external financing amount F as a function of earning Y (recall that financing only occurs when $W = 0$). And the second case in (19) corresponds to the liquidation region, $Y \leq \underline{Y}(0)$. By the firm's revealed preference, the dividing boundary $\underline{Y}(0)$ shall be higher than the first-best abandonment hurdle Y_a^* , a form of under-investment.

Should the firm seek to raise new funds, its optimal external financing amount F is given by the first-order condition (FOC)⁴ for the case with $\phi_0 > 0$:

$$P_W(F, Y) = 1 + \Phi'(F) = 1 + \phi_1, \quad \underline{Y}(0) < Y < Z. \quad (20)$$

Intuitively, conditional on the firm's issuance decision at $W = 0$, the marginal value of cash $P_W(F, Y)$ equals the marginal cost of financing $1 + \Phi'(F)$. Note that the firm's marginal value of cash $P_W(W, Y)$ depends on its revenue Y , and $P_W(W, Y)$ is greater than 1 at the moment of financing.

In each of the previous subsections, we have discussed the firm's optimal liquidation decision. The liquidation region is simply defined by the region where $Y < \underline{Y}(W)$.

Summary. In a dynamic environment with financing costs, the condition for a firm to be financially unconstrained is much tighter than in a static setting. The reason is as follows. For a firm to be financially unconstrained in a dynamic setting, the firm cannot have demand for external funds at any moment with strictly positive probability because external funding is costly and distorts corporate decisions. In our model, the firm can be financially unconstrained in one of the two ways; it either internally generates enough amount of funds at all times or it has sufficient liquidity to cover all the potential needs.

We next turn to the firm's problem in the start-up phase. Anticipating its potential financial constraints in the mature phase, a rational forward-looking firm acts accordingly in the start-up phase. This dynamic consideration has important implications on corporate financial and investment policies in the start-up phase.

5 Investment and Financing in the Start-up Phase

In the start-up phase, the firm maximizes its present value by solving the optimal investment timing problem. Let $G(W, Y)$ denote this value. When choosing its optimal investment

⁴We verify the second-order condition (SOC) to ensure that the FOC solution yields the maximal value.

timing, the firm incorporates both the one-time lumpy investment cost I and also its future operating (flow) cost Z . Before analyzing the effect of financial constraints on the firm's investment option exercising and financing decisions, we first reason how much liquidity the firm needs in order to be financially unconstrained.

5.1 The Financially Unconstrained Region: $W \geq I + \Lambda$

For a firm to be dynamically financially unconstrained, it should implement the first-best investment and abandonment decisions under all circumstances. When $\Lambda + I$, with probability one, the firm can cover both its investment cost I and its future liquidity shortfall to continue an efficient operation of its asset with liquidity amount Λ . Therefore, the firm in its start-up phase is financially unconstrained if and only if $W \geq I + \Lambda$, and the firm's value $G(W, Y)$ achieves the first-best value:

$$G(W, Y) = G^*(W, Y) = H^*(Y) + W, \quad (21)$$

where $H^*(Y)$ is given by (11) and the first-best investment hurdle Y_i^* is given by (9). Recall that $H^*(Y)$ and Y_i^* are independent of liquidity W .

When $W < I + \Lambda$, the firm is financially constrained. There are two sub-cases:

- When $I \leq W < I + \Lambda$, the firm has a sufficient liquidity hoard to fund the investment outlay I entirely out of its internal funds, but may not have sufficient funds to avoid the liquidity shortage in the mature phase and hence equity issuance or involuntary liquidation may occur with positive probability in the mature phase.
- When $W < I$, the firm cannot cover its investment cost I , and thus may require external financing to cover both the investment cost and its future liquidity need in the mature phase.

Note that a financially constrained firm has an option to building up its financial slack internally. The tradeoff between relying on internal funds and accessing external financing has significant implications on investment timing decision. More generally, the consideration of future liquidity needs and the financial flexibility (via financing timing, sources, and amount) make the financially constrained firm's decision rich and far from obvious.

5.2 The Medium Cash-holding Region: $I \leq W < I + \Lambda$

Consider now the situation of a firm with moderate financial slack. This firm has sufficient internal funds W to cover the investment cost I , but not quite enough cash to ensure that

it will never involuntarily liquidate its asset in the mature phase: $I \leq W < I + \Lambda$. In the forward-looking sense, the firm is still financially constrained and the marginal value of cash is greater than one. For such a firm it is optimal not to raise any external funds when it *chooses* to exercise its growth option, $F = 0$, in order to defer the financing cost. Note that the firm may not choose to exercise the growth option even when the earning Y reaches the first-best threshold Y^* and it has sufficient internal funds W to cover the investment cost I . The intuition is as follows. The firm realizes that exercising the investment option drains its cash holding by I and hence the firm may be led to raise external funds in the future prematurely to cover its potential operating losses in the mature phase. Therefore, the firm is still financially constrained, as it may have potential liquidity demand in the mature phase.

In the waiting region, the firm's value $G(W, Y)$ solves the following HJB equation:

$$rG(W, Y) = rWG_W(W, Y) + \mu Y G_Y(W, Y) + \frac{\sigma^2 Y^2}{2} G_{YY}(W, Y), \quad (22)$$

subject to various boundary conditions to be discussed later. Note that the first term on the right side of (22) reflects the firm's savings effect on firm value. The remaining two terms are the standard drift and volatility effects of Y on firm value.

At the endogenously chosen moment of investment, firm value $G(W, Y)$ is continuous and hence we have an implicit function for the investment threshold $\bar{Y}(W)$ defined by

$$G(W, \bar{Y}(W)) = P(W - I, \bar{Y}(W)). \quad (23)$$

The value-matching condition (23) characterizes the investment hurdle as an implicit function of liquidity W , $\bar{Y}(W)$. In this region, the investment cost I is entirely financed out of internal funds, and hence liquidity W decreases by I , as seen on the right side of (23). Additionally, because the investment hurdle $\bar{Y}(W)$ is optimally chosen, we have the following smooth-pasting condition along the earning Y dimension:

$$G_Y(W, \bar{Y}(W)) = P_Y(W - I, \bar{Y}(W)). \quad (24)$$

Finally, when the absorbing state $Y = 0$ is reached, there is no investment opportunity and the only valuable asset of the firm is its cash W , and hence

$$G(W, 0) = W. \quad (25)$$

Next, we turn to the low cash-holding region where investment cannot be financed solely with internal funds, $W < I$.

5.3 The Low Cash-holding Region, $0 \leq W < I$

In the region where internal funds are insufficient to cover the investment cost I , the firm has to raise external financing should it decide to invest. Intuitively, no matter how large its current earning Y is, the firm has to access external capital markets if choosing to *immediately* invest, as the investment cost I is lumpy while the earning Y is a flow variable. Note that the post-financing/investment liquidity is $W + F - I$. At the moment of investing, the firm's value is continuous, and hence we have an implicit function $\bar{Y}(W)$ defined by

$$G(W, \bar{Y}(W)) = P(W + F - I, \bar{Y}(W)) - F - \Phi(F). \quad (26)$$

The right side of the value-matching condition (26) gives the firm's value after it issues net amount F and incurs a cost $\Phi(F)$. The left side of (26) is the firm's value before investing.

Of course, it is quite plausible that the firm may want to wait. In this waiting region, the firm's value $G(W, Y)$ also solves the HJB equation (22) for the same argument as the one used in the previous subsection (in the medium cash-holding region). In addition to the investment hurdle $\bar{Y}(W)$, the firm also needs to choose the net equity issue amount F to at least cover the needed financing for investment $I - W$.

Let $F_g(W)$ denote the amount of external financing by the firm as a function of W in the region $0 \leq W < I$. The minimal amount of issuance required so that the post-issuance liquidity is $I - W \geq 0$. Thus, under optimal external financing the following inequalities must hold:

$$P_W(W + F_g(W) - I, \bar{Y}(W)) \leq 1 + \Phi'(F_g(W)), \text{ and } F_g(W) \geq I - W. \quad (27)$$

That is, the firm will issue equity such that the marginal value of liquidity is weakly lower than the marginal cost of issuance. (Otherwise, the firm should continue to issue equity until (27) holds, due to the fixed cost ϕ_0 .) We write the optimality conditions in this way because it is possible that the constraint $F_g(W) \geq I - W$ may bind, in which case the firm chooses to rely solely on its ability to generate sufficient liquidity from operating earnings after it has invested in the productive asset. Note that in (27), we have inequalities as opposed to equalities to characterize the optimality due to the rich tradeoffs faced by the firm.

The firm's optimality implies that the marginal value of earning Y is continuous before and after the investment option is exercised. Therefore, we have the following smooth-pasting condition:

$$G_Y(W, \bar{Y}(W)) = P_Y(W + F_g(W) - I, \bar{Y}(W)). \quad (28)$$

Finally, $G(W, 0) = W$ as $Y = 0$ is an absorbing state.

6 Analysis

As is standard in the literature (e.g., Leland, 1994), we set the risk-free interest rate $r = 5\%$, the expected earnings growth rate $\mu = 0$, and the earnings growth volatility $\sigma = 15\%$. The investment cost is set at $I = 2$ and the operating cost is $Z = 1$. When applicable, the parameter values are annualized.⁵

The first-best liquidation hurdle is $Y_a^* = 0.625$ which implies that a financially unconstrained firm will continue as a going concern even when it incurs a loss of $Z - Y_a^* = 0.375$, 37.5% of the (flow) operating cost $Z = 1$. This indicates a significant option value for a firm to continue as a going concern under perfect capital markets (the MM world). In the startup phase, the firm exercises its growth option when its earnings Y reaches the first-best investment hurdle $Y_i^* = 1.544$. At the moment of exercising, the project value (including the option value of abandonment) is $Q^*(Y_i^*) = 12.54$. As the investment cost $I = 2$, at the moment of investment, the firm's value (netting the investment cost) is $H^*(Y_i^*) = 10.54$.

With sufficiently high cash holding W , the firm is financially unconstrained at all times with probability one. In our example, the minimal amount of liquidity needed for a firm in the mature phase to be permanently financially unconstrained is

$$\Lambda = \frac{r - \gamma\mu}{r^2(1 - \gamma)}Z = 7.50, \quad (29)$$

which is 7.5 times the operating cost $Z = 1$. In the startup phase, the minimal amount of liquidity needed for the firm to be permanently financially unconstrained is thus $\Lambda + I = 9.50$ which covers both the investment cost I and the liquidity needs in the mature phase.

For the external financing cost, we choose the marginal issuance cost $\phi_1 = 0.01$ motivated by the empirical analysis in Altinkilic and Hansen (2000). The fixed equity issuance cost induces lumpy issuance, which is empirically important. We thus focus on the parameter ϕ_0 in our comparative static analysis. For our baseline case, we choose $\phi_0 = 0.4$ which implies that the fixed equity issuance cost is about 4% of $H^*(Y_i^*) = 10.54$, the first-best (net) firm value at the moment of investment and also this value is broadly in line with the empirical estimate reported in Altinkilic and Hansen (2000). To highlight the impact of the fixed financing cost ϕ_0 , we thus consider three values: $\phi_0 = 0.01, 0.4, 2$. By varying ϕ_0 we analyze the impact of financial frictions on corporate investment and financing decisions.

⁵Note that the drift μ is under the risk-neutral measure (after the appropriate risk adjustments), therefore, choosing $\mu = 0$ is consistent with a strictly positive real growth rate under the physical (data-generating) measure. Also, our numerical analysis can be implemented with a positive (risk-neutral) drift μ . Our economic analysis do not depend on the specific choice of μ .

6.1 The Mature Phase

We define the firm's *enterprise value* as its total value in excess of cash,

$$Q(W, Y) = P(W, Y) - W. \quad (30)$$

Because cash is valuable beyond its face value for a financially constrained firm, the enterprise value also depends on W . Under the MM condition, the enterprise value is independent of the firm's cash holding, and we have $Q^*(Y) = P^*(W, Y) - W$, as given by (4).

The Liquidation Decision. Figure 1 plots the optimal liquidation hurdle $\underline{Y}(W)$ for a financially constrained firm. First note that the firm becomes permanently financially unconstrained when its cash holding reaches $\Lambda = 7.50$, which is 7.5 times the (annual) operating cost $Z = 1$. For a financially unconstrained firm, it is optimal to liquidate its asset when its earning Y falls below the first-best liquidation hurdle $Y_a^* = 0.625$. And importantly, the firm will never be forced into sub-optimally abandoning its asset due to the shortage of its liquidity W as long as $W \geq \Lambda = 7.50$. Quantitatively, Figure 1 shows that the firm with liquidity W larger than 1.5 is effectively dynamically financially unconstrained.

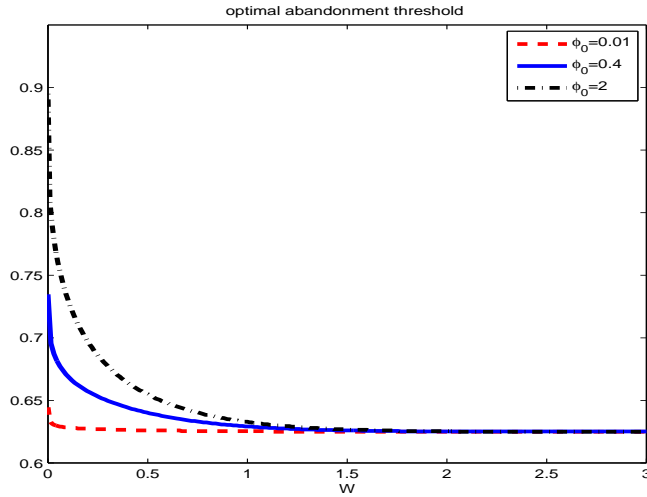


Figure 1: **The optimal liquidation hurdle $\underline{Y}(W)$ for a financially constrained firm in the mature phase.** The endogenous liquidation hurdle $\underline{Y}(W)$ is monotonically decreasing with liquidity holding W and approaches the first-best level $Y_a^* = 0.625$ independent of the financing cost ϕ_0 . For a given value of W , the larger the fixed equity issuance cost ϕ_0 , the higher the liquidation hurdle $\underline{Y}(W)$ indicating a higher degree of delayed investment timing decisions, a form of under-investment. At $W = 0$, $\underline{Y}(0)$ equals 0.645, 0.735, 0.895 for $\phi_0 = 0.01, 0.4, 2$, respectively.

When the firm exhausts its internal funds $W_\tau = 0$ and its cash flow is larger than its operating cost Z (i.e., $Y > Z = 1$), the firm is still solvent by purely relying on its internally generated cash flow. However, when its earning Y falls below its operating cost (i.e., $Y < 1$), it has to either raise external funds or will be liquidated otherwise. In our baseline case where $\phi_0 = 0.4$, if a constrained firm's earning Y lies in the region $1 > Y > \underline{Y}(0) = 0.735$, it is optimal to issue equity keeping the firm alive. Only when its earnings Y falls below 0.735, the firm will abandon its asset rather than issue equity to finance its liquidity shortfall. Recall that the first-best liquidation boundary is $Y_a^* = 0.625$. Hence, a firm with $W = 0$ will be inefficiently liquidated due to lack of internal funds in the region $0.625 < Y \leq 0.735$.

Figure 1 also illustrates that the liquidation hurdle $\underline{Y}(W)$ decreases with the firm's liquidity W in the region $[0, \Lambda] = [0, 7.50]$ for all three levels of ϕ_0 . Hence, inefficient liquidation occurs in the region $0 \leq W < \Lambda$, as $P_W > 1$ in this region and liquidity is valuable. For example, when $\phi_0 = 2$, the abandonment hurdle decreases from $\underline{Y}(0) = 0.895$ to $Y_a^* = 0.625$ as W increases from the origin to $\Lambda = 7.50$. Intuitively, the higher the liquidity holding W , the less inefficient the firm's liquidation decision. For the case with a small fixed equity issuance cost, $\phi_0 = 0.01$, the impact of financial constraints is negligible; the cash-less firm will only be abandoned inefficiently if its earning Y falls inside the tight region $0.625 < Y \leq \underline{Y}(0) = 0.645$. How important is the impact of financing costs on liquidation? At the origin $W = 0$, as we increase the fixed issuance cost ϕ_0 from 0.01 to 0.4 and then from 0.4 to 2, the abandonment hurdle $\underline{Y}(0)$ increases from 0.645 to 0.735 and then from 0.735 to 0.895, respectively. The implied real inefficiencies are significant. Finally, we note that quantitatively the effect of financial constraints essentially disappears as the firm's liquidity hoard W reaches 1.6, even when the fixed equity issuance cost is relatively high, $\phi_0 = 2$.

The Equity Issuance Decision. The firm will consider the possibility of issuing equity when it runs out of cash ($W = 0$) and its earning cannot cover its operating cost ($Y < Z$). First, it is always weakly preferable for the firm to postpone raising external funds whenever feasible, as doing so may avoid costly equity issue or at least saves the "time value" of equity issuance costs. Second, the firm will not issue equity if its earning Y falls below its first-best abandonment hurdle Y_a^* , as it must be optimal for a financially constrained firm to abandon its assets in place if it is optimal for a financially unconstrained firm to do so. Hence, in Figure 2, we only need to plot the optimal equity issuance amount $F_a(Y)$ (conditional on $W = 0$) as a function of its earning Y in the region $[Y_a^*, Z] = [0.625, 1]$ for the three cases, $\phi_0 = 0.01, 0.4, 2$.

Importantly, the amount of equity financing $F_a(Y)$ may be non-monotonic in Y . For the

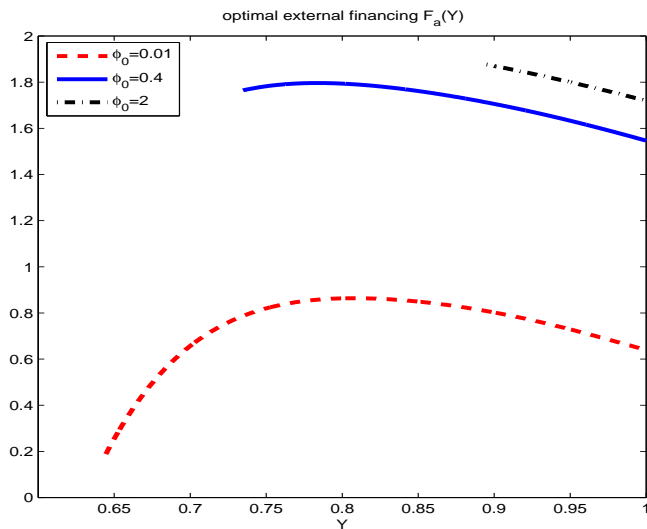


Figure 2: **The optimal external financing $F_a(Y)$ for a financially constrained firm in the mature phase.** Firms will choose to raise external funds only when it runs out of its liquidity and when its internally generated cash flow cannot cover the operating cost Z but is sufficiently high (i.e., only when $W = 0$ and $\underline{Y}(0) < Y < Z = 1$), where $\underline{Y}(0)$ is the optimal abandonment hurdle for a financially constrained firm. For $\phi_0 = 0.01, 0.4, 2$, we have shown that $\underline{Y}(0) = 0.645, 0.735, 0.895$, respectively. Interestingly, the firm's net financing amount $F_a(Y)$ is non-monotonic in its earning Y over the region $(\underline{Y}(0), 1)$. For example, for the case with $\phi_0 = 0.4$, the external financing $F_a(Y)$ first increases in the region $Y \in (0.735, 0.80)$, peaks at $Y = 0.80$ with a value of $F_a(0.80) = 1.80$, and then decreases with Y in the region $Y \in (0.80, 1)$.

baseline case with $\phi_0 = 0.4$, $F_a(Y)$ first increases with Y in the region $\underline{Y}(0) < Y < 0.80$ and then decreases with Y in the region $0.80 < Y < 1$. The net issuance amount F peaks at $Y = 0.80$ with a value of $F_a(0.80) = 1.80$. In the region $0.735 \leq Y < 0.80$, the firm's future prospects are not sufficiently encouraging for it to raise much external funding as the liquidation threat (in the future) is not low. This is the dominant consideration in this region, so that when Y increases, it is marginally worth raising more funds (paying the marginal cost $\phi_1 = 0.01$) given that the firm's survival likelihood is improving. In the region where $Y \in (0.80, 1)$ on the other hand, the dominant consideration when Y increases is the greater likelihood that the firm will be able to generate funds internally via its earning Y and interest income rW in the near future so that the firm does not need much external funds. As a result the firm optimally chooses to rely less on current external financing in the expectation of larger future internally generated funds. Intuitively, the firm's expectation about its own future ability to generate internal funds (e.g., from operations and interest incomes)

significantly influences the firm's current financing policy. This is an important difference between our paper and Bolton, Chen, and Wang (2011) where the firm's profitability and investment opportunity remains constant over time. As a result, in our model, both liquidity W and earning Y are state variables while only liquidity (scaled by firm size K) is the state variable in Bolton, Chen, and Wang (2011).

Additionally, conditional on choosing to raise funds, the firm raises more if the fixed costs of external funding ϕ_0 are higher. For example, at $Y = 0.9$, as we increase the fixed issuance cost ϕ_0 from 0.01 to 0.4 and then from 0.4 to 2, the firm's external financing $F_a(0.9)$ increases from 0.81 to 1.71, and then from 1.71 to 1.88. Intuitively, a firm that faces a larger fixed issuance cost ϕ_0 has a stronger incentive to issue more and hence capitalize on the fixed cost ϕ_0 as the cost of going back to the capital markets is greater in the future *ceteris paribus*. Importantly, this prediction is different from that based on the intuition from static models such as Froot, Scharfstein, and Stein (1993) and Kaplan and Zingales (1997). In these static models, the higher the financing cost, the more constrained the firm, and the lower the amount of equity financing, as the firm has no future financing considerations by assumption.

Figure 3 plots the firm's enterprise value $Q(W, Y)$ and its *marginal enterprise value of cash* $Q_W(W, Y)$ against liquidity W for two different levels of earning, $Y = 0.65$ and $Y = 1.5$, and for three fixed equity issuance costs, $\phi_0 = 0.01, 0.4, 2$. Intuitively, the higher the financing cost ϕ_0 , the lower the firm's enterprise value $Q(W, Y)$. Also note that the *net marginal value of cash* $Q_W(W, Y)$ is always positive implying that the firm is financially constrained and hence liquidity is valuable in the constrained region, i.e., $W < \Lambda = 7.50$.

The Firm's Value and the Marginal Value of Liquidity. A central observation emerging from Figure 3 is that the firm's marginal enterprise value of cash $Q_W(W, Y)$ can vary non-monotonically with its liquidity W . Panels A and B (with $Y = 0.65$) highlight the non-concavity of $Q(W, Y)$ in W . Specifically, $Q(W, Y)$ can be either concave or convex in liquidity W . First, consider the case with a small fixed equity issuance cost, $\phi_0 = 0.01$. Even with a low earning, e.g., $Y = 0.65$, the firm will not abandon its asset as the firm's abandonment hurdle $\underline{Y}(0) = 0.645$ and hence the firm will never liquidate. In this case, liquidity provides value by mitigating the firm's external funding needs. Hence, the firm's liquidity is valuable and firm value is concave in W with the marginal value $Q_W(W, Y)$ monotonically decreasing from 0.280 to zero as W increases from zero to $\Lambda = 7.50$.

However, as we increase the fixed cost ϕ_0 from 0.01 to 0.4, the marginal enterprise value of liquidity $Q_W(W, Y)$ is no longer concave. Indeed, $Q_W(W, Y)$ first equals zero for $W \leq 0.29$

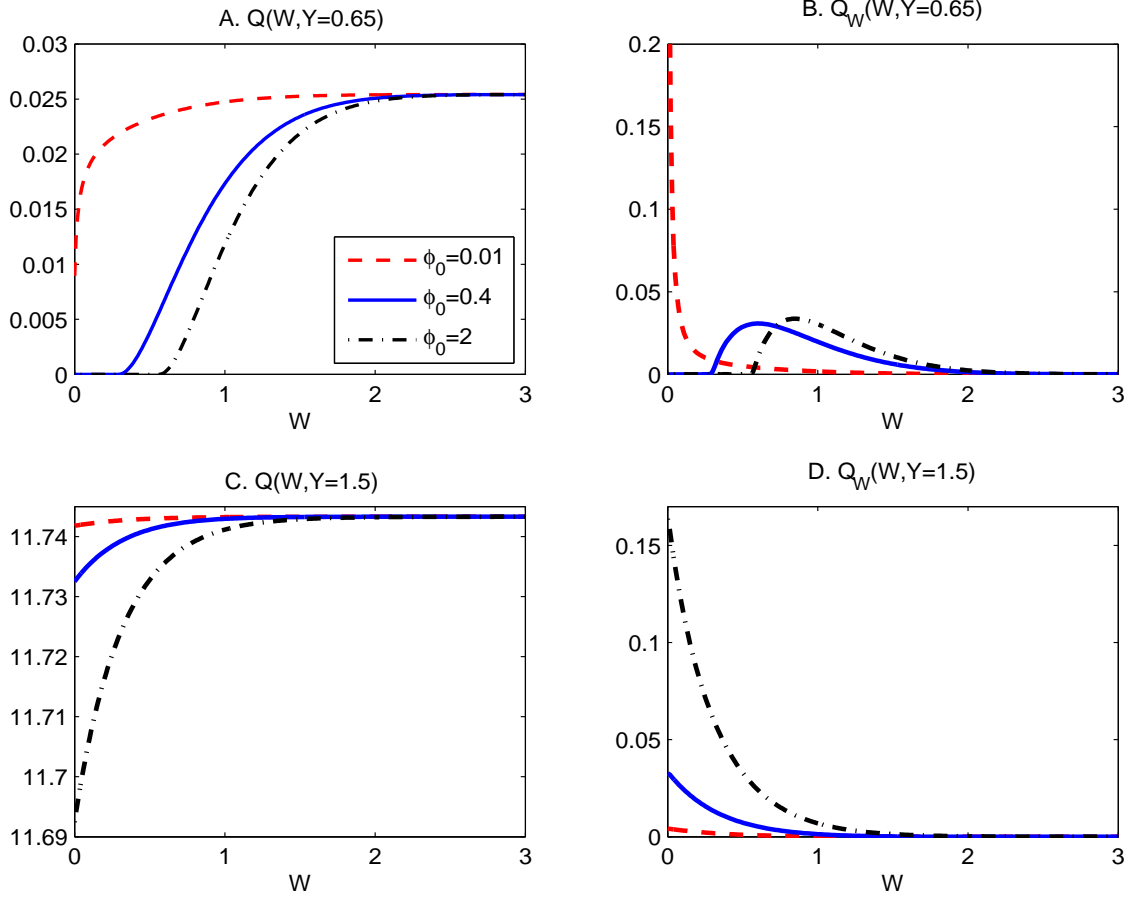


Figure 3: **Enterprise value $Q(W, Y)$ and the marginal enterprise value of cash $Q_W(W, Y)$ in the mature phase.** We plot $Q(W, Y)$ and $Q_W(W, Y)$ as functions of liquidity W for two earning levels ($Y = 0.65, 1.5$) and for three values of the fixed equity issuance cost, $\phi_0 = 0.01, 0.4, 2$.

and then increases for $0.29 < W < 0.6$ and finally decreases with W in the region with sufficiently high $W > 0.6$. The intuition is as follows. A firm with a low earning $Y = 0.65$ optimally chooses to exercise its abandonment option provided that its liquidity $W \leq 0.29$, as the abandonment hurdle $\underline{Y}(0.29) = 0.65$. Hence, in the low liquidity region $0.29 < W < 0.6$, increasing W lowers the firm's likelihood of tapping costly external financing, but the firm still has a low probability to survive. In this case, the firm may be endogenously risk-loving with respect to exercising its equity issuance option. This can only be the case if the firm faces strictly positive fixed costs of equity issuance ($\phi_0 > 0$). And hence Q_W increases with W in this low-liquidity region implying that firm value is *locally convex* in W in this region. Finally, when $W > 0.6$, the firm has sufficiently high liquidity and equity issuance becomes

much less likely causing the marginal enterprise value of liquidity Q_W to decrease as the firm becomes less financially constrained. We also see that with a higher ϕ_0 (an increase of ϕ_0 from 0.4 to 2) shifts $Q_W(W, Y)$ as a function of W to the right while retaining the general non-monotonic shape. This makes sense as a firm facing a larger external equity financing cost ϕ_0 chooses a higher abandonment hurdle $\underline{Y}(W)$.

Now, we consider the case with a relative high earning, e.g., $Y = 1.5$, the firm's enterprise value $Q(W, Y)$ is now concave in W , as its earning is significantly larger than the abandonment hurdle $\underline{Y}(W)$ and the financing option is sufficiently deeply in the money. In these cases, the firm behaves as it it were risk averse ($P_{WW} < 0$) with respect to liquidity W .

6.2 The Start-up Phase

As for the mature phase, we again define the firm's *enterprise value* in the *start-up* phase as

$$H(W, Y) = G(W, Y) - W. \quad (31)$$

For a financially unconstrained firm, its enterprise value is independent of its cash holding, i.e., $H^*(Y) = G^*(W, Y) - W$, and the closed-form solution for $H^*(Y)$ is given by (11). In general, a financially constrained firm's enterprise value depends on its cash holding W and is lower than the first-best value, $H(W, Y) \leq H^*(Y)$ due to investment timing distortions.

The Investment Timing Decision. Panels A and B of Figure 4 plot a financially constrained firm's optimal investment hurdle $\bar{Y}(W)$ for three values of the fixed cost, $\phi_0 = 0.01, 0.4, 2$ in the region $0 \leq W < I = 2$ and the region $2 \leq W$, respectively. Recall that in the start-up phase, a firm is financially unconstrained if its liquidity holding is greater than $I + \Lambda = 9.50$. We plot W only up to $W = 3$ in Panel B because quantitatively the investment threshold effectively converges as $W \rightarrow 3$. But in theory, a financially unconstrained firm (with $W \geq 9.5$) will only pay the cost of $I = 2$ to exercise the investment option if and only if the earning Y exceeds the endogenously chosen first-best time-invariant investment hurdle $Y_i^* = 1.544$. First, consider the region where $2 \leq W < 9.5$. In this medium cash-holding region, the firm's internal funds W are sufficient to cover its investment cost $I = 2$ and hence there will be no equity issuance decision. The higher the liquidity W , the less distorted investment timing decision. Panel B of Figure 4 shows the decreasing function of $\bar{Y}(W)$ against liquidity W . For the case with $\phi = 2$, the investment hurdle decreases from $\bar{Y}(2) = 1.555$ to approximately 1.544 as W increases from 2 to 3. Note that the quantitative effect of the financing friction in this medium cash-holding region is not large.

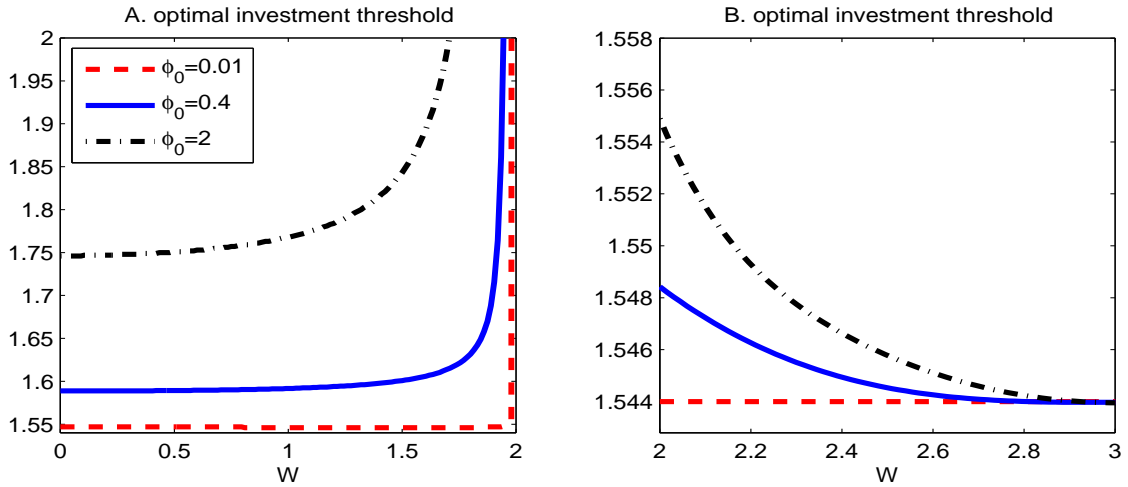


Figure 4: **The optimal investment hurdle $\bar{Y}(W)$ as a function of pre-investment liquidity W .**

In contrast, in the low cash-holding region $W \in [0, I)$, the quantitative effect of financing costs on investment is much greater. Panel A of Figure 4 plots the investment hurdle $\bar{Y}(W)$ as a function of W in the region $[0, 2)$. Consider first the case with $\phi_0 = 0.01$. In this case, $\bar{Y}(W)$ is non-monotonic in W ; it first decreases slightly from 1.547 to 1.545 as W increases from $W = 0$ to $W = 1.85$. This is intuitive: as W increases the firm can rely on a greater and greater portion of internal funds, thus lowering its overall cost of investment, so that it is willing to take on investments with lower and lower earning Y mitigating distortions due to delayed investment timing decision. But eventually \bar{W} rises with W , and *asymptotes* to ∞ , as $W \rightarrow I = 2$. Thus, far from behaving more and more like an unconstrained firm as $W \rightarrow I$, it behaves like a more and more constrained firm. What is the logic behind this behavior? In essence, as $W \rightarrow I$ the opportunity cost to the firm of entirely avoiding the fixed external financing cost $\phi_0 = 2$ by waiting until it has accumulated sufficient internal funds to pay for the investment outlay I becomes smaller and smaller. It therefore takes a larger and larger earning shock Y to get the firm to invest right away and incur the cost $\phi_0 = 2$ rather than wait a little, accumulate W at the interest rate r until $W \geq I$, and avoid this cost entirely. This is an example where dynamic considerations (via internal liquidity accumulation) significantly changes how financial constraints impact corporate real investment decisions.

The Financing Decision. Figure 5 plots the firm's post-investment liquidity at time τ_{i+} , $W_{\tau_{i+}} = W_{\tau_i} + F_g(W_{\tau_i}) - I$, implied by the financing amount $F_g(W)$ at τ_i , the moment of exercising the growth option, in the region $W < I = 2$. As W increases, the firm becomes increasingly less willing to issue equity to finance investment as the alternative of simply waiting for its internal funds to build up to cover the investment cost I is getting cheaper. As a result, the firm's investment hurdle increases significantly, approaching ∞ and hence internally generated cash flows (over time) can easily finance the firm's future liquidity demand as $W \rightarrow I$, as we discussed earlier. Therefore, the amount of post-investment liquidity $W_{\tau_{i+}}$ needed ever to cover future operating losses decreases with the firm's current liquidity W . Consider the baseline case with $\phi_0 = 0.4$. As we increase the firm's liquidity just

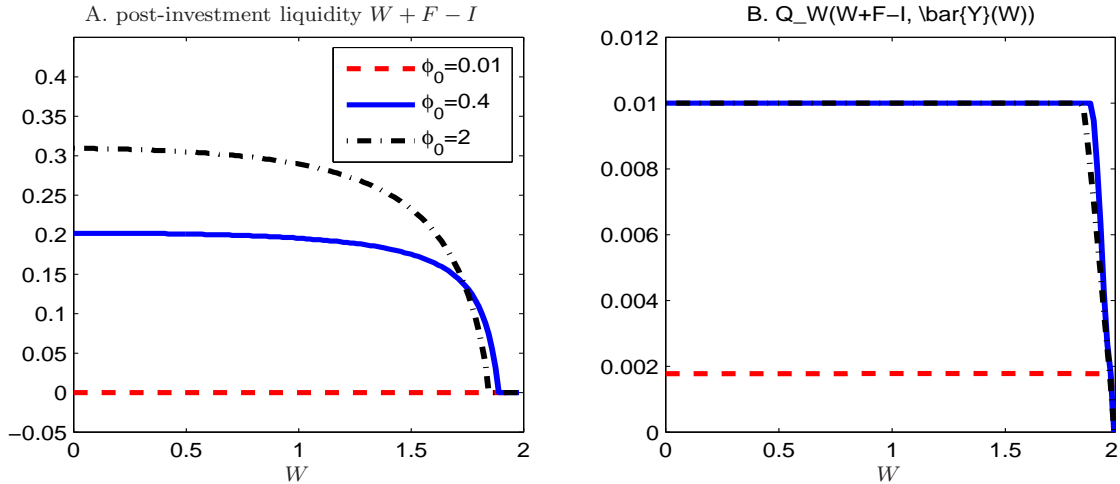


Figure 5: **Post-investment liquidity** $W_{\tau_{i+}} = W_{\tau_i} + F_g(W_{\tau_i}) - I$. Note that the post-investment liquidity decreases with the firm's current liquidity W as the firm's marginal valuation of liquidity $Q_W(W + F - I, \bar{Y}(W))$ decreases with W .

prior to investment, W_{τ_i} , from 0 to 1.89, the firm's post-investment liquidity $W_{\tau_{i+}}$ decreases from 0.202 to 0. For $W < 1.89$, should the firm decide to issue equity, it issues some more funds than needed to purely finance investment so that its post-investment liquidity is positive, e.g., $W_{\tau_{i+}} = 0.196$ when $W_{\tau_i} = 1$. For lower values of ϕ_0 , e.g., $\phi_0 = 0.01$, the post-investment liquidity is essentially zero for all levels of W , as the firm essentially has no need to hoard more liquidity than needed to cover the financing gap $I - W$.

Panel B shows that the marginal enterprise value of cash at the moment of investing, $Q_W(W_{\tau_{i+}}, \bar{Y}(W_{\tau_i}))$, is weakly lower than the proportional financing cost $\phi_1 = 0.01$. When $Q_W(W_{\tau_{i+}}, \bar{Y}(W_{\tau_i})) = \phi_1 = 0.01$, the firm's post-investment liquidity $W_{\tau_{i+}}$ is pos-

itive, as it wants to hoard liquidity to cover the operating cost in the future. When $Q_W(W_{\tau_i+}, \bar{Y}(W_{\tau_i})) < 0.01$, the firm's net marginal value of liquidity Q_W is less than its marginal issue cost $\phi_1 = 0.01$. Hence, the firm will only issue the required $F_g(W_{\tau_i}) = I - W_{\tau_i}$ to cover the investment cost $I = 2$ and defer the remaining potential equity issue for future liquidity needs to the future, and hence the post-investment liquidity $W_{\tau_i+} = 0$.

The Value of the Start-up Firm. Figure 6 plots the firm's *enterprise value* $H(W, Y)$ in the start-up phase and the corresponding marginal value $H_W(W, Y)$ for $Y = 0.65, 1.5$ and the same three fixed cost of issuing equity ($\phi_0 = 0.01, 0.4, 2$). With a small fixed financing

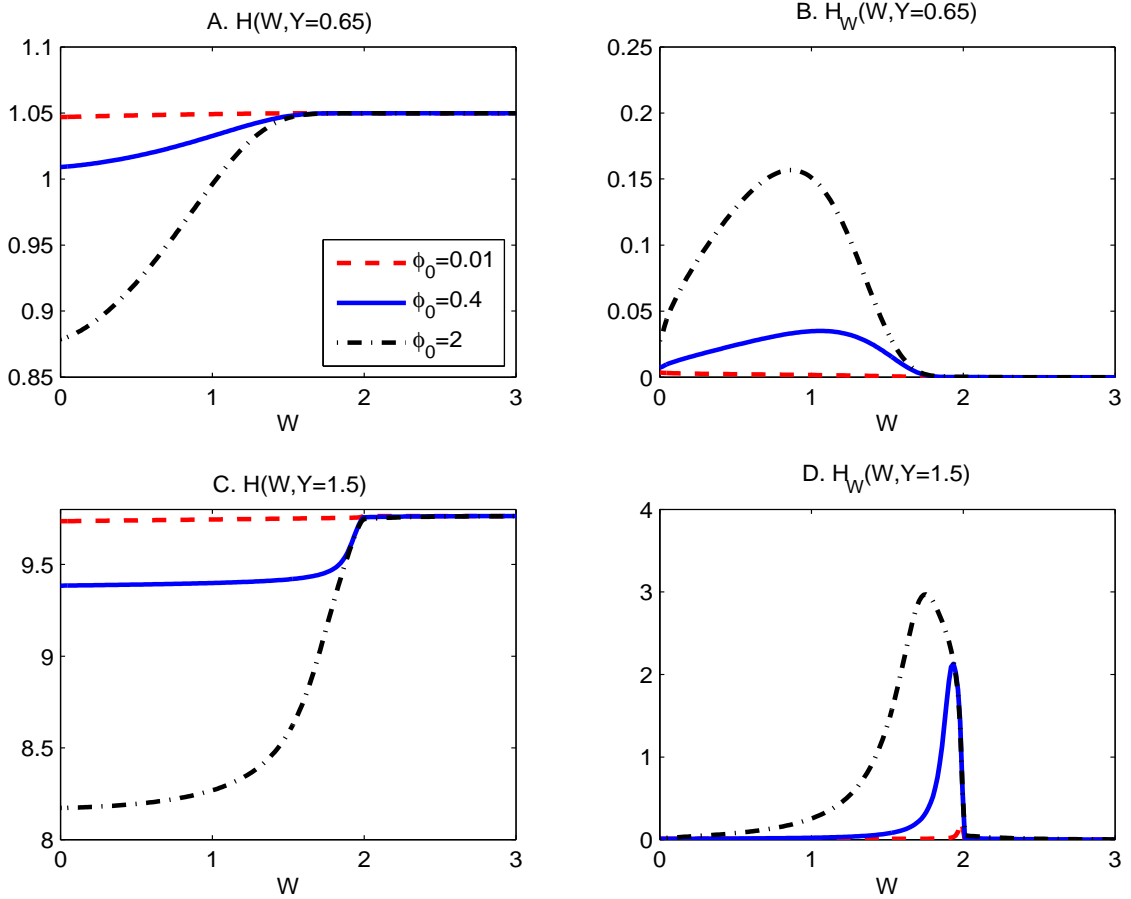


Figure 6: The firm's enterprise value in the startup phase, $H(W, Y)$, and the marginal value of liquidity $H_W(W, Y)$ for three levels of fixed costs $\phi_0 = 0.01, 0.4, 2$.

cost, $\phi_0 = 0.01$, $H(W, Y)$ is increasing and concave in W as the firm is financially constrained and needs cash in order to invest. In contrast, with fixed financing costs, for both $\phi_0 = 0.4$ and $\phi_0 = 2$ cases, the (net) marginal value of cash $H_W(W, Y)$ first increases with W and

then decreases with W as W becomes sufficiently large. For example, consider the case with $\phi_0 = 0.4$ and $Y = 0.65$, $H_W(W, 0.65)$ is upward sloping in W and reaches its highest value 0.035 around $W = 1.05$, and then declines as W decreases, effectively approaching 0 as W exceeds 1.85.

Why is the firm's growth option value $H(W, Y)$ convex in W in the region $W < 1.05$ for $Y = 0.65$? Intuitively, if the firm was able to take a mean-preserving spread with W in that region, it would be better off, as either it accumulates internal funds faster or upon incurring losses, the firm gets closer to issue external equity to finance the exercising of its investment option. This additional benefit ameliorates the costly delay of the investment exercising decision.

In contrast, in the region where $W > 1.05$, the financially constrained firm has sufficiently high liquidity and hence has much weaker incentives to tap external financing and prefers waiting until the time when it has accumulated sufficient internal funds to finance investment. Therefore, the marginal value of cash $H_W(W, Y)$ decreases with W in that region.

In summary, the firm's flexible financing options and its real (investment and abandonment) options jointly cause its value to be concave and/or convex in W depending on its financing cost ϕ_0 , liquidity holding W , and revenue Y highlighting the rich and subtle interaction between the firm's financial and real flexibility. Ignoring financial flexibility when analyzing the firm's real option exercising decisions may generate misleading insights and also overstates the value of real options. Also purely focusing on financial frictions and ignoring the firm's real flexibility may lead to a narrow interpretation of a financially constrained firm's precautionary saving motive.

7 Simulation Illustration

To provide further insight into the dynamics of cash balances, financing, investment, and abandonment decisions in our model, we simulate one sample path with two different initial values for the firm's cash holdings under our baseline parameter values. The implied time-series of corporate decisions are illustrated in Figures 7 and 8 for the two cases, the first with $W_0 = 1.8$ and the second with $W_0 = 0.5$. Table 1 reports the corporate decisions and calculates the implied equity ownership dynamics that follow from the firm's decisions over the simulated path used in Figures 7 and 8.

In the MM world where there are no external financing costs, the optimal investment time for this sample path is $t = 2.6$, just when the firm's earnings Y_t reach 1.544, and the abandonment time is $t = 29.4$, just as the firm's earnings Y_t hit the low level of 0.625. Thus,

Table 1: **Equity ownership dynamics and (investment, equity issue, and abandonment) timing decisions for a simulated path.** The parameter values are $r = 5\%$, $\mu = 0$, $\sigma = 15\%$, $I = 2$ and $Z = 1$. For the given simulated path with $Y_0 = 1$, we consider two cases with $W_0 = 1.8$ and $W_0 = 0.5$. Under the first-best case, the optimal investment time is $t = 2.6$ and the abandonment time is $t = 29.4$.

time t	firm's earning Y_t	net equity issue F_t	gross equity issue $F_t + \Phi(F_t)$	% of equity issue ω_{t+}	original investors' ownership α_{t+} (%)
Panel A: $W_0 = 1.8$					
0	1	0	0	0	100%
2.9	1.546	0	0	0	100%
7.5	0.746	1.79	2.21	94.1%	5.9%
19.8	0.656	0	0	0	liquidation
Panel B: $W_0 = 0.5$					
0	1	0	0	0	100%
3.7	1.592	1.60	2.02	14.7%	85.3%
6.3	0.789	1.80	2.22	83.8%	13.8%
18.6	0.887	1.73	2.14	58.9%	5.7%
28.6	0.692	0	0	0	liquidation

the firm is willing to fund losses of up to 37.5% of operating costs $Z = 1$ under this scenario to maximize the value of its abandonment option.

Recall that when the firm incurs an external financing cost $\Phi(F_t)$ as it raises equity F_t at time t , the firm's post-issuance value is $P(W_{t+}, Y_{t+})$. Let ω_{t+} denote the equilibrium fraction of the newly issued equity held by outside investors. By competitive market pricing, we have:

$$\omega_{t+} = \frac{F_t + \Phi(F_t)}{P(W_{t+}, Y_{t+})} = 1 - \frac{P(W_t, Y_t)}{P(W_{t+}, Y_{t+})}, \quad (32)$$

as the new investors just break even under perfectly competitive capital markets. Under the simulated sample-path we can highlight the dynamics of *equity dilution* by keeping track of the equity ownership of the original investors who have stayed with the firm since its inception. We denote by α_t the ownership share of the original equity holders at time t , with $\alpha_0 = 1$ by construction. As the firm issues equity to finance investment and/or replenish liquidity over time, the original equity investors' ownership then evolves as follows:

$$\alpha_{t+} = \alpha_t (1 - \omega_{t+}). \quad (33)$$

In other words, with no issuance $\omega_{t+} = 0$ and $\alpha_{t+} = \alpha_t$, so that α does not change. But when new equity is issued at time t , with a strictly positive ownership stake for new investors of $\omega_{t+} > 0$, the original equity investors' equity is diluted to α_{t+} from α_t according to (33).

Figure 7 plots the scenario where the firm starts with a high cash stock of $W_0 = 1.80$. Panel A plots the path of earnings Y_t starting with $Y_0 = 1$. Note first that when the firm faces external financing costs it only exercises its investment option at time $t = 2.9$ when Y_t reaches 1.546 (compared with $t = 2.6$ when the firm faces no external financing costs). It then pays the investment cost $I = 2$ solely out of internal funds, thus depleting its stock of cash $W_{2.9+}$ down to 0.08, as illustrated in Panel B. In this case, obviously there is no equity dilution and $\alpha_{2.9} = 1$. Given that the firm already starts out with a relatively high cash stock of $W_0 = 1.8$ it is optimal for the firm to wait until its internal funds have accumulated sufficiently to be able to finance its investment cost entirely out of internal funds, thus deferring costly external financing.

Next, at $t = 7.5$, when the firm's earning $Y_{7.5}$ falls to 0.746 and it has burned through its internal funds, the firm issues net equity of $F(0.746) = 1.79$ to replenish its liquidity from $W_{7.5} = 0$ to $W_{7.5+} = 1.79$, by selling $\omega_{7.5+} = 94.1\%$ of its equity. At that point the firm's original owners are nearly wiped out and only retain a stake of $\alpha_{7.5+} = 5.9\%$.

Finally, at $t = 19.8$, when the firm runs out cash for the second time and Y_t reaches 0.656, it simply abandons its asset and distributes the remaining cash of 0.07 to its shareholders.

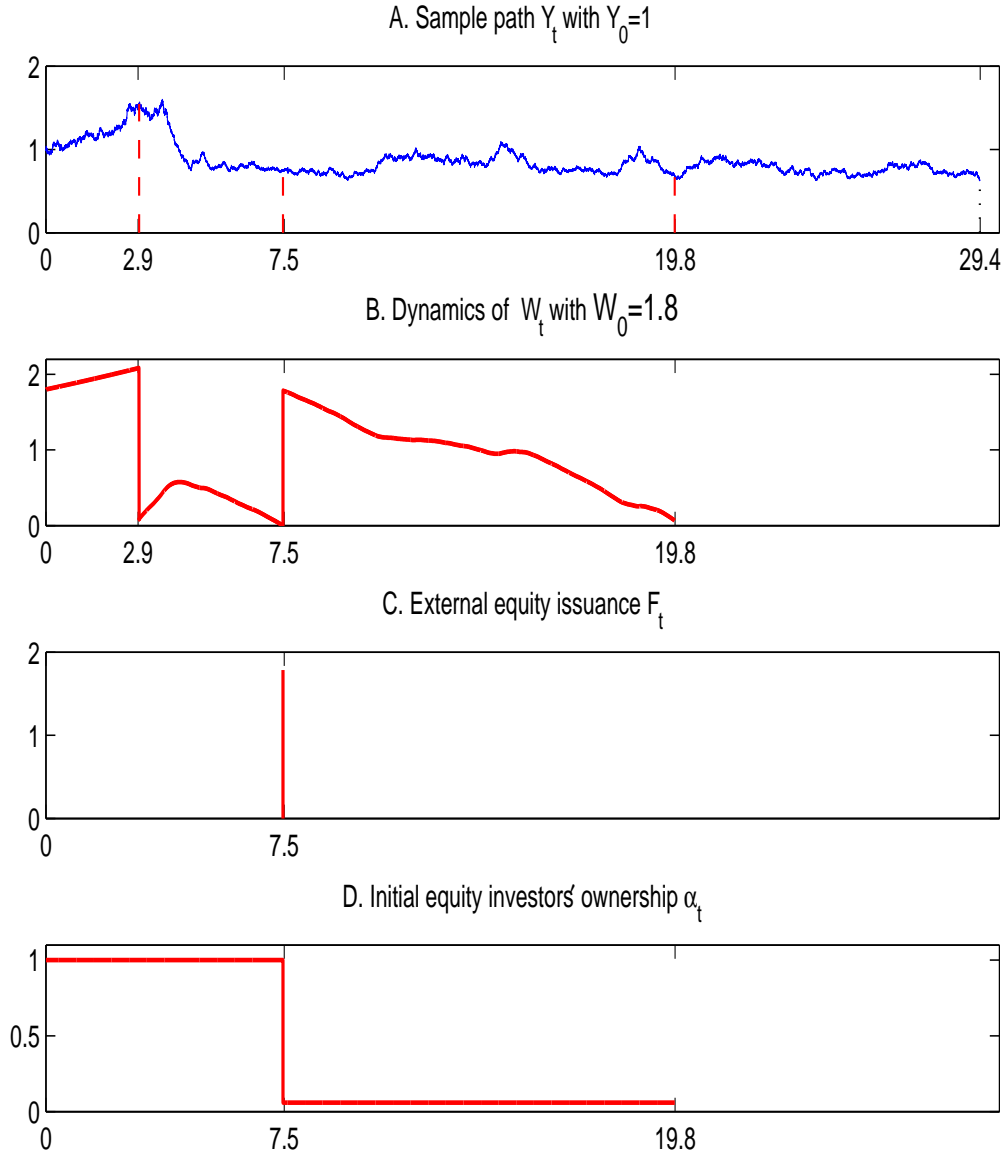


Figure 7: **Investment, liquidity accumulation, external equity issue, ownership dynamics, and liquidation for a simulated path.** The firm's initial cash holding is $W_0 = 1.80$. Panel A plots a simulated path of earning Y with $Y_0 = 1$. At $t = 2.9$, as Y reaches 1.546, the firm exercises its investment option by paying the investment cost $I = 2$ solely with internal funds leaving the post-investment liquidity 0.08 in the firm (Panel B). At $t = 7.5$, as the firm's earning falls to 0.746 and its liquidity drains out, the firm issues equity 1.80 in net to replenish liquidity W from zero to 1.80 by diluting the firm's original equity investors' ownership to 5.9%. Finally, at $t = 19.8$, as the firm runs out cash again and Y reaches 0.656, it simply abandons its asset and distributes the remaining cash to shareholders as the cost of using costly external financing outweighs the benefit of keeping the firm as a going concern.

At this low point the cost of new external financing simply outweighs the benefit of keeping the firm as a going concern.

Figure 8 plots the scenario where the firm starts with a low cash stock of $W_0 = 0.5$. Panel A again plots the identical simulated path of earning Y_t starting with $Y_0 = 1$. As the firm's earning Y_t reaches 1.592 at $t = 3.7$, the firm finances its investment cost $I = 2$ via a combination of external equity ($F(1.592) = 1.60$) by issuing a fraction of $\alpha_{3.7+} = 14.7\%$ firm's equity and internal funds (0.40), leaving the firm with a stock of post-investment cash of $W_{3.7+} = 0.20$ (as shown in Panel B). As a result, the original owners are diluted down to an ownership stake of $\omega_{3.7+} = 85.3\%$ (as shown in Panel D).

At $t = 6.3$, when earning Y_t has collapsed to 0.789 and the firm's liquidity has been drained, the firm returns to the capital markets and raises a net amount of $F = 1.80$ by selling $\alpha_{6.3+} = 83.8\%$ of the firm's equity, thus further diluting the firm's original owners' down to $\alpha_{6.3+} = 13.8\%$ (see Panel D). Next, at $t = 18.6$, as the firm's liquidity W_t is again drained out, the firm yet again issues equity by selling $\omega_{18.6+} = 58.9\%$ of the firm's equity, raising a total net amount of $F = 1.73$ and diluting the original owners down to a small stake of $\alpha_{18.6+} = 5.7\%$. Finally, at $t = 28.6$ the firm almost runs out of cash again and abandons its asset given that expected operating earnings hit the low level of $Y_t = 0.692$; it then distributes the remaining cash 0.03 to its shareholders.

Comparing the two scenarios, we can make the following observations: First, in all cases where the firm issues equity for purposes of replenishing its liquidity it chooses different financing levels because each time it faces different expected operating earnings when it exhausts its liquidity. Second, the firm with $W_0 = 1.8$ turns out to abandon its asset sooner (at $t = 19.8$) than the firm with $W_0 = 0.5$ (at $t = 28.6$). This may at first appear surprising, as one may think that a firm with more liquidity should have less distorted abandonment decisions. The intuition for this seemingly counter-intuitive result is as follows. As it is optimal for the firm with $W_0 = 1.8$ to finance its investment cost $I = 2$ purely via internal funds, this firm ends up with lower post-investment liquidity and hence an earlier inefficient liquidation decision. Put differently, a less financially constrained firm could turn out to have more severe distortions ex post due to the endogenous adjustments along other margins (in this case, the financing decision for the investment option). This is yet another example illustrating how dynamic models can provide new insights for the design of empirical tests on the implications of financial constraints on firm behavior, which are unavailable in static models such as Froot, Scharfstein, and Stein (1993) and Kaplan and Zingales (1997), and which consequently may yield imprecise empirical hypotheses.

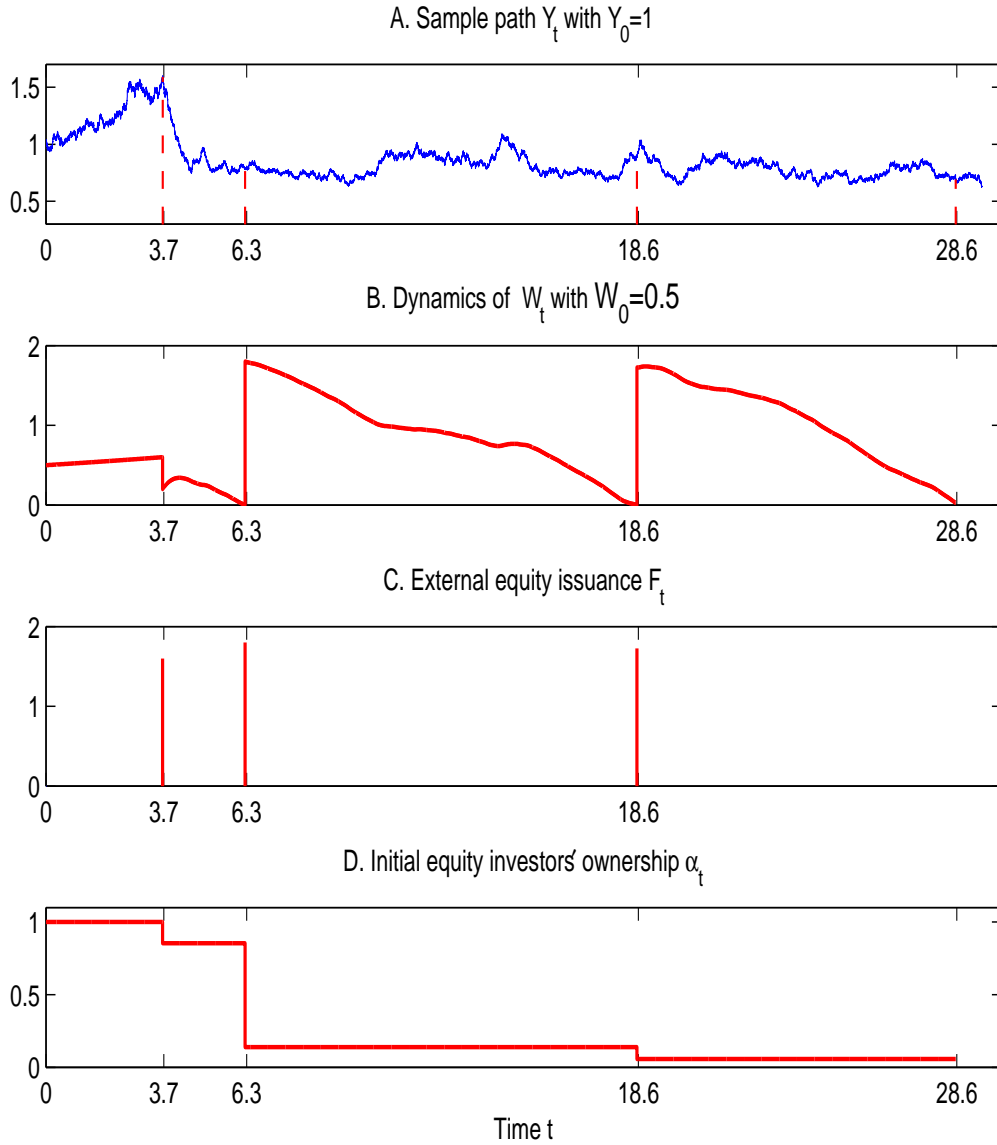


Figure 8: **Investment, liquidity accumulation, external equity issue, ownership dynamics, and liquidation for a simulated path.** The firm’s initial cash holding is $W_0 = 0.5$. Panel A plots the identical simulated path of earning Y with $Y_0 = 1$ as the one in Figure 7. At $t = 3.7$, as Y reaches 1.592, the firm raises equity $F = 1.60$ in net (Panel C), finances its investment cost $I = 2$ via a combination of external equity (1.60) and internal funds (0.40) leaving the firm with 0.20 liquidity post-investment (Panel B), and consequently dilutes the original equity investors’ ownership to 85.3% (Panel D). At $t = 6.3$, as Y falls to 0.789 and liquidity W drains out, the firm raises $F = 1.80$ in net increasing W from zero to 1.80, and consequently further dilutes the firm’s original equity investors’ ownership to 13.8% (Panel D). At $t = 18.6$, as Y again reaches 0.887 and its liquidity W drains out, the firm issues equity, again, $F = 1.73$ in net, further diluting the original equity investors’ ownership to 5.7%. Finally, at $t = 28.6$, as the firm runs out cash and $Y = 0.692$, it abandons its asset and distributes the remaining cash to shareholders.

8 Sequential Investment Opportunities

To capture how future investment opportunities may influence a firm's current investment and financing decisions, we next generalize our model to allow for two rounds of investment options. Accordingly, we refer to stage 0 as the *start-up phase* when the firm has not acquired any productive asset, stage 1 as the *growth phase* when the firm operates its first asset in place but still holds the second growth option, and stage 2 as the *mature phase* that starts after the second investment option has been exercised. As before, due to operating leverage, the firm may exercise its abandonment option in either the growth phase or the mature phase.

This extension allows us to demonstrate two novel effects. The first is the change in incentives to exercise its first investment option caused by the prospect of acquiring valuable operating cash-flows that help relax the internal funding constraint to finance the second growth option. The second is how incentives to exercise the later investment option and other corporate decisions are modified by operating cash-flow risk from the first productive asset.

A striking new result we obtain is that when the firm faces multiple rounds of growth options it may choose to invest in the earlier opportunities sooner than under the first-best benchmark. This is in sharp contrast with the standard predictions that financially constrained firms under-invest. Indeed, the firm in our model may optimally engage in a form of *over-investment* relative to the first-best as a way of building up its liquidity which helps mitigate the incentives to delay the second growth option. Accelerated timing of investment in the start-up phase may be in the firm's best interest if the cost of early investment in the initial productive asset outweighs the costs of under-investment in the later expansion phase.

Next, we provide a sketch of our model with two growth options. We denote by Z_n the firm's total operating cost (per unit of time) after exercising the n th growth option, where $n \in \{1, 2\}$. Correspondingly, we denote the stochastic revenue process generated by all existing productive assets by $x_n Y_t$, where Y_t is given in (1) and where $x_1 < x_2$ denotes the production capacity after exercising respectively the first and second investment option. That is, upon exercising the second growth option, the firm's operating revenue increases by $\Delta_x Y$ where $\Delta_x = x_2 - x_1$ and the operating cost increases by $\Delta_Z = Z_2 - Z_1$. Intuitively, we may interpret $\Delta_x Y - \Delta_Z$ as the firm's (additional) profit generated by the second asset in place.

We also denote by τ_{L_n} the abandonment time in phase n . In the growth phase, when the

firm exercises its abandonment option it releases its cash to shareholders and loses both its first asset in place and its second growth option. In the mature phase the firm distributes its cash to shareholders and liquidates all assets in place.

By backward induction, in the mature phase we essentially have the same firm value, the same dynamics for liquidity, and the same abandonment decision as in the baseline model when we replace the baseline cash flow with x_2Y and the baseline operating cost with Z_2 . The main changes to the analysis are thus in stages 0 and 1, to which we now turn. As for stage 1, the growth phase, the firm's liquidity W_t then accumulates as follows:

$$dW_t = (rW_t + x_1Y_t - Z_1)dt + dC_t, \quad W_t \geq 0, \quad (34)$$

where $\{C_t; t \geq 0\}$ denotes as before the uncoupled cumulative equity issue and dC is the net equity issued. Note that in stage 1, the revenue is x_1Y_t and the operating profit is $x_1Y_t - Z_1$.

Numerical Results. For the first growth option we set $x_1 = 1$, $Z_1 = 1$ and $I_1 = 2$, (the same as in the baseline model.) Additionally, we choose $x_2 = 3$, $Z_2 = 3$ and $I_2 = 7$, which implies that the additional asset in place (upon the exercising of the second growth option) generates $2Y - 2$ as the profit, twice as much as the first asset in place. However, the exercise cost for the second option $I_2 = 7$ is more than twice is the exercise cost $I_1 = 2$ for the first growth option. Hence, the second growth option is less valuable than the first one, a form of decreasing returns to scale. For all other parameters, we use the same values as those in the baseline model. With these parameter values, we obtain that the first-best abandonment and investment thresholds in stage 1 are given by $Y_{1,a}^* = 0.534$ and $Y_{1,i}^* = 1.586$ respectively, and the first-best abandonment and investment thresholds in stage 2 are $Y_{2,a}^* = 0.625$ and $Y_{2,i}^* = 1.678$, respectively.

For these parameter values we also obtain the following cutoffs for when the firm is forever financially unconstrained in respectively the growth and mature phases: $\Lambda_1 = 9.29$ and $\Lambda_2 = 22.5$. That is, when $W \geq \Lambda_2$ the firm is forever financially unconstrained in phase 2 (the mature phase). And in phase 1 (the growth phase) it is permanently financially unconstrained if $W \geq \max\{\Lambda_1, I_2 + \Lambda_2\} = 29.5$. Finally, in the stage 0 (the start-up phase) the firm is permanently financially unconstrained if and only if $W \geq I_1 + \max\{\Lambda_1, I_2 + \Lambda_2\} = 31.5$.

Figure 9 plots the optimal investment thresholds $\bar{Y}_1(W)$ and $\bar{Y}_2(W)$ for the first and second growth options against liquidity W , respectively. Panel A shows that $\bar{Y}_1(W)$ is non-monotonic in W in the region $W < I_1 = 2$, consistent with our earlier findings for the baseline model with a single growth option as the firm trades off the benefits and costs of using external funds or internal liquidity accumulation.

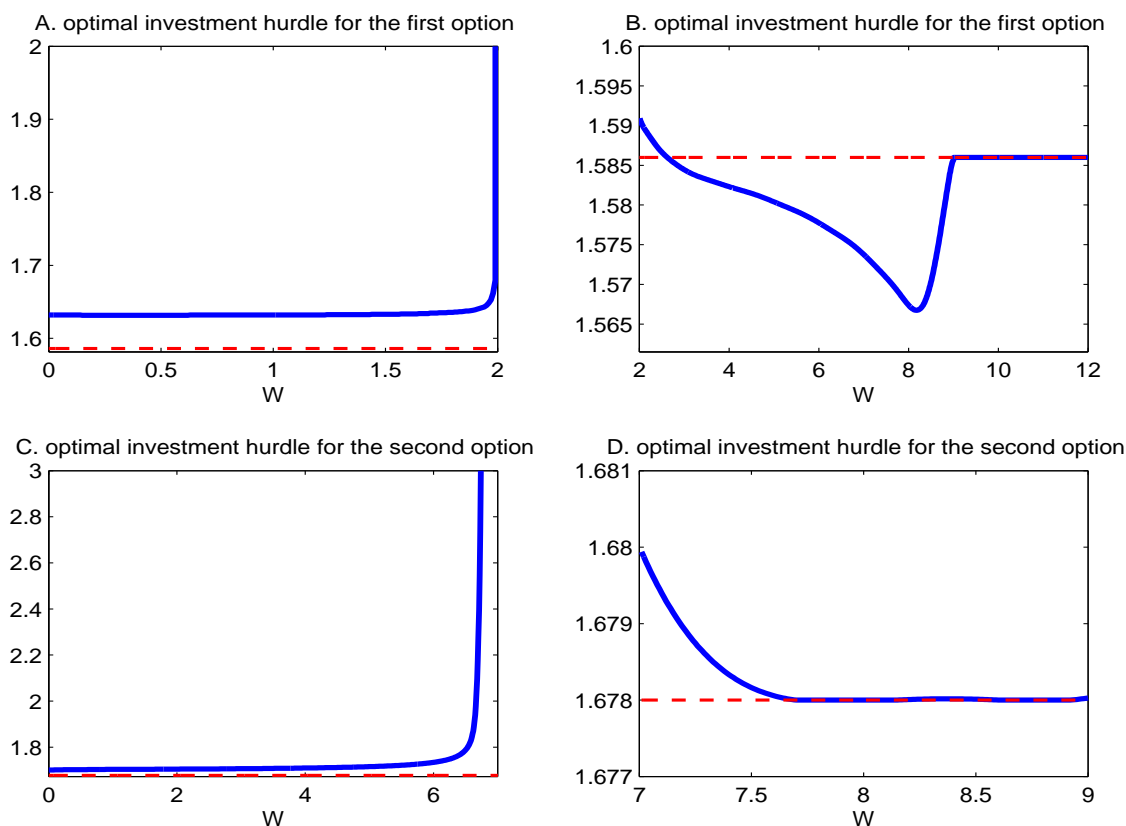


Figure 9: The optimal investment threshold $\bar{Y}_1(W)$ and $\bar{Y}_2(W)$ for a financially constrained firm in stage 0 and 1, respectively.

Importantly, Panel B shows that the investment threshold $\bar{Y}_1(W)$ is also non-monotonic in the region $W \geq I_1 = 2$ where the firm does not need external funds to finance the investment cost $I_1 = 2$ for the first growth option exercise. Surprisingly, the firm may *overinvest* by exercising its first growth option before the first-best level $Y_{1,a}^*$. Intuitively, the firm accelerates the timing of its first growth option in order to accumulate internal funds from the cash flows generated by the first asset in place. By using the internally generated cash flows from the first asset in place, the firm may finance the investment cost $I_2 = 7$ for the second growth option and also provide additional liquidity to cover the firm's current and future operating costs. This is an inter-temporal tradeoff in the interest of shareholders. By over-investing in the first growth option, the firm gains overall because the benefit of relaxing the financial constraint for the second growth option outweighs the cost of accelerating the first growth option.

Panels C and D of Figure 9 report similar patterns to those we see in our baseline model

with one growth option. That is, the investment threshold $\bar{Y}_2(W)$ is higher than the first-best level and it is non-monotonic in the firm's liquidity W .

9 Conclusion

We develop a theory of real options for a financially constrained firm to analyze how this firm dynamically manages its real and financial flexibility facing costly external financing. External financing costs create a demand for internal funds, which in turn significantly influence a financially constrained firm's investment and abandonment option decisions. As our analysis reveals, there are rich interactions between the firm's option exercising decisions and its sources of funds, including external, internal, current and future expected earnings. For example, when the firm is able to accumulate more internal funds it may well be induced to slow down its investment even further as the prospect of entirely avoiding external funding costs gets closer. Additionally, we show that a financially constrained firm may have risk-seeking motive (as the firm's value may be convex in its liquidity in certain regions), which is in contrast to the precautionary savings demand as predicted by standard corporate savings models. Finally, we find that with multiple rounds of growth options, a financially constrained firm may over-invest in earlier stages with the hope of accumulating liquidity faster in order to mitigate under-investment problems in the future. These results showcase the subtle interaction between a firm's financing and real flexibility.

Our framework is both parsimonious and sufficiently realistic to provide quantitatively plausible estimates of the value of a firm's real options, its marginal value of cash, and to provide an accurate characterization of the firm's optimal investment and external funding policy. For the sake of brevity we have left some important features out of our analysis, such as allowing for a credit line, risk management opportunities, and term debt. We plan to extend our analytical framework to incorporate these features in future work.

Appendices

Before solving for the financially constrained firm's value and optimal investment decisions in the setting with multiple growth options, it is helpful to characterize the first-best benchmark where the firm faces no external funding costs.

A The First-Best Benchmark

First, consider the firm's value in the mature phase.

A.1 The Mature Phase

The firm's value is:

$$P^*(W, Y) = Q^*(Y) + W. \quad (\text{A.1})$$

The option value $Q^*(Y)$ is the solution of the following ODE:

$$rQ(Y) = Y - Z + \mu Y Q'(Y) + \frac{\sigma^2 Y^2}{2} Q''(Y), \quad Y > Y_a^*, \quad (\text{A.2})$$

with the standard value-matching and smooth-pasting conditions:

$$Q(Y_a^*) = 0, \quad (\text{A.3})$$

$$Q'(Y_a^*) = 0. \quad (\text{A.4})$$

It is immediate to verify that the value $Q^*(Y)$ admits the following unique closed-form solution:

$$Q^*(Y) = \left(\frac{Y}{r - \mu} - \frac{Z}{r} \right) + \left(\frac{Y}{Y_a^*} \right)^\gamma \left(\frac{Z}{r} - \frac{Y_a^*}{r - \mu} \right), \quad \text{for } Y \geq Y_a^*, \quad (\text{A.5})$$

where the abandonment hurdle Y_a^* is given by :

$$Y_a^* = \frac{\gamma}{\gamma - 1} \frac{r - \mu}{r} Z, \quad (\text{A.6})$$

and the constant γ is given by

$$\gamma = \frac{1}{\sigma^2} \left[- \left(\mu - \frac{\sigma^2}{2} \right) - \sqrt{\left(\mu - \frac{\sigma^2}{2} \right)^2 + 2r\sigma^2} \right] < 0. \quad (\text{A.7})$$

A.2 The Start-up Phase

We denote the *first-best* value of a financially unconstrained firm by $G^*(W, Y)$. As for the firm's value $P^*(W, Y)$ in the mature phase, the first-best value $G^*(W, Y)$ takes the following simple additive form:

$$G^*(W, Y) = H^*(Y) + W, \quad (\text{A.8})$$

where $H^*(Y)$ is the solution of the following ODE:

$$rH(Y) = \mu Y H'(Y) + \frac{\sigma^2 Y^2}{2} H''(Y), \quad (\text{A.9})$$

subject to the value-matching and smooth-pasting boundary conditions:

$$H(Y_i^*) = Q^*(Y_i^*) - I, \quad (\text{A.10})$$

$$H'(Y_i^*) = Q'^*(Y_i^*). \quad (\text{A.11})$$

Additionally, the growth option is worthless at the origin $Y = 0$ as it is an absorbing state for a GBM process, i.e. $H(0) = 0$.

It is immediate to verify that the optimal investment hurdle Y_i^* is the solution to the following equation:

$$(\beta - \gamma) \left(\frac{Y_i^*}{Y_a^*} \right)^\gamma \left(\frac{Z}{r} - \frac{Y_a^*}{r - \mu} \right) + (\beta - 1) \frac{Y_i^*}{r - \mu} - \beta \left(\frac{Z + rI}{r} \right) = 0, \quad (\text{A.12})$$

and that the option value $H^*(Y)$ has the following closed-form solution:

$$H^*(Y) = \left(\frac{Y}{Y_i^*} \right)^\beta (Q^*(Y_i^*) - I), \quad \text{for } Y \leq Y_i^*, \quad (\text{A.13})$$

where β is a constant given by

$$\beta = \frac{1}{\sigma^2} \left[- \left(\mu - \frac{\sigma^2}{2} \right) + \sqrt{\left(\mu - \frac{\sigma^2}{2} \right)^2 + 2r\sigma^2} \right] > 1. \quad (\text{A.14})$$

A.3 Two Rounds of Investment

Let $H^*(Y)$ denote the solution for the first-best enterprise value in stage 0, $Q_n^*(Y)$ the solutions for the enterprise values in respectively stages $n = 1, 2$. Also, let $Y_{n,i}^*$ and $Y_{n,a}^*$ denote the exercising thresholds of the n th growth options and optimal abandonment boundaries in stages $n = 1, 2$, respectively.

We can then write $Q_1^*(Y)$ as the solution to the following ODE:

$$rQ_1(Y) = (x_1 Y - Z_1) + \mu Y Q_1'(Y) + \frac{\sigma^2 Y^2}{2} Q_1''(Y), \quad (\text{A.15})$$

subject to the following value-matching and smooth-pasting boundary conditions for the investment and abandonment options:

$$Q_1(Y_{2,i}^*) = Q_2^*(Y_{2,i}^*) - I_2, \quad (\text{A.16})$$

$$Q_1'(Y_{2,i}^*) = Q_2'^*(Y_{2,i}^*), \quad (\text{A.17})$$

$$Q_1(Y_{1,a}^*) = 0, \quad (\text{A.18})$$

$$Q_1'(Y_{1,a}^*) = 0. \quad (\text{A.19})$$

Note that equation (A.15) for $Q_1(Y)$ is identical to equation (A.2) for $Q(Y)$ except for the first term, which reflects the fact that the firm is not operating at full scale and only has one productive asset with capacity x_1 and operating costs Z_1 . The upper boundary $Y_{2,i}^*$ at which the firm decides to scale up to full capacity is such that the growth-phase enterprise value $Q_1(Y_{2,i}^*)$ is just equal to the mature-phase enterprise value $Q_2^*(Y_{2,i}^*)$ net of investment costs I_2 .

As we have established in Section A.1, in the mature-phase, the firm's value is given by $Q_2^*(Y; Z_2) = Q^*(x_2Y; Z = Z_2)$ and hence

$$Q_2^*(Y) = \left(\frac{x_2Y}{r - \mu} - \frac{Z_2}{r} \right) + \left(\frac{Y}{Y_{2,a}^*} \right)^\gamma \left(\frac{Z_2}{r} - \frac{x_2Y_{2,a}^*}{r - \mu} \right), \quad \text{for } Y \geq Y_{2,a}^*, \quad (\text{A.20})$$

where

$$Y_{2,a}^* = \frac{\gamma}{\gamma - 1} \frac{r - \mu}{r} \frac{Z_2}{x_2}. \quad (\text{A.21})$$

After solving for $Q_2^*(Y)$ we move backwards to the growth phase and solve for $Q_1^*(Y)$. As can be readily verified, the firm's value $Q_1^*(Y)$ in the region $Y_{2,i}^* \geq Y \geq Y_{1,a}^*$ takes the following form:

$$Q_1^*(Y) = \left(\frac{x_1Y}{r - \mu} - \frac{Z_1}{r} \right) + \Psi_i(Y) \left(Q_2^*(Y_{2,i}^*) - I_2 + \frac{Z_1}{r} - \frac{x_1Y_{2,i}^*}{r - \mu} \right) + \Psi_a(Y) \left(\frac{Z_1}{r} - \frac{x_1Y_{1,a}^*}{r - \mu} \right), \quad (\text{A.22})$$

where

$$\Psi_i(Y) = \left(\frac{Y_{1,a}^{*\beta} Y^\gamma - Y_{1,a}^{*\gamma} Y^\beta}{Y_{1,a}^{*\beta} Y_{2,i}^{*\gamma} - Y_{1,a}^{*\gamma} Y_{2,i}^{*\beta}} \right) \quad \text{and} \quad \Psi_a(Y) = \left(\frac{Y_{2,i}^{*\gamma} Y^\beta - Y_{2,i}^{*\beta} Y^\gamma}{Y_{1,a}^{*\beta} Y_{2,i}^{*\gamma} - Y_{1,a}^{*\gamma} Y_{2,i}^{*\beta}} \right), \quad (\text{A.23})$$

where β and γ are given by (10) and (6), the positive and negative roots of the fundamental quadratic equation, and the first abandonment threshold $Y_{1,a}^*$ and the second investment threshold $Y_{2,i}^*$ satisfy the boundary conditions (A.16)-(A.19).

Finally, we can move backwards to the start-up phase and solve for $H^*(Y)$, which satisfies the ODE (A.9) and the boundary conditions:

$$H(Y_{1,i}^*) = Q_1^*(Y_{1,i}^*) - I_1, \quad (\text{A.24})$$

$$H'(Y_{1,i}^*) = Q_1'^*(Y_{1,i}^*). \quad (\text{A.25})$$

Again, it is straightforward to verify that $H^*(Y)$ takes the following form:

$$H^*(Y) = \left(\frac{Y}{Y_{1,i}^*} \right)^\beta (Q_1^*(Y_{1,i}^*) - I_1) \quad \text{for } Y \leq Y_{1,i}^*, \quad (\text{A.26})$$

and $Y_{1,i}^*$ solves the following implicit equation:

$$\beta [Q_1^*(Y_{1,i}^*) - I_1] = Q_1^*(Y_{1,i}^*) Y_{1,i}^*. \quad (\text{A.27})$$

B General Solution under External Financing Costs

B.1 Value Functions: The Case of One Round of Investment

The firm's value in the start-up phase is denoted by $G(W, Y)$ and in the mature phase by $P(W, Y)$. In the start-up phase, the firm chooses the optimal investment timing τ_i to maximize the value of the growth option by solving

$$G(W_t, Y_t) = \max_{\tau_i, F \geq 0} \mathbb{E}_t \left[e^{-r(\tau_i - t)} (P(W_{\tau_i} + F - I, Y_{\tau_i}) - (F + \Phi(F)) \mathcal{I}_{F > 0}) \right],$$

where τ_i is the *endogenous* investment timing, and $\mathcal{I}_{F > 0}$ is an indicator function which takes the value of one when $F > 0$ and zero otherwise. Recall that $P(W, Y)$ is the firm's value in the mature phase. To be able to invest, the firm must have total available funds $W + F$ that cover at least the investment outlay I , i.e., $W + F \geq I$.

In the mature phase, the firm chooses the optimal abandonment timing τ_L to maximize:

$$P(W_t, Y_t) = \max_{\tau_L, dC \geq 0} \mathbb{E}_t \left[- \int_t^{\tau_U \wedge \tau_L} e^{-r(s-t)} \mathcal{I}_{dC_s > 0} [dC_s + \Phi(dC_s)] \right. \\ \left. + e^{-r(\tau_U - t)} P^*(W_{\tau_U}, Y_{\tau_U}) \mathcal{I}_{\tau_U < \tau_L} + e^{-r(\tau_L - t)} W_{\tau_L} \mathcal{I}_{\tau_U > \tau_L} \right], \quad (\text{B.1})$$

where $P^*(W_t, Y_t)$ is the first-best firm value for a financially unconstrained firm given by (4), dC is the net equity issuance and $\Phi(dC)$ is the corresponding equity issuance costs, and $\mathcal{I}_{dC_s > 0}$ is an indicator function which takes the value of one when $dC_s > 0$ and zero otherwise. Note that we have two stopping times: τ_U is the stopping time that the firm accumulates sufficient liquidity such that it permanently becomes unconstrained and hence attains the first-best firm value $P^*(W_t, Y_t)$, and τ_L is the endogenous stochastic liquidation time at which the firm is liquidated and releases its cash W_{τ_L} to shareholders. $\mathcal{I}_{\tau_U < \tau_L}$ is an indicator function which takes the value of one when the firm becomes financially unconstrained $\tau_U < \tau_L$ and zero otherwise.

If liquidation is suboptimal, the firm must raise costly external financing to be able to continue operating the project should it run out of cash. That is, at any time s when the firm incurs operating losses, $Y_s < Z$, and also is out of cash, $W_s = 0$, it has to raise funds dC_s at least sufficient to cover operating losses, if it were to continue operations.

In Section 4, we showed that a firm can be in one of four liquidity regions in the mature phase: *i*) a financially unconstrained region (when W is sufficiently large or $Y \rightarrow \infty$); *ii*) an interior financially constrained region; *iii*) an equity issuance region, and; *iv*) a liquidation region. This naturally continues to be the case for the mature phase in our more general setting.

B.2 Value Functions: The Case of Sequential Investment Opportunities

In the mature phase, the firm now chooses the optimal abandonment timing τ_{L_2} to maximize:

$$P^{(2)}(W_t, Y_t) = \max_{\tau_{L_2}, dC \geq 0} \mathbb{E}_t \left[- \int_t^{\tau_U \wedge \tau_{L_2}} e^{-r(s-t)} \mathcal{I}_{dC_s > 0} [dC_s + \Phi(dC_s)] \right. \\ \left. + e^{-r(\tau_U - t)} P^*(W_{\tau_U}, x_2 Y_{\tau_U}) \mathcal{I}_{\tau_U < \tau_{L_2}} + e^{-r(\tau_{L_2} - t)} W_{\tau_{L_2}} \mathcal{I}_{\tau_U > \tau_{L_2}} \right]. \quad (\text{B.2})$$

In the growth phase, the firm now chooses the optimal abandonment timing τ_{L_1} and investment timing decision τ_{i_2} to maximize:

$$P^{(1)}(W_t, Y_t) = \max_{\tau_{i_2}, \tau_{L_1}, F, dC \geq 0} \mathbb{E}_t \left[- \int_t^{\tau_{i_2} \wedge \tau_{L_1}} e^{-r(s-t)} \mathcal{I}_{dC_s > 0} [dC_s + \Phi(dC_s)] \right. \\ \left. + e^{-r(\tau_{i_2} - t)} [P^{(2)}(W_{\tau_{i_2}} + F - I_2, x_2 Y_{\tau_{i_2}}) - (F + \Phi(F)) \mathcal{I}_{F > 0}] \mathcal{I}_{\tau_{i_2} < \tau_{L_1}} \right. \\ \left. + e^{-r(\tau_{L_1} - t)} W_{\tau_{L_1}} \mathcal{I}_{\tau_{i_2} > \tau_{L_1}} \right]. \quad (\text{B.3})$$

The firm's value in the start-up phase is as above given by $G(W, Y)$. The firm chooses the optimal investment timing τ_{i_1} to maximize:

$$G(W_t, Y_t) = \max_{\tau_{i_1}, F \geq 0} \mathbb{E}_t \left[e^{-r(\tau_{i_1} - t)} (P^{(1)}(W_{\tau_{i_1}} + F - I_1, x_1 Y_{\tau_{i_1}}) - (F + \Phi(F)) \mathcal{I}_{F > 0}) \right]. \quad (\text{B.4})$$

B.3 The Mature Phase

The solution for the firm's value in the mature phase, $P^{(2)}(W, Y; Z_2)$, is identical to $P(W, x_2 Y; Z = Z_2)$ given in Section 4. Note that the firm is permanently financially unconstrained in this region when its liquidity $W \geq \Lambda_2$ where Λ_2 is given by

$$\Lambda_2 \equiv \frac{Z_2 - x_2 Y_{2,a}^*}{r} \equiv \frac{r - \gamma \mu}{r^2(1 - \gamma)} Z_2. \quad (\text{B.5})$$

The analysis is a bit more complicated but remains intuitive in the growth phase as the firm has both a liquidation option and also the second growth option to exercise.

B.4 The Growth Phase

Again, the firm may be in one of these four regions in the growth phase (stage 1). Let Λ_1 denote the lowest level of liquidity such that the firm will not be inefficiently liquidated in the growth phase. As long as the firm's saving rate is positive at the first-best liquidation threshold $Y_{1,a}^*$, which may be equivalently written as $W \geq \Lambda_1$, where

$$\Lambda_1 \equiv \frac{Z_1 - x_1 Y_{1,a}^*}{r}, \quad (\text{B.6})$$

the firm will not be inefficiently liquidated in phase 1. However, for a firm to be permanently financially unconstrained, it shall distort neither its liquidation decision nor its (second) growth option exercising decision. For a firm to choose the first-best investment strategy for the second growth option and to incur no further distortions, it needs at least an amount of liquidity $\Lambda_2 + I_2$. This gives the following analytical characterization for the financially unconstrained region.

The Unconstrained Region: When the firm neither satisfies

$$W \geq \max\{\Lambda_1, I_2 + \Lambda_2\} \quad (\text{B.7})$$

or its revenues approaches infinity, i.e., $Y \rightarrow \infty$, the firm will be permanently financially unconstrained with probability one, and the value of the firm will be equal to the first-best value: $P^{(1)}(W, Y) = Q_1^*(Y) + W$.

The Financially Constrained Region: When $W < \max\{\Lambda_1, \Lambda_2 + I_2\}$, the firm's value $P^{(1)}(W, Y)$ satisfies the following HJB equation:

$$rP^{(1)}(W, Y) = (rW + x_1 Y_1 - Z_1)P_W^{(1)}(W, Y) + \mu Y P_Y^{(1)}(W, Y) + \frac{\sigma^2 Y^2 P_{YY}^{(1)}(W, Y)}{2}, \quad (\text{B.8})$$

subject to the following conditions:

$$P^{(1)}(W, \underline{Y}_1(W)) = W, \quad (\text{B.9})$$

and

$$P_Y^{(1)}(W, \underline{Y}_1(W)) = 0. \quad (\text{B.10})$$

Intuitively, (B.9) states that the value of abandonment is simply equal to the firm's cash released to shareholders upon liquidation and (B.10) gives the smooth pasting condition for the optimality of the endogenously chosen abandonment threshold $\underline{Y}_1(W)$. Additionally, the firm's value also satisfies the financing boundary condition at $W = 0$,

$$P^{(1)}(0, Y) = \begin{cases} P^{(1)}(F, Y) - F - \Phi(F), & \underline{Y}_1(0) < Y < Z_1, \\ 0, & Y \leq \underline{Y}_1(0), \end{cases} \quad (\text{B.11})$$

where the optimal amount of funds raised F (net of issuing costs) is given by:

$$P_W^{(1)}(F, Y) = 1 + \Phi'(F) = 1 + \phi_1, \quad \underline{Y}_1(0) < Y < Z_1. \quad (\text{B.12})$$

Finally, $P^{(1)}(W, Y)$ also satisfies a set of boundary conditions that characterize the optimal growth option exercising decisions. Let $F_g^{(2)}(W)$ denote the amount of external funds that the firm raises. Clearly, $F_g^{(2)}(W) \geq I_2 - W$ as $W < I_2$. The additional conditions needed to characterize the investment decisions are:

1. In the region $I_2 \leq W < \max\{\Lambda_1, I_2 + \Lambda_2\}$, the firm requires no external funds and indeed will not choose to tap external funds to finance its second growth option exercising cost. Hence, we have the standard value-matching condition,

$$P^{(1)}(W, \bar{Y}_2(W)) = P^{(2)}(W - I_2, \bar{Y}_2(W)), \quad (\text{B.13})$$

and the standard smooth-pasting condition,

$$P_Y^{(1)}(W, \bar{Y}_2(W)) = P_Y^{(2)}(W - I_2, \bar{Y}_2(W)). \quad (\text{B.14})$$

2. In the region $W < I_2$, at the growth option exercising boundary $\bar{Y}_2(W)$, we have

$$P^{(1)}(W, \bar{Y}_2(W)) = P^{(2)}(W + F_g^{(2)}(W) - I_2, \bar{Y}_2(W)) - F_g^{(2)}(W) - \Phi(F_g^{(2)}(W)), \quad (\text{B.15})$$

which is the standard value-matching condition, and

$$P_Y^{(1)}(W, \bar{Y}_2(W)) = P_Y^{(2)}(W + F_g^{(2)}(W) - I_2, \bar{Y}_2(W)), \quad (\text{B.16})$$

which is the standard smooth-pasting condition. Importantly, we pin down the amount of financing $F_g^{(2)}(W)$ via the following conditions:

$$P_W^{(2)}(W + F_g^{(2)}(W) - I_2, \bar{Y}_2(W)) \leq 1 + \Phi'(F_g^{(2)}(W)) \text{ and } F_g^{(2)}(W) \geq I_2 - W. \quad (\text{B.17})$$

As in the baseline case, the constraint $F_g(W) \geq I - W$ may bind as it may be optimal for the firm to rely solely on its ability to generate sufficient liquidity from operating earnings after it has exercised its second growth option. Note that the firm's marginal value of liquidity $P_W^{(2)}(W + F_g^{(2)}(W) - I_2, \bar{Y}_2(W))$ cannot exceed the marginal cost of financing at the moment of issuing equity to finance investment cost. Otherwise, the firm should issue more and keep the proceeds inside the firm as retained earnings.

B.5 The Start-up Phase

We now characterize the solution for the start-up phase for the corresponding three regions depending on the firm's liquidity W . In the financially unconstrained region, where

$$W \geq I_1 + \max\{\Lambda_1, I_2 + \Lambda_2\}, \quad (\text{B.18})$$

we have the first-best solution for the firm's option value:

$$G(W, Y) = H^*(Y) + W. \quad (\text{B.19})$$

In the financially constrained region where $W < I_1 + \max\{\Lambda_1, I_2 + \Lambda_2\}$ the firm's value $G(W, Y)$ solves the HJB equation:

$$rG(W, Y) = rWG_W(W, Y) + \mu Y G_Y(W, Y) + \frac{\sigma^2 Y^2}{2} G_{YY}(W, Y), \quad (\text{B.20})$$

subject to the following various boundary conditions. Let $F_g^{(1)}(W)$ denote the amount of external funds that the firm raises. Clearly, $F_g^{(1)}(W) \geq I_1 - W$ as $W < I_1$. The additional conditions needed to characterize the investment decisions are:

1. In the region $I_1 \leq W < I_1 + \max\{\Lambda_1, I_2 + \Lambda_2\}$, the firm issues no equity to finance the exercising cost of the first growth option. We have the standard value-matching and smooth-pasting conditions:

$$G(W, \bar{Y}_1(W)) = P^{(1)}(W - I_1, \bar{Y}_1(W)), \quad (\text{B.21})$$

$$G_Y(W, \bar{Y}_1(W)) = P_Y^{(1)}(W - I_1, \bar{Y}_1(W)). \quad (\text{B.22})$$

2. In the region $0 \leq W < I_1$, the firm has to raise external financing, denoted by $F_g^{(1)}(W)$, if it wants to exercise its first growth option. We thus have the following value-matching and smooth-pasting conditions:

$$G(W, \bar{Y}_1(W)) = P^{(1)}(W + F_g^{(1)}(W) - I_1, \bar{Y}_1(W)) - F_g^{(1)}(W) - \Phi(F_g^{(1)}(W)), \quad (\text{B.23})$$

$$G_Y(W, \bar{Y}_1(W)) = P_Y^{(1)}(W + F_g^{(1)}(W) - I_1, \bar{Y}_1(W)). \quad (\text{B.24})$$

To pin down the external financing amount, $F_g^{(1)}(W)$, we use the following conditions:

$$P_W^{(1)}(W + F_g^{(1)}(W) - I_1, \bar{Y}_1(W)) \leq 1 + \Phi'(F_g^{(1)}(W)) \text{ and } F_g^{(1)}(W) \geq I_1 - W. \quad (\text{B.25})$$

The argument for the optimality condition (B.25) for $F_g^{(1)}(W)$ is essentially the same as the optimality condition (B.17) for $F_g^{(2)}(W)$. Finally, $G(W, Y)$ also satisfies the absorbing barrier condition $G(W, 0) = W$.

References

- Abel, A.B., and J.C. Eberly, 1994. A unified model of investment under uncertainty. *American Economic Review* 84, 1369–1384.
- Aiyagari, S. R., 1994. Uninsured idiosyncratic risk and aggregate saving. *Quarterly Journal of Economics* 99, 659–684.
- Altinkilic, O., and R.S. Hansen, 2000. Are there economies of scale in underwriting fees? Evidence of rising external financing costs. *Review of Financial Studies* 13, 191–218.
- Anderson, R., and A. Carverhill, 2012. Corporate Liquidity and Capital Structure. *Review of Financial Studies* 20, 1429–1460.
- Asvanunt, A., M. Broadie, and S. Sundaresan, 2007. Growth Options and Optimal Default under Liquidity Constraints: The Role of Corporate Cash Balances. Columbia Business School Research Paper.
- Babenko I., and Y. Tserlukevich, 2013. Embracing Risk: Hedging Policy for Firms with Real Options, working paper, Arizona State University.
- Black, F., and M. Scholes, 1973. The pricing of options and corporate liabilities. *Journal of Political Economy* 81, 637–659.
- Bolton, P., H. Chen, and N. Wang, 2011. A unified theory of Tobin's q , corporate investment, financing, and risk management. *Journal of Finance* 66, 1545–1578.
- Bolton, P., and D. Scharfstein, 1990. A theory of predation based on agency problems in financial contracting. *American Economic Review* 80, 93–106.
- Boot, A.W.A., and V. N. Vladimirov, 2014. Entrepreneurial Vision, Information, and Cash. Working paper. University of Amsterdam.
- Boyle, G., and G. Guthrie, 2003. Investment, Uncertainty, and Liquidity. *Journal of Finance* 58, 2143–2166.
- Brennan, M.J., and E. Schwartz, 1985. Evaluating natural resource investments. *Journal of Business* 58, 135–157.
- Copeland, T., and A. Lyasoff, 2013. Seeking q : the Marginal Efficiency of Liquidity and its Effect on Investment Financing and Valuation. Working paper. University of San Diego and Boston University.
- DéCamps, J.P., T. Mariotti, J.C. Rochet, and S. Villeneuve, 2011. Free cash flow, issuance costs and stock prices. *Journal of Finance* 66, 1501–1544.

- DéCamps, J.P., and S. Villeneuve, 2007. Optimal dividend policy and growth option. *Finance and Stochastics* 11, 3–27.
- DeMarzo, P.M., M.J. Fishman, Z. He, and N. Wang, 2012. Dynamic agency and the q theory of investment. *Journal of Finance* 67, 2295–2340.
- DeMarzo, P.M., and Y. Sannikov, 2006. Optimal security design and dynamic capital structure in a continuous-time agency model. *Journal of Finance* 61, 2681–2724.
- DeMarzo, P.M., and M.J. Fishman, 2007. Optimal long-term financial contracting. *Review of Financial Studies* 20, 2079–2128.
- Dixit, A.K., 1989. Entry and exit decisions under uncertainty. *Journal of Political Economy* 97, 620–638.
- Dixit, A.K., and R.S. Pindyck, 1994. *Investment Under Uncertainty*. Princeton University Press, Princeton, NJ.
- Fazzari, S., G.R. Hubbard, and B. Petersen, 1988. Finance constraints and corporate investment. *Brookings Papers on Economic Activity* 1, 141–195.
- Froot, K.A., D.S. Scharfstein, and J.C. Stein, 1993. Risk Management: Coordinating Corporate Investment and Financing Policies. *The Journal of Finance* 48, 1629–1658.
- Goldstein, R., Ju, N., and H. Leland, 2001. An EBIT-based model of dynamic capital structure. *Journal of Finance* 74, 483–512.
- Graham, J.R., and C. Harvey, 2001. The Theory and Practice of Corporate Finance: Evidence from the Field. *Journal of Financial Economics* 60, 187–243.
- Graham, J.R., and C. Harvey, 2002. How Do CFOs Make Capital Budgeting and Capital Structure Decisions?. *Journal of Applied Corporate Finance* 15, 8–23.
- Grenadier, S.R., 1996. The strategic exercise of options: development cascades and overbuilding in real estate markets. *Journal of Finance* 51, 1653–1679.
- Grenadier, S.R., and N. Wang, 2005. Investment timing, agency, and information. *Journal of Financial Economics* 75, 493–533.
- Grenadier, S.R., and N. Wang, 2007. Investment under uncertainty and time-inconsistent preferences. *Journal of Financial Economics* 84, 2–39.
- Hennessy, C.A., and T.M. Whited, 2007. How costly is external financing? Evidence from a structural estimation. *Journal of Finance* 62, 1705–1745.
- Hugonnier, J., S. Malamud, and E. Morellec, 2014. Capital supply uncertainty, cash holdings, and investment. Working paper. Swiss Finance Institute.

- Kaplan, S.N., and L. Zingales, 1997. Do investment-cash flow sensitivities provide useful measures of financing constraints? *Quarterly Journal of Economics* 112, 169–215.
- Lambrecht, B.M., and W.R. Perraudin, 2003. Real options and preemption under incomplete information. *Journal of Economic Dynamics and Control* 27, 619–643.
- Lambrecht, B.M., and S. C. Myers, 2007. A theory of takeovers and disinvestment. *Journal of Finance*. 62, 809–845.
- Lambrecht, B.M., and S. C. Myers, 2008. Debt and managerial rents in a real-options model of the firm. *Journal of Financial Economics* 89, 209–231.
- Leland, H. E., 1994. Corporate debt value, bond covenants, and optimal capital structure. *Journal of Finance* 49, 1213–1252.
- Majd, S., and R. Pindyck, 1987. Time to build, option value, and investment decision. *Journal of Financial Economics* 18, 7–28.
- Mauer, D., and A.J. Triantis, 1994. Interactions of Corporate Financing and Investment Decisions: A Dynamic Framework. *Journal of Finance* 49, 1253–1277.
- McDonald, R., and D. Siegel, 1986. The value of waiting to invest. *Quarterly Journal of Economics* 101, 707–727.
- Merton, R.C., 1973. Theory of rational option pricing. *Bell Journal of Economics and Management Science* 4, 141–183.
- Miao, J., and N. Wang, 2007. Investment, consumption, and hedging under incomplete markets. *Journal of Financial Economics* 86, 608–642.
- Miller, M. H., and D. Orr, 1966. A model of demand for money by firms. *Quarterly Journal of Economics* 80, 413–435.
- Morellec, E., 2004. Can managerial discretion explain observed leverage ratios?. *Review of Financial Studies* 17, 257–294.
- Rampini, A.A., and S. Viswanathan, 2010. Collateral, risk management, and the distribution of debt capacity. *Journal of Finance* 65, 2293–2322.
- Rampini, A.A., and S. Viswanathan, 2013. Collateral and capital structure. *Journal of Financial Economics* 109, 466–492.
- Titman, S., 1985. Urban land prices under uncertainty. *American Economic Review* 75, 505–514.
- Williams, J.T., 1991. Real estate development as an option. *Journal of Real Estate Finance and Economics* 4, 191–208.